LECTURE NOTES ON DISPERSIVE PDES

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1. INTRODUCTION

1.1. **Dispersive partial differential equations.** Let us consider the following linear PDE:

$$\begin{cases} \partial_t u = Lu \\ u|_{t=0} = u_0, \end{cases}$$
(1.1)

where $u = u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and L is a skew-adjoint constant coefficient differential operator in space. More precisely, L takes the form

$$Lu = \sum_{|\alpha| \le k} c_{\alpha} \partial_x^{\alpha} u$$

with $k \in \mathbb{N}$, $c_{\alpha} \in \mathbb{F}$, and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$ ranging over all multi-indices with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq k$, and L satisfies

$$\int_{\mathbb{R}^d} Lu(x)\overline{v(x)}dx = -\int_{\mathbb{R}^d} u(x)\overline{Lv(x)}dx$$

for all test functions u and v. We may also write $L = i \cdot h(D)$, where

$$D := \frac{1}{i} \nabla = \left(\frac{1}{i} \partial_{x_1}, \dots, \frac{1}{i} \partial_{x_d}\right)$$

and h is the polynomial

$$h(\xi_1,\ldots,\xi_d) = \sum_{|\alpha| \le k} i^{|\alpha|-1} c_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}.$$

One may easily verify (using integration by parts) that L being skew-adjoint is equivalent to the coefficients of h being real-valued. The polynomial h is referred to as the *dispersion* relation of the equation (1.1).

Let us now look at some examples of linear PDEs of the form (1.1). The transport equation is given by

$$\begin{cases} \partial_t u = -v \cdot \nabla u \\ u|_{t=0} = u_0 \end{cases}$$

for some constant vector $v \in \mathbb{R}^d$. The transport equation has the explicit solution $u(t, x) = u_0(x - tv)$ and has the dispersion relation $h(\xi) = -v \cdot \xi$. Another example is the *free* Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = 0\\ u|_{t=0} = u_0, \end{cases}$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ is the Laplacian. The free Schrödinger equation has the dispersion relation $h(\xi) = -|\xi|^2$. We also have the one-dimensional *Airy equation*:

$$\begin{cases} \partial_t u + \partial_x^3 u = 0\\ u|_{t=0} = u_0. \end{cases}$$

The Airy equation has the dispersion relation $h(\xi) = \xi^3$.

A powerful tool for solving these PDEs is the Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

Let us perform some formal computations. By taking the Fourier transform of (1.1), we have

$$\partial_t \widehat{u}(t,\xi) = ih(2\pi\xi)\widehat{u}(t,\xi),$$

which is an ordinary differential equation (ODE) with the time t as the variable. By solving this ODE, we obtain

$$\hat{u}(t,\xi) = e^{ith(2\pi\xi)}\hat{u}_0(\xi).$$
(1.2)

Thus, by applying the Fourier inversion formula, we obtain the solution

$$u(t,x) = \int_{\mathbb{R}^d} e^{ith(2\pi\xi) + 2\pi i\xi \cdot x} \widehat{u}_0(\xi) d\xi.$$

By a first order Taylor expansion at a fixed frequency $\xi_0 \in \mathbb{R}^d$, we have

$$h(2\pi\xi) \approx h(2\pi\xi_0) + 2\pi(\xi - \xi_0) \cdot \nabla h(2\pi\xi_0),$$

so that (by ignoring the constant that is independent of ξ)

$$u(t,x) \approx \int_{\mathbb{R}^d} e^{it\xi \cdot \nabla h(2\pi\xi_0) + 2\pi i\xi \cdot x} \widehat{u}_0(\xi) d\xi = u_0 \big(x + t \cdot \nabla h(2\pi\xi_0) \big).$$
(1.3)

From (1.2), we see that if u_0 has spatial frequency roughly ξ_0 (i.e. \hat{u}_0 is concentrated near ξ_0), then u(t) will have spatial frequency roughly ξ_0 for all times. Also, we see that u will oscillate in time with frequency roughly $h(\xi_0)$. From (1.3), we see that u will travel with velocity roughly $\nabla h(2\pi\xi_0)$. The quantity ∇h is called the *group velocity*.

A linear PDE of the form (1.1) is called a *dispersive PDE* if different frequencies in this equation tend to propagate at different velocities, thus dispersing the solution over time. In view of (1.3), the linear PDE (1.1) is a dispersive PDE if the group velocity $\nabla h(\xi)$ depends on ξ . In the three examples above, we see that the free Schrödinger equation and the Airy equation are dispersive PDEs, whereas the transport equation is not a dispersive PDE.

We may also consider dispersive equations that are second-order in time. An important example is the *wave equation*:

$$\begin{cases} \partial_t^2 u - \Delta u = 0\\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

By taking the Fourier transform, we obtain

$$\partial_t^2 \widehat{u}(t,\xi) + 4\pi^2 |\xi|^2 \widehat{u}(t,\xi) = 0$$

which is a second-order ODE and has the solution

$$\widehat{u}(t,\xi) = \cos(2\pi t |\xi|) \widehat{u}_0(\xi) + \frac{\sin(2\pi t |\xi|)}{2\pi |\xi|} \widehat{u}_1(\xi).$$

Since

$$\cos(2\pi t|\xi|) = \frac{e^{2\pi i t|\xi|} + e^{-2\pi i t|\xi|}}{2} \quad \text{and} \quad \sin(2\pi t|\xi|) = \frac{e^{2\pi i t|\xi|} - e^{-2\pi i t|\xi|}}{2i}.$$

we can say that the wave equation has the dispersion relation $h(\xi) = \pm |\xi|$. If the dimension $d \ge 2$, we note that $\nabla h(\xi) = \pm \frac{\xi}{|\xi|}$, which suggests that the frequency of a wave determines the direction of propagation, but not the speed. Nevertheless, we still view the wave equation as a dispersive equation. Here, we note that $|\nabla h(\xi)| = 1$, which suggests *finite speed of propagation* for wave equations.

1.2. Basics of L^p spaces and Fourier analysis. Let (X, μ) be a measure space. In most of the situations, we take $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$ and μ to be the standard Lebesgue measure. For $1 \leq p < \infty$, we recall that $L^p(X) = L^p(X, \mu)$ is the space of all complex-valued measurable functions f on X such that

$$||f||_{L^p(X)} := \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, then the space $L^{\infty}(X) = L^{\infty}(X, \mu)$ is the space of all measurable functions f on X that are bounded almost everywhere:

$$||f||_{L^{\infty}(X)} := \operatorname{ess\,sup} |f| := \inf \left\{ M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0 \right\}.$$

Here, $\|\cdot\|_{L^p(X)}$ is a complete norm that makes $L^p(X)$ a Banach space. When p = 2, the space $L^2(X)$ is a Hilbert space with the inner product

$$\langle f,g \rangle_{L^2(X)} = \int_X f(x) \overline{g(x)} d\mu(x).$$

An important tool that we will use frequently is Hölder's inequality: for any $1 \le p, q, r \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have

$$||fg||_{L^{r}(X)} \leq ||f||_{L^{p}(X)} ||g||_{L^{q}(X)}$$

Moreover, for any $1 \le p \le \infty$, one may compute the $L^p(\mathbb{R}^d)$ -norm of a function via duality:

$$||f||_{L^{p}(X)} = \sup_{||g||_{L^{p'}(X)} \le 1} \left| \int_{X} f(x)g(x)d\mu(x) \right|,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let (X, μ) and (Y, ν) be two measure spaces. For $1 \le q, r \le \infty$, the space $L^q(Y; L^r(X))$ is defined by the norm

$$\|f\|_{L^q_y L^r_x(Y \times X)} := \|\|f\|_{L^r_x(X)}\|_{L^q_u(Y)}$$

For $1 \leq r \leq q \leq \infty$, we have the following Minkowski's integral inequality:

$$\left\| \|f\|_{L^r_x(X)} \right\|_{L^q_y(Y)} \le \left\| \|f\|_{L^q_y(Y)} \right\|_{L^r_x(X)}$$

This norm is often used for space-time functions, where $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$ refers to the spatial domain and $Y = \mathbb{R}$ refers to the temporal domain. If the time variable is restricted to an interval, say [-T, T] for some T > 0, we use the abbreviation $L_T^q L_x^r(\mathbb{R}^d)$ to denote $L_t^q([-T, T]; L_x^r(\mathbb{R}^d))$. We also define the space $C(\mathbb{R}; L^r(\mathbb{R}^d))$ as the set of space-time functions f such that the map

$$\mathbb{R} \ni t \mapsto f(t) \in L^r(\mathbb{R}^d)$$

is continuous.

Let us consider \mathbb{R}^d with the standard Lebesgue measure. The convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ is defined by

$$f * g(x) := \int_{\mathbb{R}^d} f(x - y)g(y)dy = \int_{\mathbb{R}^d} f(y)g(x - y)dy.$$

An important tool is Young's convolution inequality: for any $1 \le p, q, r \le \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then f * g exists almost everywhere and

$$||f * g||_{L^r(\mathbb{R}^d)} \le ||f||_{L^p(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}$$

We now consider the Fourier transform on \mathbb{R}^d . For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, we recall the notations $x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\partial^{\alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. We define the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ as the set of smooth functions $f \in C^{\infty}(\mathbb{R}^d)$ such that for any multi-indices α, β ,

$$\sup_{x \in \mathbb{R}^d} |x^{\alpha} \partial^{\beta} f(x)| < \infty$$

The space $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for any $1 \leq p < \infty$. Given $f \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transform of f as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

We also define the inverse Fourier transform of f as

$$f^{\vee}(x) := \widehat{f}(-x) = \int_{\mathbb{R}^d} f(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

Given $f, g \in \mathcal{S}(\mathbb{R}^d)$, we list the following useful properties:

- (1) $\widehat{f} \in \mathcal{S}(\mathbb{R}^d);$
- (2) $(\hat{f})^{\vee} = f;$
- (3) $\int_{\mathbb{R}^d} \widehat{f}(x)g(x)dx = \int_{\mathbb{R}^d} f(x)\widehat{g}(x)dx;$
- (4) (Plancherel's identity) $||f||_{L^2(\mathbb{R}^d)} = ||\widehat{f}||_{L^2(\mathbb{R}^d)} = ||f^{\vee}||_{L^2(\mathbb{R}^d)};$
- (5) (Parseval's relation) $\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi;$
- (6) $\|\widehat{f}\|_{L^{\infty}(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}$ and $\|f\|_{L^{\infty}(\mathbb{R}^d)} \le \|\widehat{f}\|_{L^1(\mathbb{R}^d)};$
- (7) $(\partial^{\alpha} f)^{\wedge}(\xi) = (2\pi i\xi)^{\alpha} \widehat{f}(\xi);$
- (8) $(\partial^{\alpha}\widehat{f})^{\vee}(x) = (-2\pi i x)^{\alpha} f(x);$
- (9) With $f_{\varepsilon}(x) := \varepsilon^{-d} f(\varepsilon^{-1}x), \ \widehat{f}_{\varepsilon}(\xi) = \widehat{f}(\varepsilon\xi);$
- (10) $\widehat{f * g} = \widehat{f}\widehat{g}$ and $\widehat{fg} = \widehat{f} * \widehat{g}$.

We then define $\mathcal{S}'(\mathbb{R}^d)$ as the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$, i.e. the set of linear and continuous maps $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. Elements in $\mathcal{S}'(\mathbb{R}^d)$ are called tempered

distributions on \mathbb{R}^d . Due to property (3) above, we can define the Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ as

$$\langle f,g \rangle = \langle f,\widehat{g} \rangle$$

for any $g \in \mathcal{S}(\mathbb{R}^d)$.

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