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Functional Analysis and Partial Differential Equations

Sheet Nr.6 Due: 02.12.2016

Exercise 1

Let (X, d) be a σ compact space. Prove that there exists a sequence of compact nonempty subsets X_n so that $X = \bigcup_n X_n$ and for all n

$$\mathrm{dist}(X \backslash X_{n+1}, X_n) > 0.$$

Deduce that X_n is contained in the interior of X_{n+1}. Here we take the convention that $\mathrm{dist}(\{\}, A) = +\infty$ for any nonempty set A.

Exercise 2

Let $U \subset \mathbb{R}^d$ be a bounded open set so that there is a Barrier function at each point of the boundary. Then, given $f \in C(\partial U)$, the Perron procedure produces a unique harmonic function $u \in C(\overline{U}) \cap C^2(U)$ which satisfies $u = f$ on ∂U. Let $x \in U$. Prove that there exists a unique Radon probability measure μ on ∂U so that

$$u(x) = \int_{\partial U} f d\mu.$$

Exercise 3+4

Consider the Hilbert space of the complex valued 2π-periodic measurable functions with norm

$$\|f\|_{L^2(T^1)} = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f|^2 dx\Big)^{\frac{1}{2}}$$

and the unique inner product $\langle ., . \rangle$ with this norm. Prove:

a) The integral kernel of the map $P_N : f \to \sum_{n=-N}^{N} \langle f, e^{inx} \rangle e^{inx}$ with respect to the Lebesgue measure is the Dirichlet kernel

$$k_N(x, y) = \frac{1}{2\pi} \frac{\sin((N + \frac{1}{2})(x - y))}{\sin(\frac{1}{2}(x - y))},$$

i.e.

$$P_N f(x) = \int_0^{2\pi} k_N(x, y) f(y) dy.$$

Sketch the function $x \to k_N(x, 0)$.

b) $\int_0^{2\pi} |k_N(x, 0)| dx \to \infty$ as $N \to \infty$. Compute $\|P_N\|_{HS}$. Hint: Don't use the formula for the integral kernel for the Hilbert Schmidt norm! Give an interpretation.

c) If $f \in C^1(\mathbb{R})$ is 2π periodic then

$$\|f - P_N f\|_{C_b} \to 0 \qquad \text{as } N \to \infty$$

d) (e^{inx}) is an orthonormal basis in L^2. The span is dense in L^p for $1 \leq p < \infty$.

e) Let $f = \sum_n a_n e^{inx}$ and $g = \sum_n b_n e^{inx}$ with (square) summable coefficients . Then

$$f(x)g(x) = \sum_{n\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{n-j} b_j \right) e^{inx}$$

and

$$f * g(x) := \int_0^{2\pi} f(x-y)g(y)dy = 2\pi \sum_{n\in\mathbb{Z}} a_n b_n e^{inx}.$$