V5B1 Advanced Topics in Analysis and PDEs
Transport Equations and Fluid Dynamics

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1 Introduction

The unifying topic of this course is the transport equation: one of the simplest linear Partial Differential Equations (PDEs), which is in a way the PDE analogue of ODEs

$$\partial_t f + u \cdot \nabla f = 0,$$

where the unknown is $f(t,x)$, and where $u(t,x)$ is given a vector field. Despite being extremely simple and linear, the transport equation is ubiquitous in many models from mathematical physics, especially fluid dynamics, as it represents the evolution of a set of particles in a (fluid) flow. As such, its properties have been intensively studied throughout the past decades, are still the topic of active research, and the results concerning it are routinely used in the analysis of non-linear PDEs.

The lectures will reflect the two sides of this story, as we will alternate between studying different aspects of the theory of the linear transport equation, and pairing each of these with an application to the non-linear PDEs of fluid dynamics. This will mainly involve examining how the solution is affected by the regularity of the vector field $u(t,x)$ (if u is smooth, Lipschitz, Sobolev $W^{1,p}$, log-Lipschitz, *etc.*). In doing so, we will introduce many of the important tools of harmonic analysis that are used in the modern study of PDEs (Singular Integral Operators, commutators, Besov Spaces and Littlewood-Paley analysis, *etc.*).

2 Tentative Table of Contents

2.1 Lebesgue Space Theory

Introduction: transport of particles, incompressibility, smooth theory.

Lebesgue Solutions: weak and weak-($*$) convergence, Friedrichs commutator, non-smooth solutions in Lebesgue spaces, propagation of higher regularity.

Application 1: incompressible inviscid fluids (Euler equations), existence and uniqueness of Yudovich solutions.

DiPerna-Lions Theory: concept of renormalized solutions, pointwise stability.

Application 2: existence of global weak solutions for a Stokes-Transport problem for a fluid with non-linear viscosity.

2.2 Besov Space Theory

Introduction to Besov Spaces: motivation, Littlewood-Paley decomposition, definition, paradifferential calculus.

Besov Solutions: commutator estimates, Besov solutions.

Application 3: Euler equations in 3D, local existence and uniqueness.

Logarithmic Interpolation: logarithmic interpolation inequalities, linear growth of Besov solutions of regularity zero.

Application 4: Beale-Kato-Majda criterion for blow-up of solutions, global Besov solutions of the Euler equations in 2D.

Application 5: Magnetohydrodynamics and the lifespan of 2D solutions.

2.3 Log-Lipschitz Vector Fields

Non-Lipschitz Velocities and Regularity Loss: log-Lipschitz vector fields, estimates where the regularity depends on time.

Application 6: uniqueness of solutions for a sedimentation equation in the critical Lebesgue space.

3 Prerequisites

Although a lot of theory will be introduced (especially Fourier analysis), no prior knowledge of either PDEs or physics is required. Familiarity with the following topics is nevertheless important:

- ODEs, Cauchy-Lipschitz theorem, multi-variable calculus.
- Integration Theory: Lebesgue spaces, Hölder inequalities, approximation by smooth functions.
- Basic functional analysis, duality, weak convergence.
- Sobolev spaces $W^{k,p}$, Sobolev embeddings, compact embeddings.
- Fourier transform, Plancherel identity.

4 Exam

Information on the exam will be given during the semester.

References and Suggested Readings

No reading is required prior or during the course. However, the following books and articles cover most of the topics of the lectures, and are full of other interesting content.

- H. Bahouri, J.-Y. Chemin and R. Danchin: *"Fourier analysis and nonlinear partial differential equations"*. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Scinences), Springer, Heidelberg, 2011.
- R. Danchin and F. Fanelli: *The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces.* J. Math. Pures Appl. (9), **96** (2011), n. 3, pp. 253–278.
- R. Di Perna and P.-L. Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Inventiones Mathematicae, 98 (3), 1989, pp. 511–549.