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Functional Analysis and Partial Differential Equations

Sheet Nr.11 Due: 27.01.2017

Exercise 1

Let $U \subset \mathbb{R}^d$ be open and bounded and $f, g \in W^{1,2}(U; \mathbb{R})$ with $g \geq f$. We consider the obstacle problem

$$\begin{aligned} u =& g \qquad \text{on } \partial U \\ u \geq& f \qquad \text{in } U \\ -\Delta u = 0 \text{ or } u = f& \qquad \text{in } U \end{aligned}$$

We can encode it in a Sobolev setting in the following way. For $v \in W^{1,2}(U; \mathbb{R})$, we say $v \geq 0$ if

$$\int_U v\phi dm^d \geq 0$$

for all $\phi \in \mathcal{D}(U)$ with $\phi \geq 0$. Let

$$\mathcal{A} = \{u \in W^{1,2}(U; \mathbb{R}) : u - g \in W_0^{1,2}(U; \mathbb{R}), u - f \geq 0\}.$$

Prove that $\mathcal{A}$ is convex and nonempty, and furthermore there exists a unique minimizer of

$$E(u) = \int |\nabla u|^2 dm^d$$

in $\mathcal{A}$.

Exercise 2

Find and sketch the unique solution $u \in C^1([-1, 1])$ to

$$\begin{aligned} u(x) =& 0 \qquad \text{for } x = \pm 1 \\ u(x) \geq& 1 - 2x^2 \qquad \text{in } (-1, 1) \\ -u''(x) = 0 \text{ or } u = 1 - 2x^2& \qquad \text{in } (-1, 1). \end{aligned}$$

Show that this u is the unique minimizer in Exercise 1.

Exercise 3

Find a different proof for Theorem 5.17 relying on Lemma 3.18 instead of Hahn-Banach for the normed spaces $L^p(\mu)$ and $W^{1,p}(U)$, $1 < p < \infty$.

Exercise 4

Let $U \subset \mathbb{R}^d$ be open and convex and suppose $F : U \to \mathbb{R}$ is convex. Let $x_0 \in U$. Prove that there exists $v \in \mathbb{R}^d$ so that

$$F(x) \geq F(x_0) + \sum_{j=1}^{d} v_j (x - x_0)_j$$

by applying Theorem 5.16 to the set

$$V = \{(t, x) : x \in U, t > F(x)\}.$$