Analysis 1, Solutions to problem set 1

Problem 1 (Systems of linear equations)

(a) Assume that x and y satisfy the system of equations

$$\begin{cases} 7x + 2y = 6 \\ -2x - 6y = 7 \end{cases}$$

We subtract $7x$ on both sides of the first equation to obtain

$$2y = 6 - 7x.$$

We then divide by 2 on both sides of the equation to obtain

$$y = 3 - \frac{7}{2}x.$$

Now we apply the substitution rule, inserting this expression for y into the second equation of the system. We obtain

$$-2x - 6\Big(3 - \frac{7}{2}x\Big) = 7.$$

Simplifying the left handside yields

$$19x - 18 = 7.$$

Adding 18 on both sides and dividing by 19 gives $x = \frac{25}{19}$. Inserting this into the above equation for y gives

$$y = 3 - \frac{7}{2}\frac{25}{19} = -\frac{61}{38}.$$

We conclude that if x and y satisfy the above system, then x and y are the values above. Conversely, one observes by inserting these values into each of the equations of the system that they solve the system.

(b) We apply the substitution rule and insert the expression for y given by the third equation into the first and the second equations, thus obtaining

$$\begin{cases} 4x + z = 3x \\ x = -2 + 6x + z \end{cases}$$

Subtracting $3x + z$ from both sides of the first equation yields

$$x = -z, \tag{1}$$

which is equivalent to $z = -x$ by multiplication of both sides by -1 and the symmetry of $=$. Again by the substitution rule, we insert (1) into the second equation of the original system thus obtaining

$$x = -2 + 6x - x,$$

which can be simplified to $-4x = -2$ by adding $-5x$ to both sides of the equation. Dividing both sides of the latter equation by -4 one gets that $x = 1/2$. The third equation of the original system then tells us that $y = 3x = 3/2$. Finally, we know from (1) that $z = -x = -1/2$. We conclude that if x, y and z satisfy the system above, then

$$(x, y, z) = \Big(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\Big),$$

and one can likewise check by inserting these values into each of the equations of the system that they solve it.

Problem 2 (Discovering irrationality)

(a) Aiming at a contradiction, let us assume that there exist an integer p and an odd integer q such that

$$\left(\frac{p}{q}\right)^2 = 2. \tag{2}$$

Multiplying by q^2 on both sides of the equation yields

$$p^2 = 2q^2.$$

From the representation $p^2 = 2q^2$ we conclude that the number p^2 is even.

We claim that p is even as well. We prove this claim also by contradiction. Assume p is odd, that is there exists an integer n such that

$$p = 2n + 1.$$

Then we obtain

$$p^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1.$$

The last representation shows that p^2 is odd. This is in contradiction to p^2 being even. We thus have proved the claim by contradiction and hence p is even.

Hence p is of the form $2m$ for some integer m. We obtain

$$(2m)^2 = 2q^2.$$

Simplifying and dividing by 2 on either side gives

$$2m^2 = q^2.$$

Similarly as above this first implies that q^2 is even and then that q is even. This is a contradiction. Hence

$$\left(\frac{p}{q}\right)^2 \neq 2,$$

as we wanted to show.

(b) We split the analysis into two cases:

Case 1. q is odd: this was the content of part (a).

Case 2. p is odd: we again proceed by contradiction. Assuming p is odd and $(p/q)^2 = 2$, we conclude as before that $p^2 = 2q^2$. But this means that p^2 is even, which implies that p is even as proved in part (a). This is in contradiction to the hypothesis (under which we are working) of p being odd.

(c) We split the analysis into three cases:

Case 1. q is odd: this was the content of part (a).

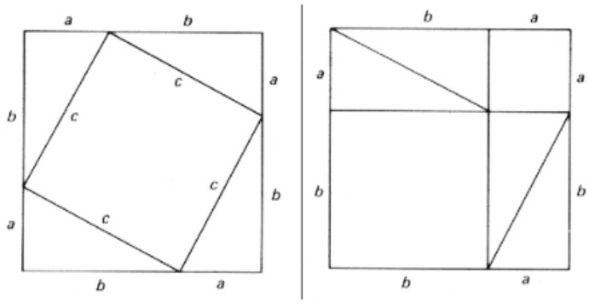
Case 2. q is even and p is odd: this was the content of part (b).

Case 3. q is even and p is even: this means that there exist positive integers p', q' such that $p = 2p'$ and $q = 2q'$. Note in particular that $p' < p$. By substitution and division by 2, it then follows that

$$\left(\frac{p}{q}\right)^2 = \left(\frac{2p'}{2q'}\right)^2 = \left(\frac{p'}{q'}\right)^2 \neq 2$$

by hypothesis. This concludes the proof.

(d) Here is a visual proof of Pythagoras theorem:



Problem 3 (Inequalities)

(a) We start by noticing that $-|x| \leq x \leq |x|$. This can be shown by noting that $|x| \geq 0$, and then splitting into cases according to whether $x \geq 0$, in which case $x = |x|$, or $x < 0$, in which case $x = -|x|$. Similarly, $-|y| \leq y \leq |y|$. Adding the two statements together, we have that

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

The desired conclusion then follows from the following observation: for any real number z and nonnegative number a, one has that:

$$\text{If} - a \leq z \leq a, \text{ then } |z| \leq a. \tag{3}$$

We prove this by considering again two cases. Start by observing that $-a \leq z \leq a$ implies (H1): $z \geq -a$ and (H2): $z \leq a$.

Case 1. If $z \geq 0$, then $|z| = z$, and then $|z| \leq a$ follows from the hypothesis (H2);

Case 2. If $z < 0$, then $|z| = -z$ and then $|z| \leq a$ is equivalent to $-z \leq a$, which in turn holds if and only if $z \geq -a$. This is precisely the content of the hypothesis (H1).

This establishes claim (3) and finishes the proof of the triangle inequality.

Alternative proof. Start by noticing that $|x| = \sqrt{x^2}$ for every real number x (this can again be proved by considering the two cases $x \geq 0$ and $x < 0$ separately). Knowing this, it suffices to show that $|x+y|^2 \leq (|x| + |y|)^2$, for afterwards one could take square roots. Let us compute:

$$|x+y|^2 = (x+y)(x+y) = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2.$$

This proof has the advantage that it sheds immediate light into the cases of equality. More precisely, $|x+y| = |x| + |y|$ if and only if $xy = |x||y| = |xy|$. By definition of absolute value, this happens if and only if $xy \geq 0$ i.e. x and y have the same sign.

(b) Without loss of generality, we may assume that $|x| \geq |y|$ (for if $|y| \geq |x|$, then we could just swap the role of x and y in the inequality). So it suffices to show that $|x| - |y| \leq |x - y|$. Write $x = (x - y) + y$ and observe that $|x| = |(x - y) + y|$. But $|(x - y) + y| \leq |x - y| + |y|$ by the triangle inequality from part (a), and so we have that $|x| \leq |x - y| + |y|$. Subtracting $|y|$ from both sides of this inequality, we are done.

(c) The first goal is to show that $x + \frac{1}{x} \geq 2$ for every positive real number $x > 0$. Note that

$$x + \frac{1}{x} \geq 2 \Leftrightarrow \frac{x^2+1}{x} \geq 2 \Leftrightarrow x^2 + 1 \geq 2x \Leftrightarrow x^2 - 2x + 1 \geq 0 \Leftrightarrow (x-1)^2 \geq 0,$$

where the last inequality is a trivial consequence of the fact, provided as a hint, that the square of every real number is nonnegative. This proof also makes transaparent the analysis of the cases of equality. Namely,

$$x + \frac{1}{x} = 2 \Leftrightarrow (x-1)^2 = 0 \Leftrightarrow x = 1.$$

(d) We have to show five different strict inequalities, all valid for arbitrary positive real numbers x, y satisfying $x < y$.

(i) $x < \frac{2}{\frac{1}{x}+\frac{1}{y}}$:

$$x < y \Leftrightarrow x+y < 2y \Leftrightarrow x(x+y) < 2xy \Leftrightarrow x < \frac{2xy}{x+y} \Leftrightarrow x < \frac{2}{\frac{1}{x}+\frac{1}{y}}.$$

(ii) $\frac{2}{\frac{1}{x}+\frac{1}{y}} < \sqrt{xy}$:

$$0 < (x-y)^2 = x^2-2xy+y^2 \Leftrightarrow 4xy < x^2+2xy+y^2 = (x+y)^2 \Leftrightarrow \frac{4xy}{(x+y)^2} < 1 \Leftrightarrow \left(\frac{2xy}{x+y}\right)^2 < xy \Leftrightarrow \frac{2}{\frac{1}{x}+\frac{1}{y}} < \sqrt{xy}.$$

(iii) $\sqrt{xy} < \frac{x+y}{2}$:

$$0 < (x-y)^2 = x^2-2xy+y^2 \Leftrightarrow 4xy < x^2+2xy+y^2 \Leftrightarrow xy < \frac{x^2+2xy+y^2}{4} \Leftrightarrow xy < \left(\frac{x+y}{2}\right)^2 \Leftrightarrow \sqrt{xy} < \frac{x+y}{2}.$$

(iv) $\frac{x+y}{2} < \sqrt{\frac{x^2+y^2}{2}}$:

$$0 < (x-y)^2 = x^2-2xy+y^2 \Leftrightarrow x^2+2xy+y^2 < 2x^2+2y^2 \Leftrightarrow \left(\frac{x+y}{2}\right)^2 < \frac{x^2+y^2}{2} \Leftrightarrow \frac{x+y}{2} < \sqrt{\frac{x^2+y^2}{2}}.$$

(v) $\sqrt{\frac{x^2+y^2}{2}} < y$:

$$x < y \Leftrightarrow x^2 < y^2 \Leftrightarrow x^2+y^2 < 2y^2 \Leftrightarrow \frac{x^2+y^2}{2} < y^2 \Leftrightarrow \sqrt{\frac{x^2+y^2}{2}} < y.$$

Problem 4 (Logic)

(a) Let us set $A =$ Akro, $D =$ Dapi, $F =$ Fipi, $G =$ Gluka, $K =$ Knull, and further set $i =$ ixen, $l =$ lüllen, $p =$ prameln, $u =$ urzen, $w =$ watzeln. We write "$S_1 \Leftrightarrow S_2$" if the statements S_1 and S_2 are equivalent. We use that for any statements X, Y we have $X \Rightarrow Y \Leftrightarrow \neg X \vee Y$ and de Morgan's law $\neg(X \wedge Y) \Leftrightarrow (\neg X) \vee (\neg Y)$. We can translate the five first sentences from the problem as follows:

$$\begin{aligned}(S1): \quad K \neq p \to (F = u \vee G = u) &\Leftrightarrow \neg(K \neq p) \vee (F = u \vee G = u)\\ &\Leftrightarrow (K = p) \vee (F = u) \vee (G = u)\end{aligned}$$

$$\begin{aligned}(S2): \quad K \neq i \to (D \neq l \to G = p) &\Leftrightarrow (K = i) \vee (D \neq l \to G = p)\\ &\Leftrightarrow (K = i) \vee (D = l) \vee (G = p)\end{aligned}$$

$$(S3): \quad A \neq u \to (F = p \vee F = w) \Leftrightarrow (A = u) \vee (F = p) \vee (F = w)$$

$$\begin{aligned}(S4): \quad (K \neq i \wedge D \neq i) \to G = u &\Leftrightarrow \neg(K \neq i \wedge D \neq i) \vee G = u\\ &\Leftrightarrow (K = i) \vee (D = i) \vee (G = u)\end{aligned}$$

$$\begin{aligned}(S5): \quad D \neq w \to (K \neq u \to F = l) &\Leftrightarrow D = w \vee (K \neq u \to F = l)\\ &\Leftrightarrow (D = w) \vee (K = u) \vee (F = l)\end{aligned}$$

Moreover we are given the information that each person did something, and that no two people did the same thing. This translates into the "action" statements

$$(S6) \quad A = i \vee A = l \vee A = p \vee A = u \vee A = w,$$
$$(S7) \quad D = i \vee D = l \vee D = p \vee D = u \vee D = w,$$
$$(S8) \quad F = i \vee F = l \vee F = p \vee F = u \vee F = w,$$
$$(S9) \quad G = i \vee G = l \vee G = p \vee G = u \vee G = w,$$
$$(S10) \quad K = i \vee K = l \vee K = p \vee K = u \vee K = w,$$

together with the "distinguishing" statements

$$(S11) \quad (A \neq D) \wedge (A \neq F) \wedge (A \neq G) \wedge (A \neq K) \wedge (D \neq F) \wedge (D \neq G) \wedge (D \neq K) \wedge (F \neq G) \wedge (F \neq K) \wedge (G \neq K),$$

$$(S12) \quad (i \neq l) \wedge (i \neq p) \wedge (i \neq u) \wedge (i \neq w) \wedge (l \neq p) \wedge (l \neq u) \wedge (l \neq w) \wedge (p \neq u) \wedge (p \neq w) \wedge (u \neq w).$$

Remark. (Thanks to David Schissler for pointing this out.) In the course of this solution we interpreted *"entweder...oder..."* to mean *"or"* and not necessarily *"exclusive or"*. The latter interpretation could be easily implemented, through which $(S1)$ would become

$$(S1') \quad [(K = p) \vee (F = u) \vee (G = u)] \wedge [(F \neq u) \vee (G \neq u)],$$

and so forth. It is straightforward to check that, thanks to the distinguishing statements $(S11)$ and $(S12)$, this modification would not affect the final solution, and as such we do not pursue this matter further.

(b) $A = l, \; D = w, \; F = p, \; G = u, \; K = i.$

(c) We shall first prove $A \neq u$. We do so by contradiction, thus assuming that $A = u$. From (S11) we know that $A \neq F$ and $A \neq G$, and so substituting u for A we have that $F \neq u$ and $G \neq u$. From (S1) we can make a case distinction into the cases (1) $K = p$, (2) $F = u$, or (3) $G = u$. The two latter cases lead to contradiction, and so $K = p$ holds. Since $K \neq G$ by (S11) and symmetry of $\neq$, this implies $G \neq p$. From (S2) we may then make a case distinction into the cases (1) $K = i$, (2) $D = l$, or (3) $G = p$. The third case leads to a contradiction as before. The first case also leads to a contradiction since $i \neq p$ from (S12). It follows that the second case, $D = l$, must hold. Since $K \neq i$ and $D \neq i$, a similar case distinction for (S4) implies that $G = u$. This however contradicts the assumption $A = u$ since by (S11) $A \neq G$. It follows that $A \neq u$.

From (S3), again by case distinction, it follows that $F = p$ or $F = w$.

Next we prove $F = p$. We do so again by contradiction: assume $F \neq p$. Similarly as before, a case distinction on (S3) yields a contradiction in the first and third cases and we conclude $F = w$. Since $D \neq F$ by (S11), we conclude that $D \neq w$. From (S5) and (S12) it follows that $K = u$. But then (S1) implies that $F = u$ or $G = u$, a contradiction to $F \neq K$ and $G \neq K$ from (S11). Thus $F = p$.

It follows that $K \neq p$ and $F \neq u$, and so $G = u$ from (S1). In particular, $K \neq u$. Since $F = p \neq l$, it follows from (S5) that $D = w$. Similarly, since $D = w \neq l$ and $G = u \neq p$, it follows from (S2) than $K = i$. So far we have that $F = p$, $G = u$, $D = w$ and $K = i$. The last remaining option, a consequence of (S11) and (S6), is to set $A = l$. It is straightforward to check that the solution thus found,

$$F = p, \quad G = u, \quad D = w, \quad K = i, \quad A = l,$$

is indeed compatible with all eleven assumptions $(S1) - (S11)$.

Remark. Another way to go about this is as follows. From $(S1), \ldots, (S5)$, we can conclude $S := (S1) \wedge (S2) \wedge (S3) \wedge (S4) \wedge (S5)$. From part (a), this is equivalent to

$$S = [(K = p) \vee (F = u) \vee (G = u)] \bigwedge [(K = i) \vee (D = l) \vee (G = p)] \bigwedge [(A = u) \vee (F = p) \vee (F = w)] \bigwedge$$
$$\bigwedge [(K = i) \vee (D = i) \vee (G = u)] \bigwedge [(D = w) \vee (K = u) \vee (F = l)].$$

Using De Morgan's laws repeatedly, we see that S is equivalent to a $\vee$-combination of $3^5 = 243$ statements of the form $(*_1 = \star_1) \wedge (*_2 = \star_2) \wedge (*_3 = \star_3) \wedge (*_4 = \star_4) \wedge (*_5 = \star_5)$ where $*_j \in \{A, D, F, G, K\}$ and $\star_k \in \{i, l, p, u, w\}$ for $j, k \in \{1, 2, 3, 4, 5\}$. A case-by-case analysis reveals the only two statements which are compatible with the remaining $(S6), \ldots, (S11)$ are

$$(G = u) \wedge (K = i) \wedge (F = p) \wedge (G = u) \wedge (D = w),$$
$$(G = u) \wedge (K = i) \wedge (F = p) \wedge (K = i) \wedge (D = w),$$

both of which lead to the solution $(F = p, \; G = u, \; D = w, \; K = i, \; A = l)$ found before.