Analysis 1, Solutions to problem set 14

Problem 1 (Integration)

Symmetry considerations show that the area A in question equals twice the area of the region

$$E_+ := \{(x,y) \in E : y \geq 0\}.$$

In the region E_+, the defining equation can be solved for y, yielding

$$E_+ = \Big\{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a \text{ and } 0 \leq y \leq b\sqrt{1-\frac{x^2}{a^2}}\Big\}.$$

It follows that

$$A = 2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}dx.$$

We simplify this integral via the substitution $x = at$:

$$A = 2ab\int_{-1}^{1}\sqrt{1-t^2}dt.$$

A further substitution $t = \sin u$ shows that

$$\int_{-1}^{1}\sqrt{1-t^2}dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u\,du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\frac{1+\cos 2u}{2}\Big)du = \frac{\pi}{2},$$

and so

$$A = \pi ab.$$

Remark. This formula is rather intuitive: start with a circle of radius b (and area πb^2) and stretch it by a factor a/b to obtain the desired ellipse. This should affect the area by the same factor; indeed, $\pi b^2(a/b) = \pi ab$.

Problem 2 (Partial fraction decomposition)

Start by noticing that the equation $x^4 + 1 = 0$ has the following four distinct (complex) solutions:

$$x_1 := e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}},\ x_2 := e^{\frac{3i\pi}{4}} = \frac{-1+i}{\sqrt{2}},\ x_3 := e^{\frac{5i\pi}{4}} = \frac{-1-i}{\sqrt{2}},\ x_4 := e^{\frac{7i\pi}{4}} = \frac{1-i}{\sqrt{2}}.$$

Further note that the polynomials

$$(x-x_1)(x-x_4) = x^2 - \sqrt{2}x + 1 \text{ and } (x-x_2)(x-x_3) = x^2 + \sqrt{2}x + 1$$

have real coefficients. As a consequence, we can easily calculate the partial fraction decomposition

$$\frac{1}{x^4+1} = \frac{\sqrt{2}x-2}{4(-x^2+\sqrt{2}x-1)} + \frac{\sqrt{2}x+2}{4(x^2+\sqrt{2}x+1)}.$$

We start by analyzing the first integrand, (a multiple of) which can be rewritten as

$$\frac{\sqrt{2}x-2}{-x^2+\sqrt{2}x-1} = -\frac{\sqrt{2}-2x}{\sqrt{2}(-x^2+\sqrt{2}x-1)} - \frac{1}{-x^2+\sqrt{2}x-1} \tag{1}$$

For the first summand on the right-hand side of (1), substitute $u = -x^2 + \sqrt{2}x - 1$. For the second summand on the right-hand side of (1), complete the square and then substitute $v = x - \frac{1}{\sqrt{2}}$. The resulting expression reads as follows:

$$\int \frac{\sqrt{2}x-2}{-x^2+\sqrt{2}x-1}dx = -\frac{\log(1-\sqrt{2}x+x^2) + 2\arctan(1-\sqrt{2}x)}{\sqrt{2}} + C. \tag{2}$$

Let us now focus on the second integrand, (a multiple of) which can be rewritten as

$$\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} = \frac{2x+\sqrt{2}}{\sqrt{2}(x^2+\sqrt{2}x+1)} + \frac{1}{x^2+\sqrt{2}x+1}. \tag{3}$$

For the first summand on the right-hand side of (3), use the substitution $u = x^2 + \sqrt{2}x + 1$. For the second summand on the right-hand side of (3), complete the square and then change variables $v = x + \frac{1}{\sqrt{2}}$. The resulting expression is

$$\int \frac{\sqrt{2}x + 2}{x^2 + \sqrt{2}x + 1} dx = \frac{\log(1 + \sqrt{2}x + x^2) + 2\arctan(1 + \sqrt{2}x)}{\sqrt{2}} + C. \tag{4}$$

Combining (2) and (4) one finally gets that

$$\int \frac{1}{x^4 + 1} dx = \frac{-2\arctan(1 - \sqrt{2}x) + 2\arctan(1 + \sqrt{2}x) - \log(1 - \sqrt{2}x + x^2) + \log(1 + \sqrt{2}x + x^2)}{4\sqrt{2}} + C.$$

Problem 3 (Stirling's formula, first part)

We follow the suggestion, and start by recalling the Wallis' integrals: for $n \in \mathbb{N}$, they are defined as

$$W_n := \int_0^{\frac{\pi}{2}} \sin^n(x) dx.$$

In Problem 3(b) of ÜB 11 we computed the value of the subsequence $\{W_{2k}\}$ exactly:

$$W_{2k} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-3}{2k-2} \cdot \frac{2k-1}{2k} \cdot \frac{\pi}{2} = \frac{(2k)!}{2^{2k}(k!)^2} \frac{\pi}{2}.$$

Integrating by parts twice as before, one shows that the recurrence relation

$$nW_n = (n-1)W_{n-2} \tag{5}$$

holds in general for every natural number $n \geq 2$. This implies that the sequence

$$\omega_n := (n+1)W_n W_{n+1}$$

does not depend on n. In particular,

$$\omega_n = \omega_0 = W_0 W_1 = \frac{\pi}{2}.$$

We now claim that $W_{n+1} \sim W_n$, in the sense that

$$\lim_{n \to \infty} \frac{W_{n+1}}{W_n} = 1. \tag{6}$$

To verify (6), start by noting that the sequence $\{W_n\}$ is decreasing in n since

$$W_n - W_{n+1} = \int_0^{\frac{\pi}{2}} \sin^n(x)(1 - \sin(x)) dx \geq 0.$$

(Recall that the function $x \mapsto \sin x$ is nonnegative on the interval $x \in [0, \pi/2]$.) In other words,

$$W_{n+2} \leq W_{n+1} \leq W_n, \text{ for every } n \in \mathbb{N}.$$

Since $W_n > 0$, this is equivalent to

$$\frac{W_{n+2}}{W_n} \leq \frac{W_{n+1}}{W_n} \leq 1,$$

which in light of identity (5) can be rewritten as

$$\frac{n+1}{n+2} \leq \frac{W_{n+1}}{W_n} \leq 1.$$

The claimed asymptotic (6) now follows by squeezing. As a consequence,

$$\lim_{n \to \infty} nW_n^2 = \lim_{n \to \infty} \left(\frac{n}{n+1} (n+1) W_n W_{n+1} \frac{W_n}{W_{n+1}} \right) = \left(\lim_{n \to \infty} \frac{n}{n+1} \right) \left(\lim_{n \to \infty} \underbrace{(n+1) W_n W_{n+1}}_{= \omega_n = \frac{\pi}{2}} \right) \left(\frac{1}{\lim_{n \to \infty} \frac{W_{n+1}}{W_n}} \right) = \frac{\pi}{2}.$$

In particular, for $n = 2k$ we see that

$$\frac{(2k)!}{2^{2k}(k!)^2} \frac{\pi}{2} = W_{2k} \sim \sqrt{\frac{\pi}{4k}}. \tag{7}$$

Let us now suppose that

$$k! \sim L\sqrt{k}\frac{k^k}{e^k}$$

for some $0 < L < \infty$, as given by the assumptions of the problem. Plugging this into the left-hand side of (7), one gets that

$$\frac{L\sqrt{2k}\frac{(2k)^{2k}}{e^{2k}}}{2^{2k}(L\sqrt{k}\frac{k^k}{e^k})^2}\frac{\pi}{2} = \frac{1}{L}\frac{\pi}{\sqrt{2k}} \sim \sqrt{\frac{\pi}{4k}},$$

whence $L = \sqrt{2\pi}$, as desired.

Problem 4 (Gaussian Integral)

(a) $(a1)$ Letting $t/n = s \in (0, 1]$, the inequality in question is seen to be equivalent to

$$(1-s)^n \leq e^{-sn}$$

which after extraction of n-th roots (this is possible since $1 - s \in [0, 1)$) amounts to

$$1 - s \leq e^{-s}. \tag{8}$$

This last inequality holds for every $s \in \mathbb{R}$, as was already shown in Problem 2(a) of ÜB 9 via Bernoulli's inequality. Further proofs include Taylor expansion with remainder, and convexity via Jensen's inequality.

$(a2)$ Reasoning as in part (a1), the desired inequality is seen to be equivalent to

$$1 + s \leq e^s, \ \forall s \geq 0.$$

That this inequality holds for every $s \in \mathbb{R}$ is *equivalent* to inequality (8) holding in the same range.

(b) Letting $t = x^2$, we have from part $(a1)$ that, for every natural number $n \geq 1$ and real number $0 \leq x \leq \sqrt{n}$,

$$\left(1 - \frac{x^2}{n}\right)^n \leq e^{-x^2}.$$

Integrating from 0 to $\sqrt{n}$,

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx \leq \int_0^{\sqrt{n}} e^{-x^2} dx \leq \int_0^{\infty} e^{-x^2} dx.$$

On the other hand, part $(a2)$ implies the inequality

$$\int_0^{\infty} e^{-x^2} dx \leq \int_0^{\infty} \left(1 + \frac{x^2}{n}\right)^{-n} dx.$$

Combining these two inequalities,

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx \leq \int_0^{\infty} e^{-x^2} dx \leq \int_0^{\infty} \left(1 + \frac{x^2}{n}\right)^{-n} dx. \tag{9}$$

The first and the last integrals in this chain of inequalities can be expressed in terms of the Wallis' integrals $\{W_n\}$ from Problem 3. Indeed, the first one equals $\sqrt{n}W_{2n+1}$ as can be seen via the substitution $x = \sqrt{n}\cos t$; the last one equals $\sqrt{n}W_{2n-2}$, and the substitution $x = \sqrt{n}\cot t$ shows. We have already shown in the course Problem 3 that

$$\lim_{n\to\infty} \sqrt{n}W_n = \sqrt{\frac{\pi}{2}}.$$

By the squeeze theorem, it then follows from (9) that

$$\int_0^{\infty} e^{-x^2} dx = \lim_{n\to\infty} \sqrt{n}W_{2n+1} = \left(\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}}\right)\left(\lim_{n\to\infty} \sqrt{2n+1}W_{2n+1}\right) = \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}.$$

Problem 5 (Leibniz series)

Start by observing that

$$\int_\pi^x \cos kt \, dt = \frac{\sin kx}{k}.$$

On the other hand, from Problem 1 of ÜB 13 we already know that, for $0 < t < 2\pi$,

$$\sum_{k=1}^n \cos kt = \frac{\sin[(n+\frac{1}{2})t]}{2\sin(\frac{t}{2})} - \frac{1}{2}.$$

It follows that, for every $n \in \mathbb{N} \setminus \{0\}$ and $x \in (0, 2\pi)$,

$$\sum_{k=1}^n \frac{\sin kx}{k} = \sum_{k=1}^n \int_\pi^x \cos kt \, dt = \int_\pi^x \Big(\sum_{k=1}^n \cos kt \Big) dt = \int_\pi^x \Big(\frac{\sin[(n+\frac{1}{2})t]}{2\sin(\frac{t}{2})} - \frac{1}{2} \Big) dt.$$

Now, the integral

$$\int_\pi^x \frac{1}{2\sin(\frac{t}{2})} \sin\Big[\Big(n + \frac{1}{2}\Big)t\Big] dt \to 0 \text{ as } n \to \infty$$

in view of the Riemann-Lebesgue lemma proved in class (one just needs to check that the function $t \mapsto (2\sin(\frac{t}{2}))^{-1}$ is continuously differentiable on the interval $(0, 2\pi)$, the details of which are straightforward and therefore omitted). It follows that

$$\sum_{k=1}^\infty \frac{\sin kx}{k} = -\frac{1}{2} \int_\pi^x dt = \frac{\pi - x}{2}, \tag{10}$$

as desired. Leibniz formula amounts to the special case of (10) for $x = \frac{\pi}{2}$.

Problem 6 (Leibniz series, revisited)

From the lectures, we know that

$$\arctan'(t) = \frac{1}{1+t^2} = \sum_{n=0}^\infty (-1)^n t^{2n}.$$

The geometric series

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots + (-1)^n t^{2n} + \ldots$$

converges uniformly in every closed interval contained in $(-1, 1)$. (Caveat: however, it does *not* converge uniformly on the closed interval $[-1, 1]$ since it diverges at both of its endpoints.) Therefore we can integrate it term by term and get that, for $|r| < 1$,

$$\arctan r = \int_0^r \frac{dt}{1+t^2} = r - \frac{r^3}{3} + \frac{r^5}{5} - \ldots + (-1)^n \frac{r^{2n+1}}{2n+1} + \ldots \tag{11}$$

This is the Taylor expansion of the function $r \mapsto \arctan r$ on the interval $(-1, 1)$. What about the endpoints? Well, the series

$$\sum_{n=0}^\infty (-1)^n \frac{r^{2n+1}}{2n+1}$$

converges at $r = -1$ and at $r = 1$ by Leibniz criterion, which makes us think that the Taylor expansion (11) should hold in the *whole* interval $[-1, 1]$. That this is indeed the case in the content of one of Abel's theorem, whose proof is somewhat lengthy and technical (but very interesting and worth the extra study!). A more elementary justification is as follows. Start with the identity

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots + (-1)^n \frac{t^{2n}}{1+t^2}.$$

Integrate it from 0 to r (with $|r| \leq 1$) to get

$$\arctan r = r - \frac{r^3}{3} + \frac{r^5}{5} - \ldots + (-1)^{n-1} \frac{r^{2n-1}}{2n-1} + R_n(r),$$

where

$$R_n(r) := (-1)^n \int_0^r \frac{t^{2n}}{1+t^2} dt.$$

For $|r| \leq 1$, we get that

$$|R_n(r)| \leq \int_0^{|r|} t^{2n} dt = \frac{|r|^{2n+1}}{2n+1} \leq \frac{1}{2n+1}.$$

It follows that $\lim_{n\to\infty} R_n(r) = 0$ if $|r| \leq 1$, and so

$$\arctan r = r - \frac{r^3}{3} + \frac{r^5}{5} - \ldots + (-1)^n \frac{r^{2n+1}}{2n+1} + \ldots$$

for *every* $r \in [-1, 1]$. In particular, for $r = 1$, we again deduce Leibniz formula.