Geometric Inverse Problems — Lecture 1

Books: 1) Paternain - Salo - Uhlmann, GIP in 2D

2) Guillarmou - Mazzucchelli, Intro to GIP

3) Lefeuvre, Microlocal Analysis in hyp. dynamics and geometry

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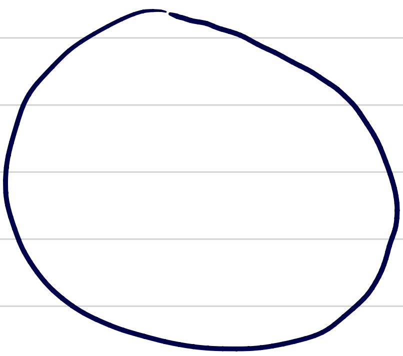


$M, \partial M$

Geometric objects:

* g Riemannian metric
* A connection
* f function / tensor

$\vdots$

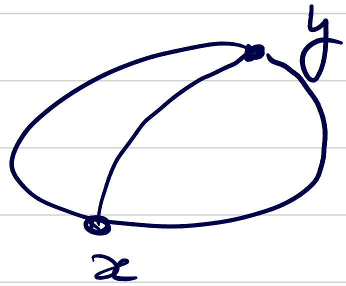
Goal: Given boundary information on geom. object, reconstruct the object in the interior!

Example Unknown g, Boundary Rigidity

$\gamma : [0,\tau] \to M$ smooth curve

$l_g(\gamma) = \int_0^\tau |\dot\gamma(t)|_g \, dt$ length

$$d_g(x,y) = \inf_{\gamma} l_g(\gamma)$$



Want to "invert" the map

$$\boxed{g \longmapsto d_g\big|_{\partial M \times \partial M}}$$

"forward map"

$$\gamma(0) = x$$
$$\gamma(\tau) = y$$

<u>First step</u>: Ask for injectivity of the forward map.

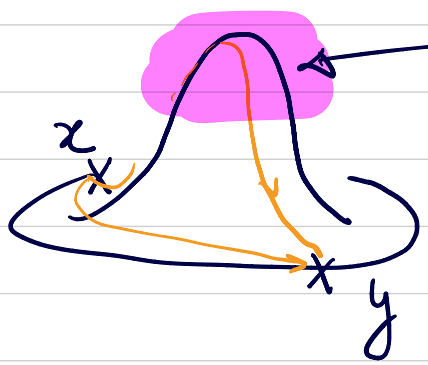
Obstructions to injectivity:

*) "Gauge": If $\varphi : M \to M$ diffeo, $\varphi|_{\partial M} = \mathrm{Id}$

$$d_g(x,y) = d_{\varphi^* g}(x,y) \quad \text{for all } x,y \in \partial M$$

↪ Only expect to recover gauge equivalence class of g

*)



local perturbations of g here do <u>not</u> affect $d_g\big|_{\partial M \times \partial M}$

→ Find conditions on (M,g) that avoid "invisible" regions

Application / Interpretation: Travel time tomography (Geophysics)

$$M = \{x \in \mathbb{R}^3 \mid |x| \leq 1\} = \text{earth}$$

$$g_{ij}(x) = \frac{1}{c(x)^2}\delta_{ij}, \qquad \underbrace{c \in C^\infty(M), \quad 0 < c(x) < \infty}_{\text{"sound speed"}}$$

$$d_g|_{\partial M \times \partial M} = \text{travel time of an earthquake}$$

<u>Rk</u>: * "Conformal BR problem" $\rightarrow$ no gauge

* general g $\rightarrow$ anisotropic sound speed

<u>Dimension count</u>:

$$\dim M = d \qquad \underset{d \text{ variables}}{g} \rightsquigarrow \underset{2d-2 \text{ variables}}{d_g|_{\partial M \times \partial M}}$$

$d = 2$: $d = 2d-2$ "formally determined"

$d \geq 3$: $d < 2d-2$ "formally overdetermined"

This lecture: mostly $d = 2$!

Outline of lecture:

§1: Geometric preliminaries

§2: Linear X-ray tomography

§3: Riemannian Rigidity Problems

§4: Non-Abelian X-ray tomography

§5: Range Characterisations

Book on differential geometry: Lee smooth manifolds

$\Omega \subset \mathbb{R}^d$ domain smooth boundary

$M = \overline{\Omega}$, $\quad g : \overline{\Omega} \longrightarrow \mathbb{R}^{d \times d}$ $\quad C^\infty$ smooth

$\left(g_{ij}(x)\right)_{ij}$ positive definite, $\forall x$

$T M = \overline{\Omega} \times \mathbb{R}^d \ni (x, v)$,

$$|v|_{g(x)} = \sqrt{\sum_{i,j=1}^{d} v^i g_{ij}(x) v^j}$$

$\gamma : \mathbb{R} \longrightarrow \overline{\Omega}$, $\quad (\gamma(t), \dot{\gamma}(t)) \in TM = \overline{\Omega} \times \mathbb{R}^d$

$$|\dot{\gamma}(t)|_g = |\dot{\gamma}(t)|_{g(\gamma(t))}$$

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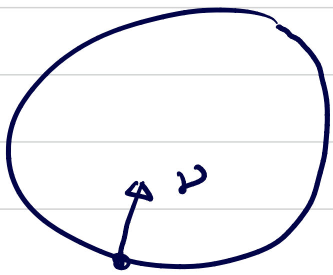
§1 Geometric Preliminaries

(M, g) compact, Riemannian mfd w/ boundary, connected, oriented, d-dimensional $(d \geq 2)$

$\nu =$ unit inner normal vector



$x \in \partial M$

$\nu(x) \in T_x M$

Def ∂M strictly convex, iff the second fundamental form $\mathrm{II} > 0$

Recall: $x \in \partial M$, $u, v \in T_x(\partial M)$, then

$$\mathrm{II}_x(u,v) = -\, g_x(\nabla_u \nu, v)$$

, $\nabla =$ Levi-Civita connection

$\mathrm{II} > 0$ means: $\mathrm{II}_x : T_x\partial M \times T_x \partial M \to \mathbb{R}$ positive definite

Unit sphere bundle (phase space)

$$SM = \{(x,v) \in TM \mid |v|_g = 1\}$$

2d−1 dimensional manifold w/ boundary

$(M = \overline{\Omega}, \quad SM = \overline{\Omega} \times \mathbb{S}^{d-1})$

Generally have fibre bundle:

$$\mathbb{S}^{d-1} \longrightarrow SM \xrightarrow{\ \pi\ } M$$

$\pi(x,v) = x$

"sphere bundle"

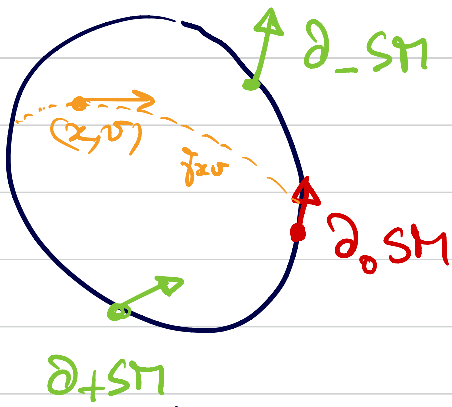
$$\partial SM = \{(x,v) \in SM \mid x \in \partial M\} \qquad \text{dim} = 2d-2$$

$$\partial_{\pm} SM = \{(x,v) \in \partial SM \mid \pm g_x(v, \nu(x)) \geq 0\}$$

influx / outflux boundaries

$$\partial_0 SM = \partial_+ SM \cap \partial_- SM = S(\partial M)$$

"glancing region"



<u>Recall</u> A geodesic is a smooth curve

$\gamma : I \to M$, s.th. $\quad \boxed{\nabla_{\dot\gamma} \dot\gamma = 0}$

$I \subset \mathbb{R}$

2nd order ODE

For $(x,v) \in SM$ there is a unique geodesic $\gamma_{x,v}$ s.th

$$\boxed{\gamma_{x,v}(0) = x, \quad \dot{\gamma}_{x,v}(0) = v}$$

Observations:

* $|\dot{\gamma}_{x,v}| \equiv 1, \quad \gamma_{x,-v}(t) = \gamma_{x,v}(-t)$

* For $(x,v) \in SM$ let $\tau(x,v)$ be the first time that $\gamma_{x,v}$ exits M,

$$\tau(x,v) \in [0, \infty]$$

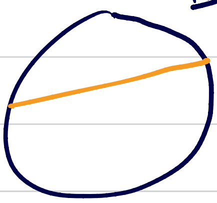
* $\gamma_{x,v}$ is defined on

$$I = [-\tau(x,-v), \tau(x,v)] \cap \mathbb{R}$$

* $\tau|_{\partial_- SM} = 0$

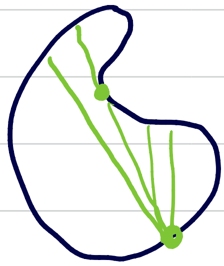
Examples:

$\mathbb{D}$ closed disk



τ smooth away from $\partial_0 SM$ (cf later)

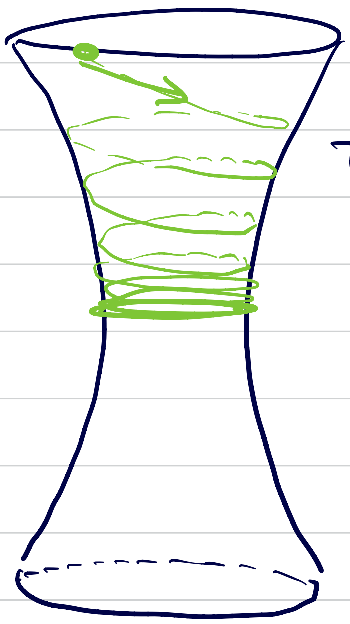
$\subset \mathbb{R}^2$



exit $\neq$ hitting

τ not continuous

but $\tau < \infty$



$\tau(x,v) = \infty$

hyperboloid

Def: (M,g) is ==non-trapping== if $\tau(x,v) < \infty$ for all $(x,v) \in SM$.

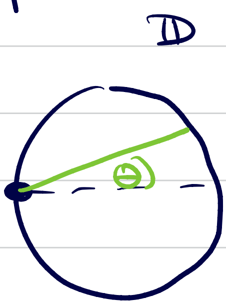
Standard setting: (M,g) non-trapping $+$ ∂M str.cvx.
($\Longrightarrow M$ contractible, see PSU, Prop 3.7.22)

<u>Mottos</u>: 1) Never want to leave SM

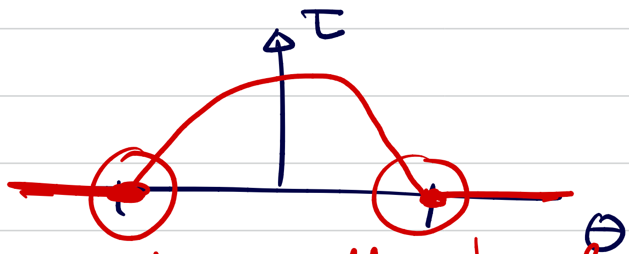
2) τ rules

We'll see: $\tau \in C(SM) \cap C^\infty(SM \setminus \partial_0 SM)$

<u>Example</u>:



$$\tau(\theta) = \begin{cases} 2\cos(\theta), & \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \text{else} \end{cases}$$



not smooth at glancing

Regularity of the exit time

(M,g) non-trapping, ∂M str.cvx.

<u>Exc</u>: Show that τ continuous.

Smoothness?

Extensions:

1) $(M,g) \subset (N,g)$, s.th. N closed
2) $(M,g) \subset (N,g)$, s.th. N complete, geodesics do not return after they exit

Then $\exists\, \rho \in C^\infty(N)$ "boundary defining fctn"

$$\begin{cases} M = \{\rho \geq 0\}, \quad \partial M = \{\rho = 0\} \\ \operatorname{grad} \rho = \nu \quad \text{on } \partial M \end{cases}$$

To analyse τ, we'll consider

$$h : SN \times \mathbb{R} \to \mathbb{R}, \quad h(x,v,t) = \rho(\gamma_{x,v}(t))$$

$\rho(x) = \operatorname{dist}(x, \partial M)$
for $x \in \mathcal{U} \cap M$