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Functional Analysis and Partial Differential Equations

Sheet Nr.4 Due: 18.11.2016

Exercise 1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p < r < q \leq \infty$.

a) Suppose that $f \in L^p(\mu) \cap L^q(\mu)$. Show that $f \in L^r(\mu)$.

b) Suppose that $g \in L^r(\mu)$. Construct $g_1 \in L^p(\mu)$ and $g_2 \in L^q(\mu)$ such that

$$\|g_1\|_{L^p} + \|g_2\|_{L^q} \leq 2\|g\|_{L^r}$$

and $g = g_1 + g_2$.

c) Suppose that $\mu(X) < \infty$. Prove that there exists a constant $C > 0$ so that

$$\|f\|_{L^p} \leq C\|f\|_{L^q} \text{ for all } f \in L^q.$$

d) Prove that

$$\|(x_j)\|_{l^q} \leq \|(x_j)\|_{l^p} \text{ for all } (x_j) \in l^p.$$

Exercise 2

(Tschebyscheff inequality). Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in L^p(\mu)$, $1 \leq p < \infty$. Show that for any positive number $t > 0$,

$$\mu\Big(\{x \in X \,:\, |f(x)| \geq t\}\Big) \leq \frac{1}{t^p} \int_X |f|^p \, d\mu.$$

Exercise 3

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ finite measure spaces. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $k(x, y)$ be $\mu \times \nu$ measurable such that

$$K := \Big\| \|k(x,y)\|_{L^q(\nu)} \Big\|_{L^p(\mu)} = \Big[\int_X \Big(\int_Y |k(x,y)|^q d\nu(y) \Big)^{p/q} d\mu(x) \Big]^{1/p} < \infty.$$

Show that the linear map

$$Tf(x) := \int_Y k(x,y) f(y) \, d\nu(y)$$

is welldefined for $f \in L^p(\nu)$ and it satisfies

$$T : L^p(\nu) \mapsto L^p(\mu) \text{ and } \|T\|_{L^p(\nu) \mapsto L^p(\mu)} \leq K.$$

Exercise 4

Let us take $p = q = 2$ in Exercise 3. Then the operators of the last exercise are called Hilbert-Schmidt operator. We define

$$\|T\|_{HS} = \|k\|_{L^2(\mu\times\nu)}.$$

Prove that

a) $\|T\|_{HS}$ defines a norm;

b) Let $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu})$ be a third σ finite measure space. Let $S : L^2(\mu) \mapsto L^2(\tilde{\mu})$ be another Hilbert-Schmidt operator defined by

$$Sf(\tilde{x}) = \int_X s(\tilde{x}, x) f(x) d\mu(x), \quad s(\tilde{x}, x) \in L^2(\tilde{\mu} \times \mu).$$

Prove that

$$\|ST\|_{HS} \leq \|S\|_{HS}\|T\|_{HS};$$

c) Let $H_1 = \mathbb{C}^{d_1}$ and $H_2 = \mathbb{C}^{d_2}$ be the Hilbert space with the inner products $\langle x, y\rangle = \sum_{j=1}^{d_l} x_j \overline{y_j}$, and let $\{e_j\}_{j=1}^{d_1}$ be an orthonormal basis of H_1. Let $T : H_1 \mapsto H_2$. Prove that

$$\|T\|_{HS} = \Big(\sum_{j=1}^{d_1} \|Te_j\|^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{d_1} \lambda_n^2\Big)^{\frac{1}{2}},$$

where $\lambda_n \geq 0$ and $\{\lambda_n^2\}_{n=1}^{d}$ are the eigenvalues (with multiplicity) of T^*T. The numbers (λ_n) different from 0 are called the singular values.