Problem 1 (Complex numbers)

 (a_1) Start by noting that

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{(1\cdot3-2\cdot4)+(1\cdot4+2\cdot3)i}{3^2+4^2} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i$$

It follows that

$$\Re\left(\frac{1+2i}{3-4i}\right) = \Re\left(-\frac{1}{5} + \frac{2}{5}i\right) = -\frac{1}{5}$$
$$\Im\left(\frac{1+2i}{3-4i}\right) = \Im\left(-\frac{1}{5} + \frac{2}{5}i\right) = \frac{2}{5},$$

and

$$\left|\frac{1+2i}{3-4i}\right| = \left|-\frac{1}{5} + \frac{2}{5}i\right| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{1^2+2^2}{5^2}} = \frac{1}{\sqrt{5}}.$$

 (a_2) Start by noting that

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1^2 - 2(1)(i) + (-i)^2}{1^2 + 1^2} = \frac{-2i}{2} = -i$$

On the other hand, the quantity $(-i)^n$ equals

1, -i, -1, or i

depending on whether n is congruent to 0, 1, 2 or $3 \pmod{4}$, respectively. Since

$$2015 = 4 \cdot 503 + 3,$$

we have that

$$\left(\frac{1-i}{1+i}\right)^{2015} = (-i)^{2015} = (-i)^{4\cdot 503+3} = (-i)^3 = i$$

a purely imaginary number. It follows that

$$\Re\left\{\left(\frac{1-i}{1+i}\right)^{2015}\right\} = 0, \ \Im\left\{\left(\frac{1-i}{1+i}\right)^{2015}\right\} = 1, \ \text{and} \ \left|\left(\frac{1-i}{1+i}\right)^{2015}\right| = 1.$$

 (a_3) This one is easy. By Euler's formula, we have that

$$\sqrt{i} = i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$

and so

$$\Re(\sqrt{i}) = \frac{1}{\sqrt{2}}, \ \Im(\sqrt{i}) = \frac{1}{\sqrt{2}}, \ \text{and} \ |\sqrt{i}| = 1.$$

 (a_4) Recall that, for every $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Plugging in z = i and recalling that $i^2 = -1$, this formula yields

$$\cos i = \frac{e^{i^2} + e^{-i^2}}{2} = \frac{e^{-1} + e}{2},$$

a real number. It follows that

$$\Re(\cos(i)) = \frac{e^{-1} + e}{2}, \Im(\cos(i)) = 0, \text{ and } |\cos(i)| = \frac{e^{-1} + e}{2}.$$

(b) Let $a, b, c, d \in \mathbb{R}$ be real numbers such that $\zeta = a + bi$ and $\eta = c + di$. Then $\overline{\zeta} = a - bi$ and

$$\begin{split} |1-\overline{\zeta}\eta|^2 - |\zeta-\eta|^2 &= |1-(a-bi)(c+di)|^2 - |(a+bi)-(c+di)|^2 \\ &= |(1-ac-bd) + (bc-ad)i|^2 - |(a-c) + (b-d)i|^2 \\ &= (1-ac-bd)^2 + (bc-ad)^2 - (a-c)^2 - (b-d)^2 \\ &= 1-a^2 - b^2 - c^2 - d^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (1-a^2-b^2)(1-c^2-d^2) \\ &= (1-|a+bi|^2)(1-|c+di|^2) \\ &= (1-|\zeta|^2)(1-|\eta|^2). \end{split}$$

 (c_1) Substituting $w = z^2$, we can start by solving the quadratic equation

$$w^2 - 2w + 4 = 0.$$

This has the solutions

$$w = \frac{2 \pm \sqrt{4 - 4 \cdot 4}}{2} = \frac{2 \pm 2\sqrt{1 - 4}}{2} = 1 \pm \sqrt{3}i.$$

It follows that the solutions z to the original equation satisfy $z^2 = 1 \pm \sqrt{3}i$. From this and from the fundamental theorem of algebra one concludes that the complete list of solutions to the original equation is

$$z_1 = \sqrt{1 + \sqrt{3}i}, \ z_2 = \sqrt{1 - \sqrt{3}i}, \ z_3 = -\sqrt{1 + \sqrt{3}i}, \ \text{and} \ z_4 = -\sqrt{1 - \sqrt{3}i}.$$

These can be put in normal form

$$z_1 = \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i, \ z_2 = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i, \ z_3 = -\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i, \ \text{and} \ z_4 = -\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i.$$

and represented (together with the unit circle) in the complex plane as follows:



(c₂) Write z = x + iy for some $x, y \in \mathbb{R}$. Then the equation $e^z = i$ can be rewritten as $e^x e^{iy} = i$, which in view of Euler's formula is equivalent to

 $(e^x \cos y) + (e^x \sin y)i = i.$

In particular,

 $e^x \cos y = 0$ and $e^x \sin y = 1$.

Since the exponential function is strictly positive on the whole real line, the first equation means that necessarily we must have $\cos y = 0$, which happens for real y if and only if

$$y = \frac{\pi}{2} + n\pi$$
, for some $n \in \mathbb{Z}$.

For such y,

$$\sin y = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n$$

and so for the second equation $e^x \sin y = 1$ to be fulfilled, one needs n to be an even integer, and x = 0. In other words, the candidates z to being a solution of the original equation $e^z = i$ are the numbers of the form

$$z_n = \left(\frac{\pi}{2} + 2n\pi\right)i, \quad (n \in \mathbb{Z}).$$
(1)

The first six elements of this sequence are depicted here (together with the unit circle):



It is straightforward to check that every element of the sequence $\{z_n\}$ defined in (1) is indeed a solution to the original equation, and we are done.

Problem 2 (Power series)

(a) We start by recognizing the first power series as a geometric series:

$$\sum_{n=1}^{\infty} 3^{\frac{n}{2}} e^{-n} x^n = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}x}{e}\right)^n.$$

This converges if

$$\left|\frac{\sqrt{3}x}{e}\right| < 1 \Leftrightarrow |x| < \frac{e}{\sqrt{3}}$$

and diverges if

$$\frac{\sqrt{3}x}{e} \Big| > 1 \Leftrightarrow |x| > \frac{e}{\sqrt{3}}$$

and so its radius of convergence equals $r_1 = e/\sqrt{3}$.

For the second example, we make use of the so-called ratio test, which states that the radius of convergence r of a power series $\sum_{n=1}^{\infty} a_n x^n$ equals

$$r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

provided this limit exists. We leave the proof of the ratio test as an exercise to the reader. In this case, we have that |(x+1)x+1| = 1

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!} = \left(1 + \frac{1}{n}\right)^n$$

which is a sequence that converges to e. Thus $r_2 = e$.

For the third and fourth examples, we make use of the Cauchy-Hadamard theorem, which states that the radius of convergence r of a power series $\sum_{n=1}^{\infty} a_n x^n$ is given by

$$r = \frac{1}{\limsup_{n \to \infty} |a_n|^{1/n}}$$

Sketch of the proof of the Cauchy-Hadamard theorem. Let $R := (\limsup_{n \to \infty} |a_n|^{1/n})^{-1}$. We will show that the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for |x| < R, and then that it diverges for |x| > R.

For the first part, let $\epsilon > 0$ be given. By definition of \limsup , there are only finitely many a_n for which

$$|a_n|^{1/n} \ge R^{-1} + \epsilon.$$

This means that

$$|a_n| < (R^{-1} + \epsilon)^n$$

for all but a finite number of a_n . In particular, by comparison with a geometric series, the power series $\sum_{n=1}^{\infty} a_n x^n$ is seen to be convergent for every $x \in \mathbb{R}$ satisfying $|x| < \frac{1}{R^{-1} + \epsilon}$. Since $\epsilon > 0$ can be chosen arbitrarily small, this means that the power series in question converges for |x| < R, as desired.

For the second part, let $\epsilon > 0$ again be given. By definition of \limsup ,

$$|a_n| \ge (R^{-1} - \epsilon)^r$$

for infinitely many a_n . It follows that, if $|x| = \frac{1}{R^{-1}-\epsilon} > R$, then the power series $\sum_{n=1}^{\infty} a_n x^n$ cannot converge because its *n*th term does not tend to 0.

For the third example,

$$|a_n|^{1/n} = \left(\frac{n^3|\sin n|}{3^n}\right)^{1/n} = \frac{n^{3/n}|\sin n|^{1/n}}{3}.$$

We already know that $n^{1/n} \to 1$ as $n \to \infty$, and so

$$\lim_{n \to \infty} n^{3/n} = (\lim_{n \to \infty} n^{1/n})^3 = 1^3 = 1.$$

On the other hand, $0 < |\sin n| \le 1$ for every $n \in \mathbb{N}_{>1}$. Moreover we have that

$$\sin x > \frac{1}{2}$$
 provided $\frac{\pi}{6} + 2\pi n < x < \frac{5\pi}{6} + 2\pi n$, for some $n \in \mathbb{Z}$.

This, together with the observation that, for each $n \in \mathbb{Z}$, the interval

$$\left[\frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n\right]$$

has length $2\pi/3 > 1$ and therefore contains an integer, allows us to show that

$$1 = \lim_{n \to \infty} \left(\frac{1}{2}\right)^{1/n} \le \limsup_{n \to \infty} |\sin n|^{1/n} \le 1$$

and therefore conclude that

$$\limsup_{n \to \infty} |\sin n|^{1/n} = 1.$$

It follows that

$$\limsup_{n \to \infty} |a_n|^{1/n} = \frac{1 \cdot 1}{3} = \frac{1}{3},$$

and, by Cauchy-Hadamard, $r_3 = 3$.

Remark. The sequence $\{|\sin n|^{1/n}\}$ actually converges (and its limit must therefore equal 1), but the proof of this fact is a bit trickier. The interested reader is urged to think about it. For the fourth example,

$$|a_n|^{1/n} = \left[n\left(1+\frac{3}{n}\right)^{n^2}\right]^{1/n} = n^{1/n}\left(1+\frac{3}{n}\right)^n.$$

As before, we have that $\lim_{n\to\infty} n^{1/n} = 1$. Moreover,

$$\lim_{n \to \infty} \left(1 + \frac{3}{n} \right)^n = e^3.$$

It follows that

$$\lim_{n \to \infty} |a_n|^{1/n} = 1 \cdot e^3 = e^3,$$

and, by Cauchy-Hadamard, $r_4 = e^{-3}$.

(b) Again by Cauchy-Hadamard we have that

$$r_a := (\limsup_{n \to \infty} |a_n|^{1/n})^{-1}$$
 and $r_b := (\limsup_{n \to \infty} |b_n|^{1/n})^{-1}$ and $r := (\limsup_{n \to \infty} |a_n b_n|^{1/n})^{-1}$

The desired inequality $r \geq r_a \cdot r_b$ is therefore equivalent to

$$(\limsup_{n \to \infty} |a_n b_n|^{1/n})^{-1} \ge (\limsup_{n \to \infty} |a_n|^{1/n})^{-1} \cdot (\limsup_{n \to \infty} |b_n|^{1/n})^{-1},$$

which can be rewritten as

$$(\limsup_{n \to \infty} |a_n|^{1/n}) \cdot (\limsup_{n \to \infty} |b_n|^{1/n}) \ge (\limsup_{n \to \infty} |a_n b_n|^{1/n}).$$

This follows from an elementary property about lim sup which we state but do not prove: if $\{\alpha_n\}$ and $\{\beta_n\}$ are arbitrary sequences of *nonnegative* real numbers, then

$$\limsup_{n \to \infty} (\alpha_n \beta_n) \le \left(\limsup_{n \to \infty} \alpha_n\right) \cdot \left(\limsup_{n \to \infty} \beta_n\right).$$

Note that strict inequality might occur in this last inequality: as a simple but instructive example, consider

$$\alpha_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \text{ and } \beta_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Problem 3 (Integration by parts)

(a) We have already proved that the product rule

$$(fg)' = f'g + fg' \tag{2}$$

holds in the case when both f and g are monotonically increasing, convex functions: this is the content of Theorem 8.2 from the Skript. In particular, (2) holds for *absolutely monotonic* functions, for these are in particular increasing (since the first derivative is nonnegative) and convex (since the second derivative is nonnegative).

We claim that (2) continues to hold if f, g are merely assumed to be real analytic as in the assumptions of the problem. This can be checked as follows: by definition, there exist absolutely monotonic functions f_1, f_2, g_1, g_2 such that

$$f = f_1 - f_2$$
 and $g = g_1 - g_2$

As such,

$$fg = (f_1 - f_2)(g_1 - g_2) = f_1g_1 - f_1g_2 - f_2g_1 + f_2g_2$$

and so

$$\begin{aligned} (fg)' &= (f_1g_1 - f_1g_2 - f_2g_1 + f_2g_2)' = (f_1g_1)' - (f_1g_2)' - (f_2g_1)' + (f_2g_2)' \\ &= (f_1'g_1 + f_1g_1') - (f_1'g_2 + f_1g_2') - (f_2'g_1 + f_2g_1') + (f_2'g_2 + f_2g_2') \\ &= (f_1'g_1 - f_1'g_2 - f_2'g_1 + f_2'g_2) + (f_1g_1' - f_1g_2' - f_2g_1' + f_2g_2') \\ &= (f_1' - f_2')(g_1 - g_2) + (f_1 - f_2)(g_1' - g_2') \\ &= (f_1 - f_2)'(g_1 - g_2) + (f_1 - f_2)(g_1 - g_2)' \\ &= f'g + fg', \end{aligned}$$

as desired.

Next, we can integrate both sides of identity (2) from a to b and obtain

$$\int_{a}^{b} (fg)'(x)dx = \int_{a}^{b} \left(f'(x)g(x) + f(x)g'(x) \right) dx.$$
(3)

Using the fundamental theorem of calculus,¹ we obtain for the left-hand side:

$$\int_{a}^{b} (fg)'(x)dx = f(b)g(b) - f(a)g(a).$$
(4)

The desired integration-by-parts formula follows from linearity of the integral together with identities (3) and (4).

 $^{^{1}}$ Caveat: the fundamental theorem of calculus as presented in class (Theorems 7.2 and 7.4 from the Skript) holds for monotonically increasing, right-continuous functions only. The adjustments to make it work for general real analytic functions are similar to what we did before, and are therefore omitted.

 (b_1) We integrate by parts once:

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \int_{-\pi}^{\pi} \left(\frac{\sin(nx)}{n}\right)' \cos(mx) dx$$

= $\frac{\sin(n\pi)}{n} \cos(m\pi) - \frac{\sin(-n\pi)}{n} \cos(-m\pi) - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n} (\cos(mx))' dx$
= $\frac{0}{n} (-1)^m - \frac{0}{n} (-1)^{-m} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n} (-m\sin(mx)) dx$
= $\frac{m}{n} \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx,$

and then once again:

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = \int_{-\pi}^{\pi} \left(-\frac{\cos(nx)}{n} \right)' \sin(mx) dx$$
$$= 0 - 0 + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} (\sin(mx))' dx$$
$$= \frac{m}{n} \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx$$

to conclude that

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \left(\frac{m}{n}\right)^2 \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx.$$

From this identity it immediate that

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = 0 \text{ unless } m = n.$$

If m = n = 0, then

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \int_{\pi}^{\pi} \underbrace{\cos 0}_{=1} \underbrace{\cos 0}_{=1} dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

Finally, if $m = n \neq 0$, then

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \int_{-\pi}^{\pi} \cos^2(nx) dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2nx)}{2} dx = \pi.$$

Here we used the trigonometric identity, valid for any real t,

$$\cos^2 t = \frac{1 + \cos 2t}{2},$$

and the fact that, for any $k \in \mathbb{N}_{\geq 1}$,

$$\int_{-\pi}^{\pi} \cos(kt) dt = \int_{-\pi}^{\pi} \left(\frac{\sin(kt)}{k}\right)' dt = \frac{\sin(k\pi)}{k} - \frac{\sin(-k\pi)}{k} = 0 - 0 = 0.$$

All in all we have that

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \begin{cases} 2\pi & \text{if } n = m = 0, \\ \pi & \text{if } n = m \neq 0, \\ 0 & \text{if } n \neq m. \end{cases}$$

This innocent-looking formula is a possible starting point for the study of Fourier analysis. (b_2) Our goal is to compute

$$I_n := \int_0^{\frac{\pi}{2}} \sin^{2n}(x) dx$$

for $n \in \mathbb{N}$. Start by computing

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0(x) dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

and

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^2(x) dx = \int_0^{\frac{\pi}{2}} \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} \cdot \frac{\pi}{2}$$

and

$$I_{2} = \int_{0}^{\frac{\pi}{2}} \sin^{4}(x) dx = \int_{0}^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right)^{2} dx = \int_{0}^{\frac{\pi}{2}} \frac{1 - 2\cos(2x) + \cos^{2}(2x)}{4} dx$$
$$= \int_{0}^{\frac{\pi}{2}} \frac{1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2}}{4} dx = \int_{0}^{\frac{\pi}{2}} \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Hopefully we start to see a pattern emerging. We might be tempted to conjecture that, for higher values of n,

$$I_n = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.$$
 (5)

To verify our conjecture, let us integrate I_n by parts once:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n}(x) dx = \int_0^{\frac{\pi}{2}} \sin(x) \sin^{2n-1}(x) dx = \int_0^{\frac{\pi}{2}} (-\cos(x))' \sin^{2n-1}(x) dx$$
$$= \int_0^{\frac{\pi}{2}} \cos(x) (\sin^{2n-1}(x))' dx = (2n-1) \int_0^{\frac{\pi}{2}} \cos^2(x) \sin^{2n-2}(x) dx.$$
(6)

This seems not to be quite enough, and so one might be tempted to integrate by parts again. However, if one just recalls the mother of all trigonometric identities,

$$\sin^2(x) + \cos^2(x) = 1,$$

valid for every $x \in \mathbb{R}$, one sees that (6) can be rewritten as

$$I_n = (2n - 1)(I_{n-1} - I_n)$$

or equivalently

$$I_n = \frac{2n-1}{2n} I_{n-1}.$$

The result (5) now follows by a straightforward induction on n, and can be equivalently rewritten as

$$I_n = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2}. \quad (n \in \mathbb{N})$$

This is known as (a special case of) Wallis' formula, and has a number of applications in classical analysis. Can you think of any?

Problem 4 (Mean value theorem and Cauchy's generalization)

(a) We provide a proof which does not minimize length but introduces and builds upon several useful results in analysis.

Let us proceed in five steps.

Step 1. Every bounded sequence $\{x_n\}$ in \mathbb{R} has a convergent subsequence.

Proof. By a slight variation of the Heine-Borel theorem seen in class (Theorem 3.11 from the Skript), the sequence $\{x_n\}$ has a monotonic subsequence. This subsequence is necessarily bounded as well. A bounded, monotonic sequence converges to a finite limit, and we are done.

Step 2. Let $g: [a, b] \to \mathbb{R}$ be a continuous function. Then g is bounded.

Proof. Aiming at a contradiction, suppose that the function g is not bounded from above on the interval [a, b]. This implies that, for every natural number $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $g(x_n) > n$. This defines a sequence $\{x_n\}$ taking values in [a, b]. Since [a, b] is bounded, Step 1 implies that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let x be its limit (which necessarily belongs to [a, b]). Since the function g is continuous at x, the sequence $\{g(x_{n_k})\}$ converges to the real number g(x). But $g(x_{n_k}) > n_k \ge k$ for every $k \in \mathbb{N}$, which implies that the sequence $\{g(x_{n_k})\}$ diverges to $+\infty$, a contradiction. The contradiction resulted from assuming that g is not bounded from above. A similar reasoning shows that g is bounded from below, and therefore g is bounded, as desired.

Step 3. If $h : [a,b] \to \mathbb{R}$ is a continuous function, then h must attain a maximum and a minimum (each at least once).

Proof. By Step 2, the function h is bounded from above. Let

$$M := \sup_{x \in [a,b]} h(x).$$

Our goal is to find $y \in [a, b]$ such that h(y) = M. Let $n \in \mathbb{N}$. Since M is the least upper bound for h on the interval [a, b], the number M - 1/n is not an upper bound for h on [a, b]. Therefore, there exists $y_n \in [a, b]$ such that

$$M - \frac{1}{n} < h(y_n).$$

This defines a sequence $\{y_n\}$ taking values in [a, b]. Since M is an upper bound for h on [a, b], we additionally have that, for every $n \in \mathbb{N}$,

$$M - \frac{1}{n} < h(y_n) \le M.$$

It follows that the sequence $\{h(y_n)\}$ converges to M. By Step 1, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to some $y \in [a, b]$. Since h is continuous at y, the sequence $\{h(y_{n_k})\}$ converges to h(y). But $\{h(y_{n_k})\}$ is a subsequence of $\{h(y_n)\}$, and the latter is already known to converge to M; therefore so does the former. It follows that M = h(y), which means that the function h attains its supremum. Proceed similarly for the infimum. \Box

Step 4. If $r : [a, b] \to \mathbb{R}$ is continuous on [a, b] and differentiable on (a, b) and such that r(a) = r(b), then there exists $c \in (a, b)$ such that r'(c) = 0.

Proof. By Step 3, the function r attains both its maximum and its minimum in [a, b]. If these are both attained at the endpoints of [a, b], then r is constant on [a, b] and so the derivative of r is identically zero, and we are done. Suppose now that the maximum is attained at an interior point $c \in (a, b)$. (The argument for the minimum is entirely analogous: one just considers the function -r instead of r.) For a real δ such that $c + \delta \in [a, b]$, the value of $r(c + \delta)$ is smaller than or equal to r(c) because r attains its maximum at c. Therefore, for every $\delta > 0$,

$$\frac{r(c+\delta) - r(c)}{\delta} \le 0,$$

and so

$$r'(c_+) := \lim_{\delta \to 0^+} \frac{r(c+\delta) - r(c)}{\delta} \le 0$$

(Note that this limit exists by our assumptions on the function r.) Similarly, for every $\delta < 0$, the inequality turns around because the denominator is now negative, and we get

$$r'(c_{-}) := \lim_{\delta \to 0^{-}} \frac{r(c+\delta) - r(c)}{\delta} \ge 0.$$

Since r is differentiable at c, we conclude that

$$r'(c) = r'(c_{-}) = r'(c_{+}) = 0.$$

Step 5. Conclude.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is continuously differentiable in the open interval (a, b). Consider the auxiliary function

$$s(x) := f(x) - cx,\tag{7}$$

where c is a constant to be chosen below. Since f is continuous on [a, b] and continuously differentiable on (a, b), the same is true for s. We now want to choose c in such a way that the function s satisfies the extra condition of Step 4 above. Namely,

$$s(a) = s(b) \Leftrightarrow f(a) - ca = f(b) - cb \Leftrightarrow c = \frac{f(b) - f(a)}{b - a}.$$

Let us take this choice of c. Since the function s satisfies all the assumptions of Step 4, we conclude the existence of a point $x_1 \in (a, b)$ for which $s'(x_1) = 0$. It follows from (7) that

$$f'(x_1) = s'(x_1) + c = 0 + c = \frac{f(b) - f(a)}{b - a},$$

and we are done.

Remark. Steps 1, 2, 3 and 4 go sometimes in the literature under the names of *Bolzano-Weierstrass* theorem, boundedness theorem, extreme value theorem and Rolle's theorem, respectively.

(b) This turns out to be an important generalization of part (a) - originally due to none other than our old friend August-Louis Cauchy - which can be proved in *almost the same way*.

Given functions f, g satisfying the conditions of the problem, define the auxiliary function

$$s(x) := f(x) - cg(x)$$

The constant c is chosen, as before, in such a way that s(a) = s(b). This happens if and only if

$$c = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

With this choice of c, the function s satisfies the hypotheses of Step 4 of part (a). In particular, there exists $x_2 \in (a, b)$ such that $s'(x_2) = 0$. This means that

$$0 = s'(x_2) = f'(x_2) - cg'(x_2) = f'(x_2) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)g'(x_2),$$

and so

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_2)}{g'(x_2)},$$

as desired.