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Winter Term 2016/2017

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Functional Analysis and Partial Differential Equations

Sheet Nr.5 Due: 25.11.2016

Exercise 1

Prove Collorary 3.20.

a) To show surjectivity consider first the case $\mu(X) < \infty$ and use that then every p integrable function is integrable.

b) Deduce the claim in the σ finite measure case.

Exercise 2

Provide precise formulations for the following assertions and prove them.

$$f * g = g * f, \tag{1}$$

$$(f * g) * h = f * (g * h), \tag{2}$$

$$\|f * g * h\|_{L^\infty} \leq \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}. \tag{3}$$

Let η be integrable with integral 1 and let $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(\varepsilon^{-1}x)$, $x \in \mathbb{R}^d$, $\varepsilon > 0$. Show that

$$f * \eta_\varepsilon \to f, \quad \varepsilon \to 0. \tag{4}$$

Exercise 3+4

A theorem of Weierstraß states the following:

Theorem *Let $a < b$, $f \in C([a,b])$ and $\varepsilon > 0$. Then there exists a polynomial p so that*

$$\|f - p\|_{C_b([a,b])} < \varepsilon.$$

Deduce that the Legendre polynomials (normalised by the factor $(n+1/2)^{1/2}$) are an orthonormal basis on $L^2([-1,1])$ and that the Hermite polynomials (normalised by the factor $(2^n n!\sqrt{\pi})^{-1/2}$) are an orthonormal basis in $L^2(\mu)$ with

$$\mu(A) = \int_A e^{-|x|^2} dm^1.$$