Prof. Dr. Herbert Koch
Dr. Xian Liao
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universitätbonn

Functional Analysis and Partial Differential Equations

Sheet Nr.10 Due: 20.01.2017

Exercise 1

Let $U \subset \mathbb{R}^d$ be open. Prove the following assertions:

a) Let $f : U \to \mathbb{R}$ Lipschitz continuous. Then there exists a Borel set $A \subset U$ so that $m^d(U \backslash A) = 0$ so that f is differentiable in A.

b) f is locally Lipschitz continuous in U if and only if $f \in W^{1,\infty}_{loc}(U)$. If U is convex then the best Lipschitz constant is

$$\| |Df| \|_{L^\infty}.$$

Here locally means that it holds true on any open subset $V \subset U$ such that its closure $\overline{V}$ is a compact subset in U. For example, $f \in W^{1,\infty}_{loc}(U)$ means that $f \in W^{1,\infty}(V)$ for any V with the above property.

Hint: Consider first the bounded Lipschitz continuous functions in $\mathbb{R}^d$ and the characterization of $W^{1,\infty}(\mathbb{R}^d)$ by finite differences and Part II of Rademacher's theorem.

Exercise 2

Let $U \subset \mathbb{R}^d$ be open and suppose that $f \in C^1(\mathbb{R})$ has bounded derivative and either U is bounded or f satisfies $f(0) = 0$. Let $1 \le p \le \infty$ and $g \in W^{1,p}(U)$. Then $f \circ g \in W^{1,p}(U)$.
Hint: Consider first the case $U = \mathbb{R}^d$ and take use of Rademacher's theorem.

Exercise 3

We consider the equation

$$-\Delta u + |x|^2 u = f$$

on $\mathbb{R}^d$. Suppose that $(1 + |x|)^{-1} f \in L^2(\mathbb{R}^d)$. Prove that there exists a unique weak solution u which satisfies $u \in H^1(\mathbb{R}^d)$ and $|x|u \in L^2(\mathbb{R}^d)$. Moreover

$$\frac{d}{2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|xu\|^2_{L^2} \le 8 \left\| \left(\sqrt{\frac{d}{2}} + |x| \right)^{-1} f \right\|^2_{L^2}.$$

Here $|x|^2 = \sum_{j=1}^d |x_j|^2$, $\|\nabla u\|^2_{L^2} = \sum_{j=1}^d \|\partial_j u\|^2_{L^2}$ and $\|xu\|^2_{L^2} = \sum_{j=1}^d \|x_j u\|^2_{L^2}$.
Hint: Use the Lemma of Lax Milgram with a suitable Hilbert space. You will need the estimate

$$\frac{d}{2}\|u\|^2_{L^2} \le \|xu\|_{L^2} \|\nabla u\|_{L^2}.$$

Prove this estimate first for $d = 1$ and smooth functions.

Exercise 4

Let $d \geq 3$. We consider the equation

$$-\Delta u = f$$

on $\mathbb{R}^d$. Suppose that $f \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$. Prove that there exists a unique weak solution u which satisfies $u \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ and $Du \in L^2(\mathbb{R}^d)$. Moreover there exists c so that

$$\|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)} \leq c\|f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}.$$