Prof. Dr. Herbert Koch
Dr. Xian Liao
Winter Term 2016/2017

universitätbonn

Functional Analysis and Partial Differential Equations

Sheet Nr.7 Due: 09.12.2016

Exercise 1

Let μ be a Radon measure on $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$ be a Borel set. Prove that there exists a μ-zero measure set B such that the limit

$$\lim_{r\to 0} a_r, \quad a_r = \begin{cases} 0 & \text{if } \mu(B_r(x)) = 0 \\ \frac{\mu(A\cap B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) \neq 0 \end{cases}$$

exists, and furthermore, takes the values 1 and 0 outside B.

Exercise 2

Let (X, d) be σ compact. We define the space of signed measure $\mathcal{M}(X)$ with the elements $\nu \in \mathcal{M}(X)$ defined by

$$\nu(A) := \mu_+(A) - \mu_-(A),$$

where A is Borel set and $\mu_\pm$ are Radon measures so that $\mu_\pm(X) < \infty$.
Let $C_0(X;\mathbb{R}) \subset C_b(X,\mathbb{R})$ be the subspace of functions satisfying for all $\varepsilon > 0$ there exists a compact set K so that $\|f\|_{C_b(X\setminus K)} < \varepsilon$.
Prove that

a) The map

$$f \to \int_X f d\nu = \int_X f d\mu_+ - \int_X f d\mu_-$$

defines an element L in $C_0(X)^*$.

b) Let μ be the variation measure defined by

$$\mu(A) := \sup\{\int_A f d\nu : f \in C_0(X), \, |f| \leq 1\}$$

for Borel set A. Prove that μ coincides with the variation measure (after Caratheodory construction) of L in the proof of Theorem 3.42, and that $\mu - \nu$ and $\mu + \nu$ are Radon measures.

c) Prove that $\mu(X) = \|L\|_{C_0(X)^*}$.

d) Deduce that the signed measure space is a complete Banach space with respect to the norm $\|\nu\| = \mu(X)$.

Exercise 3

Which of the following measures are Borel measures resp. Radon measures?

a) The counting measure on $\mathbb{R}^d$, restricted to the Borel σ algebra.

b) The number of rational numbers in a subset of $\mathbb{R}$, restricted to Borel sets.

c) The measure

$$\mu(A) = \pi^{-d/2} \int_A e^{-|x|^2} dm^d(x).$$

d) The measure defined on the sphere $\mathbb{S}^d$

$$\mu(A) = \int_A f_0 d\sigma,$$

where f_0 is a fixed continuous function defined on $\mathbb{S}^d$ and σ is the surface measure ($=$ the d dimensional Hausdorff measure).

Prove your claims.

Exercise 4

Given $f \in L^1(\mathbb{R}^d; \mathbb{R})$, prove that for almost all x

$$\limsup_{r\to 0} \sup_{y\in B_r(x)} \frac{\Gamma(\frac{d+2}{2})}{\pi^{d/2}} r^{-d} \int_{B_r(y)} f\, dm^d = \liminf_{r\to 0} \inf_{y\in B_r(x)} \frac{\Gamma(\frac{d+2}{2})}{\pi^{d/2}} r^{-d} \int_{B_r(y)} f\, dm^d.$$