GEOMETRIC INVERSE PROBLEMS

SUMMER TERM 2024

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TIME: Wednesdays 12-14 (c.t.), starting 10th April
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The *Radon transform* of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by its integrals along all lines in $\mathbb{R}^2$. Formally this yields a function

$$Rf(s,\theta) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp)dt, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1.$$

The question of whether f can be recovered from measurements of Rf is a prototypical *inverse problem* that finds application e.g. in *computerised tomography (CT)*. The first step in understanding this inverse problem is to prove *injectivity*:

$$Rf = 0 \quad \Rightarrow \quad f = 0$$

One can then further ask for *stability* (i.e. $||f|| \lesssim ||Rf||$ in appropriate norms), a characterisation of the *range* of R or explicit *inversion formulæ/algorithms*. These motives make up the field of inverse problems.

In the lecture we will consider inverse problems in a geometric context. The basic object will be the *geodesic X-ray transform*, which is defined analogously to the Radon transform, but with lines replaced by geodesics. This transform shows up as linearisation of several non-linear inverse problems. In particular we will discuss the following problems, posed on a compact manifold M with boundary:

- *Boundary rigidity:* Here one asks whether a Riemannian metric g on M is determined by its boundary distance function:

$$d_g\colon \partial M \times \partial M \to [0,\infty), \quad d_g(x,y) = \begin{array}{l}\text{Length of the } g\text{-geodesic}\\ \text{from } x \text{ to } y\end{array}$$

- *Non-Abelian X-ray tomography:* Here a (suitable) metric g is fixed and one asks whether a connection A on the vector bundle $M \times \mathbb{C}^n$ is determined by its boundary parallel transports:

$$P_A\colon \partial M \times \partial M \to \mathbb{C}^{n\times n}, \quad P_A(x,y)\xi = \begin{array}{l}\text{Parallel transport of } \xi \in \mathbb{C}^n\\ \text{along geodesic from } x \text{ to } y\end{array}$$

These problems have received a lot of attention in the last 20 years and in the setting of *simple surfaces* (2-dimensional manifolds M that are diffeomorphic to a disk and satisfy certain geometric assumptions) there is now a fairly complete understanding of the above mentioned questions. This theory will be the focus of the lecture.

Prerequisites. Some familiarity with basic differential geometry (tangent bundle, vector fields, differential forms, geodesics) will be helpful. More advanced aspects (geodesic flow, conjugate points, parallel transport, pseudodifferential operators) will be developed/sketched along the way.

Literature. Most of the course will be based on the following book:

> G.P. Paternain, M. Salo, and G. Uhlmann. *Geometric inverse problems – with an emphasis on two dimensions*, volume 204 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.
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> http://users.jyu.fi/~salomi/lecturenotes/GIP2D_driver.pdf