# THE CALDERÓN PROBLEM

## GRADUATE SEMINAR ON ANALYSIS (S4B1)
## SUMMER TERM 2023

ORGANISERS: Jan Bohr (bohr@math.uni-bonn.de) and Herbert Koch
PRELIMINARY MEETING: Tuesday 24th January, 4:15pm in room 2.008.
TIME & PLACE: Wednesdays 4:15pm, starting at 5th April, room N0.007
WEBSITE: https://www.math.uni-bonn.de/ag/ana/SoSe2023/S4B1//

### OVERVIEW

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. For $\gamma \in C^\infty(\bar{\Omega})$ with $\inf \gamma > 0$ one defines the *Cauchy data set* as

$$\mathcal{C}_\gamma = \{(u, \gamma\partial_\nu u) : u \in C^\infty(\bar{\Omega}) \text{ with } \nabla \cdot (\gamma \nabla u) = 0\} \subset C^\infty(\partial\Omega) \times C^\infty(\partial\Omega),$$

where $\partial_\nu$ is the outward normal derivative at $\partial\Omega$. The set $\mathcal{C}_\gamma$ contains information on $\gamma$ that is accessible from the boundary $\partial\Omega$. The *Calderón problem* asks whether $\gamma$ can be determined from this boundary information. This question is the mathematical basis for the imaging method of *Electrical Impedance Tomography*, where $\gamma$ models the electrical conductivity inside $\Omega$ and $\mathcal{C}_\gamma$ encodes voltage and current measurements taken at $\partial\Omega$. The Calderón problem can be posed also for other differential equations, such as the stationary Schrödinger equation or the Laplace-Beltrami equation,

$$\Delta u + qu = 0 \quad \text{or} \quad \Delta_g u \equiv -\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_i u\right) = 0,$$

and in each case either an unknown potential $q \in C^\infty(\bar{\Omega})$ or an unknown Riemannian metric $g$ on $\bar{\Omega}$ is to be determined from the corresponding Cauchy data set.

The Calderón problem is an important example of a *nonlinear inverse problem.* The term inverse problem is used in opposition to the easier direct problem of determining $\mathcal{C}_\gamma$ for a conductivity $\gamma$. Inverse problems form an active field of research that is rooted in different parts of analysis, geometry and statistics. Typical question in the field concern the injectivity of the forward map (here $\gamma \mapsto \mathcal{C}_\gamma$), possibly up to some gauge group, stability properties, and explicit reconstruction formulas. The goal of the seminar is to understand some of these questions and their resolution in the case of the Calderón problem.

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*Date*: December 21, 2022.

## List of topics

*Preliminary reading:* Introductions of [6] and [9] (optionally also of [7]).

Each topic should be covered in one 90 minute talk. Some talks can be split over multiple students/talks as indicated. Talks marked with a [†] can be left out if there are too few participants. You can make further suggestions for relevant talks.

### Part 1: General preliminaries.

(I) **Dirichlet–to–Neumann operator**: Well-posedness of the Dirichlet problem for $\nabla \cdot (\gamma \nabla)$ and $\Delta + q$ for $L^\infty$-coefficients. Definition of the Dirichlet-to-Neumann operators $\Lambda_\gamma, \Lambda_q \colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Relation to Cauchy data. (Optional: Include Riemannian metric $g$.) *References:* Section 3.1 in [6] (or Sections 2C+2D in [7])

(II) **Boundary determination**: Determination of $\gamma$ and its derivatives at $\partial\Omega$, using $\Lambda_\gamma$. Consequences in the real analytic setting. (*Optional:* $\Lambda_\gamma$ is an elliptic pseudodifferential operator, Lipschitz stability.) *Reference:* Section 6 in [8]

### Part 2: Smooth coefficients, domains in $\mathbb{R}^3$.

(III) **Complex geometric optic solutions:** Motivation and construction of CGO solutions. *Reference:* Section 3.2 in [6]

(IV) **Interior determination in $d \geq 3$:** Injectivity of $C^2(\bar{\Omega}) \ni \gamma \to \mathcal{C}_\gamma$ *References:* Sections 3.1 and 3.3 in [6]

(V) **Logarithmic stability:** Counterexample for $\gamma \in L^\infty(\Omega)$, stability estimate for sufficiently smooth $\gamma$'s. *References:* Sections 4 in [6]

(VI) [†] **Optimality of logarithmic stability:** Modulus of continuity is no better than logarithmic. *References:* [3], see also Section 1.1 in [2]

(VII) [†] **The partial data problem (2 talks):** Carleman estimates, more CGO solutions, injectivity for the partial data problem. *Reference:* Section 5 in [6]

### Part 3: Rough coefficients, domains in $\mathbb{R}^2$.

(VIII) [†] **Reduction to scattering transform:** Well-posedness for unbounded conductivities with $\log\gamma \in \dot{H}^1$, reduction to Hilbert transform type operator, determination of scattering transform. *References:* Sections 1.1, 1.3 and Section 6 in [4], modulo the proof of Theorem 1.2

(IX) [†] **Inversion of scattering transform (1-2 talks):** Estimates on integral operators, solution of d-bar problem, inversion theorem for scattering transform. *References:* Proof of Theorem 1.2 in [4] (see Sections 2,3,4 in [4])

**Part 4: Smooth coefficients, Riemannian surfaces.**

(X) [†] **Determination of conformal class:** $\Lambda_g$ determines the conformal class $[g]$, but not more. *Reference:* Section 11.6 in [5]

(XI) [†] **Schrödinger equation on Riemann surfaces (2 talks):** CGO solutions via the Riemann–Roch theorem, interior determination of potentials. *Reference:* [1]

## References


[1] Colin Guillarmou and Leo Tzou. Calderón inverse problem for the Schrödinger operator on Riemann surfaces. In *The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis*, volume 44 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 129–141. Austral. Nat. Univ., Canberra, 2010.

[2] Herbert Koch, Angkana Rüland, and Mikko Salo. On instability mechanisms for inverse problems. *Ars Inven. Anal.*, pages Paper No. 7, 93, 2021.

[3] Niculae Mandache. Exponential instability in an inverse problem for the Schrödinger equation. *Inverse Problems*, 17(5):1435–1444, 2001.

[4] Adrian Nachman, Idan Regev, and Daniel Tataru. A nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the inverse boundary value problem of Calderón. *Invent. Math.*, 220(2):395–451, 2020.

[5] Gabriel Paternain, Mikko Salo, and Gunther Uhlmann. Geometric Inverse Problems with emphasis on 2 dimensions. Lecture notes, 2022. https://www.dpmms.cam.ac.uk/%7Egpp24/GIP2D_driver.pdf.

[6] Mikko Salo. The Calderón problem. Lecture notes, 2008. http://users.jyu.fi/~salomi/lecturenotes/calderon_lectures.pdf.

[7] Mikko Salo. The Calderón problem on Riemannian manifolds. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 167–247. Cambridge Univ. Press, Cambridge, 2013. http://library.msri.org/books/Book60/files/30salo.pdf.

[8] Mikko Salo. Applications of microlocal analysis in inverse problems. *Mathematics*, 8(7), 2020.

[9] Gunther Uhlmann. 30 years of Calderón's problem. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2012–2013*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIII, 25. École Polytech., Palaiseau, 2014.