LECTURE NOTES ON DISPERSIVE PDES

RUOYUAN LIU

Contents

1. Introduction

1.1. **Dispersive partial differential equations.** Let us consider the following linear PDE:

$$\begin{cases} \partial_t u = Lu \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and L is a skew-adjoint constant coefficient differential operator in space. More precisely, L takes the form

$$Lu = \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha u$$

with $k \in \mathbb{N}$, $c_\alpha \in \mathbb{F}$, and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ ranging over all multi-indices with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq k$, and L satisfies

$$\int_{\mathbb{R}^d} Lu(x)\overline{v(x)}dx = -\int_{\mathbb{R}^d} u(x)\overline{Lv(x)}dx$$

for all test functions u and v. We may also write $L = i \cdot h(D)$, where

$$D := \frac{1}{i}\nabla = \Big(\frac{1}{i}\partial_{x_1}, \dots, \frac{1}{i}\partial_{x_d}\Big)$$

and h is the polynomial

$$h(\xi_1, \dots, \xi_d) = \sum_{|\alpha| \leq k} i^{|\alpha|-1} c_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}.$$

One may easily verify (using integration by parts) that L being skew-adjoint is equivalent to the coefficients of h being real-valued. The polynomial h is referred to as the *dispersion relation* of the equation (1.1).

Let us now look at some examples of linear PDEs of the form (1.1). The *transport equation* is given by

$$\begin{cases} \partial_t u = -v \cdot \nabla u \\ u|_{t=0} = u_0 \end{cases}$$

for some constant vector $v \in \mathbb{R}^d$. The transport equation has the explicit solution $u(t,x) = u_0(x - tv)$ and has the dispersion relation $h(\xi) = -v \cdot \xi$. Another example is the *free Schrödinger equation*:

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0, \end{cases}$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ is the Laplacian. The free Schrödinger equation has the dispersion relation $h(\xi) = -|\xi|^2$. We also have the one-dimensional *Airy equation*:

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

The Airy equation has the dispersion relation $h(\xi) = \xi^3$.

A powerful tool for solving these PDEs is the Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

Let us perform some formal computations. By taking the Fourier transform of (1.1), we have

$$\partial_t \widehat{u}(t,\xi) = ih(2\pi\xi)\widehat{u}(t,\xi),$$

which is an ordinary differential equation (ODE) with the time t as the variable. By solving this ODE, we obtain

$$\widehat{u}(t,\xi) = e^{ith(2\pi\xi)} \widehat{u}_0(\xi). \tag{1.2}$$

Thus, by applying the Fourier inversion formula, we obtain the solution

$$u(t,x) = \int_{\mathbb{R}^d} e^{ith(2\pi\xi) + 2\pi i \xi \cdot x} \widehat{u}_0(\xi) d\xi.$$

By a first order Taylor expansion at a fixed frequency $\xi_0 \in \mathbb{R}^d$, we have

$$h(2\pi\xi) \approx h(2\pi\xi_0) + 2\pi(\xi - \xi_0) \cdot \nabla h(2\pi\xi_0),$$

so that (by ignoring the constant that is independent of ξ)

$$u(t,x) \approx \int_{\mathbb{R}^d} e^{it\xi \cdot \nabla h(2\pi\xi_0) + 2\pi i \xi \cdot x} \widehat{u}_0(\xi) d\xi = u_0\big(x + t \cdot \nabla h(2\pi\xi_0)\big). \tag{1.3}$$

From (1.2), we see that if u_0 has spatial frequency roughly ξ_0 (i.e. $\widehat{u}_0$ is concentrated near ξ_0), then $u(t)$ will have spatial frequency roughly ξ_0 for all times. Also, we see that u will oscillate in time with frequency roughly $h(\xi_0)$. From (1.3), we see that u will travel with velocity roughly $\nabla h(2\pi\xi_0)$. The quantity ∇h is called the *group velocity*.

A linear PDE of the form (1.1) is called a *dispersive PDE* if different frequencies in this equation tend to propagate at different velocities, thus dispersing the solution over time. In view of (1.3), the linear PDE (1.1) is a dispersive PDE if the group velocity $\nabla h(\xi)$ depends on ξ. In the three examples above, we see that the free Schrödinger equation and the Airy equation are dispersive PDEs, whereas the transport equation is not a dispersive PDE.

We may also consider dispersive equations that are second-order in time. An important example is the *wave equation*:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

By taking the Fourier transform, we obtain

$$\partial_t^2 \widehat{u}(t,\xi) + 4\pi^2 |\xi|^2 \widehat{u}(t,\xi) = 0,$$

which is a second-order ODE and has the solution

$$\widehat{u}(t,\xi) = \cos(2\pi t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\widehat{u}_1(\xi).$$

Since

$$\cos(2\pi t|\xi|) = \frac{e^{2\pi i t|\xi|} + e^{-2\pi i t|\xi|}}{2} \quad \text{and} \quad \sin(2\pi t|\xi|) = \frac{e^{2\pi i t|\xi|} - e^{-2\pi i t|\xi|}}{2i},$$

we can say that the wave equation has the dispersion relation $h(\xi) = \pm|\xi|$. If the dimension $d \geq 2$, we note that $\nabla h(\xi) = \pm \frac{\xi}{|\xi|}$, which suggests that the frequency of a wave determines the direction of propagation, but not the speed. Nevertheless, we still view the wave equation as a dispersive equation. Here, we note that $|\nabla h(\xi)| = 1$, which suggests *finite speed of propagation* for wave equations.

1.2. **Basics of L^p spaces and Fourier analysis.** Let (X, μ) be a measure space. In most of the situations, we take $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$ and μ to be the standard Lebesgue measure. For $1 \leq p < \infty$, we recall that $L^p(X) = L^p(X,\mu)$ is the space of all complex-valued measurable functions f on X such that

$$\|f\|_{L^p(X)} := \bigg(\int_X |f(x)|^p d\mu(x) \bigg)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, then the space $L^\infty(X) = L^\infty(X,\mu)$ is the space of all measurable functions f on X that are bounded almost everywhere:

$$\|f\|_{L^\infty(X)} := \operatorname{ess\,sup} |f| := \inf \big\{ M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0 \big\}.$$

Here, $\| \cdot \|_{L^p(X)}$ is a complete norm that makes $L^p(X)$ a Banach space. When $p = 2$, the space $L^2(X)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{L^2(X)} = \int_X f(x)\overline{g(x)} d\mu(x).$$

An important tool that we will use frequently is Hölder's inequality: for any $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have

$$\|fg\|_{L^r(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}.$$

Moreover, for any $1 \leq p \leq \infty$, one may compute the $L^p(\mathbb{R}^d)$-norm of a function via duality:

$$\|f\|_{L^p(X)} = \sup_{\|g\|_{L^{p'}(X)} \leq 1} \bigg| \int_X f(x) g(x) d\mu(x) \bigg|,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let (X, μ) and (Y, ν) be two measure spaces. For $1 \le q, r \le \infty$, the space $L^q(Y; L^r(X))$ is defined by the norm

$$\|f\|_{L^q_y L^r_x(Y\times X)} := \big\| \|f\|_{L^r_x(X)} \big\|_{L^q_y(Y)}.$$

For $1 \le r \le q \le \infty$, we have the following Minkowski's integral inequality:

$$\big\| \|f\|_{L^r_x(X)} \big\|_{L^q_y(Y)} \le \big\| \|f\|_{L^q_y(Y)} \big\|_{L^r_x(X)}.$$

This norm is often used for space-time functions, where $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$ refers to the spatial domain and $Y = \mathbb{R}$ refers to the temporal domain. If the time variable is restricted to an interval, say $[-T, T]$ for some $T > 0$, we use the abbreviation $L^q_T L^r_x(\mathbb{R}^d)$ to denote $L^q_t([-T, T]; L^r_x(\mathbb{R}^d))$. We also define the space $C(\mathbb{R}; L^r(\mathbb{R}^d))$ as the set of space-time functions f such that the map

$$\mathbb{R} \ni t \mapsto f(t) \in L^r(\mathbb{R}^d)$$

is continuous.

Let us consider $\mathbb{R}^d$ with the standard Lebesgue measure. The convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ is defined by

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

An important tool is Young's convolution inequality: for any $1 \le p, q, r \le \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g$ exists almost everywhere and

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

We now consider the Fourier transform on $\mathbb{R}^d$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\ge 0}$, we recall the notations $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\partial^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d}$. We define the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ as the set of smooth functions $f \in C^\infty(\mathbb{R}^d)$ such that for any multi-indices α, β,

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty.$$

The space $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for any $1 \le p < \infty$. Given $f \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transform of f as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi\cdot x} dx.$$

We also define the inverse Fourier transform of f as

$$f^\vee(x) := \widehat{f}(-x) = \int_{\mathbb{R}^d} f(\xi) e^{2\pi i \xi\cdot x} d\xi.$$

Given $f, g \in \mathcal{S}(\mathbb{R}^d)$, we list the following useful properties:

(1) $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$;
(2) $(\widehat{f})^\vee = f$;
(3) $\int_{\mathbb{R}^d} \widehat{f}(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \widehat{g}(x) dx$;
(4) (Plancherel's identity) $\|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f^\vee\|_{L^2(\mathbb{R}^d)}$;
(5) (Parseval's relation) $\int_{\mathbb{R}^d} f(x)\overline{g(x)} dx = \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$;
(6) $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}$ and $\|f\|_{L^\infty(\mathbb{R}^d)} \le \|\widehat{f}\|_{L^1(\mathbb{R}^d)}$;

(7) $(\partial^\alpha f)^\wedge(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi)$;
(8) $(\partial^\alpha \widehat{f})^\vee(x) = (-2\pi i x)^\alpha f(x)$;
(9) With $f_\varepsilon(x) := \varepsilon^{-d} f(\varepsilon^{-1}x)$, $\widehat{f_\varepsilon}(\xi) = \widehat{f}(\varepsilon\xi)$;
(10) $\widehat{f * g} = \widehat{f}\widehat{g}$ and $\widehat{fg} = \widehat{f} * \widehat{g}$.

We then define $\mathcal{S}'(\mathbb{R}^d)$ as the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$, i.e. the set of linear and continuous maps $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. Elements in $\mathcal{S}'(\mathbb{R}^d)$ are called tempered distributions on $\mathbb{R}^d$. Due to property (3) above, we can define the Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ as

$$\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle$$

for any $g \in \mathcal{S}(\mathbb{R}^d)$.

2. Deterministic well-posedness theory

In this section, we focus on deterministic well-posedness theory of dispersive equations. We will use the following NLS on $\mathbb{R}^d$ as the guiding example:

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

At the end of the section, we will briefly mention the case of NLW on $\mathbb{R}^d$:

$$\partial_t^2 u - \Delta u \pm u^p = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

2.1. **Formulations and the L^2-based Sobolev spaces.** Let us first consider the *linear (free) Schrödinger equation*:

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

By taking the Fourier transform in the spatial variable x, we obtain

$$\begin{cases} i\partial_t \widehat{u}(\xi) - 4\pi^2|\xi|^2\widehat{u}(\xi) = 0 \\ \widehat{u}(t,\xi)|_{t=0} = \widehat{u}_0(\xi) \end{cases} \tag{2.1}$$

for any $\xi \in \mathbb{R}^d$. Note that for a fixed $\xi \in \mathbb{R}^d$, (2.1) is an ODE with t as the variable. Thus, we may solve (2.1) by writing

$$\widehat{u}(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi). \tag{2.2}$$

By taking the inverse Fourier transform of (2.2), we have

$$u(t,x) = \mathcal{F}^{-1}\big(e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi)\big)(x).$$

The operator

$$e^{it\Delta} f := \mathcal{F}^{-1}\big(e^{-4\pi^2 it|\cdot|^2}\widehat{f}(\cdot)\big)$$

is called the *linear Schrödinger operator*.

Next, we consider the *inhomogeneous linear Schrödinger equation*:

$$\begin{cases} i\partial_t u + \Delta u = F \\ u|_{t=0} = u_0 \end{cases}$$

for some $F = F(t, x)$. Again, by taking the Fourier transform in the spatial variable x, we obtain

$$\partial_t \widehat{u}(t,\xi) + 4\pi^2 i|\xi|^2 \widehat{u}(t,\xi) = -i\widehat{F}(t,\xi) \tag{2.3}$$

for any $\xi \in \mathbb{R}^d$. Multiplying (2.3) by an integrating factor $e^{4\pi^2 i|\xi|^2}$, we get

$$\partial_t\big(e^{4\pi^2 it|\xi|^2}\widehat{u}(t,\xi)\big) = -ie^{4\pi^2 it|\xi|^2}\widehat{F}(t,\xi). \tag{2.4}$$

Integrating (2.4) from 0 to time t, we obtain

$$e^{4\pi^2 it|\xi|^2}\widehat{u}(t,\xi) - \widehat{u}_0(\xi) = -i\int_0^t e^{4\pi^2 it'|\xi|^2}\widehat{F}(t',\xi)dt',$$

which gives

$$\widehat{u}(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi) - i\int_0^t e^{-4\pi^2 i(t-t')|\xi|^2}\widehat{F}(t',\xi)dt'. \tag{2.5}$$

By taking the inverse Fourier transform of (2.5), we obtain

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}F(t')dt'. \tag{2.6}$$

We now come back to NLS:

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^{p-1}u \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d) \end{cases} \tag{2.7}$$

with $p \in 2\mathbb{N}+1$. In view of (2.6), we introduce the following notion of a solution to (2.7).

Definition 2.1. We say that a function $u \in C(\mathbb{R}; X)$ is a *mild solution* to NLS (2.7) if u satisfies the *Duhamel formulation*:

$$u(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt', \tag{2.8}$$

where X is a suitable function space.

We now introduce the function spaces that are suitable for solving NLS (2.7). Let us introduce the following L^2-based Sobolev spaces.

Definition 2.2. Let $d \in \mathbb{N}$ and $s \in \mathbb{R}$. We define $H^s(\mathbb{R}^d)$ as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^s(\mathbb{R}^d)} := \big\|(1+|\xi|^2)^{\frac{s}{2}}\widehat{f}(\xi)\big\|_{L^2_\xi(\mathbb{R}^d)}$$

is finite.

Later on, we adopt the notation of the Japanese bracket $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. The $H^s(\mathbb{R}^d)$ space is in fact a Hilbert space with the inner product

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \big\langle \langle\cdot\rangle^s\widehat{f}, \langle\cdot\rangle^s\widehat{f}\big\rangle_{L^2(\mathbb{R}^d)}.$$

Example 1. (i) When $s = 0$, by Plancherel's identity, we have

$$\|f\|_{H^0(\mathbb{R}^d)} = \|\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

Thus, $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

(ii) When $s=1$, since $|\widehat{\nabla f}(\xi)| = 2\pi|\xi||\widehat{f}(\xi)|$, we have

$$\|f\|^2_{H^1(\mathbb{R}^d)} = \|\widehat{f}\|^2_{L^2_\xi(\mathbb{R}^d)} + \big\||\xi|\widehat{f}(\xi)\big\|^2_{L^2_\xi(\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{4\pi^2}\|\nabla f\|^2_{L^2(\mathbb{R}^d)}.$$

In order words, $H^1(\mathbb{R}^d)$ is precisely the space of functions f such that $f \in L^2(\mathbb{R}^d)$ and $\nabla f \in L^2(\mathbb{R}^d)$.

(iii) For a general $s \in \mathbb{R}$, we also write

$$\langle \nabla \rangle^s f = \mathcal{F}^{-1}\big(\langle \cdot \rangle^s \widehat{f}(\cdot)\big).$$

When $s > 0$, $\langle \nabla \rangle^s$ means taking the fractional derivative of order s. When $s < 0$, $\langle \nabla \rangle^s$ can be viewed as a fractional integration of order $-s$.

Let us show the following algebra property of the $H^s(\mathbb{R}^d)$-norm with $s > \frac{d}{2}$.

Lemma 2.3 (Algebra property of $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$)**.** *Let $d \in \mathbb{N}$ and $s > \frac{d}{2}$. Then, there exists $C = C(s,d) > 0$ such that for any $f, g \in H^s(\mathbb{R}^d)$, we have*

$$\|fg\|_{H^s(\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)}.$$

Notation 2.1. For two quantities $A, B > 0$, we use the notation

$$A \lesssim B$$

if $A \le CB$ for some constant $C > 0$ uniform with respect to the set where A and B are allowed to vary. We also use subscripts such as $\lesssim_a$ to denote dependence on parameters.

Proof of Lemma 2.3. We first recall that

$$\widehat{fg}(\xi) = \widehat{f} * \widehat{g}(\xi) = \int_{\mathbb{R}^d} \widehat{f}(\xi - \xi_1)\widehat{g}(\xi_1)d\xi_1.$$

Also, using the fact that $(a+b)^\theta \le a^\theta + b^\theta$ for $\theta \in (0,1]$ and $(a+b)^\theta \le Ca^\theta + Cb^\theta$ for $\theta > 1$ for some constant $C = C(\theta) > 0$, we have

$$\langle \xi \rangle^s \lesssim_s \langle \xi - \xi_1 \rangle^s + \langle \xi_1 \rangle^s. \tag{2.9}$$

Thus, we have

$$\begin{aligned}
\|fg\|_{H^s(\mathbb{R}^d)} &= \big\|\langle \xi \rangle^s \widehat{fg}(\xi)\big\|_{L^2_\xi(\mathbb{R}^d)} \\
\text{(2.9)} \qquad &\lesssim \bigg\| \int_{\mathbb{R}^d} \langle \xi - \xi_1 \rangle^s |\widehat{f}(\xi - \xi_1)||\widehat{g}(\xi_1)|d\xi_1 \bigg\|_{L^2_\xi(\mathbb{R}^d)} \\
&\quad + \bigg\| \int_{\mathbb{R}^d} |\widehat{f}(\xi - \xi_1)|\langle \xi_1 \rangle^s |\widehat{g}(\xi_1)|d\xi_1 \bigg\|_{L^2_\xi(\mathbb{R}^d)} \qquad (2.10) \\
&= \big\|(\langle \cdot \rangle^s |\widehat{f}|) * |\widehat{g}|\big\|_{L^2_\xi(\mathbb{R}^d)} + \big\||\widehat{f}| * (\langle \cdot \rangle^s |\widehat{g}|)\big\|_{L^2_\xi(\mathbb{R}^d)} \\
\text{(Young's convolution)} \qquad &\le \|\langle \cdot \rangle^s \widehat{f}\|_{L^2_\xi(\mathbb{R}^d)}\|\widehat{g}\|_{L^1_\xi(\mathbb{R}^d)} + \|\widehat{f}\|_{L^1_\xi(\mathbb{R}^d)}\|\langle \cdot \rangle^s \widehat{g}\|_{L^2_\xi(\mathbb{R}^d)},
\end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\widehat{f}\|_{L^1_\xi(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \langle\xi\rangle^{-s}\langle\xi\rangle^{s}|\widehat{f}(\xi)|d\xi \\ &\le \left(\int_{\mathbb{R}^d} \frac{1}{\langle\xi\rangle^{2s}}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &= C(s,d)\|f\|_{H^s(\mathbb{R}^d)}.\end{aligned} \tag{2.11}$$

Thus, the desired inequality follows from (2.10) and (2.11). □

The following lemma shows why the $H^s(\mathbb{R}^d)$ space is important for Schrödinger equations.

Lemma 2.4 (Properties of the linear Schrödinger operator)**.** *Let $d \in \mathbb{N}$ and $s \in \mathbb{R}$.*
(i) *For any $t \in \mathbb{R}$, the operator $e^{it\Delta}$ is unitary on $H^s(\mathbb{R}^d)$:*

$$\|e^{it\Delta}f\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^s(\mathbb{R}^d)}.$$

(ii) *For any $f \in H^s(\mathbb{R}^d)$, the function $t \mapsto e^{it\Delta}f$ lies in $C(\mathbb{R}; H^s(\mathbb{R}^d))$.*

Proof. (i) Using the definition, we get

$$\begin{aligned}\|e^{it\Delta}f\|_{H^s(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d}\langle\xi\rangle^{2s}\left|e^{-4\pi^2 it|\xi|^2}\widehat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^d}\langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &= \|f\|_{H^s(\mathbb{R}^d)},\end{aligned}$$

which gives the desired property.
(ii) For any $t_1, t_2 \in \mathbb{R}$, we have the semigroup property of the linear Schrödinger operator:

$$e^{i(t_1+t_2)\Delta} = e^{it_1\Delta}e^{it_2\Delta}.$$

Fix $t \in \mathbb{R}$. By the semigroup property and part (i), we have that for any $h \in \mathbb{R}$,

$$\begin{aligned}\|e^{i(t+h)\Delta}f - e^{it\Delta}f\|^2_{H^s(\mathbb{R}^d)} &= \|e^{it\Delta}(e^{ih\Delta}-1)f\|^2_{H^s(\mathbb{R}^d)} \\ &= \|(e^{ih\Delta}-1)f\|^2_{H^s(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d}\langle\xi\rangle^{2s}\left|(e^{-4\pi^2 ih|\xi|^2}-1)\widehat{f}(\xi)\right|^2 d\xi.\end{aligned} \tag{2.12}$$

Since $|e^{-4\pi^2 ih|\xi|^2} - 1| \le 2$ and $f \in H^s(\mathbb{R}^d)$, we can use the dominated convergence theorem to obtain

$$\lim_{h\to 0}\|e^{i(t+h)\Delta}f - e^{it\Delta}f\|_{H^s(\mathbb{R}^d)} = 0.$$

This proves the continuity-in-time property. □

2.2. **Local well-posedness of NLS.** Our goal in this subsection is to construct a unique mild solution to NLS (2.7) in the function space $C([-T,T];H^s(\mathbb{R}^d))$ with $s > \frac{d}{2}$ for some $T > 0$.

A important tool for proving well-posedness of PDEs is the Banach fixed-point theorem.

Theorem 2.5 (Banach fixed-point theorem)**.** *Let (X,d) be a non-empty complete metric space. Let $F : X \to X$ be a contraction mapping: there exists $q \in [0,1)$ such that*

$$d(F(x),F(y)) \leq qd(x,y)$$

for all $x,y \in X$. Then, there exists a unique $x^* \in X$ such that $F(x^*) = x^*$.

Proof. Let $x_0 \in X$ be arbitrary and we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = F(x_{n-1})$. A simple induction on n gives

$$d(x_{n+1},x_n) \leq q^n d(x_1,x_0).$$

Then, for any $m,n \in \mathbb{N}$ with $m > n$, we have

$$d(x_m,x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1},x_j) \leq q^n d(x_1,x_0) \sum_{j=0}^{m-n-1} q^j \leq \frac{q^n}{1-q} d(x_1,x_0). \tag{2.13}$$

Since $0 \leq q < 1$, the inequality (2.13) shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete metric space, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a limit x^* in X. The fact that F is a contraction mapping implies that F is continuous, so that

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}) = F(x^*),$$

so that x^* is a fixed point of F. To show uniqueness, let $x_1^*, x_2^* \in X$ be two fixed points of F. Then, we have

$$d(x_1^*,x_2^*) = d(F(x_1^*),F(x_2^*)) \leq qd(x_1^*,x_2^*),$$

which implies that $d(x_1^*,x_2^*) = 0$ since $0 \leq q < 1$. This finishes the proof of the theorem. □

We are now ready to prove local well-posedness of NLS (2.8) in $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$.

Proposition 2.6 (Local well-posedness of NLS in $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$)**.** *Let $d \in \mathbb{N}$, $s > \frac{d}{2}$, and $p \in 2\mathbb{N}+1$. Then, for any $u_0 \in H^s(\mathbb{R}^d)$, there exist $T = T(s,d,p,\|u_0\|_{H^s(\mathbb{R}^d)}) > 0$ and a unique solution u to NLS (2.8) in a closed ball of the space $C([-T,T];H^s(\mathbb{R}^d))$, and the solution $u \in C([-T,T];H^s(\mathbb{R}^d))$ depends continuously on $u_0 \in H^s(\mathbb{R}^d)$.*

Proof. Fix $u_0 \in H^s(\mathbb{R}^d)$. Let us define

$$\Gamma_{u_0}[u](t) := e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt'.$$

Our goal is to show that there exists a unique u such that

$$u = \Gamma_{u_0}[u]$$

in $C([-T,T];H^s(\mathbb{R}^d))$ for some small $T > 0$.

Let $T > 0$ be chosen later. We compute that

$$\begin{aligned}
\|\Gamma_{u_0}[u]\|_{C_T H^s_x(\mathbb{R}^d)} &\le \|e^{it\Delta}u_0\|_{C_T H^s_x(\mathbb{R}^d)} + \left\| \int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt' \right\|_{C_T H^s_x(\mathbb{R}^d)} \\
\text{(Lm 2.4 \& Mink)} \quad &\le \|u_0\|_{H^s(\mathbb{R}^d)} + \left\| \int_0^t \left\|e^{i(t-t')\Delta}(|u|^{p-1}u)(t')\right\|_{H^s_x(\mathbb{R}^d)} dt' \right\|_{C_t([-T,T])} \\
\text{(Lm 2.4)} \quad &\le \|u_0\|_{H^s(\mathbb{R}^d)} + \int_0^T \||u|^{p-1}u\|_{C_T H^s_x(\mathbb{R}^d)} dt' \\
\text{(Lm 2.3)} \quad &\le \|u_0\|_{H^s(\mathbb{R}^d)} + CT\|u\|^p_{C_T H^s_x(\mathbb{R}^d)}
\end{aligned} \tag{2.14}$$

for some constant $C = C(s,d) > 0$. Let $R = 2\|u_0\|_{H^s(\mathbb{R}^d)}$ and define

$$\overline{B}_R := \{u \in C([-T,T]; H^s(\mathbb{R}^d)) : \|u\|_{C_T H^s_x(\mathbb{R}^d)} \le R\}.$$

Then, for any $u \in \overline{B}_R$, by choosing $T \le \frac{1}{2CR^{p-1}}$, we get

$$\|\Gamma_{u_0}[u]\|_{C_T H^s_x(\mathbb{R}^d)} \le \frac{R}{2} + CTR^p \le R,$$

which shows that Γ_{u_0} maps from $\overline{B}_R$ to $\overline{B}_R$.

We still need to show that Γ_{u_0} is a contraction mapping on $\overline{B}_R$. Since $p \in 2\mathbb{N}+1$, we have the following telescopic sum:

$$|u|^{p-1}u - |v|^{p-1}v = (u-v)\overline{u}\cdots u + v(\overline{u}-\overline{v})\cdots u + \cdots + v\overline{v}\cdots(u-v).$$

Thus, for any $u, v \in \overline{B}_R$, we use similar steps to obtain

$$\begin{aligned}
&\|\Gamma_{u_0}[u] - \Gamma_{u_0}[v]\|_{C_T H^s_x(\mathbb{R}^d)} \\
&\quad = \left\| \int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u - |v|^{p-1}v)(t')dt' \right\|_{C_T H^s_x(\mathbb{R}^d)} \\
\text{(Lm 2.4 \& Minkowski)} \quad &\quad \le T\||u|^{p-1}u - |v|^{p-1}v\|_{C_T H^s_x(\mathbb{R}^d)} \\
\text{(Lm 2.3)} \quad &\quad \le CT\|u-v\|_{C_T H^s_x(\mathbb{R}^d)} \sum_{j=0}^{p-1} \|u\|^{p-1-j}_{C_T H^s_x(\mathbb{R}^d)} \|v\|^j_{C_T H^s_x(\mathbb{R}^d)} \\
(u, v \in \overline{B}_R) \quad &\quad \le C'TR^{p-1}\|u-v\|_{C_T H^s_x(\mathbb{R}^d)}
\end{aligned} \tag{2.15}$$

for some constant $C' = C'(s,d,p) > 0$. Then, by choosing $T \le \frac{1}{2C'R^{p-1}}$, we get

$$\|\Gamma_{u_0}[u] - \Gamma_{u_0}[v]\|_{C_T H^s_x(\mathbb{R}^d)} \le \frac{1}{2}\|u-v\|_{C_T H^s_x(\mathbb{R}^d)}.$$

In summary, by choosing $T = \min\{\frac{1}{2CR^{p-1}}, \frac{1}{2C'R^{p-1}}\}$, we obtain a contraction mapping Γ_{u_0} from $\overline{B}_R$ to $\overline{B}_R$. By the Banach fixed-point theorem, there exists a unique $u \in \overline{B}_R$ such that $u = \Gamma_{u_0}[u]$.

It remains to show continuity of the solution with respect to the initial data. Let u be the solution to NLS (2.8) with initial data $u_0 \in H^s(\mathbb{R}^d)$ and let v be the solution to NLS (2.8) with initial data $v_0 \in H^s(\mathbb{R}^d)$. Then, by using Lemma 2.4 (i) and (2.15), we obtain

$$\begin{aligned}\|u-v\|_{C_T H^s_x(\mathbb{R}^d)} &= \|\Gamma_{u_0}[u]-\Gamma_{v_0}[v]\|_{C_T H^s_x(\mathbb{R}^d)}\\ &\le \|e^{it\Delta}(u_0-v_0)\|_{C_T H^s_x(\mathbb{R}^d)} + \bigg\|\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u-|v|^{p-1}v)(t')dt'\bigg\|_{C_T H^s_x(\mathbb{R}^d)}\\ &\le \|u_0-v_0\|_{H^s(\mathbb{R}^d)} + C'TR^{p-1}\|u-v\|_{C_T H^s_x(\mathbb{R}^d)}.\end{aligned}$$

Thus, with $T \le \frac{1}{2C'R^{p-1}}$, we absorb the $\|u-v\|_{C_T H^s_x}$ term to the left-hand-side to obtain

$$\|u-v\|_{C_T H^s_x(\mathbb{R}^d)} \le 2\|u_0-v_0\|_{H^s_x(\mathbb{R}^d)}.$$

This shows the continuity of the solution with respect to the initial data. □

Remark 2.7. (i) In the proof, we omitted checking the continuity of

$$t \mapsto \mathcal{I}(t) := \int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt'.$$

This is achieved via the following observation:

$$\begin{aligned}\mathcal{I}(t+h)-\mathcal{I}(t) &= \int_0^{t+h} e^{i(t+h-t')\Delta}(|u|^{p-1}u)(t')dt' - \int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt'\\ &= \int_t^{t+h} e^{i(t+h-t')\Delta}(|u|^{p-1}u)(t')dt' + \int_0^t (e^{i(t+h-t')\Delta}-e^{i(t-t')\Delta})(|u|^{p-1}u)(t')dt'.\end{aligned}$$

In the first integral, the domain of integration becomes small as $|h|$ gets small. In the second integral, we have

$$e^{i(t+h-t')\Delta}-e^{i(t-t')\Delta} = e^{i(t-t')\Delta}(e^{ih\Delta}-1),$$

which can be treated in a similar manner as in the proof of Lemma 2.4 (ii). The rest of the steps are left as an exercise.

(ii) The method of using the Banach fixed-point theorem for solving a PDE is also referred to as the Picard iteration scheme. Let us follow the proof of the Banach fixed-point theorem and see what the solution should look like. Let $v_0 \in \overline{B}_R$ be arbitrarily chosen. We then define $v_1 \in \overline{B}_R$ as $v_1 = \Gamma_{u_0}[v_0]$, which gives

$$v_1(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-t')\Delta}(|v_0|^{p-1}v_0)(t')dt'.$$

We then define $v_2 = \Gamma_{u_0}[v_1]$, which becomes more complicated but we can write out the first two terms (the linear evolution and the Picard second iterate):

$$v_2(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-t')\Delta}(|e^{it'\Delta}u_0|^{p-1}e^{it'\Delta}u_0)(t')dt' + \cdots.$$

As this process moves on, we see that the solution u is given by $u=\sum_{k=1}^\infty u_k$, where u_k's are iteratively defined by

$$\begin{aligned}u_1(t) &:= e^{it\Delta}u_0,\\ u_k(t) &:= \mp i \sum_{\substack{k_1,\dots,k_p\ge 1\\ k_1+\cdots+k_p=k}} \int_0^t e^{i(t-t')\Delta}\prod_{j=1}^p u_{k_j}(t')dt', \quad k\ge 2.\end{aligned}$$

This shows that u is given by a power series in $e^{it\Delta}u_0$. We say that this form of the solution u as an *analytic solution* of NLS (2.8).

In the proof of Proposition 2.6, we only showed uniqueness of the solution to NLS (2.8) in a closed ball $\overline{B}_R$. Let us show uniqueness of the solution in the entire space $C([-T,T];H^s(\mathbb{R}^d))$. This is also referred to as *unconditional uniqueness*, which does not rely on any auxiliary sets or spaces. The argument presented below is often called a continuity argument or a bootstrap argument.

Proposition 2.8 (Unconditional uniqueness of NLS in $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$)**.** *The solution u in Proposition 2.6 is unique in the space $C([-T,T];H^s(\mathbb{R}^d))$ (with the value of $T > 0$ possibly shrinked by a constant factor).*

Proof. We only focus on the time interval $[0,T]$, since the analysis on $[-T,0]$ is the same. Let $R = 2\|u_0\|_{H^s(\mathbb{R}^d)}$. Our goal is to show that, whenever u is a solution to NLS (2.8) in $C([0,T];H^s(\mathbb{R}^d))$, we must have $\|u\|_{C([0,T];H^s(\mathbb{R}^d))} \leq R$. Thus, $u \in \overline{B}_R$, and so we can conclude from the uniqueness part proved in Proposition 2.6.

We first note that the function $t \mapsto \|u\|_{C([0,t];H^s(\mathbb{R}^d))}$ is uniformly continuous on $[0,T]$. Then, there exists $\varepsilon > 0$ such that

$$\|u\|_{C([0,\varepsilon];H^s(\mathbb{R}^d))} \leq 2R.$$

By using (2.14), we get

$$\begin{aligned}\|u\|_{C([0,\varepsilon];H^s(\mathbb{R}^d))} &= \|\Gamma_{u_0}[u]\|_{C([0,\varepsilon];H^s(\mathbb{R}^d))} \\ &\leq \|u_0\|_{H^s(\mathbb{R}^d)} + CT\|u\|^p_{C([0,\varepsilon];H^s(\mathbb{R}^d))} \\ &\leq \frac{R}{2} + 2^p CTR^p \\ &\leq R\end{aligned} \tag{2.16}$$

as long as we have $0 < T \leq \frac{1}{2^{p+1}CR^{p-1}}$.

Suppose that we now have

$$\|u\|_{C([0,t];H^s(\mathbb{R}^d))} \leq R$$

for some $0 < t < T$. Then, by continuity, there exists $\varepsilon_0 > 0$ with $0 < t + \varepsilon_0 \leq T$ such that

$$\|u\|_{C([0,t+\varepsilon_0];H^s(\mathbb{R}^d))} \leq 2R.$$

By proceeding as in (2.16), we get

$$\begin{aligned}\|u\|_{C([0,t+\varepsilon_0];H^s(\mathbb{R}^d))} &\leq \|u_0\|_{H^s(\mathbb{R}^d)} + CT\|u\|^p_{C([0,t+\varepsilon_0];H^s(\mathbb{R}^d))} \\ &\leq \frac{R}{2} + 2^p CTR^p \\ &\leq R\end{aligned}$$

given $0 < T \leq \frac{1}{2^{p+1}CR^{p-1}}$. By induction, we conclude that

$$\|u\|_{C([0,T];H^s(\mathbb{R}^d))} \leq R,$$

as desired. □

We are interested in establishing local well-posedness for NLS in $H^s(\mathbb{R}^d)$ with $s \leq \frac{d}{2}$. Before that, we need to know how rough we can go (i.e. how low the value of s can be) for local well-posedness.

Let us define the following homogeneous L^2-based Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$ via

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \big\||\xi|^s \widehat{f}(\xi)\big\|_{L^2_\xi(\mathbb{R}^d)}$$

given $d \in \mathbb{N}$ and $s \in \mathbb{R}$. Let u be a solution to the following NLS on $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u \\ u|_{t=0} = u_0. \end{cases} \tag{2.17}$$

Then, one can easily check that for any $\lambda > 0$,

$$u^\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

is a solution to NLS (2.17) with scaled initial data

$$u_0^\lambda(x) = \lambda^{\frac{2}{p-1}} u_0(\lambda x).$$

Let us compute the $\dot{H}^s(\mathbb{R}^d)$-norm of u_0^λ. Note that for any $\xi \in \mathbb{R}^d$, we have

$$\widehat{u_0^\lambda}(\xi) = \lambda^{\frac{2}{p-1}-d} \widehat{u_0}(\lambda^{-1}\xi).$$

Thus, by using a change of variable, we have

$$\begin{aligned} \|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} &= \bigg(\int_{\mathbb{R}^d} |\xi|^{2s} \big|\widehat{u_0^\lambda}(\xi)\big|^2 d\xi \bigg)^{\frac{1}{2}} \\ &= \lambda^{\frac{2}{p-1}-d+s+\frac{d}{2}} \bigg(\int_{\mathbb{R}^d} |\lambda^{-1}\xi|^{2s} \big|\widehat{u_0}(\lambda^{-1}\xi)\big|^2 d(\lambda^{-1}\xi) \bigg)^{\frac{1}{2}} \\ &= \lambda^{\frac{2}{p-1}-\frac{d}{2}+s} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)}. \end{aligned}$$

We define

$$s_c := \frac{d}{2} - \frac{2}{p-1},$$

which is the index that achieves

$$\|u_0^\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$$

for any $\lambda > 0$. The index s_c is called the *scaling-critical Sobolev index* and gives the following three regimes for the Cauchy problem of NLS (2.17) given $u_0 \in H^s(\mathbb{R}^d)$:

- NLS is *subcritical* (with respect to scaling) if $s > s_c$. In this case, we expect good behaviors of the equation, such as local well-posedness. Indeed, we have

  $$\|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s-s_c} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)},$$

  and the solution u on $[0,T]$ corresponds to the solution u^λ on $[0, \lambda^{-2}T]$. As λ gets smaller, $\|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)}$ gets smaller and the time interval $[0, \lambda^{-2}T]$ gets larger. This means that smaller data implies that the solution lives longer, which makes sense intuitively.

- NLS is *supercritical* (with respect to scaling) if $s < s_c$. In this case, we expect bad behaviors of the equation, such as ill-posedness. Indeed, as λ gets smaller, $\|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)}$ gets larger and the time interval $[0, \lambda^{-2}T]$ gets larger. This means that larger data implies that the solution lives longer, which does not sound reasonable.
- NLS is *critical* (with respect to scaling) if $s = s_c$. This case corresponds to a delicate balance between the linear dispersion and the nonlinear concentration. We usually need more information than the $\dot{H}^s(\mathbb{R}^d)$-norm of initial data.

2.3. **Strichartz estimates for Schrödinger equations.** An important tool for establishing low regularity local well-posedness of NLS, or dispersive PDEs in general, is the *Strichartz estimates*. Let us briefly mention the heuristics. For a fixed time $t \in \mathbb{R}$, we have the isometry

$$\|e^{it\Delta} f\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^s(\mathbb{R}^d)}.$$

This basically tells us that the linear Schrödinger operator $e^{it\Delta}$ did nothing at any fixed time $t \in \mathbb{R}$. Nevertheless, we can gain from $e^{it\Delta}$ by taking an "average" over $t \in \mathbb{R}$.

Before moving on, we first recall the following interpolation theorem.

Theorem 2.9 (Riesz-Thorin interpolation theorem)**.** *Let $d \in \mathbb{N}$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let T be a bounded linear operator on functions on $\mathbb{R}^d$ such that*

$$\|T(f)\|_{L^{q_0}(\mathbb{R}^d)} \leq C_0 \|f\|_{L^{p_0}(\mathbb{R}^d)},$$
$$\|T(f)\|_{L^{q_1}(\mathbb{R}^d)} \leq C_1 \|f\|_{L^{p_1}(\mathbb{R}^d)}$$

for some $C_0, C_1 > 0$. Then, for all $0 < \theta < 1$, we have

$$\|T(f)\|_{L^q(\mathbb{R}^d)} \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^p(\mathbb{R}^d)},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

We now prove the following dispersive estimates.

Lemma 2.10 (Dispersive estimates for Schrödinger equations)**.** *Let $d \in \mathbb{N}$.*

(i) *For any $t \neq 0$, we have*

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi|t|)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.18}$$

(ii) *For any $t \neq 0$ and $2 \leq p \leq \infty$, we have*

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L^{p'}(\mathbb{R}^d)}, \tag{2.19}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. We claim that

$$\int_{\mathbb{R}} e^{-ix^2} dx = \sqrt{\frac{\pi}{i}}, \tag{2.20}$$

which is called the Fresnel integral. Taking (2.20) as granted, we compute by using a change of variable that

$$\begin{aligned}(e^{-4\pi^2 it|\xi|^2})^\vee(x) &= \int_{\mathbb{R}^d} e^{-4\pi^2 it|\xi|^2} e^{2\pi i\xi\cdot x} d\xi \\ &= \prod_{j=1}^d \left(e^{\frac{ix_j^2}{4t}} \int_{\mathbb{R}} e^{-4\pi^2 it(\xi_j - \frac{x_j}{4\pi t})^2} d\xi_j \right) \\ &= \prod_{j=1}^d e^{\frac{ix_j^2}{4t}} \cdot \sqrt{\frac{1}{4\pi it}} \\ &= \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{\frac{i|x|^2}{4t}}.\end{aligned}$$

Thus, we have

$$e^{it\Delta}f = \left(e^{-4\pi^2 it|\xi|^2}\widehat{f}(\xi)\right)^\vee = (e^{-4\pi^2 it|\xi|^2})^\vee * f = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y)dy,$$

so that the desired estimate (2.18) follows directly from this formula.

For (2.19), we first note that the case $p = 2$ follows from Lemma 2.4 (i) and the case $p = \infty$ follows from (2.18). For $2 < p < \infty$, we let $0 < \theta < 1$ be such that

$$\frac{1}{p} = \frac{\theta}{\infty} + \frac{1-\theta}{2} \quad \Longleftrightarrow \quad \theta = 1 - \frac{2}{p}.$$

Thus, since $\frac{1}{p'} = 1 - \frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$, by using the Riesz-Thorin interpolation theorem with $q_0 = 2$, $p_0 = 2$, $q_1 = \infty$, and $p_1 = 1$, we obtain

$$\|e^{it\Delta}f\|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{\frac{d}{2}\theta}} \|f\|_{L^{p'}(\mathbb{R}^d)} = \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L^{p'}(\mathbb{R}^d)},$$

as desired.

We now consider (2.20), which we prove using complex analysis. Let $R > 0$ be a large number and let γ_R be line segment from 0 to $Re^{i\frac{\pi}{4}}$. Let us consider the line integral

$$I := \lim_{R\to\infty} \int_{\gamma_R} e^{-z^2} dz.$$

On the one hand, by computing the integral directly using the parametrization $t \mapsto te^{i\frac{\pi}{4}}$, we have

$$I = e^{i\frac{\pi}{4}} \int_0^\infty e^{-it^2} dt = \frac{\sqrt{i}}{2} \int_{\mathbb{R}} e^{-ix^2} dx.$$

On the other hand, since $z \mapsto e^{-z^2}$ does not have any poles, by using Cauchy's theorem, we have

$$I = I_1 + I_2 := \lim_{R\to\infty} \int_{\gamma_{R,1}} e^{-z^2} dz + \lim_{R\to\infty} \int_{\gamma_{R,2}} e^{-z^2} dz,$$

where $\gamma_{R,1}$ is the line segment from 0 to R and $\gamma_{R,2}$ is the arc (of radius R centered at 0) from R to $Re^{i\frac{\pi}{4}}$. For I_1, we have

$$I_1 = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

For I_2, by using the parametrization $\theta \mapsto Re^{i\theta}$, we have

$$\begin{aligned}
\left|\int_{\gamma_{R,2}} e^{-z^2}dz\right| &= \left|iR\int_0^{\frac{\pi}{4}} e^{-R^2(\cos(2\theta)+i\sin(2\theta))}d\theta\right| \\
&\leq \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\cos\theta}d\theta \\
&= \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin\theta}d\theta \\
\Big(\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}\Big) \qquad &\leq \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R^2\theta}d\theta \lesssim \frac{1}{R} \longrightarrow 0
\end{aligned}$$

as $R \to \infty$, so that $I_2 = 0$. Thus, we obtain

$$\int_{\mathbb{R}} e^{-ix^2}dx = \frac{2}{\sqrt{i}}I = \frac{2}{\sqrt{i}}I_1 + \frac{2}{\sqrt{i}}I_2 = \sqrt{\frac{\pi}{i}}\,,$$

as desired. □

Before stating and proving the Strichartz estimates, we need the following two lemmas.

For any vector spaces A and B, we denote by $L(A;B)$ as the space of linear maps from A to B. We also denote the dual space $A^* = L(A;\mathbb{C})$. For any $f \in A$ and $\varphi \in A^*$, we write $\langle \varphi, f\rangle_A$ as the duality pairing between A^* and A. For an operator T, we denote by $\mathrm{Ran}(T)$ as the range of T.

Lemma 2.11 (T^*T argument)**.** *Let H be a Hilbert space, X a Banach space, and $D \subset X$ a vector space dense in X. Let $T \in L(D;H)$ and let $T^* \in L(H;D^*)$ be its adjoint defined by*

$$\langle T^*v, f\rangle_D := \langle v, Tf\rangle_H$$

for any $f \in D$ and $v \in H$. Then, the following three statements are equivalent.

(i) *For all $f \in D$,*

$$\|Tf\|_H \lesssim \|f\|_X.$$

(ii) $\mathrm{Ran}(T^*) \subset X^*$ *and for all $v \in H$,*

$$\|T^*v\|_{X^*} \lesssim \|v\|_H.$$

(iii) $\mathrm{Ran}(T^*T) \subset X^*$ *and for all $f \in D$,*

$$\|T^*Tf\|_{X^*} \lesssim \|f\|_X.$$

If the above conditions are satisfied, then the operators T and T^*T can be extended to bounded operators from X to H and from X to X^*, respectively.

Proof. We first show that (i) implies (ii). For any $v \in H$ and $f \in D$, we have

$$|\langle T^*v, f\rangle_D| = |\langle v, Tf\rangle_H| \leq \|v\|_H\|Tf\|_H \lesssim \|v\|_H\|f\|_X.$$

By density of D in X and continuity, we see that $T^*v \in X^*$ and $\|T^*v\|_{X^*} \lesssim \|v\|_H$.

We then show that (ii) implies (i). For any $f \in D$ and $v \in H$, we have

$$|\langle v, Tf\rangle_H| = |\langle T^*v, f\rangle_D| \leq \|T^*v\|_{X^*}\|f\|_X \lesssim \|v\|_H\|f\|_X,$$

which gives the desired bound by duality.

We also see that clearly (i) and (ii) imply (iii), so that (i) or (ii) implies (iii). It remains to show that (iii) implies (i). For any $f \in D$, we have

$$\langle Tf, Tf\rangle_H = \langle T^*Tf, f\rangle_D \le \|T^*Tf\|_{X^*}\|f\|_X \lesssim \|f\|_X^2,$$

which gives the desired bound. □

Lemma 2.12 (Hardy-Littlewood-Sobolev inequality)**.** *Let $d \in \mathbb{N}$ and $1 < p, q, r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then, we have*

$$\left\| \frac{1}{|\cdot|^{\frac{d}{p}}} * f \right\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^q(\mathbb{R}^d)}.$$

Proof. Let $N > 0$ be chosen later. We write

$$\left(\frac{1}{|\cdot|^{\frac{d}{p}}} * f\right)(x) = \int_{\{|y-x|\le N\}} \frac{f(y)}{|x-y|^{\frac{d}{p}}}dy + \int_{\{|y-x|>N\}} \frac{f(y)}{|x-y|^{\frac{d}{p}}}dy.$$

By decomposing into dyadic annuli:

$$\{|y-x| \le N\} = \bigcup_{j=0}^{\infty}\{2^{-j-1}N < |y-x| \le 2^{-j}N\},$$

we obtain

$$\left| \int_{\{|y-x|\le N\}} \frac{f(y)}{|x-y|^{\frac{d}{p}}}dy \right| \le N^{d-\frac{d}{p}} \sum_{j=0}^{\infty} 2^{-j(d-\frac{d}{p})} Mf(x) \lesssim N^{d-\frac{d}{p}} Mf(x) = N^{\frac{d}{q}-\frac{d}{r}} Mf(x),$$

where Mf denotes the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{\substack{B \text{ ball} \\ B \ni x}} \frac{1}{|B|}\int_B |f(y)|dy.$$

Also, by Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\{|y-x|>N\}} \frac{f(y)}{|x-y|^{\frac{d}{p}}}dy \right| &\le \left(\int_{\{|y|>N\}} \frac{1}{|y|^{\frac{dq'}{p}}}dy \right)^{\frac{1}{q'}} \|f\|_{L^q(\mathbb{R}^d)} \\ &\lesssim N^{d-\frac{d}{q}-\frac{d}{p}}\|f\|_{L^q(\mathbb{R}^d)} \\ &= N^{-\frac{d}{r}}\|f\|_{L^q(\mathbb{R}^d)}, \end{aligned}$$

where $\frac{1}{q'} + \frac{1}{q} = 1$. Thus, we have the pointwise bound

$$\left(\frac{1}{|\cdot|^{\frac{d}{p}}} * f\right)(x) \lesssim N^{\frac{d}{q}-\frac{d}{r}} Mf(x) + N^{-\frac{d}{r}}\|f\|_{L^q(\mathbb{R}^d)}.$$

By optimizing with $N = (\|f\|_{L^q(\mathbb{R}^d)}/Mf(x))^{\frac{q}{d}}$, taking the L^r-norm in x, and using the boundedness of the Hardy-Littlewood maximal operator in $L^q(\mathbb{R}^d)$, we obtain

$$\left\| \frac{1}{|\cdot|^{\frac{d}{p}}} * f \right\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^q(\mathbb{R}^d)}^{1-\frac{q}{r}} \|Mf\|_{L^q(\mathbb{R}^d)}^{\frac{q}{r}} \lesssim \|f\|_{L^q(\mathbb{R}^d)},$$

as desired. □

Remark 2.13. The Hardy-Littlewood-Sobolev inequality is a stronger inequality than Young's convolution inequality, and it can be seen as an "endpoint case" of Young's convolution inequality.

We now use the dispersive estimates in Lemma 2.10 to establish the Strichartz estimates.

Theorem 2.14 (Strichartz estimates for Schrödinger equations)**.** *Let $d \in \mathbb{N}$. Let $2 \le q, r \le \infty$ be such that*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \textit{and} \quad (q, r, d) \neq (2, \infty, 2).$$

We call such (q, r) an admissible pair.

(i) *We have the following homogeneous Strichartz estimate*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) *We have the following dual homogeneous Strichartz estimate*

$$\left\| \int_{\mathbb{R}} e^{-it'\Delta} F(t') dt' \right\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.$$

(iii) *For any admissible pairs (q, r) and $(\widetilde{q}, \widetilde{r})$, we have the following inhomogeneous Strichartz estimate*

$$\left\| \int_0^t e^{i(t-t')\Delta} F(t') dt' \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Proof. We only prove the non-endpoint case $2 < q \le \infty$.

We first consider (i) and (ii). When $q = \infty$, we must have $r = 2$, so that (i) and (ii) follow easily. Let us now focus on the case $2 < q < \infty$. For any $F \in L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)$, we define the operator T as

$$TF = \int_{\mathbb{R}} e^{-it\Delta} F(t) dt.$$

For any $f \in L^2(\mathbb{R}^d)$ and $F \in L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\begin{aligned} \langle e^{it\Delta} f, F\rangle_{L_t^2 L_x^2(\mathbb{R}\times\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{it\Delta} f(x) \overline{F(t,x)} dt dx \\ &= \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}} \overline{e^{-it\Delta} F(t,x)} dt dx \\ &= \left\langle f, \int_{\mathbb{R}} e^{-it\Delta} F(t) dt \right\rangle_{L_x^2(\mathbb{R}^d)}, \end{aligned}$$

where we used

$$\langle e^{it\Delta} f, g\rangle_{L_x^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} e^{-it|\xi|^2} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi = \int_{\mathbb{R}^d} \widehat{f}(\xi) \overline{e^{it|\xi|^2} \widehat{g}(\xi)} d\xi = \langle f, e^{-it\Delta} g\rangle_{L_x^2(\mathbb{R}^d)} \tag{2.21}$$

from Parseval's relation. This shows that

$$T^* f = e^{it\Delta} f.$$

Thus,

$$T^* T F = \int_{\mathbb{R}} e^{i(t-t')\Delta} F(t') dt'.$$

We now compute that

$$
\begin{aligned}
&\|T^*TF\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \\
\text{(Minkowski)} \quad &\leq \Big\| \int_{\mathbb{R}} \|e^{i(t-t')\Delta}F(t')\|_{L_x^r(\mathbb{R}^d)} dt' \Big\|_{L_t^q(\mathbb{R})} \\
\text{(Lm 2.10)} \quad &\lesssim \Big\| \int_{\mathbb{R}} \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \|F(t')\|_{L_x^{r'}(\mathbb{R}^d)} dt' \Big\|_{L_t^q(\mathbb{R})} \\
\text{(H-L-S)} \quad &\lesssim \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}
\tag{2.22}
$$

One can check that the conditions for using the Hardy-Littlewood-Sobolev inequality are satisfied given

$$
d\Big(\frac{1}{2}-\frac{1}{r}\Big)+\frac{1}{q'}=1+\frac{1}{q} \quad \Longleftrightarrow \quad \frac{2}{q}+\frac{d}{r}=\frac{d}{2}
$$

and

$$
0<d\Big(\frac{1}{2}-\frac{1}{r}\Big), \frac{1}{q'}, \frac{1}{q}<1 \quad \Longleftrightarrow \quad 1<q<\infty \text{ and } 2<r<\frac{2d}{d-2}.
$$

The condition for r is satisfied given $2<q<\infty$. Then, the T^*T argument gives (i) and (ii).

It remains to prove (iii). From (i) and (ii), we have

$$
\begin{aligned}
\Big\| \int_{\mathbb{R}} e^{i(t-t')\Delta}F(t')dt' \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} &= \Big\| e^{it\Delta} \int_{\mathbb{R}} e^{-it'\Delta}F(t')dt' \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \int_{\mathbb{R}} e^{-it'\Delta}F(t')dt' \Big\|_{L_x^2(\mathbb{R}^d)} \\
&\lesssim \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}
\tag{2.23}
$$

Using similar steps, for any fixed $-\infty \leq a < b \leq \infty$, we also have

$$
\begin{aligned}
\Big\| \int_a^b e^{i(t-t')\Delta}F(t')dt' \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} &= \Big\| \int_{\mathbb{R}} e^{i(t-t')\Delta}\mathbf{1}_{(a,b)}(t')F(t')dt' \Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \|\mathbf{1}_{(a,b)}(t')F(t')\|_{L_{t'}^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)} \\
&\leq \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}
\tag{2.24}
$$

We need to insert a time interval depending on t. When $(q,r)=(\widetilde{q},\widetilde{r})$, we use similar steps as in (2.22) to obtain

$$
\begin{aligned}
\Big\| \int_{-\infty}^t e^{i(t-t')\Delta}F(t')dt' \Big\|_{L_t^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{R}\times\mathbb{R}^d)} &= \Big\| \int_{\mathbb{R}} e^{i(t-t')\Delta}\mathbf{1}_{(-\infty,t]}(t')F(t')dt' \Big\|_{L_t^{\widetilde{q}} L_x^{\widetilde{r}}(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \Big\| \int_{\mathbb{R}} \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \|\mathbf{1}_{(-\infty,t]}F(t')\|_{L_x^{\widetilde{r}'}(\mathbb{R}^d)} dt' \Big\|_{L_t^q(\mathbb{R})} \\
&\leq \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}
\tag{2.25}
$$

Then, when $(q,r)=(\infty,2)$, for any fixed $t\in\mathbb{R}$, we perform the following computations:

$$\begin{aligned}
\Big\| \int_{-\infty}^{t} e^{i(t-t')\Delta}F(t')dt' \Big\|^2_{L^2_x(\mathbb{R}^d)} &= \Big\langle \int_{-\infty}^{t} e^{i(t-t_1)\Delta}F(t_1)dt_1, \int_{-\infty}^{t} e^{i(t-t_2)\Delta}F(t_2)dt_2 \Big\rangle_{L^2_x(\mathbb{R}^d)}\\
(2.21)\quad &= \Big\langle \int_{-\infty}^{t} e^{-it_1\Delta}F(t_1)dt_1, \int_{-\infty}^{t} e^{-it_2\Delta}F(t_2)dt_2 \Big\rangle_{L^2_x(\mathbb{R}^d)}\\
(2.21)\quad &= \int_{-\infty}^{t} \Big\langle F(t_1), \int_{-\infty}^{t} e^{i(t_1-t_2)\Delta}F(t_2)dt_2 \Big\rangle_{L^2_x(\mathbb{R}^d)} dt_1\\
(\text{Hölder})\quad &\le \int_{\mathbb{R}} \|F(t_1)\|_{L^{\widetilde{r}'}_x(\mathbb{R}^d)} \Big\| \int_{-\infty}^{t} e^{i(t_1-t_2)}F(t_2)dt_2 \Big\|_{L^{\widetilde{r}}_x(\mathbb{R}^d)} dt_1\\
(\text{Hölder})\quad &\le \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)} \Big\| \int_{-\infty}^{t} e^{i(t_1-t_2)}F(t_2)dt_2 \Big\|_{L^{\widetilde{q}}_{t_1} L^{\widetilde{r}}_x(\mathbb{R}\times\mathbb{R}^d)}\\
(2.24)\quad &\lesssim \|F\|^2_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}$$

This gives

$$\Big\| \int_{-\infty}^{t} e^{i(t-t')\Delta}F(t')dt' \Big\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.26}$$

To prove (iii), thanks to (2.24), we write $\mathbf{1}_{[0,t]}=\mathbf{1}_{(-\infty,t]}-\mathbf{1}_{(-\infty,0)}$, so that our goal is to show

$$\Big\| \int_{-\infty}^{t} e^{i(t-t')\Delta}F(t')dt' \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.27}$$

If $q\ge\widetilde{q}$, (2.27) follows from interpolating (2.25) and (2.26). If $2<q<\widetilde{q}$, we have

$$\begin{aligned}
&\Big\| \int_{t}^{\infty} e^{i(t-t')\Delta}F(t')dt' \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)}\\
(\text{duality})\quad &= \sup_{\|G\|_{L^{q'}_t L^{r'}_x}\le 1} \Big| \int_{\mathbb{R}} \Big\langle \int_{t}^{\infty} e^{i(t-t')\Delta}F(t')dt', G(t) \Big\rangle_{L^2_x(\mathbb{R}^d)} dt \Big|\\
((2.21)\ \&\ \text{Fubini})\quad &= \sup_{\|G\|_{L^{q'}_t L^{r'}_x}\le 1} \Big| \int_{\mathbb{R}} \Big\langle F(t'), \int_{-\infty}^{t'} e^{i(t'-t)\Delta}G(t)dt \Big\rangle_{L^2_x(\mathbb{R}^d)} dt' \Big|\\
(\text{Hölder})\quad &\le \sup_{\|G\|_{L^{q'}_t L^{r'}_x}\le 1} \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)} \Big\| \int_{-\infty}^{t} e^{i(t-t')\Delta}G(t')dt' \Big\|_{L^{\widetilde{q}}_t L^{\widetilde{r}}_x(\mathbb{R}\times\mathbb{R}^d)}\\
((2.27)\text{ with } q\ge\widetilde{q})\quad &\lesssim \sup_{\|G\|_{L^{q'}_t L^{r'}_x}\le 1} \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)} \|G\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^d)}\\
&= \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}.
\end{aligned}$$

This gives (2.27) by writing $\mathbf{1}_{(-\infty,t]}=1-\mathbf{1}_{(t,\infty)}$ and using (2.23). □

2.4. **More on local well-posedness of NLS.** In this subsection, we consider two examples of local well-posedness of NLS with low regularity.

Our first example is the 1-d cubic NLS in $L^2(\mathbb{R})$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \pm|u|^2 u \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}). \end{cases} \tag{2.28}$$

The scaling critical regularity for the 1-d cubic NLS (2.28) is $s = -\frac{1}{2}$, and so we expect (2.28) to be locally well-posed.

Proposition 2.15 (Local well-posedness of cubic NLS in $L^2(\mathbb{R})$)**.** *For any $u_0 \in L^2(\mathbb{R})$, there exists $T = T(\|u_0\|_{L^2(\mathbb{R})}) > 0$ such that there exists a unique solution u to NLS (2.28) in the space $C([-T,T]; L^2(\mathbb{R})) \cap L^8([-T,T]; L^4(\mathbb{R}))$, and the solution u depends continuously on $u_0 \in L^2(\mathbb{R})$.*

Proof. We note that $(q,r) = (8,4)$ is an admissible pair for the Strichartz estimates with $d = 1$. We construct the solution in the space $X_T = C([-T,T]; L^2(\mathbb{R})) \cap L^8([-T,T]; L^4(\mathbb{R}))$ equipped with the norm

$$\|u\|_{X_T} = \|u\|_{C_T L^2_x(\mathbb{R})} + \|u\|_{L^8_T L^4_x(\mathbb{R})}.$$

We define

$$\Gamma_{u_0}[u](t) := e^{it\partial_x^2} u_0 \mp i \int_0^t e^{i(t-t')\partial_x^2} (|u|^2 u)(t') dt'. \tag{2.29}$$

By Lemma 2.4 (i) and the homogeneous Strichartz estimate in Proposition 2.14 (i), we have

$$\|e^{it\partial_x^2} u_0\|_{X_T} = \|e^{it\partial_x^2} u_0\|_{C_T L^2_x(\mathbb{R})} + \|e^{it\partial_x^2} u_0\|_{L^8_T L^4_x(\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})}. \tag{2.30}$$

By using the inhomogeneous Strichartz estimate in Proposition 2.14 (iii) with $(\widetilde{q}, \widetilde{r}) = (8,4)$ and Hölder's inequality in time, we have

$$\begin{aligned} &\bigg\| \int_0^t e^{i(t-t')\partial_x^2} (|u|^2 u)(t') dt' \bigg\|_{X_T} \\ &= \bigg\| \int_0^t e^{i(t-t')\partial_x^2} (|u|^2 u)(t') dt' \bigg\|_{L^\infty_T L^2_x(\mathbb{R})} + \bigg\| \int_0^t e^{i(t-t')\partial_x^2} (|u|^2 u)(t') dt' \bigg\|_{L^8_T L^4_x(\mathbb{R})} \\ &\lesssim \big\| |u|^2 u \big\|_{L_T^{\frac{8}{7}} L_x^{\frac{4}{3}}(\mathbb{R})} \lesssim T^{\frac{1}{2}} \big\| |u|^2 u \big\|_{L_T^{\frac{8}{3}} L_x^{\frac{4}{3}}(\mathbb{R})} \\ &= T^{\frac{1}{2}} \|u\|^3_{L^8_T L^4_x(\mathbb{R})} \le T^{\frac{1}{2}} \|u\|^3_{X_T}. \end{aligned} \tag{2.31}$$

Thus, from (2.30) and (2.31), we get

$$\|\Gamma_{u_0}[u]\|_{X_T} \lesssim \|u_0\|_{L^2(\mathbb{R})} + T^{\frac{1}{2}} \|u\|^3_{X_T}.$$

Using similar steps, we get the following difference estimate:

$$\|\Gamma_{u_0}[u] - \Gamma_{u_0}[v]\|_{X_T} \lesssim T^{\frac{1}{2}} \big(\|u\|_{X_T} + \|v\|_{X_T} \big) \|u - v\|_{X_T}.$$

Thus, by choosing $T = T(\|u_0\|_{L^2(\mathbb{R})}) > 0$ sufficiently small, we obtain that Γ_{u_0} is a contraction on a ball in X_T, which gives the desired local well-posedness result. □

Our second example is the 2-d cubic NLS in $L^2(\mathbb{R}^2)$:

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^2 u \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^2). \end{cases} \tag{2.32}$$

The scaling critical regularity for the 2-d cubic NLS (2.32) is $s = 0$, and so we are in the scaling critical case. It turns out that (2.32) is locally well-posed.

Proposition 2.16 (Local well-posedness of cubic NLS in $L^2(\mathbb{R}^2)$)**.** *For any $u_0 \in L^2(\mathbb{R}^2)$, there exists $T = T(u_0) > 0$ such that there exists a unique solution u to NLS* (2.28) *in the space $C([-T,T]; L^2(\mathbb{R}^2)) \cap L^4([-T,T]; L^4(\mathbb{R}^2))$, and the solution u depends continuously on $u_0 \in L^2(\mathbb{R})$.*

Proof. We note that $(q,r) = (4,4)$ is as admissible pair for the Strichartz estimate with $d = 2$. We construct the solution in the space $X_T = C([-T,T]; L^2(\mathbb{R})) \cap L^4([-T,T]; L^4(\mathbb{R}))$ equipped with the norm

$$\|u\|_{X_T} = \|u\|_{C_T L^2_x(\mathbb{R})} + \|u\|_{L^4_T L^4_x(\mathbb{R})}.$$

Let Γ_{u_0} be as in (2.29). If we proceed as in the 1-d case, we get

$$\|\Gamma_{u_0}[u]\|_{X_T} \le C\|u_0\|_{L^2(\mathbb{R})} + C\|u\|^3_{X_T}.$$

Note that there is no power of T in front of $\|u\|^3_{X_T}$. This argument still works, but only for small enough initial data.

To cover arbitrary $L^2(\mathbb{R}^2)$-initial data, we do not apply the Strichartz estimate for the linear Schrödinger propagator and arrive at

$$\|\Gamma_{u_0}(u)\|_{L^4_T L^4_x(\mathbb{R}^2)} \le \|e^{it\Delta}u_0\|_{L^4_T L^4_x(\mathbb{R}^2)} + C\|u\|^3_{L^4_T L^4_x(\mathbb{R}^2)}.$$

Let $\eta > 0$ be a small number to be specified later. Then, since $u_0 \in L^2(\mathbb{R}^2)$, there exists a small $T = T(u_0) > 0$ such that

$$\|e^{it\Delta}u_0\|_{L^4_T L^4_x(\mathbb{R}^2)} \le \frac{\eta}{2}.$$

We define

$$\overline{B}_\eta := \{u \in L^4([-T,T]; L^4(\mathbb{R}^2)); \|u\|_{L^4_T L^4_x(\mathbb{R}^2)} \le \eta\}.$$

Then, for any $u \in \overline{B}_\eta$, we have

$$\|\Gamma_{u_0}(u)\|_{L^4_T L^4_x(\mathbb{R}^2)} \le \frac{\eta}{2} + C\eta^3 \le \eta$$

given $\eta > 0$ sufficiently small. Also, for any $u, v \in \overline{B}_\eta$, we have

$$\begin{aligned}\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{L^4_T L^4_x(\mathbb{R}^2)} &\le C'\big(\|u\|^2_{L^4_T L^4_x(\mathbb{R}^2)} + \|v\|^2_{L^4_T L^4_x(\mathbb{R}^2)}\big)\|u - v\|_{L^4_T L^4_x(\mathbb{R}^2)} \\ &\le 2C'\eta^2\|u - v\|_{L^4_T L^4_x(\mathbb{R}^2)},\end{aligned}$$

where we may further shrink η so that $2C'\eta^2 \le \frac{1}{2}$. We then run the contraction argument on $\overline{B}_\eta$ to obtain a unique solution u to (2.32). We also note that $u \in C([-T,T]; L^2(\mathbb{R}^2))$ via

$$\|u\|_{C_T L^2_x(\mathbb{R}^2)} = \|\Gamma_{u_0}[u]\|_{C_T L^2_x(\mathbb{R}^2)} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)} + \|u\|^3_{L^4_T L^4_x(\mathbb{R}^2)} \le \|u_0\|_{L^2(\mathbb{R}^2)} + \eta^3 < \infty.$$

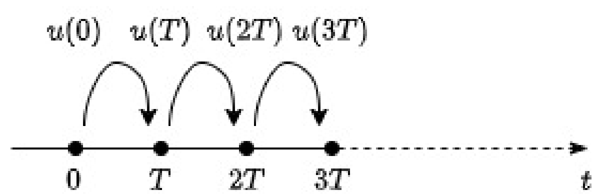
□

Remark 2.17. For the 2-d cubic NLS, the local existence time T depends on the profile of the initial data u_0 (but not just on $\|u_0\|_{L^2(\mathbb{R}^2)}$ as in the scaling subcritical cases). Such phenomenon is the critical nature of the problem.

2.5. **Global well-posedness of NLS.** In this subsection, we show that in some desired situations, we can upgrade local-in-time well-posedness result of NLS into a global-in-time well-posedness result.

For simplicity, let us focus on the positive time line. By iterating the local well-posedness argument, we are able to extend the solution u to the time interval $[0, T_{\max})$, which means that $u \in C([0, T_{\max}); H^s(\mathbb{R}^d))$. If $T_{\max} = \infty$, we obtain a global-in-time solution, which means that the equation is globally well-posed. If $T_{\max} < \infty$, then we must have $\lim_{t \nearrow T_{\max}} \|u(t)\|_{H^s(\mathbb{R}^d)} = \infty$ (otherwise we have a contradiction in view of local well-posedness), which refers to the finite-time blowup phenomenon.

Let us focus on good behaviors of the solution, i.e. when we are able to obtain global-in-time solutions. In the local well-posedness result we have already seen (except for the scaling critical case), the local existence time $T > 0$ depends proportionally on $\|u_0\|_{H^s(\mathbb{R}^d)}^{-\theta}$ for some $\theta > 0$. Thus, if we can provide an upper bound for $\|u(t)\|_{H^s(\mathbb{R}^d)}$ for all $t \in \mathbb{R}$, we are able to iterate the local well-posedness argument and obtain a global-in-time solution. The idea is illustrated in the following figure:



For NLS, one can use *conservation laws* to obtain upper bounds for $\|u(t)\|_{H^s(\mathbb{R}^d)}$. There are two particularly useful conservation laws for NLS:

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = u_0. \end{cases} \tag{2.33}$$

One is the conservation of *mass*:

$$M[u](t) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

The other one is the conservation of *energy* (or *Hamiltonian*):

$$E[u](t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx \pm \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx.$$

By conservation, we mean $M[u](t) = M[u](0)$ and $E[u](t) = E[u](0)$ for any $t \in \mathbb{R}$, provided that all quantities are well-defined and finite. Let us assume that u is a smooth solution to

(2.33). Then, by using the equation with an integration by parts, we get

$$\begin{aligned} \frac{d}{dt}M[u](t) &= \frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2 dx \\ &= 2\operatorname{Re}\int_{\mathbb{R}^d} \partial_t u(t,x)\overline{u(t,x)}dx \\ &= 2\operatorname{Re} i\int_{\mathbb{R}^d} \Delta u(t,x)\overline{u(t,x)}dx \mp 2\operatorname{Re} i\int_{\mathbb{R}^d} |u(t,x)|^{p+1}dx \\ &= -2\operatorname{Re} i\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx \\ &= 0, \end{aligned} \tag{2.34}$$

which shows the conservation of mass. Similarly, one can also verity the conservation of energy.

However, the computation in (2.34) is only formal, since at the moment our solution constructed using the Banach-fixed point theorem is only in the sense of the Duhamel formulation. To show conservation of mass and energy for general solutions (as opposed to smooth solutions), some extra work is needed.

Let us first mention the following product lemma, whose proof can be found in [4, Lemma A.8].

Lemma 2.18 (Product lemma)**.** *Let $d \in \mathbb{N}$ and $s \geq 0$. Then, we have*

$$\|fg\|_{H^s(\mathbb{R}^d)} \lesssim_{s,d} \|f\|_{H^s(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)}.$$

Let us now show an important property called the persistence of regularity. Namely, under certain conditions, if the initial data lies in some higher order Sobolev space, then the solution remains bounded in the same Sobolev space.

Proposition 2.19 (Persistence of regularity)**.** *Let $d \in \mathbb{N}$, $s \geq 0$, $T > 0$, and u be a solution to* (2.33) *with $u_0 \in H^s(\mathbb{R}^d)$. If $u \in L^{p-1}([-T,T]; L^\infty(\mathbb{R}^d))$, then $u \in C([-T,T]; H^s(\mathbb{R}^d))$.*

Proof. From the Duhamel formulation, we have

$$u(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)(t')dt'.$$

Then, for any $t \in [-T,T]$, we have

$$\begin{aligned} \|u(t)\|_{H^s(\mathbb{R}^d)} &\leq \|u_0\|_{H^s(\mathbb{R}^d)} + \int_0^t \big\|(|u|^{p-1}u)(t')\big\|_{H^s(\mathbb{R}^d)}dt' \\ \text{(Lm 2.18)} \quad &\leq \|u_0\|_{H^s(\mathbb{R}^d)} + C_{s,d}\int_0^t \|u(t')\|_{L^\infty(\mathbb{R}^d)}^{p-1}\|u(t')\|_{H^s(\mathbb{R}^d)}dt'. \end{aligned}$$

Thus, by Grönwall's inequality, we get

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \leq \|u_0\|_{H^s(\mathbb{R}^d)} \exp\Big(C_{s,d}\|u\|_{L_T^{p-1}L_x^\infty(\mathbb{R}^d)}^{p-1}\Big).$$

Together with the continuity of u in time, this bound gives the desired property. □

Let us mention another very useful tool in the study of PDEs.

Lemma 2.20 (Sobolev's inequality)**.** *Let $d \in \mathbb{N}$.*
(i) *If $2 < p < \infty$ and $s > 0$ satisfy*

$$\frac{s}{d} \geq \frac{1}{2} - \frac{1}{p},$$

then we have

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim_{s,d,p} \|f\|_{H^s(\mathbb{R}^d)}.$$

(ii) *If $s > 0$ satisfies*

$$\frac{s}{d} > \frac{1}{2},$$

then we have

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim_{s,d} \|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. We only prove the inequality with $\frac{s}{d} > \frac{1}{2} - \frac{1}{p}$. We have the following Hausdorff-Young's inequality:

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, which follows from interpolating

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \|\widehat{f}\|_{L^1(\mathbb{R}^d)} \quad \text{and} \quad \|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)}.$$

Thus, by Hölder's inequality, we get

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|\langle \cdot \rangle^s \widehat{f}\|_{L^2(\mathbb{R}^d)} \|\langle \cdot \rangle^{-s}\|_{L^{\frac{2p'}{2-p'}}(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)},$$

where we used the fact that

$$\frac{2sp'}{2-p'} = \frac{s}{\frac{1}{2} - \frac{1}{p}} > d.$$

This finishes the proof. □

Remark 2.21. Here is a more general version of Sobolev's inequality. For a proof, we refer the interested readers to [2, Theorem 1.3.5]. Let $d \in \mathbb{N}$, $1 < p \leq q < \infty$, and $s > 0$ be such that $\frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}$. Then, we have

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim_{s,d,p,q} \|\langle \nabla \rangle^s f\|_{L^p(\mathbb{R}^d)}.$$

If $q = \infty$ and $\frac{s}{d} > \frac{1}{p}$, then

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim_{s,d,p} \|\langle \nabla \rangle^s f\|_{L^p(\mathbb{R}^d)}.$$

The heuristic for Sobolev's ienquality is that we can trade regularity s for higher integrability.

Let us now consider the 1-d cubic NLS in $H^1(\mathbb{R})$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}). \end{cases} \tag{2.35}$$

The equation is known to be locally well-posed from the regime $s > \frac{d}{2}$.

Proposition 2.22 (Conservation of mass and energy)**.** *Let u be the local-in-time solution to* (2.35) *with $u_0 \in H^1(\mathbb{R})$ in $C([-T,T];H^1(\mathbb{R}))$ for some $T > 0$. Then, $M[u](t) = M[u](0)$ and $E[u](t) = E[u](0)$ for all $t \in [-T,T]$.*

Proof. Let us first consider smooth initial data $u_0 \in H^\infty(\mathbb{R}) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R})$. Our goal is to show that u satisfies (2.35) in the classical sense. By Hölder's inequality and Sobolev's inequality, we know that

$$\|u\|_{L^2_T L^\infty_x(\mathbb{R})} \lesssim \|u\|_{C_T H^1_x(\mathbb{R})} < \infty.$$

Thus, by the persistence of regularity, we know that

$$\|u\|_{C_T H^s_x(\mathbb{R})} < \infty$$

for any $s \geq 0$. From the Duhamel formulation, we note that

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} e^{-4\pi^2 it|\xi|^2} \widehat{u}_0(\xi) e^{2\pi i\xi\cdot x} d\xi \\ &\quad - i\int_0^t \int_{\mathbb{R}} e^{-4\pi^2 i(t-t')|\xi|^2} (|u|^2 u)^\wedge(t',\xi) e^{2\pi i\xi\cdot x} d\xi dt'. \end{aligned} \tag{2.36}$$

In order to take the time derivative of (2.36), we need to justify the switching of ∂_t with $\int_{\mathbb{R}} d\xi$, which can be done by using the Lebesgue dominated convergence theorem after checking that

$$\int_{\mathbb{R}} |\xi|^2 |\widehat{u}_0(\xi)| d\xi \leq \bigg(\int_{\mathbb{R}} \frac{|\xi|^4}{\langle\xi\rangle^6} d\xi \bigg)^{\frac{1}{2}} \bigg(\int_{\mathbb{R}} \langle\xi\rangle^6 |\widehat{u}_0(\xi)|^2 \bigg)^{\frac{1}{2}} \lesssim \|u_0\|_{H^3(\mathbb{R})} < \infty$$

and

$$\int_{\mathbb{R}^d} |\xi|^2 \big|(|u|^2 u)^\wedge(t,\xi)\big| d\xi \lesssim \big\||u|^2 u\big\|_{C_T H^3_x(\mathbb{R})} \lesssim \|u\|^3_{C_T H^3_x(\mathbb{R})} < \infty,$$

where we used the Cauchy-Schwarz inequalities and the algebraic property of $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$. Similarly, one can justify the switching of ∂_x^2 with $\int_{\mathbb{R}} d\xi$ and $\int_0^t dt'$. One can then compute that

$$\begin{aligned} \partial_t u(t,x) &= -4\pi^2 i|\xi|^2 \int_{\mathbb{R}^d} e^{-4\pi^2 it|\xi|^2} \widehat{u}_0(\xi) e^{2\pi i\xi\cdot x} d\xi \\ &\quad - i\int_{\mathbb{R}} (|u|^{p-1}u)^\wedge(t,\xi) e^{2\pi i\xi\cdot x} d\xi \\ &\quad - 4\pi^2|\xi|^2 \int_0^t \int_{\mathbb{R}} e^{-4\pi^2 i(t-t')|\xi|^2} (|u|^{p-1}u)^\wedge(t',\xi) e^{2\pi i\xi\cdot x} d\xi dt' \\ &= i\partial_x^2 u(t,x) - i(|u|^2 u)(t,x). \end{aligned}$$

We also check that

$$M[u](t) = \|u(t)\|^2_{L^2(\mathbb{R})} \leq \|u(t)\|^2_{H^1(\mathbb{R})} < \infty$$

and

$$E[u](t) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4}\|u(t)\|^4_{L^4(\mathbb{R})} \lesssim \|u(t)\|^2_{H^1(\mathbb{R})} + \|u(t)\|^4_{H^1(\mathbb{R})} < \infty$$

for any $t \in [-T,T]$, where we used Sobolev's inequality in the last step. Thus, we can perform the computation in (2.34) and get $M[u](t) = M[u](0)$ and $E[u](t) = E[u](0)$ for all $t \in [-T,T]$

We now consider rough initial data $u_0 \in H^1(\mathbb{R})$. We take a sequence of smooth functions $\{u_{0,n}\}_{n\in\mathbb{N}} \subset H^\infty(\mathbb{R})$ such that $u_{0,n} \to u_0$ in $H^1(\mathbb{R})$. Let u_n be the local-in-time solution to (2.33) with initial data $u_{0,n}$. Then, by the continuity of the solution map with respect to the initial data, we have $u_n \to u$ in $C([-T,T]; H^1(\mathbb{R}))$ for some $T > 0$. Thus, for any $t \in [-T,T]$, we have

$$M[u](t) = \lim_{n\to\infty} M[u_n](t) = \lim_{n\to\infty} M[u_n](0) = M[u](0)$$

and

$$E[u](t) = \lim_{n\to\infty} E[u_n](t) = \lim_{n\to\infty} E[u_n](0) = E[u](0),$$

as desired. □

References

[1] Á. Bényi, T. Oh, *The Sobolev inequality on the torus revisited.* Publ. Math. Debrecen 83 (2013), no. 3, 359–374.

[2] L. Grafakos, *Modern Fourier analysis.* Third edition. Grad. Texts in Math., 250 Springer, New York, 2014. xvi+624 pp.

[3] C. Sun, N. Tzvetkov, W. Xu, *Weak universality results for a class of nonlinear wave equations*, arXiv:2206.05945 [math.AP], to appear in Ann. Inst. Fourier.

[4] T. Tao, *Nonlinear dispersive equations. Local and global analysis.* CBMS Reg. Conf. Ser. Math., 106 Published for the Conference Board of the Mathematical Sciences, Washington, DC; by theAmerican Mathematical Society, Providence, RI, 2006. xvi+373 pp.

[5] N. Tzvetkov, *Random data wave equations.* arXiv:1704.01191v2 [math.AP].

[6] M. Vişan, *Oberwolfach Seminar: Dispersive Equations.* In "Dispersive Equations and Nonlinear Waves", Oberwolfach Seminars, 45, Birkhauser/Springer Basel AG, Basel, 2014.