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Functional Analysis and Partial Differential Equations

Sheet Nr.9 Due: 23.12.2016

Exercise 1

Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$. We can define $\phi * T(x) = T(\phi(x - \cdot)) \in C^\infty(\mathbb{R}^d)$. Prove that

- If $\psi \in L^1(\mathbb{R}^d)$ then
$$\phi * T_\psi(x) = \phi * \psi(x);$$
- If $\mathrm{supp}\phi = K_1$ and $\mathrm{supp}T = K_2$, then $\mathrm{supp}(\phi * T) \subset K_1 + K_2$;
- If we identify $\phi * T$ with its associated distribution, then it can be equivalently defined as
$$(\phi * T)(f) = T(\tilde{\phi} * f), \forall f \in \mathcal{D}(\mathbb{R}^d), \quad \tilde{\phi}(x) = \phi(-x).$$

Exercise 2

Let $S \in \mathcal{D}'(\mathbb{R}^d)$ with compact support and $T \in \mathcal{D}'(\mathbb{R}^d)$ and we define the convolution $S * T \in \mathcal{D}'(\mathbb{R}^d)$ as

$$(S * T)(f) = T(\tilde{S} * f), \quad \forall f \in \mathcal{D}(\mathbb{R}^d),$$

where $\tilde{S}(g) = S(\tilde{g})$, $g \in \mathcal{D}(\mathbb{R}^d)$. Formulate and prove the reasonable properties of the convolution of distributions.

Exercise 3

Prove the following statement: Let $U \subset \mathbb{R}^d$ be open and connected. If $T \in \mathcal{D}'(U)$ satisfies $\partial_{x_j} T = 0$ for $j = 1, \ldots, d$ then there exists $c \in \mathbb{K}$ so that

$$T(\phi) = c \int_U \phi \, dm^d = T_c(\phi), \quad \forall \phi \in \mathcal{D}(U).$$

Prove the following steps one by one:

a) If $\phi \in \mathcal{D}(U)$ satisfies $\int_U \phi dm^d = 0$ and $\mathrm{supp}\phi$ is convex, then there exists $F_j \in \mathcal{D}(U)$ with $\phi = \sum_{j=1}^d \partial_{x_j} F_j$.
Hint: Begin with $d = 1$ and use by induction with respect to d.

b) Remove the restriction that $\mathrm{supp}\phi$ is convex above.

c) $T(\phi) = 0$ if $\phi \in \mathcal{D}(U)$ such that $\int_U \phi dm^d = 0$.

d) Pick a fixed function $\phi_0 \in \mathcal{D}(U)$ with $\int \phi_0 dm^d = 1$. Then

$$T(\phi) = T(\phi - \int_U \phi dm^d \, \phi_0) + \int_U \phi dm^d \, T(\phi_0) = \int_U \phi dm^d \, T(\phi_0).$$

Exercise 4

Let $f \in W^{k,p}(U)$, $k \in \mathbb{N}$, $1 \leq p < \infty$. Prove that

- If $g \in C_b^k(U)$ then $gf \in W^{k,p}(U)$.
- If $\phi : V \to U$ is a C^k diffeomorphism then there exists $C > 1$ so that

$$\|f \circ \phi\|_{W^{k,p}(V)} \leq C\|f\|_{W^{k,p}(U)}.$$

- If $1 < p < \infty$, then the map J: $L^p(U \times \Sigma_k) \to W^{-k,p}(U)$ defined by

$$J((f_\alpha))(u) = \sum_{|\alpha| \leq k} \int_U f_\alpha \partial^\alpha u \, dm^d, \quad \forall u \in W_0^{k,p'}(U)$$

  has norm 1. Here we define the space $W^{-k,p}(U) = (W_0^{k,p'}(U))^*$.