V5B8 – Selected Topics in Analysis
The vector field method and quasilinear wave equations

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Contents

1 Introduction

In this course, we would like to study the lifespans of solutions to some special types of quasilinear wave equations with small, smooth and localized initial data.

Lifespan: Given some initial data at $t = 0$, what is the supremum of all $T \geq 0$, such that a solution to a certain wave equation with the given data exists for $0 \leq t \leq T$?

Quasilinear wave equations (studied in this course): Consider

$$\Box u = (-\partial_t^2 + \Delta_x)u = F(u, u', u'') \qquad \text{in } \mathbb{R}_+^{1+d} = (0,\infty) \times \mathbb{R}^d.$$

Here u', u'' denote the first and second derivatives of u, respectively. An equation of this form is a nonlinear wave equation. If $F \equiv 0$, then we get a linear wave equation. If $F = F(u, u')$ (independent of u''), then we get a semilinear wave equation. If $F = c(u, u') \cdot u'' + f(u, u')$, then we get a quasilinear wave equation.

Small, smooth and localized initial data: initial data in $C_c^\infty(\mathbb{R}^d)$ of size $\varepsilon \ll 1$.

Here are some examples. Unless specified otherwise, all the unknown functions in this note are $\mathbb{R}$-valued.

Example 1.1. Linear wave equation (with constant coefficients).

$$\begin{cases} \Box u = -\partial_t^2 u + \Delta_x u = 0 \qquad \text{in } \mathbb{R}_+^{1+3}; \\ (u, \partial_t u)|_{t=0} = (u^0, u^1) \in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{1.1}$$

Fact. We have a global existence result. That is,

$$\forall (u^0, u^1) \in C_c^\infty(\mathbb{R}^3), \quad \exists! \text{ a global solution } u \in C^\infty(\mathbb{R}_+^{1+3}) \text{ to to the Cauchy problem (1.1).} \tag{1.2}$$

Example 1.2. Semilinear wave equations.

$$\begin{cases} \Box u = (\partial_t u)^2 \qquad \text{in } \mathbb{R}_+^{1+3}; \\ (u, \partial_t u)|_{t=0} = (\varepsilon u^0, \varepsilon u^1) \in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{1.3}$$

$$\begin{cases} \Box u = (\partial_t u)^2 - |\nabla_x u|^2 \qquad \text{in } \mathbb{R}_+^{1+3}; \\ (u, \partial_t u)|_{t=0} = (\varepsilon u^0, \varepsilon u^1) \in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{1.4}$$

Here (u^0, u^1) is an arbitrary pair of functions in $C_c^\infty(\mathbb{R}^3)$ (which are independent of ε), and $\varepsilon \in (0,1)$ is a sufficiently small constant depending on (u^0, u^1).

Question 1. Does there exist a global solution in $\mathbb{R}_+^{1+3}$ to (1.3)?

Answer. No. It is even worse. In fact, any nontrivial solution to (1.3) blows up in finite time. This result was proved by Fritz John [Joh81, Joh85]. In general, the best result is an *almost global existence* result:

$$\forall (u^0, u^1) \in C_c^\infty(\mathbb{R}^3), \quad \exists \varepsilon_0 \in (0,1), \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \exists \text{ a } C^\infty \text{ solution to (1.3) for } t \in [0, \exp(c/\varepsilon)]. \tag{1.5}$$

Here ε_0 and c are two small constants depending on the data.

Question 2. Does there exist a global solution in $\mathbb{R}_+^{1+3}$ to (1.4)?

Answer. Yes. We have

$$\forall (u^0,u^1)\in C_c^\infty(\mathbb{R}^3),\quad \exists \varepsilon_0\in(0,1),\quad \forall\varepsilon\in(0,\varepsilon_0),\quad \exists \text{ a global } C^\infty \text{ solution to (1.4) in } \mathbb{R}^{1+3}_+. \tag{1.6}$$

We sometimes call it a *small data global existence* result. This result was proved by Klainerman [Kla85, Kla84] and Christodoulou [Chr86].

Example 1.3. Quasilinear wave equations.

$$\begin{cases} \Box u = \partial_t u \partial_t^2 u & \text{in } \mathbb{R}^{1+3}_+; \\ (u,\partial_t u)|_{t=0} = (\varepsilon u^0, \varepsilon u^1)\in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{1.7}$$

$$\begin{cases} \Box u = \partial_t u \partial_t^2 u - \nabla_x u\cdot\nabla_x\partial_t u & \text{in } \mathbb{R}^{1+3}_+; \\ (u,\partial_t u)|_{t=0} = (\varepsilon u^0, \varepsilon u^1)\in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{1.8}$$

Here (u^0,u^1) is an arbitrary pair of functions in $C_c^\infty(\mathbb{R}^3)$ (which are independent of ε), and $\varepsilon\in(0,1)$ is a sufficiently small constant depending on (u^0,u^1).

Question. Does there exist a global solution in $\mathbb{R}^{1+3}_+$ to (1.7) or (1.8)?

Answer. No for (1.7), and yes for (1.8). In fact, the results for (1.7) and (1.8) are exactly the same as those for (1.3) and (1.4), respectively.

In this course, we seek to prove all the results listed above except the finite time blowup.

Remark. Some remarks.

1) Why are the results different?

   In fact, (1.4) and (1.8) satisfy the *null condition*. Klainerman [Kla85, Kla84] and Christodoulou [Chr86] proved that the null condition is sufficient for the small data global existence. If time permitted, I would also introduce the *Hörmander's asymptotic equations* (introduced by Hörmander [H97, H87, H91]) which are closely related to this question.

2) Why do we consider C_c^∞ data of size $\varepsilon \ll 1$?

   C_c^∞: We will use the *energy method* which requires us to use integration by parts. Of course, C_c^∞ is usually too strong and not necessary.

   Size ε: We hope to view the nonlinear wave equations studied in this course as perturbations of $\Box u = 0$, so that solutions to those NLW behave as a linear solution as $t\to\infty$. Extreme and bad case: $\Box u = u\Delta u$, $u\approx 2 \Longrightarrow$ closer to the Laplace's equation $\Delta_{t,x}u\approx 0$ instead of $\Box u = 0$.

3) Motivation. Why are we interested in the lifespan problem with small, smooth and localized data?

   - The Einstein vacuum equations in the wave coordinates become a system of quasilinear wave equations (with $\mathbb{R}^{4\times4}$-valued unknowns). In this case, the lifespan problem is closely related to the global stability problem. See Lindblad-Rodnianski [LR03, LR05].
   - The 3D compressible Euler equations can be written as a system of quasilinear wave equations (with $\mathbb{R}^{4\times4}$-valued unknowns) coupled to some transport equations. In this case, the lifespan problem is closely related to the shock formation. See Speck [Spe19], Luk-Speck [LS20], Christodoulou-Miao [CM14], etc.

2 The linear wave equation

Let us first have a review on the linear wave equation in three space dimensions:

$$\begin{cases} \Box u = -\partial_t^2 u + \Delta_x u = 0 & \text{in } \mathbb{R}^{1+3}_+; \\ (u, \partial_t u)|_{t=0} = (u^0, u^1) \in C_c^\infty(\mathbb{R}^3). \end{cases} \tag{2.1}$$

Notation. Let us explain some notations used in (2.1).

1) $\mathbb{R}^{1+3}_+ := (0,\infty) \times \mathbb{R}^3$. A point in $\mathbb{R}^{1+3}_+$ is denoted by $(x_\alpha)_{\alpha=0}^3 = (t,x) = (t, x_1, x_2, x_3)$ where $t \in (0,\infty)$ and $x \in \mathbb{R}^3$. Sometimes we also write $x_0 = t$.

2) $\Box = -\partial_t^2 + \Delta_x = -\partial_t^2 + \sum_{j=1}^3 \partial_j^2$ is the usual d'Alembertian in $\mathbb{R}^{1+3}$. We can also write $\Box = m^{\alpha\beta}\partial_\alpha\partial_\beta$, where we use $\partial_0 = \partial_t$, the Einstein summation convention (so the sum is taken over all $\alpha, \beta \in \{0,1,2,3\}$) and the Minkowski metric $(m^{\alpha\beta}) = (m_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1) \in \mathbb{R}^{4\times 4}$.

3) $C_c^\infty(\mathbb{R}^3)$ denotes the set of all C^∞ $\mathbb{R}$-valued functions f in $\mathbb{R}^3$ which are also compactly supported. In this note, "C^∞ functions" is the same as "smooth functions".

Now we discuss the following four topics related to (2.1): existence, uniqueness, pointwise decays and energy conservation.

2.1 Existence of a global smooth solution

Proposition 2.1. *For each $r \in \mathbb{R}$ and $x \in \mathbb{R}^3$, we set*

$$A_r h(x) := \frac{1}{4\pi}\int_{\mathbb{S}^2} h(x + r\omega)\, dS_\omega.$$

Then

$$u(t,x) = \partial_t(tA_t u_0) + tA_t u^1 \tag{2.2}$$

is a solution to (2.1) *which belongs to* $C^\infty(\mathbb{R}^{1+3}_+)$. *The formula* (2.2) *is called the* Kirchoff's formula.

"Proof". It is easy to show that the function u defined by (2.2) is indeed a solution to (2.1) and that $u \in C^\infty(\mathbb{R}^{1+3}_+)$. Instead, I would like to explain how to derive (2.2).

I. Spherical mean. Suppose that $u \in C^2(\mathbb{R}^{1+3}_+)$ is a solution to (2.1). For each $r \in \mathbb{R}$ and $(t,x) \in \mathbb{R}^{1+3}_+$, we set

$$U(r;t,x) = A_r u(t,\cdot) = \frac{1}{4\pi}\int_{\mathbb{S}^2} u(t, x + r\omega)\, dS_\omega.$$

Then,

$$\begin{aligned} U_r &= \frac{1}{4\pi}\int_{\mathbb{S}^2} \omega \cdot \nabla_x u(t, x + r\omega)\, dS_\omega \\ &= \frac{r}{4\pi}\int_{B(0,1)} \Delta_x u(t, x + ry)\, dy && \text{(divergence's theroem)} \\ &= \frac{1}{4\pi r^2}\Delta_x \int_{B(x,r)} u(t,z)\, dz && \text{(substitute } z = x + ry\text{)}, \end{aligned}$$

$$
\begin{aligned}
\partial_r(r^2 U_r) &= \frac{1}{4\pi}\Delta_x \int_{\partial B(x,r)} u(t,z)\, dS_z = \frac{r^2}{4\pi}\int_{\mathbb{S}^2} \Delta_x u(t,x+r\omega)\, dS_\omega \qquad \text{(substitute } z = x+ry\text{)} \\
&= \frac{r^2}{4\pi}\int_{\mathbb{S}^2} \partial_t^2 u(t,x+r\omega)\, dS_\omega = r^2 U_{tt} \qquad (u \text{ solves } \Box u = 0).
\end{aligned}
$$

As a result, we have

$$U_{tt} = r^{-2}\partial_r(r^2 U_r) = 2r^{-1}U_r + U_{rr} \Longrightarrow (rU)_{tt} = 2U_r + rU_{rr} = (rU)_{rr}.$$

As a result, $v = rU$ is a solution to the linear wave equation in one space dimension:

$$v_{tt} = v_{rr}, \qquad (t,r) \in \mathbb{R}^{1+1}; \qquad (v, v_t)|_{t=0} = (rU(r;0,x), rU_t(r;0,x)) = (rA_r u^0, rA_r u^1)(x).$$

Solving this equation (left as an exercise), we obtain

$$v(t,r) = \frac{1}{2}[(r+t)A_{r+t}u^0(x) + (r-t)A_{r-t}u^0(x)] + \frac{1}{2}\int_{r-t}^{r+t} \rho A_\rho u^1(x)\, d\rho.$$

Note that $A_r h$ is defined for all $r \in \mathbb{R}$ and that $A_r h = A_{-r}h$. Thus, we have

$$
\begin{aligned}
u(t,x) &= \lim_{r\to 0} U(r;t,x) \\
&= \lim_{r\to 0}\left(\frac{1}{2r}[(t+r)A_{t+r}u^0(x) - (t-r)A_{t-r}u^0(x)] + \frac{1}{2r}\int_{t-r}^{t+r} \rho A_\rho u^1(x)\, d\rho\right) \\
&\qquad\qquad (\rho A_\rho \text{ is an odd function of } \rho) \\
&= \partial_t(tA_t u^0) + tA_t u^1.
\end{aligned}
$$

II. Fourier transform. Recall that the Fourier transform of a function in $\mathbb{R}^d$ is defined by

$$\mathcal{F}w(\xi) = \widehat{w}(\xi) := \int_{\mathbb{R}^d} w(x)e^{-ix\cdot\xi}\, dx.$$

We then have

1) The map $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a bijective map. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions.
2) The inverse $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ of $\mathcal{F}$ is given by the Fourier inversion formula
$$\mathcal{F}^{-1}h(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} h(\xi)e^{ix\cdot\xi}\, d\xi.$$
3) $\mathcal{F}(\partial_j w)(\xi) = i\xi_j\widehat{w}(\xi)$, so $\mathcal{F}(\Delta w)(\xi) = -|\xi|^2\widehat{w}(\xi)$.
4) For any $w, v \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{F}(w * v) = \widehat{w}\widehat{v}$. Here $*$ is the convolution.

The proofs of these facts can be found in any standard textbook of Fourier analysis.

Assume that $u(t,\cdot) \in \mathcal{S}(\mathbb{R}^3)$ for each $t \geq 0$. By taking the Fourier transform (with respect to x but not t), we obtain from the wave equation that

$$\partial_t^2\widehat{u}(t,\xi) = \mathcal{F}(\Delta u)(t,\xi) = -|\xi|^2\widehat{u}(t,\xi).$$

This is an ODE for $\widehat{u}(t,\xi)$ with initial data $(\widehat{u},\partial_t\widehat{u})|_{t=0}=(\widehat{u}^0,\widehat{u}^1)$. Solve this ODE and we obtain

$$\widehat{u}(t,\xi)=\cos(t|\xi|)\widehat{u}^0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}^1(\xi). \tag{2.3}$$

We now need to show that (2.3) implies (2.2). Note that (the computation is left as an exercise)

$$\int_{\mathbb{S}^2} e^{-i\omega\cdot\xi}dS_\omega=4\pi\cdot\frac{\sin(|\xi|)}{|\xi|}.$$

Thus,

$$\begin{aligned}\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}^1(\xi)&=\frac{t}{4\pi}\int_{\mathbb{S}^2}e^{-i\omega\cdot t\xi}\widehat{u}^1(\xi)\ dS_\omega=\frac{t}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}e^{-i\omega\cdot t\xi-ix\cdot\xi}u^1(x)\ dxdS_\omega\\&=\frac{t}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}e^{-iy\cdot\xi}u^1(y-t\omega)\ dydS_\omega=\frac{t}{4\pi}\int_{\mathbb{R}^3}e^{-iy\cdot\xi}\int_{\mathbb{S}^2}u^1(y-t\omega)\ dS_\omega dy\\&=\mathcal{F}(\frac{t}{4\pi}\int_{\mathbb{S}^2}u^1(\cdot-t\omega)\ dS_\omega)(\xi)=\mathcal{F}(tA_tu^1)(\xi).\end{aligned}$$

It follows that

$$\cos(t|\xi|)\widehat{u}^0(\xi)\partial_t(\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}^0)=\mathcal{F}(\partial_t(tA_tu^0))$$

and therefore

$$u(t,x)=\mathcal{F}^{-1}(\cos(t|\xi|)\widehat{u}^0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}^1(\xi))=\partial_t(tA_tu^0)+tA_tu^1.$$

$\square$

Remark 2.1.1. In this proposition we assume $u^0,u^1\in C_c^\infty$, but it is obvious that these assumptions are too strong and not necessary. In general, if $k\geq 2$ is an integer, if $(u^0,u^1)\in C^{k+1}\times C^k(\mathbb{R}^3)$, then (2.2) gives a global C^k solution to (2.1).

Using the Fourier transform, we can relax the assumptions on the initial data even further. For example, if $u^0,u^1\in L^2(\mathbb{R}^3)$, then the formula (2.3) is well defined, so we still obtain a solution (not necessarily a C^2 solution but a solution in some weak sense).

Remark 2.1.2. The formula (2.3) is closely related to the *Fourier multiplier*. That is, given a function $m=m(\xi)\in L^\infty(\mathbb{R}^d)$, we can define a bounded linear operator $m(D):L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ (here L^2 can be replaced with any L^2-based Sobolev space $H^s,\dot{H}^s$) by

$$m(D)f(x):=\mathcal{F}^{-1}(m(\cdot)\widehat{f}(\cdot))(x).$$

This definition is motivated by the case when m is a polynomial of ξ (in which case $m(D)$ is a linear differential operator since $D=\nabla/i$). Now (2.3) implies that

$$u(t)=\cos(t|D|)u^0+\frac{\sin(t|D|)}{|D|}u^1. \tag{2.4}$$

You might check Appendix A in [Tao06] if you are interested in this topic.

2.2 Uniqueness

Proposition 2.2. *If u and $\widetilde{u}$ are two $C^2(\mathbb{R}_+^{1+3})$ solutions to* (2.1) *with the same data, then $u = \widetilde{u}$.*

Proof. We have already proved this proposition in "I. Spherical mean" in the derivation of (2.2). There we showed that a $C^2(\mathbb{R}_+^{1+3})$ solution to (2.1) must satisfy (2.2). $\square$

Remark 2.2.1. Combining Proposition 2.1 and 2.2, we conclude that there exists a unique $C^\infty(\mathbb{R}_+^{1+3})$ solution to (2.1) and that this unique solution is given by (2.2).

In the rest of this section, when we say "the solution to (2.1)", we always mean this unique $C^\infty(\mathbb{R}_+^{1+3})$ solution defined by(2.2).

2.3 Pointwise decays

Before we state the results in this subsection, we first introduce a useful notation.

Notation. In this note, we use C to denote universal positive constants. We write $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ if $|A| \leq CB$ for some $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We use C_v, $\lesssim_v$ or $\gtrsim_v$ if we want to emphasize that the constant depends on a parameter v.

Let us rewrite the Kirchoff's formula (2.2): for each $(t, x) \in \mathbb{R}_+^{1+3}$,

$$u(t, x) = \frac{1}{4\pi} \int_{\mathbb{S}^2} u^0(x + t\omega) + t\omega \cdot \nabla u^0(x + t\omega) + tu^1(x + t\omega)\ dS_\omega. \tag{2.5}$$

Since $(u^0, u^1) \in C_c^\infty$, there exists $R > 0$ such that $u^0 \equiv u^1 \equiv 0$ whenever $|x| \geq R$. So the integrand in (2.5) is nonzero only if $|x + t\omega| < R$.

It follows that

1) Suppose $||x| - t| \geq R$. In this case $|x + t\omega| \geq ||x| - t| \geq R$, so the integrand in (2.5) is zero everywhere on $\mathbb{S}^2$. Thus, $u = 0$.

2) Suppose $||x| - t| < R$ and $t \geq 2$. Using substitution $y = x + t\omega$, we write (2.5) as

$$u(t, x) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} u^0(y) + (y - x) \cdot \nabla u^0(y) + tu^1(y)\ dS_y$$

and (since $\operatorname{supp}(u^0, u^1) \subset B(0, R)$)

$$\begin{aligned} |u(t, x)| &\leq \frac{1}{4\pi t^2} \int_{\partial B(x,t)} |u^0(y)| + t|\nabla u^0(y)| + t|u^1(y)|\ dS_\omega \\ &\lesssim t^{-2} |\partial B(x, t) \cap B(0, R)| \cdot \left(\|u^0\|_{L^\infty(\mathbb{R}^3)} + t(\|\nabla u^0\|_{L^\infty(\mathbb{R}^3)} + \|u^1\|_{L^\infty(\mathbb{R}^3)}) \right). \end{aligned}$$

Here $|\partial B(x, t) \cap B(0, R)|$ is the surface area of a spherical cap with both height and width $\leq 2R$, so $|\partial B(x, t) \cap B(0, R)| \lesssim R^2$ and

$$\begin{aligned} |u(t, x)| &\lesssim t^{-2} R^2 (1 + t) \left(\|u^0\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u^0\|_{L^\infty(\mathbb{R}^3)} + \|u^1\|_{L^\infty(\mathbb{R}^3)} \right) \\ &\lesssim_R (1 + t)^{-1} \left(\|u^0\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u^0\|_{L^\infty(\mathbb{R}^3)} + \|u^1\|_{L^\infty(\mathbb{R}^3)} \right). \end{aligned}$$

3) Suppose $t \leq 2$. By (2.5), we have

$$
\begin{aligned}
|u(t,x)| &\leq \frac{1}{4\pi}\int_{\mathbb{S}^2} |u^0(x+t\omega)| + |t\omega||\nabla u^0(x+t\omega)| + t|u^1(x+t\omega)|\ dS_\omega \\
&\lesssim \left\|u^0\right\|_{L^\infty(\mathbb{R}^3)} + t(\left\|\nabla u^0\right\|_{L^\infty(\mathbb{R}^3)} + \left\|u^1\right\|_{L^\infty(\mathbb{R}^3)}) \\
&\lesssim \left\|u^0\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\nabla u^0\right\|_{L^\infty(\mathbb{R}^3)} + \left\|u^1\right\|_{L^\infty(\mathbb{R}^3)}.
\end{aligned}
$$

In summary, we obtain the following proposition.

Proposition 2.3. *Let u be the solution to* (2.1)*. Suppose that there exists $R > 0$ such that $u^0 \equiv u^1 \equiv 0$ whenever $|x| \geq R$. Then,*

a) *We have $u(t,x) = 0$ whenever $||x| - t| \geq R$.*

b) *If we write $\langle s\rangle := \sqrt{1+s^2} \sim 1 + |s|$, then $|u(t,x)| \lesssim_{R,u^0,u^1} \langle t\rangle^{-1}$ for each $(t,x) \in \mathbb{R}^{1+3}_+$.*

Remark 2.3.1. Part a) of this proposition is a corollary of the *strong Huygens' principle* which states that the value of u at (t,x) is determined by the initial data on the sphere $\partial B(x,t)$ at time 0. In fact, the strong Huygens' principle holds for in each odd space dimension $d \geq 3$.

If we consider the linear wave equation in an even space dimension, then the best result is the *weak Huygens' principle* or the *finite speed of propagation*. Here we simply replace "the sphere $\partial B(x,t)$" with "the ball $B(x,t)$" in the statement above.

In contrast, for the linear heat equation or the linear Schördinger equation, we have the infinite speed of propagation.

For a nonlinear problem, the best result we expect is the finite speed of propagation, even if $d = 3$ or d is an odd integer.

2.4 Energy conservation

Proposition 2.4. *Suppose that u is the solution to* (2.1)*. Define the energy*

$$
E(t) := \frac{1}{2}\int_{\mathbb{R}^3} \sum_{\alpha=0}^{3} |\partial_\alpha u(t,x)|^2\ dx, \qquad t \geq 0.
$$

Then, $E(t) = E(0)$ for all $t \geq 0$.

Proof. We have

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\mathbb{R}^3} \sum_{\alpha=0}^{3} u_\alpha u_{t\alpha}\ dx = \int_{\mathbb{R}^3} u_t u_{tt} + \sum_{j=1}^{3} u_j u_{jt}\ dx \\
&= \int_{\mathbb{R}^3} u_t u_{tt} - \sum_{j=1}^{3} u_{jj} u_t\ dx \qquad \text{(Integrations by parts and finite speed of propagation)} \\
&= \int_{\mathbb{R}^3} -u_t \Box u\ dx = 0.
\end{aligned}
$$

□

2.5 Other space dimensions

In this note, we focus on the case when the space dimension is 3. However, most results above are still valid for general space dimensions. Let us state these results without proofs.

Consider the linear wave equation (2.1) in $\mathbb{R}^{1+d}_+$ with C_c^∞ data.

(A) Existence. There exists a global smooth solution to (2.1). We can write down an explicit formula for each fixed d, but I prefer not to do so here for simplicity. See Section 2.4 in [Eva10] or Section I.1 in [Sog08].

However, the formula (2.4) derived from the Fourier transform holds in all space dimensions.

(B) Uniqueness. Nothing is changed.

(C) Pointwise decays. If $d \geq 3$ is odd, we have the strong Huygens' principle and $u = O(\langle t\rangle^{-(d-1)/2})$. If $d \geq 2$ is even, we have the weak Huygens' principle and $u = O((\langle t\rangle\langle |x| - t\rangle)^{-(d-1)/2})$. If $d = 1$, then we still have $u = O(1)$ but the strong Huygens' principle does not hold.

Reason: For odd $d \geq 3$, there is an integral on the sphere $\mathbb{S}^{d-1}$. For even $d \geq 2$, there is an integral in the ball $B(0, 1)$.

(D) Energy conservation. Nothing is changed.

2.6 Difficulty in a nonlinear problem

In the previous subsections, we make use of the Kirchoff's formula to prove most properties of the solution to (2.1). In a nonlinear problem (e.g. $\Box u = u_t^2 - |\nabla_x u|^2$), we no longer have an explicit formula for a solution. How do we prove existence, uniqueness, pointwise decays and energy estimates without the help of the Kirchoff's formula? That is what we would like to know in the future classes.

3 Energy estimate

In the previous section, we have proved that a solution to (2.1) satisfies the energy conservation law. In this section, we extend it to a general case.

Proposition 3.1 (Energy estimate)**.** *Let $u \in C^2([0, T] \times \mathbb{R}^d)$ vanish for large $|x|$ and satisfy*

$$g^{\alpha\beta}(t, x)\partial_\alpha\partial_\beta u(t, x) = F(t, x), \qquad \forall (t, x) \in [0, T) \times \mathbb{R}^d. \tag{3.1}$$

Suppose that $g^{\alpha\beta} = g^{\beta\alpha}$ and that

$$\sum_{\alpha,\beta=0}^{d} |r^{\alpha\beta}(t, x)| \leq \frac{1}{2}, \qquad \forall (t, x) \in [0, T) \times \mathbb{R}^d. \tag{3.2}$$

Here $r^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$, $(m^{\alpha\beta}) = \mathrm{diag}(-1, 1, \dots, 1) \in \mathbb{R}^{(d+1)\times(d+1)}$.

Then, for each $t \in [0, T)$, we have

$$\left\|u'(t, \cdot)\right\|_{L^2(\mathbb{R}^d)} \lesssim (\left\|u'(0, \cdot)\right\|_{L^2(\mathbb{R}^d)} + \int_0^t \|F(\tau, \cdot)\|_{L^2(\mathbb{R}^d)}\ d\tau) \exp(2\int_0^t \sum_{\alpha,\beta,\gamma=0}^{d} \left\|\partial_\alpha g^{\beta\gamma}(\tau, \cdot)\right\|_{L^\infty(\mathbb{R}^d)}\ d\tau). \tag{3.3}$$

Here $|u'|^2 = \sum_{\alpha=0}^d |\partial_\alpha u|^2$.

Proof. Denote the left hand side of (3.1) as $\square_g u$. Our goal is to write

$$u_t \square_g u = \sum_{i=1}^{d} \underbrace{\partial_i(\dots)}_{\text{divergence in } \mathbb{R}^d,\text{ IBP}} + \underbrace{\partial_t(\dots)}_{\text{included in energy}} + \underbrace{\text{remainders}}_{\text{contains no second derviatives of } u}.$$

We have

$$u_t \square_g u = u_t g^{\alpha\beta}\partial_\alpha\partial_\beta u = \underbrace{\partial_\alpha(u_t g^{\alpha\beta} u_\beta)}_{\text{good terms}} - u_{t\alpha} g^{\alpha\beta} u_\beta - \underbrace{u_t(\partial_\alpha g^{\alpha\beta})u_\beta}_{\text{remainders}},$$

$$-u_{t\alpha} g^{\alpha\beta} u_\beta = \underbrace{\partial_t(-u_\alpha g^{\alpha\beta} u_\beta)}_{\text{good terms}} + \underbrace{u_\alpha(\partial_t g^{\alpha\beta})u_\beta}_{\text{remainders}} + u_\alpha g^{\alpha\beta} u_{\beta t}.$$

Since $g^{\alpha\beta} = g^{\beta\alpha}$, we have $u_\alpha g^{\alpha\beta} u_{\beta t} = u_\alpha g^{\beta\alpha} u_{\beta t} = -\text{LHS}$, so

$$-u_{t\alpha} g^{\alpha\beta} u_\beta = \frac{1}{2}(\underbrace{\partial_t(-u_\alpha g^{\alpha\beta} u_\beta)}_{\text{good terms}} + \underbrace{u_\alpha(\partial_t g^{\alpha\beta})u_\beta}_{\text{remainders}}).$$

Combining all the computations above, we have

$$u_t \square_g u = \partial_\alpha(u_t g^{\alpha\beta} u_\beta) + \frac{1}{2}\partial_t(-u_\alpha g^{\alpha\beta} u_\beta) - u_t(\partial_\alpha g^{\alpha\beta})u_\beta + \frac{1}{2}u_\alpha(\partial_t g^{\alpha\beta})u_\beta.$$

Set (we add a negative sign to make $e_0 \geq 0$)

$$e_0(t,x) = -(g^{0\beta}u_t u_\beta - \frac{1}{2}g^{\alpha\beta}u_\alpha u_\beta); \quad e_j(t,x) = -g^{j\beta}u_t u_\beta; \quad R = (\partial_\alpha g^{\alpha\beta})u_t u_\beta - \frac{1}{2}(\partial_t g^{\alpha\beta})u_\alpha u_\beta.$$

We thus have

$$-u_t \square_g u = \partial_t e_0 + \sum_{j=1}^{d} \partial_j e_j + R.$$

Integrate this identity, and we have

$$\int_{\mathbb{R}^d} -u_t \square_g u \; dx = \int_{\mathbb{R}^d} \partial_t e_0 + \sum_{j=1}^{d} \partial_j e_j + R \; dx = \partial_t \int_{\mathbb{R}^d} e_0(t,x) \; dx + \int_{\mathbb{R}^d} R \; dx. \tag{3.4}$$

To finish our proof, we need to estimate e_0 and R. Note that

$$\begin{aligned} R &= (\partial_\alpha g^{\alpha 0} - \frac{1}{2}\partial_t g^{00})u_t^2 + (\partial_\alpha g^{\alpha j} - \frac{1}{2}\partial_t g^{0j} - \frac{1}{2}\partial_t g^{j0})u_t u_j - \frac{1}{2}(\partial_t g^{ij})u_i u_j \\ &= (\partial_j g^{j0} + \frac{1}{2}\partial_t g^{00})u_t^2 + (\partial_i g^{ij})u_t u_j - \frac{1}{2}(\partial_t g^{ij})u_i u_j. \end{aligned}$$

Here the sum is taken over all $i, j = 1, \dots, d$. It is then clear that

$$|R| \leq |u'|^2(\sum_j |\partial_j g^{j0}| + \frac{1}{2}|\partial_t g^{00}| + \sum_{i,j} |\partial_i g^{ij}| + \frac{1}{2}\sum_{i,j} |\partial_t g^{ij}|) \leq |u'|^2 |\partial g| \qquad \text{(no } \partial_* g^{**} \text{ appear twice!)}$$

where $|u'|^2 = \sum_{\alpha=0}^d |\partial_\alpha u|^2$ and $|\partial g| = \sum_{\alpha,\beta,\gamma=0}^d |\partial_\alpha g^{\beta\gamma}|$. To estimate e_0, we note that

$$e_0 = -(m^{0\beta}u_t u_\beta - \frac{1}{2}m^{\alpha\beta}u_\alpha u_\beta) - (r^{0\beta}u_t u_\beta - \frac{1}{2}r^{\alpha\beta}u_\alpha u_\beta) = \frac{1}{2}|u'|^2 + Q.$$

Here

$$Q = (-r^{00} + \frac{1}{2}r^{00})u_t^2 + (-r^{0j} + \frac{1}{2}r^{0j} + \frac{1}{2}r^{j0})u_t u_j + \frac{1}{2}r^{ij}u_i u_j = -\frac{1}{2}r^{00}u_t^2 + \frac{1}{2}r^{ij}u_i u_j$$

and thus

$$|Q| \le \frac{1}{2}\sum_{\alpha,\beta=0}^d |r^{\alpha\beta}| \cdot |u'|^2 \le \frac{1}{4}|u'|^2, \qquad \text{(by (3.2))}.$$

As a result, we have $\frac{1}{4}|u'|^2 \le e_0 \le \frac{3}{4}|u'|^2$. In summary, by (3.4) we have

$$\partial_t \int_{\mathbb{R}^d} e_0(t,x)\ dx = \int_{\mathbb{R}^d} -u_t F - R\ dx \le \int_{\mathbb{R}^d} |u'||F| + |\partial g||u'|^2\ dx \le \int_{\mathbb{R}^d} (2\sqrt{e_0}|F| + 4|\partial g|e_0)(t,x)\ dx.$$

If we set $E(t) = \int_{\mathbb{R}^d} e_0(t,x)\ dx$, then by the Hölder's inequality,

$$\frac{d}{dt}E(t) \le 2E(t)^{1/2}\|F(t)\|_{L^2(\mathbb{R}^d)} + 4\|\partial g(t)\|_{L^\infty(\mathbb{R}^d)} E(t),$$
$$\frac{d}{dt}E(t)^{1/2} \le \|F(t)\|_{L^2(\mathbb{R}^d)} + 2\|\partial g(t)\|_{L^\infty(\mathbb{R}^d)} E(t)^{1/2}.$$

We thus have

$$\begin{aligned}&\frac{d}{dt}\left(E(t)^{1/2}\exp(-2\int_0^t \|\partial g(\tau)\|_{L^\infty(\mathbb{R}^d)}\ d\tau)\right)\\ &\le \|F(t)\|_{L^2(\mathbb{R}^d)}\exp(-2\int_0^t \|\partial g(\tau)\|_{L^\infty(\mathbb{R}^d)}\ d\tau) \le \|F(t)\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

As a result, we have

$$E(t)^{1/2}\exp(-2\int_0^t \|\partial g(\tau)\|_{L^\infty(\mathbb{R}^d)}\ d\tau) \le E(0)^{1/2} + \int_0^t \|F(\tau)\|_{L^2(\mathbb{R}^d)}\ d\tau$$

and therefore

$$\begin{aligned}\|u'(t)\|_{L^2(\mathbb{R}^d)} &\le 2\sqrt{E(t)} \le (2E(0)^{1/2} + \int_0^t \|F(\tau)\|_{L^2(\mathbb{R}^d)}\ d\tau)\exp(2\int_0^t \|\partial g(\tau)\|_{L^\infty(\mathbb{R}^d)}\ d\tau)\\ &\le (\sqrt{3}\|u'(0)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|F(\tau)\|_{L^2(\mathbb{R}^d)}\ d\tau)\exp(2\int_0^t \|\partial g(\tau)\|_{L^\infty(\mathbb{R}^d)}\ d\tau).\end{aligned}$$

□

Remark 3.1.1. This proof is an example of the "multiplier method". In general, for a vector field X, we write $Xu\Box_g u$ = divergence + remainder and derive some estimates. We can even add a weight (i.e. compute $Xu\Box_g u \cdot w$). This method will appear again in the future notes.

It turns out that the proof of the energy estimate can be used to prove a finite speed of propagation result.

Proposition 3.2 (Finite speed of propagation)**.** *Let u be a C^2 solution to $\Box u = F(u, u', u'')$ in the backward light cone through (t_0, x_0):*

$$\Lambda^-_{(t_0,x_0)} = \{(t,x) \in [0,t_0) \times \mathbb{R}^d : \ |x - x_0| < t_0 - t\}. \tag{3.5}$$

Assume that $F(0,0,u'') = 0$. If $u = \partial_t u = 0$ whenever $t = 0$ and $|x - x_0| < t_0$, then $u \equiv 0$ in $\Lambda^-_{(t_0,x_0)}$.

Proof. Let $\phi = \phi(s,x)$ be a C^1 function defined for $s \in [0,t_0)$ and $x \in \mathbb{R}^d$ with $|x - x_0| < t_0$. At this moment we do not give an explicit formula for ϕ, but we assume that

a) $\phi(0,x) = 0$ and $\lim_{s\to t_0} \phi(s,x) = t_0 - |x - x_0|$.

b) $s \mapsto \phi(s,x)$ is nondecreasing for each fixed x.

c) There exists a nondecreasing function θ defined on $[0,t_0)$, such that for each $0 \le s_0 < t_0$, we have

$$|\nabla_x \phi(s,x)| \le \theta(s_0) < 1, \qquad \text{whenever } s \in [0,s_0] \text{ and } |x - x_0| < t_0. \tag{3.6}$$

Set

$$\begin{aligned} R_s &= \{(t,x) : \ 0 \le t \le \phi(s,x), \ |x - x_0| < t_0.\}, \\ \Lambda_s &= \{(t,x) : \ t = \phi(s,x), \ |x - x_0| < t_0.\}. \end{aligned}$$

Then, we have

$$\Lambda^-_{(t_0,x_0)} = \bigcup_{s\in[0,t_0)} R_s, \qquad \partial R_s = \Lambda_s \cup \underbrace{\{t = 0, \ |x - x_0| < t_0\}}_{\text{where } u = u_t = 0}.$$

Since the outward unit normal at $(\phi(s,x),x) \in \Lambda_s$ is $(1, -\nabla_x \phi)/\sqrt{1 + |\nabla_x \phi|^2}$, so by the divergence theorem, we have

$$\begin{aligned} \int_{R_s} u_t F \ dtdx = \int_{R_s} u_t \Box u \ dtdx &= \int_{R_s} \underbrace{-\frac{1}{2}\partial_t |u'|^2 + \sum_{j=1}^d \partial_j (u_j u_t)}_{\text{from the proof of the energy estimate with } g = m} \ dtdx \\ &= \int_{\Lambda_s} (-\frac{1}{2}|u'|^2 - \sum_{j=1}^d \phi_j u_j u_t) \ \frac{dS}{\sqrt{1 + |\nabla_x \phi|^2}} \\ &\le \int_{\Lambda_s} \frac{1}{2}(-1 + \theta(s))|u'|^2 \ \frac{dS}{\sqrt{1 + |\nabla_x \phi|^2}}. \end{aligned}$$

To get the last estimate, we notice that

$$|\sum_{j=1}^d \phi_j u_j u_t| \le |\nabla_x \phi \cdot \nabla_x u||u_t| \le |\nabla_x \phi||\nabla_x u||u_t| \le \frac{1}{2}|\nabla_x \phi|(|\nabla_x u|^2 + |u_t|^2) = \frac{1}{2}|\nabla_x \phi||u'|^2.$$

Meanwhile, since $F(0,0,u'')=0$, we have $|F(u,u',u'')| \lesssim |u|+|u'|$ (since u,u',u'' remains bounded in the closure of $\Lambda^-_{(t_0,x_0)}$) and thus

$$u_tF \geq -|u_tF| \geq -C|u_t|(|u|+|u'|) \geq -C(|u|^2+|u'|^2).$$

By the Minkowski inequality

$$\begin{aligned}\int_0^{\phi(s,x)} |u(t,x)|^2\ dt &= \int_0^{\phi(s,x)} |\int_0^t \partial_\tau u(\tau,x)\ d\tau|^2\ dt \qquad\qquad (u(0,x)=0)\\ &\leq t_0^2 \int_0^{\phi(s,x)} |u'(\tau,x)|^2\ d\tau,\end{aligned}$$

we have

$$\begin{aligned}\int_{R_s} u_tF\ dtdx \geq -C\int_{R_s}(|u'|^2+|u|^2)\ dtdx &\geq -C(t_0^2+1)\int_{R_s}|u'|^2\ dtdx\\ &= -C(t_0^2+1)\int_0^s\int_{\Lambda_{s'}} |u'|^2\cdot\frac{\phi_t}{\sqrt{1+|\nabla_x\phi|^2}}\ dSds'.\end{aligned}$$

The last identity comes from the following lemma (which is Theorem 6 in Appendix C.4, [Eva10]).

Lemma 3.3. *Consider a family of smooth bounded regions $U(\tau)\subset\mathbb{R}^d$ depending on $\tau\in\mathbb{R}$ smoothly. Write v for the velocity of the moving boundary $\partial U(\tau)$ and ν for the outward pointing unit normal. If $f=f(x,\tau)$ is a smooth function, then*

$$\frac{d}{d\tau}\int_{U(\tau)} f dx = \int_{\partial U(\tau)} fv\cdot\nu\ dS + \int_{U(\tau)}\partial_\tau f\ dx.$$

In this proof, $|u'|^2$ does not depend on s', so there is only one integral on $\Lambda_{s'}$. Morever, $\nu = (1,-\nabla_x\phi)/\sqrt{1+|\nabla_x\phi|^2}$ and $v=\frac{d}{ds'}(\phi(s',x),x)=(\phi_t,0)$.

In summary, if we set $I(s)=\int_{\Lambda_s}|u'|^2\frac{dS}{\sqrt{1+|\nabla_x\phi|^2}}$, for each $0\leq s\leq s_0<t_0$ we have

$$\frac{1}{2}(1-\theta(s_0))I(s) \leq -\int_{R_s} u_tF\ dtdx \leq C(t_0^2+1)\int_0^s \sup_{\substack{t\in[0,s_0]\\ |x-x_0|<t_0}} |\partial_t\phi(t,x)|\cdot I(s')\ ds'.$$

We can prove that $I(s)=0$ for all $0\leq s\leq s_0<t_0$ by applying the Gronwall's inequality.

Lemma 3.4 (Gronwall's inequality)**.** *Suppose that A,E,r are bounded nonnegative functions on $[0,T]$ and that E is increasing there. If*

$$A(t)\leq E(t)+\int_0^t r(s)A(s)\ ds, \qquad t\in[0,T],$$

it follows that

$$A(t)\leq E(t)\exp(\int_0^t r(s)\ ds). \tag{3.7}$$

Proof. It suffices to prove this inequality at $t = T$, in which case we can replace $E(t)$ with $E := E(T)$ in the assumption. If we set

$$B(t) = E + \int_0^t r(s)A(s)\ ds,$$

then

$$B'(t) = r(t)A(t) \le r(t)B(t).$$

Thus,

$$\partial_t (B(t) \exp(-\int_0^t r(s)\ ds)) \le 0$$

and thus

$$B(T) \exp(-\int_0^T r(s)\ ds)) \le B(0) = E.$$

This ends the proof. $\square$

Thus

$$u' \equiv 0 \text{ in } \qquad \bigcup_{s_0 \in [0,t_0)} \bigcup_{s \in [0,s_0]} \Lambda_s = \bigcup_{s_0 \in [0,t_0)} R_{s_0} = \Lambda^-_{(t_0,x_0)}.$$

It follows from $u \equiv 0$ whenever $t = 0$ and $|x - x_0| < t_0$ that $u \equiv 0$ in $\Lambda^-_{(t_0,x_0)}$.

Finally, we give an explicit formula for ϕ. We can set

$$\phi(s,x) := t_0 - \left((t_0 - s)^2 - t_0^{-2}(s^2 - 2ts)|x - x_0|^2\right)^{1/2}.$$

We can check (left as an exercise) that the assumptions a)-c) listed at the beginning of this proof hold for this ϕ.

$\square$

Remark 3.4.1. The Gronwall's inequality (3.7) is an important tool in the proof of the (almost) global existence results in this course.

An obvious corollary of Proposition 3.2 is as follows.

Corollary 3.5. *Let u be a C^2 solution to $\Box u = F(u, u', u'')$ in $[0,T) \times \mathbb{R}^d$. Assume that $F(0,0,u'') = 0$. If $u = \partial_t u = 0$ whenever $t = 0$ and $|x| > R$, then $u \equiv 0$ whenever $t \in [0,T)$ and $|x| > t + R$.*

By sending $R \to 0$ and using the continuity, we also get the uniqueness result in the previous section.

4 Local existence and blowup criteria

In order to prove a global existence result, we need to apply a local existence result.

Consider the Cauchy problem

$$\begin{cases} g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta u = F(u,u') & \text{in } \mathbb{R}^{1+d}_+; \\ (u,\partial_t u)|_{t=0} = (u^0,u^1). \end{cases} \tag{4.1}$$

In this section, we consider general space dimension $d \geq 1$. Moreover, we assume that $g^{\alpha\beta}, F$ are given C^∞ functions with all derivatives $O(1)$, such that $F(0,0) = 0$ and

$$\sum_{\alpha,\beta=0}^{3} |g^{\alpha\beta} - m^{\alpha\beta}| < 1/2. \tag{4.2}$$

In the following local existence result, we assume that our initial data belong to some L^2-based Sobolev space. Here is the definition.

Definition 4.1. Fix $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{R}^d)$ consists of $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\widehat{u} \in L^2_{loc}(\mathbb{R}^d)$ and

$$\|u\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\widehat{u}(\xi)|^2 \, d\xi < \infty. \tag{4.3}$$

Here we recall that $\mathcal{S}'(\mathbb{R}^d)$ is the space of *tempered distributions*, i.e. bounded (or equivalently, continuous) linear functionals of $\mathcal{S}(\mathbb{R}^d)$. If you have never seen $\mathcal{S}'(\mathbb{R}^d)$ before, you can simply view $H^s(\mathbb{R}^d)$ as the closure of $\mathcal{S}(\mathbb{R}^d)$ in some larger space under the H^s norm.

By the Plancherel's theorem, if $u \in H^s(\mathbb{R}^d)$, then using the Fourier multiplier, we can write

$$\|u\|_{H^s(\mathbb{R}^d)} = (2\pi)^{d/2} \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^d)}. \tag{4.4}$$

Moreover, since $\mathcal{F}(\partial_j u) = i\xi_j \widehat{u}$ and since $\langle\xi\rangle^2 \sim 1 + \sum_{j=1}^d \xi_j^2$, we have

$$\|\langle D\rangle u\|_{L^2(\mathbb{R}^d)} \sim_d \|u\|_{L^2(\mathbb{R}^d)} + \sum_{j=1}^d \|\partial_j u\|_{L^2(\mathbb{R}^d)}.$$

Thus, if s is a nonnegative integer, then

$$\|u\|_{H^s(\mathbb{R}^d)} \sim_{s,d} \sum_{|\alpha|\leq s} \|\partial^\alpha u\|_{L^2(\mathbb{R}^d)}. \tag{4.5}$$

We can now state the local existence theorem.

Theorem 4.2 (Local existence, Theorem 6.4.11 in [H97])**.** *Let $s > (d+2)/2$ be an integer. If $(u^0,u^1) \in H^{s+1} \times H^s(\mathbb{R}^d)$, then there exists $T > 0$, depending on the norm of the initial data, such that the Cauchy problem* (4.1) *has a unique solution*

$$u \in L^\infty H^{s+1} \cap C^{0,1} H^s([0,T] \times \mathbb{R}^d). \tag{4.6}$$

Here $C^{0,1}$ denotes the space of Lipschitz continuous function, so $u_t \in H^s$. It follows that $u \in C^2([0,T] \times \mathbb{R}^d)$.

Moreover, if T_* is the supremum over all such times T, then either $T_* = \infty$ or

$$\sum_{|\alpha|\leq 2} |\partial^\alpha u| \notin L^\infty([0,T_*) \times \mathbb{R}^d). \tag{4.7}$$

Remark 4.2.1. The second half of this theorem is a *blowup criterion*. It tells us what happens if there is a finite time blowup. Obviously, we need to use this criterion if we hope to prove global existence.

In this note, I prefer not to give a complete proof of Theorem 4.2. For simplicity, I plan to follow the proof in [Sog08] which gives a weaker version of Theorem 4.2.

Theorem 4.3 (Local existence, Theorem I.4.1 in [Sog08])**.** *Let $s \geq (d+2)$ be an integer. If $(u^0, u^1) \in H^{s+1} \times H^s(\mathbb{R}^d)$, then there exists $T > 0$, depending on the norm of the initial data, such that the Cauchy problem* (4.1) *has a unique solution satisfying*

$$\sum_{|\alpha| \leq s+1} \|\partial^\alpha u\|_{L^2(\mathbb{R}^d)} < \infty, \qquad \forall t \in [0, T]. \tag{4.8}$$

Moreover, if T_* is the supremum over all such times T, then either $T_* = \infty$ or

$$\sum_{|\alpha| \leq (s+3)/2} |\partial^\alpha u| \notin L^\infty([0, T_*) \times \mathbb{R}^d). \tag{4.9}$$

In this note, we are mainly concerned with Cauchy problems with C_c^∞ data. Thus, we prefer to use the following theorem.

Theorem 4.4 (Theorem I.4.2 in [Sog08])**.** *If $(u^0, u^1) \in C_c^\infty(\mathbb{R}^d)$, then there exists $T > 0$ such that* (4.1) *has a solution $u \in C^\infty([0, T] \times \mathbb{R}^d)$. If T_* is the supremum over all such times T, then either $T_* = \infty$ or*

$$\sum_{|\alpha| \leq (d+6)/2} |\partial^\alpha u| \notin L^\infty([0, T_*) \times \mathbb{R}^d). \tag{4.10}$$

4.1 Existence and uniqueness for linear equations

The proof of Theorem 4.3 is based on local existence of a linear problem. Set a linear differential operator L by

$$Lu = g^{\alpha\beta}(t, x)\partial_\alpha\partial_\beta u + b^\alpha(t, x)\partial_\alpha u + a(t, x)u. \tag{4.11}$$

Here we assume that g^{**}, b^*, a are all given C^∞ functions with uniform bounds on each derivative if $(t, x) \in [0, T] \times \mathbb{R}^d$. Moreover, we assume that

$$\sum_{\alpha,\beta=0}^{3} |g^{\alpha\beta}(t, x) - m^{\alpha\beta}| < 1/2, \qquad \text{in } [0, T] \times \mathbb{R}^d.$$

We now set L^* as the $L^2(\mathbb{R}^{1+d})$-adjoint of L. In other words, given $v \in C_c^\infty(\mathbb{R}^{1+d})$, we expect

$$\langle Lu, v\rangle_{L^2(\mathbb{R}^{1+d})} (= \int_{\mathbb{R}^{1+d}} Lu \cdot v \ dtdx) = \langle u, L^* v\rangle_{L^2(\mathbb{R}^{1+d})}.$$

Using integration by parts, we can see that

$$L^* v = \partial_\alpha \partial_\beta (g^{\alpha\beta} v) - \partial_\alpha (b^\alpha v) + av. \tag{4.12}$$

Here (4.12) is written in divergence form, and we can write it in non-divergence form (so L^* has the same form as L does with different b^* and a).

We can now state the main theorem for this subsection.

Theorem 4.5. *Let $s \in \mathbb{R}$. Then, for each $(u^0, u^1) \in H^{s+1} \times H^s(\mathbb{R}^d)$ and $F \in L^1 H^s([0,T] \times \mathbb{R}^d)$, there is a unique*

$$u \in CH^{s+1} \cap C^1 H^s([0,T] \times \mathbb{R}^d) \tag{4.13}$$

solving

$$\begin{cases} Lu = F & \text{in } (0,T) \times \mathbb{R}^d; \\ (u, \partial_t u)|_{t=0} = (u^0, u^1). \end{cases} \tag{4.14}$$

In this theorem, when we say u solves (4.14), we mean that u is a weak solution to (4.14) in the following sense: for each $\psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)$, we have

$$\begin{aligned} \int_{[0,T]\times\mathbb{R}^d} \psi F \; dtdx = &\int_{[0,T]\times\mathbb{R}^d} L^* \psi u \; dtdx - \int_{\mathbb{R}^d} \psi(0,x) g^{00}(0,x) u^1(x) \; dx \\ &+ \int_{\mathbb{R}^d} \left(\partial_t(g^{00}\psi) - b^0 \psi + 2 \sum_{j=1}^d \partial_j(\psi g^{j0}) \right) (0,x) u^0(x) \; dx. \end{aligned} \tag{4.15}$$

One can derive this formula assuming that $u \in C^2$ solves (4.14) pointwisely by applying integration by parts.

We start our proof with the following estimate.

Theorem 4.6. *Let $s \in \mathbb{R}$, $T \in (0, \infty)$ and assume L is as above. If*

$$u \in CH^{s+1} \cap C^1 H^s([0,T] \times \mathbb{R}^d),$$

and if

$$Lu \in L^1 H^s([0,T] \times \mathbb{R}^d),$$

then for $t \in (0,T)$ we have

$$\sum_{|\alpha| \le 1} \|\partial^\alpha u(t, \cdot)\|_{H^s(\mathbb{R}^d)} \lesssim_{s,T} \sum_{|\alpha| \le 1} \|\partial^\alpha u(0, \cdot)\|_{H^s(\mathbb{R}^d)} + \int_0^t \|Lu(\tau, \cdot)\|_{H^s(\mathbb{R}^d)} \; d\tau. \tag{4.16}$$

Proof. We start with the case $s = 0$. Since the coefficients of L are bounded, we have

$$\left\| g^{\alpha\beta} \partial_\alpha \partial_\beta u(t) \right\|_{L^2} \le \|Lu(t)\|_{L^2} + C \sum_{|\alpha| \le 1} \|\partial^\alpha u(t)\|_{L^2}.$$

By the energy estimate (3.3) in Proposition 3.1, we have

$$\left\| u'(t) \right\|_{L^2} \lesssim \left(\left\| u'(0) \right\|_{L^2} + \int_0^t (\|Lu(\tau)\|_{L^2} + \sum_{|\alpha| \le 1} \|\partial^\alpha u(\tau)\|_{L^2}) \; d\tau \right) \exp(CT).$$

By the fundamental theorem of calculus, we have

$$\|u(t)\|_{L^2} \le \|u(0)\|_{L^2} + \int_0^t \left\| u'(\tau) \right\|_{L^2} \; d\tau.$$

In summary, if we set $A(t) = \sum_{|\alpha|\leq 1} \|\partial^\alpha u(t)\|_{L^2}$, then we have

$$A(t) \leq C_{s,T}(A(0) + \int_0^t Lu(\tau)\ d\tau + \int_0^t A(\tau)\ d\tau).$$

By the Gronwall's inequality ((3.7)), we conclude (4.16) with $s = 0$.

For general $s \in \mathbb{R}$, we apply (4.16) with $s = 0$ to $\langle D\rangle^s u$ (recall that $\langle D\rangle^s$ is a Fourier multiplier defined in Remark 2.1.2). We thus have

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha \langle D\rangle^s u(t)\|_{L^2} \lesssim_{s,T} \sum_{|\alpha|\leq 1} \|\partial^\alpha \langle D\rangle^s u(0)\|_{L^2} + \int_0^t \|L\langle D\rangle^s u(\tau)\|_{L^2}\ d\tau.$$

Since we can commute ∂^α and $\langle D\rangle^s$, by (4.4) we have

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha \langle D\rangle^s u(t)\|_{L^2} \sim \sum_{|\alpha|\leq 1} \|\partial^\alpha u(t)\|_{H^s}.$$

Moreover, we have

$$\|L\langle D\rangle^s u(\tau)\|_{L^2} \lesssim \|\langle D\rangle^s Lu(\tau)\|_{L^2} + \|[L, \langle D\rangle^s]u(\tau)\|_{L^2} \lesssim \|Lu(\tau)\|_{H^s} + \|[L, \langle D\rangle^s]u(\tau)\|_{L^2}.$$

By replacing L with $|g^{00}|^{-1}L$, we assume without loss of generality that $g^{00} \equiv -1$. In this case, we claim that for each $s \in \mathbb{R}$,

$$\|[L, \langle D\rangle^s]u(\tau)\|_{L^2} \lesssim_s \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\tau)\|_{H^s}.$$

If this claim holds, then we finish the proof by applying the Gronwall's inequality.

The proof of this claim follows from the boundedness of pseudodifferential operators. For simplicity, I will not prove this result in this note. Instead, let us consider the simplest case when s is a positive even number. In this case, $\langle D\rangle^s = (I - \Delta)^{s/2}$ is a linear differential operator of order s. In this case, the commutator $[L, \langle D\rangle^s]$ is a linear differential operator of order $s+1$ with $O(1)$ coefficients, and we notice that there is at most one t-derivative because $g^{00} \equiv -1$. As a result,

$$\|[L, \langle D\rangle^s]u(\tau)\|_{L^2} \lesssim \sum_{\substack{|\alpha|+|\beta|\leq 1+s \\ |\alpha|\leq 1}} \left\|\partial_t^\alpha \partial_x^\beta u(\tau)\right\|_{L^2} \lesssim \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\tau)\|_{H^s}.$$

Note that in the last step we use (4.5). □

We can now prove a local existence result for (4.14).

Proof of Theorem 4.5. We first prove the uniqueness part. If u and $\widetilde{u}$ are two solutions to (4.14) with the same initial data, then $w := u - \widetilde{u}$ solves $Lw = 0$ and $(w, w_t)|_{t=0} = 0$. By applying (4.16) to w, we conclude that $w = 0$.

Now we prove the existence part. We start with the zero data case ($u^0, u^1 \equiv 0$). For each $\psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)$, by applying (4.16) to ψ with t replaced by $T - t$ and L replaced by L^*, we have for each $t \in [0, T]$

$$\|\psi(t)\|_{H^{-s}} \lesssim \|\psi(t)\|_{H^{-s-1}} + \left\|\psi'(t)\right\|_{H^{-s-1}} \lesssim \int_0^T \|L^*\psi(\tau)\|_{H^{-s-1}}\ d\tau.$$

Here we recall that $\psi \equiv 0$ near $t = T$. We have

$$\begin{aligned}
|\langle F, \psi\rangle_{L^2(\mathbb{R}^{1+d}_+)}| &\lesssim \int_0^T |\langle F, \psi\rangle_{L^2(\mathbb{R}^d)}| \, dt \lesssim \int_0^T \|F(t)\|_{H^s} \|\psi(t)\|_{H^{-s}} \, dt \\
&\lesssim \int_0^T \int_0^T \|F(t)\|_{H^s} \|L^*\psi(\tau)\|_{H^{-s-1}} \, d\tau dt \\
&\lesssim \|F\|_{L^1H^s} \int_0^T \|L^*\psi(t)\|_{H^{-s-1}} \, dt.
\end{aligned}$$

Note that if u is a weak solution to (4.14) with zero data, then we should have $\langle u, L^*\psi\rangle_{L^2(\mathbb{R}^{1+d}_+)} = \langle F, \psi\rangle_{L^2(\mathbb{R}^{1+d}_+)}$. The proof above shows that if we set $W(v) := \langle F, v\rangle$, then W is a bounded linear functional of $\{L^*\psi : \ \psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)\} \subset L^1H^{-s-1}([0,T] \times \mathbb{R}^d)$. So by the Hahn-Banach theorem, we can extend W to a bounded linear functional of $L^1H^{-s-1}([0,T] \times \mathbb{R}^d)$ with the same norm. And since the dual of L^1H^{-s-1} is $L^\infty H^{s+1}$, we obtain

$$u \in L^\infty H^{s+1}([0,T] \times \mathbb{R}^d), \qquad \text{such that } W(v) = \langle u, v\rangle.$$

Since $\langle u, L^*\psi\rangle = W(L^*\psi) = \langle F, \psi\rangle$, we conclude that u is a weak solution to (4.14) with zero data.

Next we need to show (4.13). First we assume that $F \in C_c^\infty([0,T] \times \mathbb{R}^d)$. Since $C_c^\infty \subset L^1H^s$ for each $s \in \mathbb{R}$ and since the initial data are zero, the weak solution from the previous paragraph belongs to $L^\infty H^N([0,T] \times \mathbb{R}^d)$ for each $N \in \mathbb{Z}$. We fix a large integer $N \geq 2$ whose value will be chosen later, and we claim that in this case

$$u \in CH^{N-1} \cap C^1H^{N-2}([0,T] \times \mathbb{R}^d).$$

To see this, we notice that $v = u_t$ solves

$$g^{00}v_t + 2\sum_{j=1}^d g^{j0}v_j + b^0v = -\sum_{j,k\geq 1} g^{jk}u_{jk} - \sum_{j=1}^d b^ju_j - au + F.$$

The right hand side belongs to $L^\infty H^{N-1}([0,T] \times \mathbb{R}^d)$ because of (4.5) and the Leibniz's rule; the proof is left as an exercise. Without loss of generality, we assume that $g^{00} \equiv -1$. We claim that $u_t = v \in L^\infty H^{N-1}([0,T] \times \mathbb{R}^d)$. In fact, for each $|\alpha| \leq N-1$, we have

$$\begin{aligned}
\frac{d}{dt} \|\partial_x^\alpha v(t)\|_{L^2}^2 &= 2\int \partial_x^\alpha v \partial_x^\alpha v_t \, dx = 2\int \partial_x^\alpha v \partial_x^\alpha (2g^{j0}\partial_j v + b^0 v + Q) \, dx \\
&\leq 2\int g^{j0}\partial_j[(\partial_x^\alpha v)^2] \, dx + C\int |\partial_x^\alpha v|(|\partial_x^\alpha Q| + \sum_{|\beta|\leq|\alpha|} |\partial_x^\beta v|) \, dx \\
&\leq C\|Q\|_{L^\infty H^{N-1}} \cdot \|v(t)\|_{H^{N-1}} + C\|v(t)\|_{H^{N-1}}^2.
\end{aligned}$$

Here $Q = -(-\sum_{j,k\geq 1} g^{jk}u_{jk} - \sum_{j=1}^d b^ju_j - au + F) \in L^\infty H^{N-1}([0,T] \times \mathbb{R}^d)$. To get the last step, we note that (using integration by parts and the density of C_c^∞ in H^N)

$$\int g^{j0}\partial_j[(\partial_x^\alpha v)^2] \, dx = \int \partial_j[g^{j0}(\partial_x^\alpha v)^2] \, dx - \int (\partial_j g^{j0})(\partial_x^\alpha v)^2 \, dx \leq C\|v(t)\|_{H^{N-1}}^2.$$

As a result, we have

$$\frac{d}{dt}\|v(t)\|_{H^{N-1}} \leq C\|Q\|_{L^\infty H^{N-1}} + C\|v(t)\|_{H^{N-1}}$$

and thus $\|v(t)\|_{H^{N-1}} \lesssim_T \|Q\|_{L^\infty H^{N-1}}$ by the Gronwall's inequality. This finishes the proof of the claim. Using the equation $Lu = F$ again, we obtain $u_{tt} = g \cdot \nabla_x u' + b \cdot u' + au - F \in L^\infty H^{N-2}([0,T] \times \mathbb{R}^d)$. In summary, we have

$$u \in CH^{N-1} \cap C^1 H^{N-2}([0,T] \times \mathbb{R}^d).$$

We now choose $N \geq s+2$ and thus obtain (4.13).

For general $F \in L^1 H^s([0,T] \times \mathbb{R}^d)$, we choose a sequence of F_m in $C_c^\infty([0,T] \times \mathbb{R}^d)$ such that $F_m \to F$ in $L^1 H^s$. For each F_m, we have obtain a unique $u_m \in CH^{s+1} \cap C^1 H^s([0,T] \times \mathbb{R}^d)$ such that $Lu_m = F_m$ and $(u_m, \partial_t u_m)|_{t=0} = 0$. By applying (4.16) to $u_m - u_n$, we obtain

$$\sup_{t\in[0,T]}(\|(u_m-u_n)(t)\|_{H^s} + \left\|(u_m-u_n)'(t)\right\|_{H^s}) \lesssim \int_0^T \|(F_m-F_n)(\tau)\|_{H^s}\ d\tau \to 0, \quad m,n\to\infty.$$

So, $\{u_m\}$ is a Cauchy sequence in $CH^{s+1} \cap C^1 H^s([0,T] \times \mathbb{R}^d)$, so it has a limit u in $CH^{s+1} \cap C^1 H^s([0,T] \times \mathbb{R}^d)$. It is easy to check that u is the solution to (4.14) with zero data.

To solve the equation with general Cauchy data, we first assume that the initial data belong to C_c^∞. We then set $u_0(t,x) = u^0(x) + tu^1(x)$. If v solves $Lv = F - Lu_0$ with zero data, then $u = v + u_0$ solves (4.14) with data (u^0, u^1). Here we need to assume that the initial data belong to C_c^∞ because we need $Lu_0 \in L^1 H^s$. For general data $(u^0, u^1) \in H^{s+1} \times H^s$, we use a sequence of C_c^∞ data to approximate it and apply (4.16). The proof here is very similar to that in the previous paragraph. $\square$

4.2 Local existence for quasilinear equations

We now return to the proof of Theorem 4.3.

4.2.1 Uniqueness

Suppose that u and $\widetilde{u}$ solve (4.1) with the same data and that

$$u, \widetilde{u} \in L^\infty H^{s+1} \cap C^{0,1} H^s([0,T] \times \mathbb{R}^d).$$

Then, we obtain

$$g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta(u-\widetilde{u}) = (g^{\alpha\beta}(\widetilde{u},\widetilde{u}') - g^{\alpha\beta}(u,u'))\partial_\alpha\partial_\beta\widetilde{u} + F(u,u') - F(\widetilde{u},\widetilde{u}').$$

We hope to apply the energy estimate (3.3) to $u - \widetilde{u}$, so we need to estimate the L^2 norm of the right hand side at time t. It is easy to see that

$$|(g^{\alpha\beta}(\widetilde{u},\widetilde{u}') - g^{\alpha\beta}(u,u'))\partial_\alpha\partial_\beta\widetilde{u} + F(u,u') - F(\widetilde{u},\widetilde{u}')| \lesssim \sum_{|\alpha|\leq 1} |\partial^\alpha(u-\widetilde{u})| \cdot (1 + |\widetilde{u}''|).$$

Then, by the energy estimate (3.3) and the fundamental theorem of calculus

$$\|w(t)\|_{L^2} \lesssim \|w(0)\|_{L^2} + \int_0^t \|\partial_t w(\tau)\|_{L^2} \, d\tau, \tag{4.17}$$

we conclude that

$$\begin{aligned}\sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(t)\|_{L^2} &\lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(0)\|_{L^2} + \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(\tau)\|_{L^2} \cdot (1+\left\|\widetilde{u}''(\tau)\right\|_{L^\infty}) \, d\tau \\ &= \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(\tau)\|_{L^2} \cdot (1+\left\|\widetilde{u}''(\tau)\right\|_{L^\infty}) \, d\tau.\end{aligned}$$

The second estimate holds because u and $\widetilde{u}$ have equal initial data. By the Sobolev embedding, we also have

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \sum_{|\alpha|\le \lfloor \frac{d+2}{2} \rfloor} \|\partial^\alpha f\|_{L^2(\mathbb{R}^d)} .$$

Here we remind our readers that $\lfloor \frac{d+2}{2} \rfloor$ is the smallest integer larger than $d/2$. Since $s \ge d+2 \ge 2 + \lfloor \frac{d+2}{2} \rfloor$, we have

$$\left\|\widetilde{u}''(t)\right\|_{L^\infty} \lesssim \sum_{|\alpha|\le 2+\lfloor \frac{d+2}{2} \rfloor} \|\partial^\alpha \widetilde{u}(t)\|_{L^2} \lesssim \|\widetilde{u}\|_{L^\infty H^{s+1}} .$$

It follows that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(t)\|_{L^2} \lesssim \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha (u-\widetilde{u})(\tau)\|_{L^2} \cdot (1+\left\|\widetilde{u}''\right\|_{L^\infty H^{s+1}}) \, d\tau, \qquad t \in [0,T].$$

By the Gronwall's inequality (3.7), we conclude that $u \equiv \widetilde{u}$ in $[0,T] \times \mathbb{R}^d$.

The argument in the uniqueness proof will also be used in the existence proof below. There we also need to apply the energy estimate (3.3) and the estimate (4.17) to control the H^{s+1} norm of the solution.

4.2.2 Existence

We now prove the existence part. We assume that $u^0, u^1 \in C_c^\infty$ for simplicity. For general data, we can use an approximation argument which is similar to that used in the proof of Theorem 4.5; we skip the details in this note. To construct a solution, we use the method of Picard iteration. Set $u_{-1} \equiv 0$ and define u_m for $m \ge 0$ inductively by

$$\begin{cases} g^{\alpha\beta}(u_{m-1}, u'_{m-1}) \partial_\alpha \partial_\beta u_m = F(u_{m-1}, u'_{m-1}), \\ (u_m, \partial_t u_m)|_{t=0} = (u^0, u^1). \end{cases} \tag{4.18}$$

Since u_{m-1} is known before we solve (4.18), the equation (4.18) is a *linear* Cauchy problem for u_m. Here we hope to prove that

a) The solution u_m to (4.18) exists and belongs to $C_c^\infty([0,T]\times\mathbb{R}^d)$ (in both t,x). Here T is a fixed number independent of m to be chosen later in b).

b) There exists a sufficiently small time $T>0$ and a sufficiently large constant $A>1$, both independent of m, such that

$$A_m(t) := \sum_{|\alpha|\le s+1} \|\partial^\alpha u_m(t)\|_{L^2} \le A < \infty, \qquad t\in[0,T]. \tag{4.19}$$

Here we remind our readers that $A_m(t)$ is not equivalent to the H^{s+1} norm of $u_m(t)$, because we allow time derivatives in (4.19).

c) We have

$$C_m(t) := \|u_m(t)-u_{m-1}(t)\|_{L^2} + \left\|u'_m(t)-u'_{m-1}(t)\right\|_{L^2} \lesssim 2^{-m}. \tag{4.20}$$

We will prove a)-c) by induction. If $m=-1$, there is nothing to prove. Now we fix $m\ge 0$ and suppose a)-c) above hold with m replaced by $m-1$.

Let us first prove part a). By the induction hypotheses, we have $u_{m-1}\in C_c^\infty([0,T]\times\mathbb{R}^d)$, so $g^{**}(u_{m-1},u'_{m-1})$ and $F(u_{m-1},u'_{m-1})$ are C^∞ with uniform bounds on each derivative. Since $F(0,0)=0$, we also have $F(u_{m-1},u'_{m-1})\in C_c^\infty([0,T]\times\mathbb{R}^d)$. Thus, we can apply Theorem 4.5 to obtain a unique solution

$$u_m \in \bigcap_{s\ge 0}(CH^{s+1}\cap C^1H^s)([0,T]\times\mathbb{R}^d) \subset C_t^1C_x^\infty([0,T]\times\mathbb{R}^d)$$

to (4.18). Here we use the Sobolev embedding $\bigcap_{s\ge0}H_x^s\subset C_x^\infty$. To show $u_m\in C_{t,x}^\infty$, we use (4.18) (or (4.21) below) to lower the order of time derivatives. For example, we have

$$\partial_t^2 u_m = (-g^{00})^{-1}(g^{**}\cdot\partial\nabla_x u_m - F).$$

It follows that $\partial_t^2 u_m\in C_x^\infty$. We continue this process and conclude that $u_m\in C_{t,x}^\infty$. Since $u_{m-1}\in C_c^\infty$, we can choose some $R=R_{m-1}>0$ such that $u_{m-1}\equiv 0$ whenever $t\in[0,T]$ and $|x|>R$. In other words, we have

$$g^{\alpha\beta}(0,0)\partial_\alpha\partial_\beta u_m = 0, \quad \text{whenever } |x|\ge R; \qquad u_m=\partial_t u_m = 0, \quad \text{whenever } t=0,\ |x|\ge R.$$

If $\chi\in C^\infty(\mathbb{R})$ is a function such that $\chi|_{(-\infty,R+1)}=0$ and $\chi|_{(R+2,\infty)}=1$, then $w:=\chi(|x|)u_m$ is a solution to $g^{\alpha\beta}(0,0)\partial_\alpha\partial_\beta w=0$ with zero initial data. With the help of the energy estimate (3.3), we conclude that $u\equiv 0$ whenever $|x|\ge R+2$. Thus, $u_m\in C_c^\infty([0,T]\times\mathbb{R}^d)$.

Next we prove part b). For each $l\le s$, we have

$$\begin{aligned} g^{\alpha\beta}(u_{m-1},u'_{m-1})\partial_\alpha\partial_\beta\partial^l u_m &= [g^{\alpha\beta}(u_{m-1},u'_{m-1})\partial_\alpha\partial_\beta,\partial^l]u_m + \partial^l(g^{\alpha\beta}(u_{m-1},u'_{m-1})\partial_\alpha\partial_\beta u_m) \\ &= [g^{\alpha\beta}(u_{m-1},u'_{m-1}),\partial^l]\partial_\alpha\partial_\beta u_m + \partial^l F(u_{m-1},u'_{m-1}). \end{aligned} \tag{4.21}$$

To avoid ambiguity, we use ∂^l to denote any ∂^α with $|\alpha|=l$. We seek to apply the energy estimate (3.3) to (4.21), so we need the following lemma.

Lemma 4.7. *Let F and g^{**} be as above and assume that $v \in C^\infty(\mathbb{R}^{1+d})$. For $l \leq s$, we have*

$$|\partial^l F(v,v')| \lesssim (1+\sum_{|\beta|\leq\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta v(t,\cdot)\right\|_{L^\infty})^{s-1}\sum_{|\beta|\leq s+1}|\partial^\beta v(t,x)|, \tag{4.22}$$

$$\begin{aligned}|[g^{\alpha\beta}(v,v'),\partial^l]\partial_\alpha\partial_\beta w| &\lesssim (1+\sum_{|\beta|\leq\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta v(t,\cdot)\right\|_{L^\infty})^{s}\sum_{|\beta|\leq s+1}|\partial^\beta w(t,x)|\\ &+(1+\sum_{|\beta|\leq\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta v(t,\cdot)\right\|_{L^\infty})^{s-1}\sum_{|\beta|\leq\lfloor\frac{s+3}{2}\rfloor}\left\|\partial^\beta w(t,\cdot)\right\|_{L^\infty}\cdot\sum_{|\beta|\leq s+1}|\partial^\beta v(t,x)|.\end{aligned} \tag{4.23}$$

Proof. For simplicity, for each $N \geq 0$ we write $|v_{\leq N}| := \sum_{|\alpha|\leq N}|v|$.

If $l = 0$, it follows from $F(0,0) = 0$ and $|F^{(1)}| \lesssim 1$ that $|F(v,v')| \lesssim |v| + |v'|$. We also have $[g^{\alpha\beta}(v,v'),\partial^l] = 0$. Now we assume $0 < l \leq s$. By the Leibniz's rule and chain rule, we can write $\partial^l F(v,v')$ as a linear combination (with real constant coefficients) of terms of the form

$$F^{(r)}(v,v')\cdot\prod_{j=1}^r\partial^{l_j}\partial^{k_j}v,\qquad 1\leq r\leq l,\ \sum l_* = l\leq s,\ k_j = 0,1.$$

So, there is at most one l_j with $l_j > s/2$ (or equivalently, $l_j \geq \lfloor s/2\rfloor + 1 = \lfloor (s+2)/2\rfloor$). We use $|v_{\leq s+1}|$ to estimate the term $\partial^{l_j}\partial^{k_j}v$ with the largest l_j, and use $|v_{\leq\lfloor\frac{s+2}{2}\rfloor}|$ to control the rest $r-1$ terms. We conclude that

$$|\partial^l F(v,v')| \lesssim (1+|v_{\leq\lfloor\frac{s+2}{2}\rfloor}|)^{s-1}|v_{\leq s+1}|.$$

This finishes the proof of (4.22). Note that the assumption $F(0,0) = 0$ is not used in the proof above with $l > 0$, so for each $0 < l \leq s$, we have also proved that

$$|\partial^l g(v,v')| \lesssim (1+|v_{\leq\lfloor\frac{s+2}{2}\rfloor}|)^{s-1}|v_{\leq s+1}|. \tag{4.24}$$

Moreover, we have

$$[g^{\alpha\beta}(v,v'),\partial^l]\partial_\alpha\partial_\beta w = g^{\alpha\beta}(v,v')\partial^l\partial_\alpha\partial_\beta w - \partial^l(g^{\alpha\beta}(v,v')\partial_\alpha\partial_\beta w).$$

If we apply the Leibniz's rule and chain rule to expand $-\partial^l(g^{\alpha\beta}(v,v')\partial_\alpha\partial_\beta w)$, again we can write $[g^{\alpha\beta}(v,v'),\partial^l]\partial_\alpha\partial_\beta w$ as a linear combination (with real constant coefficients) of terms of the form

$$\partial^{l_0}(g(v,v'))\cdot\partial^{l_1}\partial^2 w,\qquad l_0+l_1 = l,\ l_1 < l.$$

If $l_0 = 0$, then we obtain an upper bound

$$|\partial^{l_1}\partial^2 w| \lesssim |w_{\leq s+1}|.$$

If $l_0 > 0$, then we apply (4.24) (with s replaced by $l_0 \leq s$) to obtain an upper bound

$$(1+|v_{\leq\lfloor\frac{l_0+2}{2}\rfloor}|)^{l_0-1}|v_{\leq l_0+1}|\cdot|\partial^{l_1+2}w| \lesssim (1+|v_{\leq\lfloor\frac{s+2}{2}\rfloor}|)^{s-1}|v_{\leq l_0+1}|\cdot|w_{\leq l_1+2}|.$$

If $l_1 \geq l_0$, then $l_1+2 < l+2 \leq s+2$ and $l_0 \leq l/2 \leq s/2$, so in this case we have $|v_{\leq l_0+1}|\cdot|w_{\leq l_1+2}| \lesssim |v_{\leq\lfloor\frac{s+2}{2}\rfloor}|\cdot|w_{\leq s+1}|$. If $l_1 < l_0$, then $l_1 < s/2$ and $l_1+2 \leq \lfloor\frac{s+3}{2}\rfloor$ and $l_0+1 \leq s+1$, so in this case we have $|v_{\leq l_0+1}|\cdot|w_{\leq l_1+2}| \lesssim |v_{\leq s+1}|\cdot|w_{\leq\lfloor\frac{s+2}{2}\rfloor}|$. We thus finish the proof. ∎

Apply Lemma 4.7 to the right hand side of (4.21). Thus, the $L^2(\mathbb{R}^d)$ norm of the right hand side of (4.21) at time t is bounded by

$$
\begin{aligned}
&(1+\sum_{|\beta|\le\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^\infty})^{s-1}\sum_{|\beta|\le s+1}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^2}+(1+\sum_{|\beta|\le\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^\infty})^{s}\sum_{|\beta|\le s+1}\left\|\partial^\beta u_{m}(t)\right\|_{L^2}\\
&\quad+(1+\sum_{|\beta|\le\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^\infty})^{s-1}\sum_{|\beta|\le\lfloor\frac{s+3}{2}\rfloor}\left\|\partial^\beta u_{m}(t)\right\|_{L^\infty}\cdot\sum_{|\beta|\le s+1}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^2}.
\end{aligned}
$$

Moreover, by the Sobolev embedding $H^{\lfloor(d+2)/2\rfloor}\subset L^\infty$, we have

$$
\begin{aligned}
\sum_{|\beta|\le\lfloor\frac{s+2}{2}\rfloor}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^\infty}&\lesssim\sum_{|\beta|\le\lfloor\frac{s+2}{2}\rfloor+\lfloor\frac{d+2}{2}\rfloor}\left\|\partial^\beta u_{m-1}(t)\right\|_{L^2},\\
\sum_{|\beta|\le\lfloor\frac{s+3}{2}\rfloor}\left\|\partial^\beta u_{m}(t)\right\|_{L^\infty}&\lesssim\sum_{|\beta|\le\lfloor\frac{s+3}{2}\rfloor+\lfloor\frac{d+2}{2}\rfloor}\left\|\partial^\beta u_{m}(t)\right\|_{L^2}.
\end{aligned}
\tag{4.25}
$$

Note that

$$\lfloor\frac{s+3}{2}\rfloor+\lfloor\frac{d+2}{2}\rfloor\le s+1\Longleftrightarrow s\ge d+2.$$

By the induction hypotheses $A_{m-1}(t)\le A$, we have

$$
\begin{aligned}
&\sum_{l\le s}\left\|g^{\alpha\beta}(u_{m-1},u'_{m-1})\partial_\alpha\partial_\beta\partial^l u_m(t)\right\|_{L^2}\\
&\lesssim(1+A_{m-1}(t))^{s-1}A_{m-1}(t)+(1+A_{m-1}(t))^{s}A_m(t)+(1+A_{m-1}(t))^{s-1}A_{m-1}(t)A_m(t)\\
&\lesssim(1+A)^s(1+A_m(t)).
\end{aligned}
$$

We can now apply the energy estimate to $\partial^l u$ with $l\le s$. It follows from (3.3) and (4.17) that for each $t\in[0,T]$,

$$
\begin{aligned}
&\sum_{l\le s}(\left\|\partial^l u_m(t)\right\|_{L^2}+\left\|\partial^l u'_m(t)\right\|_{L^2})\\
&\lesssim(\sum_{l\le s}(\left\|\partial^l u_m(0)\right\|_{L^2}+\left\|\partial^l u'_m(0)\right\|_{L^2})+C_A\int_0^t(A_m(\tau)+1)\ d\tau)\exp(2\int_0^t\sum\left\|\partial g^{**}(u_{m-1},u'_{m-1})(\tau)\right\|_{L^\infty}\ d\tau).
\end{aligned}
$$

By the chain rule and the Sobolev embedding, we have

$$|\partial g^{**}(u_{m-1},u'_{m-1})(t,x)|\lesssim|u_{m-1}(t,x)|+|u'_{m-1}(t,x)|\lesssim A_{m-1}(t)\le A.$$

As a result, we have

$$A_m(t)\le Ce^{CAt}(A_m(0)+C_At+\int_0^t C_AA_m(\tau)\ d\tau).$$

By the Gronwall's inequality (3.7) we have

$$A_m(t)\le Ce^{CAt}(A_m(0)+C_At)\exp(CC_Ae^{CAt}),\qquad t\in[0,T].\tag{4.26}$$

All the constants in this inequality are independent of m. Besides, we claim that $A_m(0)$ can be controlled by a constant A_0 independent of m. To see this, we fix $m \geq s$. If ∂^l contains at most one time derivative, then we have nothing to prove since the initial data are (u^0, u^1). In general, we replace ∂^l with ∂_t^l where $l \geq 0$ in (4.21). It follows that at $t = 0$

$$g^{00}(u^0, u^1)\partial_t^{l+2} u_m = -g^{**}(u^0, u^1) \cdot \partial\nabla_x \partial_t^l u_m + [g^{\alpha\beta}(u_{m-1}, u'_{m-1}), \partial_t^l]\partial_\alpha\partial_\beta u_m + \partial_t^l F(u_{m-1}, u'_{m-1}).$$

If we expand $[g^{\alpha\beta}(u_{m-1}, u'_{m-1}), \partial_t^l]\partial_\alpha\partial_\beta u_m + \partial_t^l F(u_{m-1}, u'_{m-1})$ by applying the chain rule, we notice that the order of time derivatives of each term on the right hand side must be $\leq l+1$. For all those terms with time derivatives of order ≥ 2, we again apply (4.21) to lower the order of their time derivatives. Then, after at most s such actions, every term on the right hand side can be expressed in terms of (u^0, u^1) and their (spatial) derivatives. We also note that this expression is independent of m. Thus, $A_m(0) \lesssim A_0$ for some A_0 independent of m (but depending on s). This finishes the proof of our claim. In conclusion, by choosing $A \gg 1$ and $T \ll 1$ (both independent of m), we can make $A_m(t) \leq CA_0 < A$. This finishes the proof of part b).

Finally we prove part c). If c) is true, then the sequence (u_m, u'_m) converges to some $(u, u') \in CH^1 \cap C^1L^2([0,T] \times \mathbb{R}^d)$. We can check that (u, u') is indeed a weak solution to (4.1) (related to taking limit in (4.15)). For each $t \in [0,T]$, the sequence $(u_m, u'_m)(t)$ is bounded in $H^{s+1} \times H^s$, so it has a subsequence converging weakly to some $(\widetilde{u}, \widetilde{w})$ in $H^{s+1} \times H^s$ (Banach-Alaoglu). At the same time, $(u_m, u'_m)(t) \to (u, u')(t)$ in $H^1 \times L^2$, so we must have $(u, u')(t) = (\widetilde{u}, \widetilde{w}) \in H^{s+1} \times H^s$. Using part b), we have

$$\|u(t)\|_{H^{s+1}} + \|u_t(t)\|_{H^s} \leq \liminf_{m\to\infty} \|u_m(t)\|_{H^{s+1}} + \liminf_{m\to\infty} \|\partial_t u_m(t)\|_{H^s} \leq 2A < \infty.$$

Using the equation (4.1) to lower the order of time derivatives, we conclude that

$$A(t) := \sum_{|\alpha| \leq s+1} \|\partial^\alpha u(t)\|_{L^2} < \infty, \qquad t \in [0,T]. \tag{4.27}$$

To show c), we notice that

$$\begin{aligned} &g^{\alpha\beta}(u_{m-1}, u'_{m-1})\partial_\alpha\partial_\beta(u_m - u_{m-1}) \\ &= (g^{\alpha\beta}(u_{m-2}, u'_{m-2}) - g^{\alpha\beta}(u_{m-1}, u'_{m-1}))\partial_\alpha\partial_\beta u_{m-1} + F(u_{m-1}, u'_{m-1}) - F(u_{m-2}, u'_{m-2}). \end{aligned} \tag{4.28}$$

The right hand side is bounded by

$$(|u_{m-1} - u_{m-2}| + |u'_{m-1} - u'_{m-2}|)(1 + |u''_{m-1}|).$$

Also recall that u_m and u_{m-1} have the same Cauchy data at $t = 0$. Thus, by the energy estimate (3.3), the Sobolev embedding (4.25) and the estimate (4.17), we have

$$C_m(t) \lesssim (1 + A)\int_0^t C_m(\tau) + C_{m-1}(\tau)\ d\tau, \qquad t \in [0,T)$$

and by the Gronwall's inequality

$$C_m(t) \leq C_A e^{CAT} \int_0^t C_{m-1}(\tau)\ d\tau, \qquad t \in [0,T).$$

The constant here is independent of m. By iteration, we have

$$C_m(t) \le (C_A e^{CAT})^m \int_{0\le\tau_1\le\cdots\le\tau_m\le t} C_0(\tau_1)\ d\tau_1\cdots d\tau_m \le \frac{(C_A e^{CAT}t)^m}{m!}\sup_{t\in[0,T]} C_0(t).$$

By choosing sufficiently small T, we have $C_A e^{CAT} T \le 1$. And since $1/m! \lesssim 2^{-m}$, we obtain c). Note that the proof shows that T can be bounded from below by a fixed small constant if one assumes that the $H^{s+1}\times H^s$ norm of the data is smaller than a fixed constant.

4.2.3 Blowup criterion

Suppose that $T_* < \infty$ where T_* comes from the statement of Theorem 4.3. We claim that if

$$\sup_{(t,x)\in[0,T_*)\times\mathbb{R}^d} \sum_{|\alpha|\le(s+3)/2} |\partial^\alpha u(t,x)| \le A < \infty, \tag{4.29}$$

then

$$\sup_{t\in[0,T_*)} \sum_{|\alpha|\le s+1} \|\partial^\alpha u(t)\|_{L^2} < \infty. \tag{4.30}$$

In fact, if $A(t) = \sum_{|\alpha|\le s+1}\|\partial^\alpha u(t)\|_{L^2}$, then arguing as in the proof in Section 4.2.2, we can show that (4.30) implies

$$A(t) \le C_{T_*,A}(A(0) + C_A\int_0^t (A(\tau)+1)\ d\tau), \qquad t\in[0,T_*).$$

Now (4.30) follows from an application of the Gronwall's inequality.

With this claim and the last sentence in Section 4.2.2, we can show that (4.29) implies that u can extend to a function in

$$L^\infty H^{s+1}\cap C^{0,1}H^s([0,T_*]\times\mathbb{R}^d).$$

Hence we can use the existence part of Theorem 4.3 to see that u extends to a solution verifying the bounds in Theorem 4.3 for some $T > T_*$.

4.3 Proof of Theorem 4.4

We finish this section by proving Theorem 4.4.

By the Sobolev embedding, to prove the first part, we only need to show that there exists $T > 0$ such that

$$\sum_{|\alpha|\le s+1} \|\partial^\alpha u(t)\|_{L^2} \le C_s, \qquad \forall t\in[0,T], \forall s\in\mathbb{Z}_+. \tag{4.31}$$

By Theorem 4.3, there exists such a T for $s = d+3$.

Next, by the Sobolev embedding, we notice that if (4.31) holds for some s, then

$$\sup_{[0,T]\times\mathbb{R}^d} \sum_{|\alpha|\le s+1-\lfloor\frac{d+2}{2}\rfloor} |\partial^\alpha u| < \infty.$$

Also note that

$$\lfloor \frac{s+4}{2} \rfloor \le s+1-\lfloor \frac{d+2}{2} \rfloor \Longleftrightarrow s \ge d+3.$$

Here we have $(s+4)/2$ instead of $(s+3)/2$ on the left hand side because we now want to estimate $\sum_{|\alpha|\le s+2} \|\partial^\alpha u(t)\|_{L^2}$ instead of $\sum_{|\alpha|\le s+1} \|\partial^\alpha u(t)\|_{L^2}$ as in Section 4.2.2. Using the proof in Section 4.2.2, we obtain (4.31) with $s+1$ replaced by $s+2$. By induction, we prove (4.31) for all s.

It remains to prove the second part of Theorem 4.4. By Theorem 4.3, the T_* in our theorem is exactly the supremum over all T such that (4.31) holds with $s = d+3$. By the above induction, it also has to be the supremum over all T such that there is a $C^\infty([0,T]\times\mathbb{R}^d)$ solution.

5 Commuting vector fields

In Section 2, we have proved that a solution to (2.1) in $\mathbb{R}^{1+3}$ has a pointwise decay rate $O(\langle t\rangle^{-1})$. In order to prove this decay without using the Kirchoff's formula (2.2), we introduce *commuting vector fields* Z and a related Sobolev type inequality (called the *Klainerman-Sobolev inequality*).

5.1 Definition

We now give the definition of the commuting vector fields.

Definition 5.1. In $\mathbb{R}^{1+d}$, we consider the following vector fields:

$$\begin{array}{lr} \partial_\alpha, \quad \alpha = 0,1,\dots,d & \text{translations;} \\ S := t\partial_t + \sum_{j=1}^d x_j\partial_j & \text{scaling;} \\ \Omega_{ij} := x_i\partial_j - x_j\partial_i, \quad 1\le i<j\le d & \text{rotations;} \\ \Omega_{0i} := t\partial_i + x_i\partial_t, \quad i=1,\dots,d & \text{Lorentz boosts.} \end{array} \tag{5.1}$$

We use $Z_0,\dots,Z_{(d+1)(d+2)/2}$ to denote the $(d+1)(d+2)/2+1$ vector fields in (5.1) respectively, and we call each Z_j a *commuting vector field*.

If $I = (i_1,\dots,i_r)$ is a multiindex (of length $|I| = r$) where $0 \le i_* \le (d+1)(d+2)/2$, we shall write

$$Z^I := Z_{i_1}\cdots Z_{i_r}. \tag{5.2}$$

If $|I| = 1$, then we may also omit the superscript and write Z only.

Remark 5.1.1. It is convenient to set $\Omega_{ii} = 0$ and $\Omega_{ij} = -\Omega_{ji}$ if $d \ge i > j \ge 1$.

Remark 5.1.2. The notations in this note is different from those in other texts. For example, in [Sog08], the author uses L_0 to denote the S here. He also uses Γ to denote the vector fields in (5.1), and uses Z to denote a proper subset of (5.1).

Remark 5.1.3. Each commuting vector field Z is related to a symmetry of the linear wave equation $\Box u = 0$ in $\mathbb{R}^{1+d}$. Let us take the scaling S as an example. If $u = u(t,x)$ solves $\Box u = 0$, then so does $u_\lambda = u_\lambda(t,x) = u(\lambda t,\lambda x)$ for each constant $\lambda\in\mathbb{R}$. By differentiating u_λ with respect to λ and setting $\lambda = 0$, we get another solution to the linear wave equation:

$$\frac{d}{d\lambda}u_\lambda|_{\lambda=0} = Su.$$

Similarly for other commuting vector fields.

5.2 Basic properties

We first list several commutation properties. These properties more or less explain why we call them "commuting vector fields". For simplicity, we use $C \cdot Z$ to denote a linear combination of the commuting vector fields with real constant coefficients. In other words,

$$C \cdot Z = \sum_{j=0}^{(d+1)(d+2)/2} C_j Z_j, \qquad \text{where the } C_* \text{ are real constants.}$$

Similarly, we use $C \cdot \partial$ to denote a linear combination of partial derivatives with real constant coefficients.

We have

1. For any two commuting vector fields Z_j and Z_k, we have $[Z_j, Z_k] = C \cdot Z$.
2. For any commuting vector field Z_j and any partial derivative ∂_α, we have $[Z_j, \partial_\alpha] = C \cdot \partial$ (or simply $[Z, \partial] = C \cdot \partial$). Note that a corollary of this property is that for any $k, i \geq 0$, we have
$$\sum_{|I| \leq i} |\partial^k Z^I \phi| \sim \sum_{|I| \leq i} |Z^I \partial^k \phi|, \tag{5.3}$$
where ∂^l denotes any partial derivatives of order l.
3. For any commuting vector field Z_j, we have $[Z_j, \Box] = 0$ whenever $Z_j \neq S$, and $[S, \Box] = -2\Box$.

The proofs of these commutation properties are left as an exercise.

In addition, we have the following pointwise estimates.

Lemma 5.2. *For any function $\phi = \phi(t, x)$ with $t \geq 0$, we have*

$$|\partial^k \phi| \lesssim \langle |x| - t \rangle^{-k} \sum_{|I| \leq k} |Z^I \phi|, \qquad \forall k \geq 0; \tag{5.4}$$

$$\sum_{i=1}^{d} |(\partial_i + \omega_i \partial_t)\phi| \lesssim \langle |x| + t \rangle^{-1} \sum_{|I|=1} |Z^I \phi|. \tag{5.5}$$

Here recall that ∂^k denotes any partial derivatives of order k, and $\omega_i := x_i/|x|$.

Proof. We first prove (5.4) with $k = 1$ (and there is nothing to prove when $k = 0$). Since $\partial \in \{Z_0, \ldots, Z_{(d+1)(d+2)/2}\}$, we already have $|\partial \phi| \lesssim \sum_{|I| \leq 1} |Z^I \phi|$. Moreover, we notice that

$$\sum_{i=1}^{d} \omega_i \Omega_{0i} = \sum_{i=1}^{d} \omega_i (t\partial_i + x_i \partial_t) = t\partial_r + |x|\partial_t, \qquad S = t\partial_t + |x|\partial_r.$$

Here $\partial_r := \sum_{j=1}^d |x|^{-1}x_j\partial_j$. As a result, we can express ∂_t, ∂_r in terms of $\sum_{i=1}^d \omega_i\Omega_{0i}$ and S:

$$\partial_t + \partial_r = \frac{S + \sum_{i=1}^d \omega_i\Omega_{0i}}{|x| + t}, \qquad \partial_t - \partial_r = \frac{\sum_{i=1}^d \omega_i\Omega_{0i} - S}{|x| - t};$$
$$2\partial_t = (\frac{1}{|x|+t} - \frac{1}{|x|-t})S + (\frac{1}{|x|+t} + \frac{1}{|x|-t})\sum_{i=1}^d \omega_i\Omega_{0i} = \frac{-2tS + 2\sum_{i=1}^d x_i\Omega_{0i}}{|x|^2 - t^2},$$
$$2\partial_r = (\frac{1}{|x|+t} + \frac{1}{|x|-t})S + (\frac{1}{|x|+t} - \frac{1}{|x|-t})\sum_{i=1}^d \omega_i\Omega_{0i} = \frac{2|x|S - 2t\sum_{i=1}^d \omega_i\Omega_{0i}}{|x|^2 - t^2}.$$

Since $\frac{|x|+t}{|x|^2-t^2} = \frac{1}{|x|-t}$, we have

$$|\phi_t| + |\phi_r| \lesssim ||x| - t|^{-1}(|S\phi| + \sum_{i=1}^d |\Omega_{0i}\phi|) \lesssim ||x| - t|^{-1}\sum_{|I|=1} |Z^I\phi|.$$

Moreover, since

$$\sum_{i=1}^d \omega_i\Omega_{ij} = \sum_{i=1}^d \omega_i(x_i\partial_j - x_j\partial_i) = |x|\partial_j - x_j\partial_r, \qquad \Omega_{0j} = x_j\partial_t + t\partial_j,$$

for each $j = 1, \dots, d$ we have

$$\partial_j = (t + |x|)^{-1}(\sum_{i=1}^d \omega_i\Omega_{ij} + \Omega_{0j} - x_j(\partial_t - \partial_r)).$$

It thus follows that

$$\begin{aligned}|\phi_j| &\lesssim (t + |x|)^{-1}(\sum_{i=1}^d |\Omega_{ij}\phi| + |\Omega_{0j}\phi| + |x||\phi_t - \phi_r|)\\ &\lesssim (|x| + t)^{-1}\sum_{|I|=1} |Z^I\phi| + |\phi_t| + |\phi_r| \lesssim ||x| - t|^{-1}\sum_{|I|=1} |Z^I\phi|.\end{aligned}$$

In the last estimate we use the triangle inequality $||x| - t| \leq |x| + t$. By noticing that $\min\{1, ||x| - t|\} \lesssim \langle |x| - t\rangle^{-1}$, we obtain (5.4) with $k = 1$.

To prove (5.4) with $k > 1$, we use induction. Suppose we have proved (5.4) for each $k < k_0$. Then,

$$\begin{aligned}|\partial^{k_0}\phi| = |\partial^{k_0-1}\partial\phi| &\lesssim \langle |x| - t\rangle^{1-k_0}\sum_{|I|\leq k_0-1} |Z^I\partial\phi|\\ &\lesssim \langle |x| - t\rangle^{1-k_0}\sum_{|I|\leq k_0-1} |\partial Z^I\phi| \qquad \text{(by (5.3))}\\ &\lesssim \langle |x| - t\rangle^{-k_0}\sum_{|I|\leq k_0} |\partial Z^I\phi|.\end{aligned}$$

Finally we prove (5.5). By the computations above, we have

$$\begin{aligned}\partial_j + \omega_j\partial_t &= (t+|x|)^{-1}(\sum_{i=1}^d \omega_i\Omega_{ij} + \Omega_{0j} - x_j(\partial_t - \partial_r)) + \omega_j\partial_t \\ &= \frac{\sum_{i=1}^d \omega_i\Omega_{ij} + \Omega_{0j} - x_j(\partial_t-\partial_r) + (x_j + t\omega_j)\partial_t}{|x|+t} = \frac{\sum_{i=1}^d \omega_i\Omega_{ij} + \Omega_{0j} + \omega_j S}{|x|+t}.\end{aligned}$$

Thus, $|\phi_j + \omega_j\phi_t| \lesssim (|x|+t)^{-1}\sum_{|I|=1}|Z^I\phi|$. And since $|\phi_j+\omega_j\phi_t| \lesssim \sum_{|I|=1}|Z^I\phi|$, we are done. $\square$

Remark 5.2.1. Note that $\partial_i + \omega_i\partial_t$, $i=1,2,3$ span the tangent space of the light cone $|x|-t=C$, and that $\partial_t - \partial_r$ is orthogonal to the light cone $|x|-t=C$. Thus, sometimes we call $\partial_t - \partial_r$ *normal derivative*, and call $\partial_i + \omega_i\partial_t$ *tangential derivatives*.

Lemma 5.2 tells us that heuristically the tangential derivatives have better decays than normal derivative.

5.3 The Klainerman-Sobolev inequality

Recall the Sobolev embedding $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, $s > d/2$. Since $\lfloor (d+2)/2\rfloor$ is the smallest integer larger than $d/2$, we have

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim_d \|f\|_{H^{\lfloor (d+2)/2\rfloor}(\mathbb{R}^d)} \lesssim \sum_{|\alpha|\le (d+2)/2} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}. \tag{5.6}$$

In this subsection, we prove the Klainerman-Sobolev inequality (see (5.7) below). We ask our readers to compare (5.7) with (5.6).

Theorem 5.3 (Klainerman-Sobolev)**.** *Let $u \in C^\infty(\mathbb{R}^{1+d})$ vanish when $|x|$ is large. Then, for all $t>0$ and $x\in\mathbb{R}^d$,*

$$\langle |x|+t\rangle^{\frac{d-1}{2}}\langle |x|-t\rangle^{\frac12}|u(t,x)| \lesssim_d \sum_{|I|\le\frac{d+2}{2}} \left\|Z^I u(t)\right\|_{L^2(\mathbb{R}^d)}. \tag{5.7}$$

Proof. If $|x|+t\le 1$, then $\langle |x|+t\rangle^{\frac{d-1}{2}}\langle |x|-t\rangle^{\frac12} \lesssim 1$, so (5.7) follows directly from (5.6). So we can assume that $|x|+t>1$ from now on.

Let us first assume that $||x|-t| > t/2$, or equivalently $|x| \notin [t/2, 3t/2]$. In this case, we have (left as an exercise)

$$\frac{|x|+t}{6} \le ||x|-t| \le t+|x|. \tag{5.8}$$

Now define a function f in $\mathbb{R}^d$ by

$$f(y) = u(t, x+(t+|x|)y).$$

We now recall a localized version of the Sobolev embedding (5.6): for each $\delta>0$, we have

$$|f(x)|^2 \lesssim_{\delta,d} \sum_{|\alpha|\le(d+2)/2}\int_{B(0,\delta)} |\partial_y^\alpha f(x+y)|^2\, dy, \qquad \forall x\in\mathbb{R}^d. \tag{5.9}$$

To prove this, we simply apply (5.6) to $y \mapsto f(x+y)\cdot\chi(x+y)$ where $\chi \in C_c^\infty(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(0)=1$ and $\chi|_{\mathbb{R}^d\setminus B(0,\delta)} = 0$. Apply (5.9) with $x=0$ and $\delta = 1/12$ (or any number in $(0,1/6)$), and we obtain

$$\begin{aligned}|u(t,x)|^2 = |f(0)|^2 &\lesssim \sum_{|\alpha|\le (d+2)/2} \int_{B(0,\delta)} |\partial_y^\alpha f(y)|^2\ dy \\ &\lesssim \sum_{|\alpha|\le (d+2)/2} \int_{B(0,\delta)} (t+|x|)^{2|\alpha|} |u^{(\alpha)}(t,x+(t+|x|)y)|^2\ dy \\ &\lesssim \sum_{|\alpha|\le (d+2)/2} \int_{B(x,\delta(t+|x|))} (t+|x|)^{2|\alpha|-d} |u^{(\alpha)}(t,z)|^2\ dz.\end{aligned}$$

To avoid ambiguity, we use $u^{(\alpha)}(t,z) := (\partial^\alpha u)(t,z)$. In the second row, we use the chain rule and Leibniz's rule. In the third row, we make a substitution $z = x+(t+|x|)y$. By (5.4) in Lemma 5.2, we have

$$|u^{(\alpha)}(t,z)| \lesssim \sum_{|I|\le|\alpha|} \langle |z|-t\rangle^{-|\alpha|} |Z^I u(t,z)| \lesssim (|x|+t)^{-|\alpha|} \sum_{|I|\le|\alpha|} |Z^I u(t,z)|, \qquad \text{whenever } ||z|-t| \ge \frac{t+|x|}{6}.$$

The second estimate holds because $||z|-t| \ge (t+|x|)/12$ whenever $|z-x| < (|x|+t)/12$ and $||x|-t| > t/2$. To see this, we note that by (5.8) and the triangle inequality

$$||z|-t| \ge ||x|-t| - ||z|-|x|| \ge (|x|+t)/6 - |z-x| \ge (1/6-1/12)(t+|x|).$$

In summary, we have

$$|u(t,x)|^2 \lesssim (t+|x|)^{-d} \sum_{|I|\le (d+2)/2} \left\| Z^I u(t)\right\|^2_{L^2(\mathbb{R}^d)}.$$

This gives us (5.7).

Now we suppose $||x|-t| \le t/2$, or equivalently $|x| \in [t/2, 3t/2]$. Note that now $t > 2/5$ because $|x|+t > 1$. It suffices to prove the following two estimates:

$$|u(t,x)| \lesssim t^{-(d-1)/2} \sum_{|I|\le (d+2)/2} \left\| Z^I u(t)\right\|_{L^2(\mathbb{R}^d)}; \tag{5.10}$$

$$|u(t,x)| \lesssim t^{-(d-1)/2} ||x|-t|^{-1/2} \sum_{|I|\le (d+2)/2} \left\| Z^I u(t)\right\|_{L^2(\mathbb{R}^d)}, \qquad \text{whenever } ||x|-t| \ge 1. \tag{5.11}$$

We first prove (5.10). Define a function f in $\mathbb{R}\times\mathbb{S}^{d-1}$ by

$$f(q,\omega) = u(t,(t+q)\omega).$$

We need a new Sobolev type inequality in $\mathbb{R}\times\mathbb{S}^{d-1}$: for each $\delta > 0$, we have

$$|f(q,\omega)|^2 \lesssim_{\delta,d} \sum_{j+k\le (d+2)/2} \int_{-\delta}^{\delta} \int_{\mathbb{S}^{d-1}} |\partial_s^j \partial_\nu^k f(q+s,\nu)|^2\ d\nu ds, \qquad \forall (q,\omega) \in \mathbb{R}\times\mathbb{S}^{d-1}. \tag{5.12}$$

Here ∂_ν denotes any rotation Ω_{ij}, $1 \le i < j \le 3$ restricted to the sphere $\mathbb{S}^{d-1}$, and ∂_ν^k denotes any product of k such vector fields. We shall not prove this inequality here, but we remark that it follows from (5.6) and partitions of unity. Now, apply (5.12) with $q = |x| - t$, $\omega = x/|x|$ and $\delta = 1/10 < t/4$. Because of the definition of f, we have

$$\partial_s^j \partial_\nu^l f(|x| - t + s, \nu) = (\partial_r^j \Omega^l u)(t, (|x| + s)\nu).$$

Thus, by (5.12) we have

$$\begin{aligned}
|u(t,x)|^2 = |f(|x|-t,\omega)|^2 &\lesssim \sum_{j+k\le(d+2)/2} \int_{-\delta}^{\delta} \int_{\mathbb{S}^{d-1}} |(\partial_r^j \Omega^k u)(t,(|x|+s)\nu)|^2 \, d\nu ds \\
&\lesssim \sum_{j+|I|\le(d+2)/2} \int_{|x|-\delta}^{|x|+\delta} \int_{\mathbb{S}^{d-1}} r^{1-d} |(\partial_r^j Z^I u)(t,r\nu)|^2 r^{d-1} \, d\nu dr \\
&\lesssim \sum_{|I|\le(d+2)/2} \int_{t/4}^{2t} \int_{\mathbb{S}^{d-1}} t^{1-d} |(Z^I u)(t,r\nu)|^2 r^{d-1} \, d\nu dr \\
&\lesssim t^{1-d} \sum_{|I|\le(d+2)/2} \int_{\mathbb{R}^d} |(Z^I u)(t,y)|^2 \, dy.
\end{aligned}$$

This gives us (5.10). In the third row, we use $|x| - \delta \ge t/2 - t/4$ and $|x| + \delta \le 3t/2 + t/4$. We also use $Z^I(y/|y|) = O(1)$ whenever $|y| \sim t \gtrsim 1$.

Now we prove (5.11). Fix $(t, x_0) \in \mathbb{R}_+^{1+d}$ with $t/2 \ge ||x_0| - t| \ge 1$ and $|x_0| + t \ge 1$. Set $q_0 := |x_0| - t$, so $1 \le |q_0| \le t/2$. We now set

$$v(s,\nu) := u(t, (t + q_0 + q_0 s)\nu).$$

It is clear that $v(0,\omega) = u(t, x_0)$ where $\omega = x_0/|x_0|$. Moreover, by the chain rule and Leibniz's rule, we have

$$\sum_{j+k\le(d+2)/2} |\partial_s^j \partial_\nu^k v(s,\nu)| \lesssim \sum_{j+k\le(d+2)/2} |((q_0\partial_r)^j \Omega^k u)(t,(|x_0| + q_0 s)\nu)|.$$

Thus, by (5.12) with $q = 0$, $\omega = x_0/|x_0|$ and $\delta = 1/4$, we have

$$\begin{aligned}
|u(t,x_0)|^2 = |v(0,\omega)|^2 &\lesssim \sum_{j+k\le(d+2)/2} \int_{-\delta}^{\delta} \int_{\mathbb{S}^{d-1}} |\partial_s^j \partial_\nu^k v(s,\nu)|^2 \, d\nu ds \\
&\lesssim \sum_{j+k\le(d+2)/2} \int_{-\delta}^{\delta} \int_{\mathbb{S}^{d-1}} |((q_0\partial_r)^j \Omega^k u)(t,(|x_0|+q_0 s)\nu)|^2 \, d\nu ds \\
&= |q_0|^{-1} \sum_{j+k\le(d+2)/2} \int_{|x_0|-\delta|q_0|}^{|x_0|+\delta|q_0|} \int_{\mathbb{S}^{d-1}} r^{1-d} \cdot |((q_0\partial_r)^j \Omega^k u)(t,r\nu)|^2 r^{d-1} \, d\nu dr.
\end{aligned}$$

In the third row, we make a substitution $r = |x_0| + q_0 s$. To continue, we notice that

$$|x_0| - \delta|q_0| \ge t/2 - \delta \cdot t/2 \ge t/8,$$

so $r^{1-d} \lesssim t^{1-d}$ in the integral. Moreover, whenever $r \in [|x_0| - \delta|q_0|, |x_0| + \delta|q_0|]$, we have

$$|r - t| \geq ||x_0| - t| - |r - |x_0|| \geq |q_0| - \delta|q_0| = 3|q_0|/4.$$

It follows that

$$\begin{aligned}
|u(t,x_0)|^2 &\lesssim |q_0|^{-1}t^{1-d} \sum_{j+k\leq(d+2)/2} \int_{|x_0|-\delta|q_0|}^{|x_0|+\delta|q_0|} \int_{\mathbb{S}^{d-1}} |(((r-t)\partial_r)^j\Omega^k u)(t,r\nu)|^2 r^{d-1}\ d\nu dr \\
&\lesssim |q_0|^{-1}t^{1-d} \sum_{j+k\leq(d+2)/2} \int_{|y|\in[|x_0|-\delta|q_0|,|x_0|+\delta|q_0|]} |(((|y|-t)\partial_r)^j\Omega^k u)(t,y)|^2\ dy \\
&\lesssim |q_0|^{-1}t^{1-d} \sum_{|I|\leq(d+2)/2} \int_{\mathbb{R}^d} |Z^I u(t,y)|^2\ dy.
\end{aligned}$$

This gives us (5.11). $\square$

Remark 5.3.1. The key idea is that we combine the usual Sobolev embedding in $\mathbb{R}^d$ (and some of its variants) with a change of variables.

5.4 Application to the linear wave equation

Using the commuting vector fields and the Klainerman-Sobolev inequality, we can now show some pointwise estimates for solutions to the linear wave equation without solving it explicitly.

Suppose that $u \in C^\infty(\mathbb{R}^{1+d}_+)$ (which is of course not necessary) is a global solution to

$$\begin{cases} \Box u = 0 \qquad \text{in } \mathbb{R}^{1+d}_+; \\ (u, \partial_t u)|_{t=0} = (u^0, u^1) \in C^\infty_c(\mathbb{R}^d). \end{cases} \tag{5.13}$$

Since $[\Box, Z] = C \cdot \Box$ for some constant C, for each multiindex I the function $Z^I u$ also satisfies $\Box Z^I u = 0$ and we have $(Z^I u, \partial_t Z^I u)|_{t=0} \in C^\infty_c(\mathbb{R}^d)$. By the energy conservation law and the finite speed of propagation from Theorem 3.2, we have

$$\left\|(Z^I u)'(t)\right\|_{L^2(\mathbb{R}^d)} \lesssim \left\|(Z^I u)'(0)\right\|_{L^2(\mathbb{R}^d)} \lesssim_I 1, \qquad \forall \text{ multiindex } I.$$

By the Klainerman-Sobolev inequality and by (5.3), we have

$$\begin{aligned}
|u'(t,x)| &\lesssim \langle |x|+t\rangle^{-(d-1)/2}\langle |x|-t\rangle^{-1/2} \sum_{|I|\leq(d+2)/2} \left\|Z^I \partial u(t)\right\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \langle |x|+t\rangle^{-(d-1)/2}\langle |x|-t\rangle^{-1/2} \sum_{|I|\leq(d+2)/2} \left\|(Z^I u)'(t)\right\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \langle |x|+t\rangle^{-(d-1)/2}\langle |x|-t\rangle^{-1/2}.
\end{aligned}$$

By the finite speed of propagation, we have $u(t,x) = 0$ whenever $|x| - t \geq C$ for some constant C. Since

$$\int_r^{t+C} \langle \rho + t\rangle^{-(d-1)/2}\langle \rho - t\rangle^{-1/2}\ d\rho \lesssim \langle t\rangle^{-(d-1)/2}\langle r-t\rangle^{1/2},$$

we conclude that

$$|u(t,x)| \lesssim \langle t\rangle^{-(d-1)/2}\langle |x|-t\rangle^{1/2}.$$

This is weaker than the estimates given in Section 2.5, but such estimates are usually enough in the study of nonlinear problems.

If d is odd and if we assume that the solution vanishes for $||x|-t| \geq R$, then the proof above does recover the estimate $u = O(\langle t\rangle^{-(d-1)/2})$ in Section 2.5. However, proving that the solution vanishes for $||x|-t| \geq R$ seems to require the use of the Kirchoff's formula.

6 Almost global existence in three space dimensions and global existence in higher dimensions

Using the tools developed in the previous sections, we can now prove the first long time existence result in this course.

Consider the Cauchy problem

$$\begin{cases} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = F(u') \qquad \text{in } \mathbb{R}^{1+d}_+; \\ \\ (u,\partial_t u)|_{t=0} = (\varepsilon u^0, \varepsilon u^1). \end{cases} \tag{6.1}$$

Here we have

1) The unknown u is $\mathbb{R}$-valued, and $u' = (\partial_\alpha u)_{\alpha=0}^d$. Of course, all the results proved in this section also hold for $\mathbb{R}^N$-valued functions.

2) We have $(u^0,u^1) \in C_c^\infty(\mathbb{R}^d)$ and $0 < \varepsilon \ll 1$ is sufficiently small. By "sufficiently small", we mean there exists some $\varepsilon_0 \in (0,1)$ depending on (u^0,u^1) such that ε is an arbitrary number in $(0,\varepsilon_0)$.

3) g^{**}, F are given C^∞ functions such that $g^{\alpha\beta} = g^{\beta\alpha}$, $g^{\alpha\beta}(0) = m^{\alpha\beta}$, $F(0) = 0$ and $dF(0) = 0$. As a result, we have $g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \square$ and $F(u') = O(|u'|^2)$.

Note that the derivatives of g^{**}, F do not need to be $O(1)$ everywhere, and that the assumption (4.2) is not necessary. This is because u' is expected to be small.

The main result for this section is the following theorem.

Theorem 6.1. *Fix a dimension $d \geq 1$ and fix $(u^0,u^1) \in C_c^\infty(\mathbb{R}^d)$. Then, for all sufficiently small $0 < \varepsilon \ll 1$ (depending on (u^0,u^1)), the Cauchy problem* (6.1) *has a (unique) C^∞ solution for all $0 \leq t < T_d$, where*

$$T_d := \begin{cases} \infty, & d \geq 4; \\ \exp(c/\varepsilon), & d = 3; \\ c/\varepsilon^2, & d = 2; \\ c/\varepsilon, & d = 1. \end{cases}$$

When $d = 1,2,3$, the constant c is a small constant in $(0,1)$ depending only on (u^0,u^1) (and not on ε).

Remark 6.1.1. When $d = 3$, this existence result (called *almost global existence*) is sharp. For example, it is known that any nontrivial solutions (1.3) and (1.7) with C_c^∞ data must blow up in finite time, and we can show that the lifespan is $e^{c/\varepsilon}$ for some constant $c > 0$ (not necessarily small).

Before we start the proof, I would like to explain why we have T_d in the result. Using the Klainerman-Sobolev inequality, the energy estimate and the Gronwall's inequality, to end the proof we need to show that

$$I_d := \int_0^{T_d} CA_0\varepsilon\langle\tau\rangle^{-(d-1)/2}\, d\tau < 1.$$

Here C, A_0 are two large constants and ε is a sufficiently small constant chosen after C, A_0 are chosen.

If $d \geq 4$, then $(d-1)/2 \geq 3/2$ and thus $I_d \leq CA_0\varepsilon$. Thus, by choosing $\varepsilon \ll 1$ we do have $CA_0\varepsilon < 1$. If $d = 1, 2, 3$, then we have $I_d \leq CA_0\varepsilon T, CA_0\langle T_d\rangle^{1/2}, CA_0\varepsilon \ln\langle T_d\rangle$, respectively. By setting T_d as above with c sufficiently small (depending on A_0, C), we can make $I_d < 1$.

6.1 Continuity arguments: an introduction

To prove Theorem 6.1, of course we need to apply Theorem 4.2–4.4. Meanwhile we also need to apply a continuity argument (also called a bootstrap argument). Such an argument is based on the following easy fact.

Proposition 6.2 (Proposition 1.21 in [Tao06])**.** *Let I be a time interval (bounded or unbounded). For each $t \in I$, we have two statements, a "hypothesis" $H(t)$, and a "conclusion" $C(t)$. Suppose we can verify the following four assertions:*

(a) *If $H(t)$ holds for some $t \in I$, then $C(t)$ holds for the same t.*

(b) *If $C(t)$ holds for some $t \in I$, then there exists an open set $O \subset I$ containing t such that $H(t')$ holds for all $t' \in O$.*

(c) *If $t_1, t_2, \ldots$ is a sequence in I which converges to some $t \in I$, and if $C(t_n)$ holds for each n, then $C(t)$ holds.*

(d) *There exists $t_0 \in I$ such that $H(t_0)$ holds.*

Then $C(t)$ holds for all $t \in I$.

Proof. Let $A := \{t \in I : \ C(t) \text{ is true}\}$. Then, (a)–(d) tell us that A is a nonempty set in I which is both open and closed. Since I is connected, we conclude that $A = I$ by using basic topology. $\square$

You can check Section 1.6 of [Tao06] for some simple applications of this argument if you are interested in it.

6.2 Setup of the continuity argument

In the proof of Theorem 6.1, we set $I := [0, T_d)$ (if $d \leq 3$, then we first fix a small $c \in (0,1)$ without choosing its explicit value at this moment). For each $T \in [0, T_d)$, our hypothesis $H(T)$ is that there exists a C^∞ solution u for all $t \in [0,T]$ such that

$$A(t) := \sum_{|I| \leq N} \left\| Z^I u'(t) \right\|_{L^2(\mathbb{R}^d)} \leq A_0 \varepsilon, \qquad \forall t \in [0,T]. \tag{6.2}$$

Our conclusion $C(T)$ is that there exists a C^∞ solution u for all $t \in [0,T]$ such that

$$A(t) \leq \frac{1}{2} A_0 \varepsilon, \qquad \forall t \in [0,T]. \tag{6.3}$$

Here $N \geq d+4$ is a large integer and $A_0 > 1$ is a large constant. We will choose their values later in the proof.

Let us briefly explain how we apply Proposition 6.2. The assertion (a) states that $H(T) \implies C(T)$, i.e. the estimate (6.2) implies (6.3). Checking this assertion would be the most difficult step in the proof of Theorem 6.1. We will prove (a) in the rest of this section. Here we emphasize that we can prove $H(T) \implies C(T)$ only holds for large A_0. If it holds for all $A_0 > 0$, then by applying $H(T) \implies C(T)$ repeatedly, we get $A(T) = 0$ which is absurd.

The assertion (b) follows from the local existence result for (6.1) and the continuity of $A(t)$. To see this, we suppose that $C(T)$ is true. By the Klainerman-Sobolev inequality, we have

$$\sum_{|I| \leq (d+6)/2} \left\| Z^I u'(t) \right\|_{L^\infty} \lesssim \langle t \rangle^{-(d-1)/2} \sum_{|I| \leq \lfloor \frac{d+6}{2} \rfloor + \lfloor \frac{d+2}{2} \rfloor} \left\| Z^I u'(t) \right\|_{L^2}.$$

If we choose $N \geq \lfloor \frac{d+6}{2} \rfloor + \lfloor \frac{d+2}{2} \rfloor$ (e.g. $N = d+4$), then we conclude that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \sum_{1 \leq |\alpha| \leq (d+6)/2} |\partial^\alpha u(t,x)| \lesssim \sup_{t \in [0,T]} A(t) \leq A_0 \varepsilon / 2.$$

And since $u(t,x) = u(0,x) + \int_0^t (\partial_\tau u)(\tau, x) \, d\tau$, we conclude that

$$\sum_{|\alpha| \leq (d+6)/2} |\partial^\alpha u| \in L^\infty([0,T] \times \mathbb{R}^d).$$

By Theorem 4.4, we can extend u to a C^∞ solution in $[0, T+\delta] \times \mathbb{R}^d$ for some positive $\delta > 0$. We also need to apply Theorem 4.3 to see that $A(t) < \infty$ for each $t \in [0, T+\delta]$. Since $A(t)$ is a continuous function, and since $A(T) \leq A_0 \varepsilon / 2$, we have

$$A(t) \leq A_0 \varepsilon, \qquad t \in [0, T+\delta]$$

by shrinking δ if necessary. This gives us $H(t)$ in a neighborhood of T.

The assertion (c) is an easy consequence of the continuity of $A(t)$.

The assertion (d) follows if we can show $H(0)$ holds. In fact, since $(u^0, u^1) \in C_c^\infty$, we have $(Z^I u, \partial_t Z^I u)|_{t=0} \in C_c^\infty$ by using the equation (6.1) to lower the order of time derivatives. Thus, if we choose A_0 sufficiently large (depending on (u^0, u^1)), we have (6.2) for $T = 0$.

In summary, we check the four assertions in Proposition 6.2. We can thus apply this proposition to show that there exists a solution to (6.1) for $t \in [0, T_d)$ satisfying (6.3). This finishes the proof of Theorem 6.1.

In future we will keep using Proposition 6.2 to prove long time existence results. At that time, we will only state that we apply a continuity argument without referring to Proposition 6.2, and we will only check the assertions (a) and (d) and take the other two assertions for granted.

6.3 Proof of assertion (a)

As explained in the previous subsection, we need to prove that $H(T)$ implies $C(T)$, or (6.2) implies (6.3). We need to apply the energy estimate (3.3).

Let us first prove some pointwise bounds. By the Klainerman-Sobolev inequality, we have

$$\sum_{|I| \leq N - \lfloor \frac{d+2}{2} \rfloor} |Z^I u'(t,x)| \lesssim \langle |x| + t \rangle^{-(d-1)/2} \langle |x| - t \rangle^{-1/2} A(t) \lesssim A_0 \varepsilon \langle t \rangle^{-(d-1)/2}. \tag{6.4}$$

Since ε is chosen after A_0 is chosen, we have

$$\sum_{|I| \leq N - \lfloor \frac{d+2}{2} \rfloor} |Z^I u'(t,x)| \leq C A_0 \varepsilon \leq 1, \qquad \text{as long as } \varepsilon \ll 1. \tag{6.5}$$

Fix a multiindex I with $|I| \leq N$. We first derive an equation for $Z^I u$. In fact,

$$\begin{aligned} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u &= [g^{\alpha\beta}(u')\partial_\alpha\partial_\beta, Z^I]u + Z^I(F(u')) \\ &= [\Box, Z^I]u + [(g^{\alpha\beta}(u') - m^{\alpha\beta})\partial_\alpha\partial_\beta, Z^I]u + Z^I(F(u')) \\ &= [\Box, Z^I]u + [(g^{\alpha\beta}(u') - m^{\alpha\beta}), Z^I]\partial_\alpha\partial_\beta u + (g^{\alpha\beta}(u') - m^{\alpha\beta})[\partial_\alpha\partial_\beta, Z^I]u + Z^I(F(u')). \end{aligned} \tag{6.6}$$

We write the right hand side of (6.6) as $R_1^I + R_2^I + R_3^I + R_4^I$. To apply the energy estimate, we need to control the $L^2(\mathbb{R}^d)$ norm of R_*^I at time t.

Let us first estimate R_4^I. For simplicity we write $|v_{\leq M}| = \sum_{|I| \leq M} |Z^I v|$ (this notation is different from that used in the proof of Lemma 4.22). Since $F(0) = 0$ and $dF(0) = 0$, and since $|u'| \leq 1$ (by (6.5)), we have

$$|F(u')| \lesssim |u'|^2.$$

In general, if $|I| \geq 1$, by the chain rule and Leibniz's rule, we can write $Z^I(F(u'))$ as a linear combination of terms of the form

$$F^{(r)}(u') \cdot \prod_{j=1}^r Z^{J_j} u', \qquad r > 0, \ \sum |J_*| = |I|, \ |J_j| > 0 \text{ for each } j.$$

If $r = 1$, then since $dF(0) = 0$ and $|u'| \leq 1$, we have $|F^{(1)}(u')| \lesssim |u'|$. In this case

$$|F^{(r)}(u') \cdot \prod_{j=1}^r Z^{J_j} u'| \lesssim |u'||(u')_{\leq N}|.$$

If $r > 1$, then we have $|F^{(r)}(u')| \lesssim 1$ since $|u'| \leq 1$. As in the proof of Lemma 4.22, we have $|J_j| > |I|/2$ for at most one J_j. For the other j, since $N \geq d+4$ we have

$$|J_j| \leq |I|/2 \leq N/2 \leq N - \lfloor \frac{d+2}{2} \rfloor.$$

In this case

$$|F^{(r)}(u') \cdot \prod_{j=1}^{r} Z^{J_j} u'| \lesssim |(u')_{\leq N - \lfloor \frac{d+2}{2} \rfloor}|^{r-1} |(u')_{\leq N}| \leq |(u')_{\leq N - \lfloor \frac{d+2}{2} \rfloor}| |(u')_{\leq N}|.$$

In summary, we have

$$|R_4^I| \lesssim |(u')_{\leq N - \lfloor \frac{d+2}{2} \rfloor}| |(u')_{\leq N}| \lesssim A_0 \varepsilon \langle t \rangle^{-(d-1)/2} |(u')_{\leq N}|;$$

and

$$\left\| R_4^I(t) \right\|_{L^2} \lesssim A_0 \varepsilon \langle t \rangle^{-(d-1)/2} A(t).$$

Next let us estimate R_2^I. We have $R_2^I = 0$ if $|I| = 0$, so suppose that $|I| > 0$. By the Leibniz's rule, we can write R_2^I as a linear combination of terms of the form

$$Z^J \partial^2 u \cdot Z^K (g^{\alpha\beta}(u') - m^{\alpha\beta}), \qquad |J| + |K| = |I|,\ |J| < |I|,\ |K| > 0.$$

By the chain rule and Leibniz's rule, we can follow the proof for $Z^I(F(u'))$ above to show that

$$|Z^K (g^{**}(u') - m^{\alpha\beta})| = |Z^K (g^{**}(u'))| \lesssim |(u')_{\leq |K|}|.$$

Thus,

$$|R_2^I| \lesssim \sum_{0 \leq j < |I|} |(u'')_{\leq j}| |(u')_{\leq |I| - j}| \lesssim \sum_{0 \leq j < |I|} |(u')_{\leq j+1}| |(u')_{\leq |I| - j}|.$$

For each $0 \leq j < |I|$, at most one of $j+1$ and $|I| - j$ is larger than $(N+1)/2$. And since

$$N \geq d+4 \Longrightarrow \frac{N+1}{2} \leq N - \lfloor \frac{d+2}{2} \rfloor,$$

we have

$$|R_2^I| \lesssim |(u')_{\leq N - \lfloor \frac{d+2}{2} \rfloor}| |(u')_{\leq N}|.$$

Again, we obtain

$$\left\| R_2^I(t) \right\|_{L^2} \lesssim A_0 \varepsilon \langle t \rangle^{-(d-1)/2} A(t).$$

We remark that it is even simpler to estimate R_3^I than to estimate R_2^I. Again we would get

$$\left\| R_3^I(t) \right\|_{L^2} \lesssim A_0 \varepsilon \langle t \rangle^{-(d-1)/2} A(t).$$

The proof for this estimate is left as an exercise.

Finally let us estimate R_1^I. If $|I| = 0$, we have $R_1^I = 0$, so suppose that $|I| > 0$. Since $[\Box, Z] = C\Box$ for each commuting vector field Z, $[\Box, Z^I]u$ can be written as a linear combination of terms of the form

$$\Box Z^J u, \qquad |J| < |I|.$$

Meanwhile, we have

$$\Box Z^J u = g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^J u - (g^{\alpha\beta}(u') - m^{\alpha\beta})\partial_\alpha\partial_\beta Z^J u = \sum_{j=1}^{4} R_j^J + O(|u'||(u')_{\leq N}|).$$

It follows that

$$|R_1^I| \lesssim \sum_{|J|<|I|} |\Box Z^J u| \lesssim \sum_{|J|<|I|} |R_j^J| + |u'||(u')_{\leq N}|.$$

Thus, we can use induction to prove that

$$|R_1^I| \lesssim \sum_{|J|<|I|} |R_j^J| + |u'||(u')_{\leq N}|.$$

This inequality and the proofs above show that

$$\left\|R_1^I(t)\right\|_{L^2} \lesssim A_0\varepsilon\langle t\rangle^{-(d-1)/2}A(t).$$

Now we apply the energy estimate (3.3) to $Z^I u$. Since the L^2 norm of the right hand side of (6.6) at time t is bounded above by $A_0\varepsilon\langle t\rangle^{-(d-1)/2}A(t)$, we have

$$A(t) \lesssim (A(0) + \int_0^t A_0\varepsilon\langle\tau\rangle^{-(d-1)/2}A(\tau)\ d\tau) \cdot \exp(2\int_0^t \left\|\partial(g^{**}(u'))(\tau)\right\|_{L^\infty}\ d\tau), \qquad t \in [0, T] \subset [0, T_d).$$

Meanwhile, since

$$|\partial(g^{**}(u'))| \lesssim |u''| \lesssim |(u')_{\leq N-\lfloor\frac{d+2}{2}\rfloor}| \lesssim A_0\varepsilon\langle t\rangle^{-(d-1)/2},$$

we have

$$\int_0^t \left\|\partial(g^{**}(u'))(\tau)\right\|_{L^\infty}\ d\tau \lesssim \begin{cases} A_0\varepsilon, & d \geq 4; \\ A_0\varepsilon\ln\langle T\rangle, & d = 3; \\ A_0\varepsilon\langle T\rangle^{1/2}, & d = 2; \\ A_0\varepsilon T, & d = 1. \end{cases} \tag{6.7}$$

Recall that $T < T_d$ and that

$$T_d := \begin{cases} \infty, & d \geq 4; \\ \exp(c/\varepsilon), & d = 3; \\ c/\varepsilon^2, & d = 2; \\ c/\varepsilon, & d = 1. \end{cases}$$

By choosing sufficiently small ε when $d \geq 4$, and by choosing $c \in (0,1)$ sufficiently small when $d \leq 3$, we can make the left hand side of (6.7) smaller than 1. It follows that

$$A(t) \leq CA(0) + \int_0^t CA_0\varepsilon\langle\tau\rangle^{-(d-1)/2}A(\tau)\ d\tau, \qquad t \in [0,T].$$

By the Gronwall's inequality, we have

$$A(t) \leq CA(0)\exp(\int_0^t CA_0\varepsilon\langle\tau\rangle^{-(d-1)/2}\ d\tau), \qquad t \in [0,T].$$

By choosing ε or c sufficiently small, again we can make the second exponential here $\lesssim 1$. We conclude that $A(t) \leq CA(0)$. Since $A(0) \leq C_0\varepsilon$ and since C, C_0 are known before we choose A_0, by choosing A_0 sufficiently large we have $CA(0) \leq \frac{A_0\varepsilon}{2}$. This finishes the proof of the assertion (a).

7 The null condition

In this section we would focus on the lifespan of (6.1) when $d = 3$. Given C_c^∞ data of size $\varepsilon \ll 1$, by Theorem 6.1 we know that this equation has a solution for $t \in [0, \exp(c/\varepsilon)]$. We also know that not all equations of the form (6.1) have global solutions (John's examples: $\Box u = u_t^2$ and $\Box u = u_t u_{tt}$). In contrast, we know that $\Box u = 0$ admits a global solution for all C_c^∞ data. The following question arises naturally: is there a sufficient condition for a small data global existence result for (6.1)? This is why we introduce the *null condition*. In this section, our main result is the following theorem.

Theorem 7.1. *Fix $(u^0, u^1) \in C_c^\infty(\mathbb{R}^3)$ and consider the Cauchy problem* (6.1) *in $\mathbb{R}_+^{1+3}$. Suppose that the equation also satisfies the null condition. Then, for all sufficiently small $0 < \varepsilon \ll 1$ (depending on (u^0, u^1)), the Cauchy problem* (6.1) *has a (unique) C^∞ solution for all $t \geq 0$.*

Remark 7.1.1. This theorem was first proved by Klainerman [Kla85, Kla84] and Christodoulou [Chr86]. To my knowledge, it has at least three different proofs. One is from [H97] (or the first edition of [Sog08]). In both books, the authors allow the coefficients to depend on the unknown function u itself, so their methods actually work for a larger class of equations. The second one is from [Sog08] (the second edition). This proof can be adapted to other multi-speed systems where the Lorentz boosts are not available. The third one from [Ali10]. There the author makes use of the Alinhac's *ghost weight*.

To prevent from making this note too long, here I cannot discuss all these three proofs above. In this section, I will use the last proof from [Ali10]. At the end of this section, I will also briefly discuss how the proofs from [H97, Sog08] work.

7.1 Definition and basic properties

Definition 7.2. Suppose that the Taylor expansions of g^{**} and F in (6.1) at 0 are

$$\begin{aligned} g^{\alpha\beta}(u') &= m^{\alpha\beta} + g_0^{\alpha\beta\lambda}\partial_\lambda u + O(|u'|^2), \\ F(u') &= f_0^{\alpha\beta}\partial_\alpha u \partial_\beta u + O(|u'|^3). \end{aligned} \tag{7.1}$$

We say the equation (6.1) (or the coefficients g_0^{***}, f_0^{**}) satisfies the *null condition* if

$$g_0^{\alpha\beta\lambda}\xi_\alpha\xi_\beta\xi_\gamma = f_0^{\alpha\beta}\xi_\alpha\xi_\beta = 0, \qquad \text{whenever } \xi \in \mathbb{R}^{1+3},\ |\xi_0|^2 = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2. \tag{7.2}$$

It is easy to check that (1.4) and (1.8) satisfy the null condition.

In general, we can also define the null condition for

$$g^{\alpha\beta}(u, u')\partial_\alpha\partial_\beta u = F(u, u')$$

where u is an $\mathbb{R}$-valued unknown. In this case, we simply replace the remainders in (7.1) with $O(|u|^2 + |u'|^2)$ or $O(|u|^3 + |u'|^3)$. We can also define the null condition if (6.1) is replaced by a system of quasilinear wave equations for an $\mathbb{R}^N$-valued unknown. For simplicity, we shall not discuss these general cases in this note.

Let us briefly explain the motivation behind (7.2). To apply the energy estimate, we need to estimate the L^2 norm of $F(u')$. The cubic terms in the Taylor expansion of F is good, so let us focus on the quadratic terms. For simplicity, we assume that $F(u') = f_0^{\alpha\beta}u_\alpha u_\beta$. For each partial derivative ∂_α, we can decompose it as the sum of a normal derivative (with respect to the light cone $|x| - t = C$) and a tangential derivative. By easy computations, the normal derivative is equal to $\frac{1}{2}q_\alpha(\partial_t - \partial_r)$ where $q = |x| - t$. We can check that $q_t^2 = |\nabla_x q|^2$. Thus, if we expand the quadratic form $F(u')$ by using the decomposition above, we get

$$F(u') = \frac{1}{4}f_0^{\alpha\beta}q_\alpha q_\beta(u_t - u_r)^2 + \partial u \cdot \bar{\partial} u = \partial u \cdot \bar{\partial} u$$

where $\bar{\partial} u$ denotes one of the tangential derivatives $\partial_i + \omega_i\partial_t$. By (5.5), we expect tangential derivatives to have better decays, so that is good. Similarly, we can make a similar discussion for $g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta u$.

We now make the discussions above rigorous. Since $\partial_t - \partial_r = 2\partial_t +$ tangential derivatives, it would not affect our proof if we replace $\partial_t - \partial_r$ with ∂_t in our decomposition above. This would make our computations a little simpler.

Lemma 7.3. *Suppose that g_0^{***} and f_0^{**} satisfy the null condition* (7.2)*. Then,*

$$|g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v| + |g_0^{\alpha\beta\lambda}\partial_\alpha u\partial_\beta\partial_\lambda v| + |g_0^{\alpha\beta\lambda}\partial_\beta u\partial_\alpha\partial_\lambda v| \lesssim |Tu||\partial^2 v| + |\partial u||T\partial v|,$$

$$|g_0^{\alpha\beta\lambda}\partial_\alpha u\partial_\beta v\partial_\lambda w| \lesssim |Tu||\partial v||\partial w| + |\partial u||Tv||\partial w| + |\partial u||\partial v||Tw|,$$

$$|f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v| \lesssim |\partial u||Tv| + |Tu||\partial v|.$$

Here $|Tu| := \sum_{i=1}^3 |u_i + \omega_i u_t|$. We can then apply (5.5) *in Lemma 5.2 to control the right hand sides of these two inequalities.*

Proof. Set $q = |x| - t$ and $T_\alpha = \partial_\alpha + q_\alpha\partial_t$. We now have $\partial_\alpha = T_\alpha - q_\alpha\partial_t$ and thus

$$\begin{aligned}
g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v &= -g_0^{\alpha\beta\lambda}q_\lambda u_t\partial_\alpha\partial_\beta v + T_\lambda u \cdot \partial^2 v \\
&= g_0^{\alpha\beta\lambda}q_\alpha q_\lambda u_t\partial_t\partial_\beta v + Tu \cdot \partial^2 v + \partial u \cdot T\partial v \\
&= -g_0^{\alpha\beta\lambda}q_\alpha q_\beta q_\lambda u_t \cdot \partial_t^2 v + Tu \cdot \partial^2 v + \partial u \cdot T\partial v + \partial u \cdot T\partial v \\
&= Tu \cdot \partial^2 v + \partial u \cdot T\partial v.
\end{aligned}$$

Here $Tu \cdot \partial^2 v$ is a linear combination of $T_\lambda u \cdot \partial_\alpha\partial_\beta v$ with $O(1)$ coefficients. Similarly for other terms. Also note that the last identity follows from the null condition. The same proof applies to other terms on the left hand side of the first inequality. The second and third inequality can be proved in a similar way. ☐

Remark 7.3.1. If there is no null condition, then the best result we have is

$$|g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v| \lesssim |\partial u||\partial^2 v|, \qquad |f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v| \lesssim |\partial u||\partial v|.$$

By Lemma 5.2, we expect $|\partial u| \lesssim \langle r-t\rangle^{-1}|Zu|$ and $|Tu| \lesssim \langle r+t\rangle^{-1}|Zu|$. The second estimate is better than the first one. That explains why the null condition improves the almost global existence result in Theorem 6.1.

We now state another useful property for the null condition. For simplicity, we call $f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v$ or $g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v$ a *null form* (for (u,v)) if g_0^{***}, f_0^{**} satisfy the null condition (7.2). We will see in the next lemma that if we apply Z^I to a null form, then we will get a sum of several null forms (not necessarily for (u,v)).

Lemma 7.4. *Suppose that g_0^{***} and f_0^{**} satisfy the null condition* (7.2)*. For each multiindex I,*

$$\begin{aligned} Z^I(g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v) &= \sum_{|J_1|+|J_2|\leq|I|} \widetilde{g}_{J_1,J_2}^{\alpha\beta\lambda}\partial_\lambda Z^{J_1}u\partial_\alpha\partial_\beta Z^{J_2}v, \\ Z^I(f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v) &= \sum_{|J_1|+|J_2|\leq|I|} \widetilde{f}_{J_1,J_2}^{\alpha\beta}\partial_\alpha Z^{J_1}u\partial_\beta Z^{J_2}v. \end{aligned}$$

Here for each pair of multiindices (J_1,J_2) with $|J_1|+|J_2|\leq|I|$, the coefficients $\widetilde{g}_{J_1,J_2}^{***}$ and $\widetilde{f}_{J_1}^{**}$ are constants satisfying the null condition (7.2)*.*

Moreover, we have $\widetilde{g}_{0,I}^{\alpha\beta\lambda} = g_0^{\alpha\beta\lambda}$ and $\widetilde{g}_{J_1,J_2}^{\alpha\beta\lambda} = 0$ if $|J_2|=|I|$, $|J_1|=0$ but $J_2\neq I$.

Proof. By induction, we only need to compute $Z(f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v)$ and $Z(g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v)$ for an arbitrary commuting vector field Z. By the product rule, we have

$$\begin{aligned} Z(f_0^{\alpha\beta}\partial_\alpha u\partial_\beta v) &= f_0^{\alpha\beta}Z\partial_\alpha u\partial_\beta v + f_0^{\alpha\beta}\partial_\alpha uZ\partial_\beta v \\ &= \underbrace{f_0^{\alpha\beta}\partial_\alpha Zu\partial_\beta v + f_0^{\alpha\beta}\partial_\alpha u\partial_\beta Zv}_{\text{null form}} + f_0^{\alpha\beta}[Z,\partial_\alpha]u\partial_\beta v + f_0^{\alpha\beta}\partial_\alpha u[Z,\partial_\beta]v, \end{aligned}$$

$$\begin{aligned} Z(g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta v) &= g_0^{\alpha\beta\lambda}Z\partial_\lambda u\partial_\alpha\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda uZ\partial_\alpha\partial_\beta v \\ &= \underbrace{g_0^{\alpha\beta\lambda}\partial_\lambda Zu\partial_\alpha\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta Zv}_{\text{null form}} \\ &\quad + g_0^{\alpha\beta\lambda}[Z,\partial_\lambda]u\partial_\alpha\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda u[Z,\partial_\alpha]\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha[Z,\partial_\beta]v. \end{aligned}$$

Moreover, for each vector field $Z = z^\alpha(t,x)\partial_\alpha$, we have $[Z,\partial_\beta] = -(\partial_\beta z^\alpha)\partial_\alpha$. As a result,

$$f_0^{\alpha\beta}[Z,\partial_\alpha]u\partial_\beta v + f_0^{\alpha\beta}\partial_\alpha u[Z,\partial_\beta]v = -f_0^{\alpha\beta}(\partial_\alpha z^\sigma)\partial_\sigma u\partial_\beta v - f_0^{\alpha\beta}\partial_\alpha u(\partial_\beta z^\sigma)\partial_\sigma v,$$

$$\begin{aligned} &g_0^{\alpha\beta\lambda}[Z,\partial_\lambda]u\partial_\alpha\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda u[Z,\partial_\alpha]\partial_\beta v + g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha[Z,\partial_\beta]v \\ &= -g_0^{\alpha\beta\lambda}(\partial_\lambda z^\sigma)\partial_\sigma u\partial_\alpha\partial_\beta v - g_0^{\alpha\beta\lambda}\partial_\lambda u(\partial_\alpha z^\sigma)\partial_\sigma\partial_\beta v - g_0^{\alpha\beta\lambda}\partial_\lambda u(\partial_\beta z^\sigma)\partial_\alpha\partial_\sigma v. \end{aligned}$$

In order to check the null condition, we need to check that

$$f_0^{\alpha\beta}(\partial_\alpha z^\sigma)\xi_\sigma\xi_\beta + f_0^{\alpha\beta}\xi_\alpha(\partial_\beta z^\sigma)\xi_\sigma = 0,$$

$$g_0^{\alpha\beta\lambda}(\partial_\lambda z^\sigma)\xi_\sigma\xi_\alpha\xi_\beta + g_0^{\alpha\beta\lambda}\xi_\lambda(\partial_\alpha z^\sigma)\xi_\sigma\xi_\beta + g_0^{\alpha\beta\lambda}\xi_\lambda(\partial_\beta z^\sigma)\xi_\alpha\xi_\sigma = 0,$$

whenever $\xi \in \mathbb{R}^{1+3}$ with $|\xi_0|^2 = \sum_{j=1,2,3}|\xi_j|^2$. If $Z = \partial$, then $\partial z = 0$ so these two identities hold trivially. If $Z = S$, then $\partial_\alpha z^\sigma \xi_\sigma = \xi_\alpha$, so we reduce these two identities to (7.2). If $Z = \Omega_{ij}$, then $\partial_\alpha z^\sigma \xi_\sigma = \delta_{\alpha i}\xi_j - \delta_{\alpha j}\xi_i$, so the left hand sides of these two identities reduce to

$$f_0^{i\beta}\xi_j\xi_\beta - f_0^{j\beta}\xi_i\xi_\beta + f_0^{\alpha i}\xi_\alpha\xi_j - f_0^{\alpha j}\xi_\alpha\xi_i = (\xi_j\partial_{\xi_i} - \xi_i\partial_{\xi_j})(f_0^{\alpha\beta}\xi_\alpha\xi_\beta),$$

$$g_0^{\alpha\beta i}\xi_j\xi_\alpha\xi_\beta - g_0^{\alpha\beta j}\xi_i\xi_\alpha\xi_\beta + g_0^{i\beta\lambda}\xi_\lambda\xi_j\xi_\beta - g_0^{j\beta\lambda}\xi_\lambda\xi_i\xi_\beta + g_0^{\alpha i\lambda}\xi_\lambda\xi_j\xi_\alpha - g_0^{\alpha j\lambda}\xi_\lambda\xi_i\xi_\alpha = (\xi_j\partial_{\xi_i} - \xi_i\partial_{\xi_j})(g_0^{\alpha\beta\lambda}\xi_\alpha\xi_\beta\xi_\lambda).$$

Since $\xi_j\partial_{\xi_i} - \xi_i\partial_{\xi_j}$ is tangent to the cone $|\xi_0|^2 = \sum_{j=1,2,3}|\xi_j|^2$, and since $g_0^{\alpha\beta\lambda}\xi_\alpha\xi_\beta\xi_\lambda = f_0^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ on this cone, we conclude these two identities. Finally, if $Z = \Omega_{0i}$, then $\partial_\alpha z^\sigma\xi_\sigma = \delta_{\alpha i}\xi_0 + \delta_{\alpha 0}\xi_i$, so the left hand sides of these two identities reduce to

$$f_0^{i\beta}\xi_0\xi_\beta + f_0^{0\beta}\xi_i\xi_\beta + f_0^{\alpha i}\xi_\alpha\xi_0 + f_0^{\alpha 0}\xi_\alpha\xi_i = (\xi_i\partial_{\xi_0} + \xi_0\partial_{\xi_i})(f_0^{\alpha\beta}\xi_\alpha\xi_\beta),$$

$$g_0^{\alpha\beta i}\xi_0\xi_\alpha\xi_\beta + g_0^{\alpha\beta 0}\xi_i\xi_\alpha\xi_\beta + g_0^{i\beta\lambda}\xi_\lambda\xi_0\xi_\beta + g_0^{0\beta\lambda}\xi_\lambda\xi_i\xi_\beta + g_0^{\alpha i\lambda}\xi_\lambda\xi_0\xi_\alpha + g_0^{\alpha 0\lambda}\xi_\lambda\xi_i\xi_\alpha = (\xi_i\partial_{\xi_0} + \xi_0\partial_{\xi_i})(g_0^{\alpha\beta\lambda}\xi_\alpha\xi_\beta\xi_\lambda).$$

Since $\xi_i\partial_{\xi_0} + \xi_0\partial_{\xi_i}$ is tangent to the cone $|\xi_0|^2 = \sum_{j=1,2,3}|\xi_j|^2$, and since $g_0^{\alpha\beta\lambda}\xi_\alpha\xi_\beta\xi_\lambda = f_0^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ on this cone, we conclude these two identities. $\square$

7.2 Preliminary estimates

In this subsection, we seek to prove the following estimate.

Lemma 7.5. *Fix an integer $N \geq 2$. Suppose that u is a solution to* (6.1) *which satisfies the null condition. Also assume that*

$$|(u')_{\leq N/2+1}| \leq 1, \qquad \text{where } |v_{\leq M}| := \sum_{|J|\leq M}|Z^J v|. \tag{7.3}$$

Then, we have

$$\sum_{|I|\leq N}|g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u| \lesssim |T(u_{\leq N})||(u')_{\leq N/2+1}| + |(u')_{\leq N}||T(u_{\leq N/2+1})| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|. \tag{7.4}$$

Here we set $|T(v_{\leq M})| := \sum_{|J|\leq M}|TZ^J v|$ with $T = (T_\alpha) = (\partial_\alpha + \partial_\alpha(r-t)\partial_t)$.

Recall from the previous section that for each multiindex I, we have

$$\begin{aligned} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u &= [\Box, Z^I]u + [(g^{\alpha\beta}(u') - m^{\alpha\beta}), Z^I]\partial_\alpha\partial_\beta u + (g^{\alpha\beta}(u') - m^{\alpha\beta})[\partial_\alpha\partial_\beta, Z^I]u + Z^I(F(u')) \\ &=: R_1^I + R_2^I + R_3^I + R_4^I. \end{aligned} \tag{7.5}$$

We start with R_4^I. Note that $F(u') = f_0^{\alpha\beta} u_\alpha u_\beta + F_c(u')$ where $F_c(u')$ vanishes of third order at 0 (i.e. $d^k F_c(0) = 0$, $k = 0,1,2$). By the chain rule and Leibniz's rule, and because of (7.3), we have

$$\sum_{|I|\le N} |Z^I(F_c(u'))| \lesssim |(u')_{\le N/2}|^2 |(u')_{\le N}|.$$

The proof of this estimate is left as an exercise. Moreover, by Lemma 7.3 and 7.4, we have

$$\sum_{|I|\le N} |Z^I(f_0^{\alpha\beta} u_\alpha u_\beta)| \lesssim \sum_{|J_1|+|J_2|\le N} (|TZ^{J_1}u||\partial Z^{J_2}u|) \lesssim |T(u_{\le N})||(u')_{\le N/2}| + |(u')_{\le N}||T(u_{\le N/2})|.$$

In summary,

$$\sum_{|I|\le N} |R_4^I| \lesssim |T(u_{\le N})||(u')_{\le N/2}| + |(u')_{\le N}||T(u_{\le N/2})| + |(u')_{\le N/2}|^2 |(u')_{\le N}|. \tag{7.6}$$

Next we consider $R_2^I + R_3^I$. Write

$$g^{\alpha\beta}(u') - m^{\alpha\beta} = g_0^{\alpha\beta\lambda} u_\lambda + g_c^{\alpha\beta}(u')$$

where $g_c^{\alpha\beta}(u')$ vanishes of second order at 0. We can prove that (exercise)

$$\sum_{|J|\le N} |Z^J(g_c^{\alpha\beta}(u'))| \lesssim |(u')_{\le N/2}||(u')_{\le N}|.$$

Now,

$$\begin{aligned}
R_2^I &= [g_0^{\alpha\beta\lambda} u_\lambda, Z^I]\partial_\alpha\partial_\beta u + [g_c^{\alpha\beta}, Z^I]\partial_\alpha\partial_\beta u \\
&= g_0^{\alpha\beta\lambda}\partial_\lambda u Z^I \partial_\alpha\partial_\beta u - Z^I(g^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta u) + [g_c^{\alpha\beta}(u'), Z^I]\partial_\alpha\partial_\beta u, \\
R_3^I &= g_0^{\alpha\beta\lambda}\partial_\lambda u [\partial_\alpha\partial_\beta, Z^I]u + g_c^{\alpha\beta}(u')[\partial_\alpha\partial_\beta, Z^I]u.
\end{aligned}$$

As a result,

$$R_2^I + R_3^I = g_0^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta Z^I u - Z^I(g^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta u) - Z^I(g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta u) + g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u.$$

By Lemma 7.4, we can write $g_0^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta Z^I u - Z^I(g^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta u)$ as a sum of null forms

$$\widetilde{g}_{J_1,J_2}^{\alpha\beta\lambda}\partial_\lambda Z^{J_1}u \cdot \partial_\alpha\partial_\beta Z^{J_2}u, \qquad |J_1| + |J_2| \le |I|,\ |J_2| < |I|.$$

In particular, we emphasize $|J_2| < |I|$ because of the second half of Lemma 7.4. It follows from Lemma 7.3 and from $[\partial, Z] = C \cdot \partial$ that

$$\begin{aligned}
&|g_0^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta Z^I u - Z^I(g^{\alpha\beta\lambda}\partial_\lambda u \partial_\alpha\partial_\beta u)| \\
&\lesssim \sum_{\substack{|J_1|+|J_2|\le N \\ |J_2|<N}} (|TZ^{J_1}u||\partial^2 Z^{J_2}u| + |\partial Z^{J_1}u||T\partial Z^{J_2}u|) \\
&\lesssim |T(u_{\le N})||(u')_{\le N/2+1}| + |T(u_{\le N/2})||(u')_{\le N}| + |(u')_{\le N}||T(u_{\le N/2+1})| + |(u')_{\le N/2}||T(u_{\le N})| \\
&\lesssim |T(u_{\le N})||(u')_{\le N/2+1}| + |(u')_{\le N}||T(u_{\le N/2+1})|.
\end{aligned}$$

Similarly, we can write $-Z^I(g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta u) + g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u$ as a linear combination of

$$Z^{J_1}(g_c^{\alpha\beta}(u'))Z^{J_2}\partial_\alpha\partial_\beta u, \qquad |J_1| + |J_2| \leq |I|,\ |J_2| < |I|.$$

It follows that

$$\begin{aligned}
&| - Z^I(g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta u) + g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u| \\
&\lesssim \sum_{\substack{|J_1|+|J_2|\leq N \\ |J_2|<N}} |(u')_{\leq |J_1|/2}||(u')_{\leq |J_1|}||(u')_{\leq |J_2|+1}| \\
&\lesssim |(u')_{\leq N/2}||(u')_{\leq N}||(u')_{\leq N/2+1}| + |(u')_{\leq N/2}|^2|(u')_{\leq N}| \lesssim |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|.
\end{aligned}$$

In summary, we have

$$|R_2^I + R_3^I| \lesssim |T(u_{\leq N})||(u')_{\leq N/2+1}| + |(u')_{\leq N}||T(u_{\leq N/2+1})| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|. \tag{7.7}$$

Finally we estimate R_1^I. It is clear that $R_1^I = 0$ if $|I| = 0$. Since $[\Box, Z] = C\Box$, we can write R_1^I as a linear combination of terms of the form $\Box Z^J u$ with $|J| < |I|$. Meanwhile, we have

$$\begin{aligned}
\Box Z^J u &= g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^J u - (g^{\alpha\beta}(u') - m^{\alpha\beta})\partial_\alpha\partial_\beta Z^J u \\
&= \sum_{j=1}^4 R_j^J - g_0^{\alpha\beta\lambda}\partial_\lambda u\partial_\alpha\partial_\beta Z^J u - g_c^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^J u \\
&= \sum_{j=1}^4 R_j^J + O(|Tu||\partial^2 Z^J u| + |\partial u||T\partial Z^J u| + |u'|^2|\partial^2 Z^J u|) \qquad \text{(by Lemma 7.3)}.
\end{aligned}$$

We can estimate $R_2^J + R_3^J + R_4^J$ using (7.6) and (7.7). And since $|J| < N$, we conclude that

$$|R_1^I| \lesssim \sum_{|J|<|I|} |R_1^J| + |T(u_{\leq N})||(u')_{\leq N/2+1}| + |(u')_{\leq N}||T(u_{\leq N/2+1})| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|.$$

By induction, we conclude that

$$|R_1^I| \lesssim |T(u_{\leq N})||(u')_{\leq N/2+1}| + |(u')_{\leq N}||T(u_{\leq N/2+1})| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|. \tag{7.8}$$

This finishes the proof of the lemma.

Remark 7.5.1. Since $|T\phi| \lesssim |\phi'|$, it follows from (7.4) that

$$\sum_{|I|\leq N} |g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u| \lesssim |(u')_{\leq N}||(u')_{\leq N/2+1}| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}|. \tag{7.9}$$

This is in fact the estimate we shall get without assuming the null condition.

7.3 The energy estimate

I would like to start this subsection with an explanation on why the usual energy estimate is not enough in the proof of Theorem 7.1. In the proof, we need to estimate $E(t) = \|(u')_{\leq N}\|_{L^2}$. So, what is the upper bound for $E(t)$ in our continuity argument here? If $E(t) \leq C\varepsilon$, then by the Klainerman-Sobolev inequality, we shall obtain

$$E(t) \lesssim E(0) + \int_0^t \varepsilon\tau^{-1}E(\tau)\ d\tau.$$

By the Gronwall's inequality we have $E(t) \lesssim \varepsilon t^{C\varepsilon}$ which does not end the continuity argument. How about $E(t) \leq C\varepsilon t^{C\varepsilon}$ in the continuity argument? It is even worse because we shall get

$$E(t) \lesssim E(0) + \int_0^t \varepsilon\tau^{-1+C\varepsilon}E(\tau)\ d\tau.$$

By the Gronwall's inequality we have $E(t) \lesssim \varepsilon t^{Ct^{C\varepsilon}}$. Even worse! This is why we need to introduce some new energy estimates here.

We need an energy estimate from [Ali10].

Proposition 7.6. *Let $u \in C^2([0,T] \times \mathbb{R}^3)$ vanish for large $|x|$ and satisfy*

$$Pu := g^{\alpha\beta}(w')\partial_\alpha\partial_\beta u = F, \qquad \forall (t,x) \in [0,T) \times \mathbb{R}^d. \tag{7.10}$$

Suppose that $g^{\alpha\beta} = g^{\beta\alpha}$, that w vanishes for all $|x| \geq t + C$, that

$$g^{\alpha\beta}(w') = m^{\alpha\beta} + g_0^{\alpha\beta\gamma}\partial_\gamma w + O(|w'|^2) \tag{7.11}$$

where the coefficients g_0^{***} are constants satisfying the null condition (7.2), and that

$$\sum_{|J|\leq 3} \left\|\partial Z^J w\right\|_{L^2(\mathbb{R}^d)} \leq C_0\varepsilon, \qquad 0 < \varepsilon \ll 1. \tag{7.12}$$

Then, for each small $\eta \in (0,1)$, we have

$$\begin{aligned}&\left\|u'(t)\right\|_{L^2(\mathbb{R}^3)} + \left(\int_0^t \int_{\mathbb{R}^3} \langle |x| - t\rangle^{-1-\eta}|Tu(\tau)|^2\ dxd\tau\right)^{1/2} \\ &\lesssim_\eta e^{CC_0\varepsilon}(\left\|u'(0)\right\|_{L^2(\mathbb{R}^3)} + \int_0^t \|F(\tau)\|_{L^2(\mathbb{R}^3)}\ d\tau).\end{aligned} \tag{7.13}$$

Here recall that $|Tu|^2 = \sum_{j=1}^3 |u_j + \omega_j u_t|^2$.

In the proof, we make use of the method of *ghost weight* which was introduced by Alinhac. Let $a \in C^1(\mathbb{R})$ be a function to be chosen later. Now we shall compute $e^{a(|x|-t)}u_t Pu$. Let $g_\gamma^{\alpha\beta} = \partial_{w^\gamma}g^{\alpha\beta}$ be the derivative of $g^{\alpha\beta}$ with respect to the γ-th component. Now,

$$\begin{aligned}e^a u_t g^{\alpha\beta}(w')\partial_\alpha\partial_\beta u &= \partial_\alpha(e^a u_t g^{\alpha\beta}(w')u_\beta) - \partial_\alpha(e^a u_t g^{\alpha\beta}(w'))u_\beta \\ &= \partial_\alpha(e^a u_t g^{\alpha\beta}(w')u_\beta) - e^a a' q_\alpha u_t g^{\alpha\beta}(w')u_\beta \\ &\quad - e^a(\partial_\alpha\partial_t u)g^{\alpha\beta}(w')u_\beta - e^a u_t g_\gamma^{\alpha\beta}(w')\cdot(\partial_\alpha\partial_\gamma w)u_\beta.\end{aligned}$$

Here recall that $q = r - t$. Next,

$$\begin{aligned}
-e^a(\partial_\alpha\partial_t u)g^{\alpha\beta}(w')u_\beta &= -\partial_t(e^a g^{\alpha\beta}(w')u_\alpha u_\beta) + \partial_t(e^a g^{\alpha\beta}(w')u_\beta)u_\alpha \\
&= -\partial_t(e^a g^{\alpha\beta}(w')u_\alpha u_\beta) - e^a a' g^{\alpha\beta}(w')u_\beta u_\alpha \\
&\quad + e^a g_\gamma^{\alpha\beta}(w')\cdot(\partial_t\partial_\gamma w)u_\beta u_\alpha + \underbrace{e^a g^{\alpha\beta}(w')(\partial_t\partial_\beta u)u_\alpha}_{=-\text{LHS}}.
\end{aligned}$$

Thus,

$$-e^a(\partial_\alpha\partial_t u)g^{\alpha\beta}(w')u_\beta = -\frac{1}{2}\partial_t(e^a g^{\alpha\beta}(w')u_\alpha u_\beta) - \frac{1}{2}e^a a' g^{\alpha\beta}(w')u_\beta u_\alpha + \frac{1}{2}e^a g_\gamma^{\alpha\beta}(w')\cdot(\partial_t\partial_\gamma w)u_\beta u_\alpha.$$

In summary,

$$\begin{aligned}
e^a u_t P u = \partial_\alpha(e^a u_t g^{\alpha\beta}(w')u_\beta) - e^a a' q_\alpha u_t g^{\alpha\beta}(w')u_\beta - e^a u_t g_\gamma^{\alpha\beta}(w')\cdot(\partial_\alpha\partial_\gamma w)u_\beta \\
- \frac{1}{2}\partial_t(e^a g^{\alpha\beta}(w')u_\alpha u_\beta) - \frac{1}{2}e^a a' g^{\alpha\beta}(w')u_\beta u_\alpha + \frac{1}{2}e^a g_\gamma^{\alpha\beta}(w')\cdot(\partial_t\partial_\gamma w)u_\beta u_\alpha.
\end{aligned}$$

Thus we can write $-e^a u_t P u = \sum_{\beta=0}^{3}\partial_\beta e_\beta + e^a R_1 + e^a a' R_2$ where

$$\begin{aligned}
e_0 &= -e^a u_t g^{0\beta}(w')u_\beta + \frac{1}{2}e^a g^{\alpha\beta}(w')u_\alpha u_\beta, \\
R_1 &= g_\gamma^{\alpha\beta}(w')\cdot(\partial_\alpha\partial_\gamma w)u_t u_\beta - \frac{1}{2}g_\gamma^{\alpha\beta}(w')\cdot(\partial_t\partial_\gamma w)u_\beta u_\alpha, \\
R_2 &= q_\alpha g^{\alpha\beta}(w')u_t u_\beta + \frac{1}{2}g^{\alpha\beta}(w')u_\alpha u_\beta.
\end{aligned}$$

We first estimate e_0. By (7.11), (7.12) and the Klainerman-Sobolev inequality, we have $|g^{\alpha\beta}(w') - m^{\alpha\beta}| \lesssim C_0\varepsilon$. It follows that

$$e_0 = e^a(\frac{1}{2}|u'|^2 + O(C_0\varepsilon|u'|^2)), \tag{7.14}$$

and therefore

$$(\frac{1}{2} - CC_0\varepsilon)e^a|u'|^2 \le e_0 \le (\frac{1}{2} + CC_0\varepsilon)e^a|u'|^2.$$

Next let us estimate R_1. By (7.11), we have

$$g_\gamma^{\alpha\beta}(w') = g_0^{\alpha\beta\gamma} + O(|w'|)$$

and therefore

$$\begin{aligned}
R_1 &= g_0^{\alpha\beta\gamma}(u_t u_\beta\partial_\alpha\partial_\gamma w - \frac{1}{2}u_\alpha u_\beta\partial_t\partial_\gamma w) + O(|u'|^2|w'||w''|) \\
&= O(|u'||Tu||w''| + |u'|^2|T\partial w| + |u'|^2|w'||w''|) \qquad \text{by Lemma 7.3}.
\end{aligned}$$

By (7.12), the Klainerman-Sobolev inequality and Lemma 5.2, we have

$$\sum_{|J|\le 1}|\partial Z^J w| \lesssim C_0\varepsilon\langle t+|x|\rangle^{-1}\langle |x|-t\rangle^{-1/2}, \quad |w''| \lesssim C_0\varepsilon\langle t+|x|\rangle^{-1}\langle |x|-t\rangle^{-3/2}, \quad |T\partial w| \lesssim C_0\varepsilon\langle t+|x|\rangle^{-2}.$$

In summary,

$$|R_1| \lesssim C_0\varepsilon\langle t+|x|\rangle^{-1}\langle |x|-t\rangle^{-1/2}|u'||Tu| + C_0\varepsilon\langle |x|+t\rangle^{-2}|u'|^2.$$

Finally we estimate R_2. Since

$$g^{\alpha\beta}(w') = m^{\alpha\beta} + g_0^{\alpha\beta\lambda}\partial_\lambda w + O(|w'|^2),$$

we have

$$\begin{aligned}
R_2 &= q_\alpha m^{\alpha\beta}u_t u_\beta + \frac{1}{2}m^{\alpha\beta}u_\alpha u_\beta + q_\alpha g_0^{\alpha\beta\lambda}w_\lambda u_t u_\beta + \frac{1}{2}g_0^{\alpha\beta\lambda}w_\lambda u_\alpha u_\beta + O(|w'|^2|u'|^2) \\
&= u_t u_r + \frac{1}{2}|u'|^2 + q_\alpha g_0^{\alpha\beta\lambda}(T_\lambda w - q_\lambda w_t)u_t(T_\beta u - q_\beta u_t) + O(|u'|^2|Tw| + |u'||Tu||w'| + |w'|^2|u'|^2) \\
&= u_t u_r + \frac{1}{2}|u'|^2 + O(|u'|^2|Tw| + |u'||Tu||w'| + |w'|^2|u'|^2).
\end{aligned}$$

Note that $u_t u_r + \frac{1}{2}|u'|^2 = \frac{1}{2}\sum_{j=1}^3 |T_j u|^2$ and that

$$\begin{aligned}
|Tw| &\lesssim \langle |x|+t\rangle^{-1}\sum_{|J|=1}|Z^J w| \lesssim \langle |x|+t\rangle^{-1}\Big(\int_{[|x|,t+C]}\sum_{|J|=1}|\partial_\rho Z^J w(t,\rho x/|x|)|\; d\rho\Big) \qquad w \text{ vanishes for } |x|-t\geq C \\
&\lesssim \langle |x|+t\rangle^{-1}\Big(\int_{[|x|,t+C]} C_0\varepsilon\langle \rho+t\rangle^{-1}\langle \rho-t\rangle^{-1/2}\; d\rho\Big) \lesssim C_0\varepsilon\langle |x|+t\rangle^{-2}\langle |x|-t\rangle^{1/2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
R_2 &\geq \frac{1}{2}|Tu|^2 - CC_0\varepsilon\langle |x|+t\rangle^{-2}\langle |x|-t\rangle^{1/2}|u'|^2 - CC_0\varepsilon\langle |x|+t\rangle^{-1}\langle |x|-t\rangle^{-1/2}|u'||Tu| \\
&\geq \frac{1}{4}|Tu|^2 - CC_0\varepsilon\langle |x|+t\rangle^{-2}\langle |x|-t\rangle^{1/2}|u'|^2.
\end{aligned}$$

We now choose a so that $\lim_{s\to\infty} a(s) = 0$ and that $a'(s) = 8\langle s\rangle^{-1-\eta}$. Note that $\eta > 0$ implies that $|a| \lesssim 1$ and thus $e^a \sim 1$ everywhere. Now,

$$\begin{aligned}
a'R_2 + R_1 &\geq 2\langle |x|-t\rangle^{-1-\eta}|Tu|^2 - CC_0\varepsilon\langle |x|+t\rangle^{-2}|u'|^2 - CC_0\varepsilon\langle t+|x|\rangle^{-1}\langle |x|-t\rangle^{-3/2}|u'||Tu| \\
&\geq \langle |x|-t\rangle^{-1-\eta}|Tu|^2 - CC_0\varepsilon\langle |x|+t\rangle^{-2}|u'|^2.
\end{aligned}$$

We now integrate $-e^a u_t Pu = \sum_{\beta=0}^3 \partial_\beta e_\beta + e^a R_1 + e^a a' R_2$. By setting $E(t) = \int e_0(t,x)\; dx$, we have

$$\begin{aligned}
E'(t) &= \int -e^a u_t Pu - e^a(R_1 + a'R_2)\; dx \\
&\leq \int e^a|u'||F| - e^a\langle |x|-t\rangle^{-1-\eta}|Tu|^2 + CC_0\varepsilon\langle |x|+t\rangle^{-2}e^a|u'|^2\; dx \\
&\leq CE(t)^{1/2}\left\|F(t)\right\|_{L^2} - \left\|\langle |\cdot| - t\rangle^{-(1+\eta)/2}|Tu|\right\|_{L^2}^2 + CC_0\varepsilon\langle t\rangle^{-2}E(t).
\end{aligned}$$

If we set

$$H(t) = E(t) + \int_0^t \left\|\langle |\cdot| - \tau\rangle^{-(1+\eta)/2}|Tu|\right\|_{L^2}^2\; d\tau,$$

then

$$H'(t) \lesssim H(t)^{1/2} \|F(t)\|_{L^2} + CC_0\varepsilon\langle t\rangle^{-2}H(t).$$

We finish the proof by dividing both sides by $H(t)^{1/2}$, applying the Gronwall's inequality and noticing $e^a \sim 1$.

Remark 7.6.1. From this proof, we can see why e^a is called a ghost weight. Since $e^a \sim 1$, the energy defined with this weight is equivalent to that defined without this weight. This is very different from the energy used in [H97]. It thus seems useless to introduce this weight. However, using this weight would introduce an extra term in the energy estimate. This extra term will be necessary in the proof of global existence.

7.4 Continuity argument

We can now set up the continuity argument used for the proof. For each $T \in [0,\infty)$, our hypothesis is that there exists a C^∞ solution u for all $t \in [0,T]$, such that

$$A(t) := \sum_{|I|\leq N} \left\|Z^I u'(t)\right\|_{L^2(\mathbb{R}^3)} \leq A_1\varepsilon, \qquad t \in [0,T]. \tag{7.15}$$

Here $N \geq 8$ and $A_1 > 1$ are large constants, and $0 < \varepsilon < 1$ is a sufficiently small constant depending on N and A_1. All these constants are to be chosen later later. We would like to prove (7.15) with A_1 replaced by $A_1/2$. For simplicity, we would only check the assertion (a) in Proposition 6.2. The proofs of other assertions are the same as those in Section 6.2.

By the Klainerman-Sobolev inequality, we first notice that whenever $0 \leq t \leq T$,

$$\sum_{|I|\leq N-2} |\partial Z^I u(t,x)| \sim |(u')_{\leq N-2}| = \sum_{|I|\leq N-2} |Z^I u'(t,x)| \leq CA_1\varepsilon\langle |x|+t\rangle^{-1}\langle |x|-t\rangle^{-1/2}. \tag{7.16}$$

By the finite speed of propagation, we have $Z^I u = 0$ whenever $|x| - t \geq R$ for some constant $R > 0$ depending on the initial data. Thus,

$$|u_{\leq N-2}| \lesssim \int_{[|x|,t+R]} |(u')_{\leq N-2}(t,\rho x/|x|)|\, d\rho \lesssim \int_{[|x|,t+R]} A_1\varepsilon\langle \rho+t\rangle^{-1}\langle \rho-t\rangle^{-1/2}\, d\rho \lesssim A_1\varepsilon\langle |x|+t\rangle^{-1}\langle |x|-t\rangle^{1/2}. \tag{7.17}$$

By choosing $\varepsilon \ll_{A_1} 1$ and noticing that $N/2+2 \leq N-2$ whenever $N \geq 8$, we have $|(u')_{\leq N/2+1}| \leq 1$. Thus, by Lemma 7.5, we have

$$\begin{aligned}
&\sum_{|I|\leq N} |g^{\alpha\beta}(u')\partial_\alpha\partial_\beta Z^I u| \\
&\lesssim |T(u_{\leq N})||(u')_{\leq N/2+1}| + |(u')_{\leq N}||T(u_{\leq N/2+1})| + |(u')_{\leq N/2+1}|^2|(u')_{\leq N}| \\
&\lesssim A_1\varepsilon\langle |x|+t\rangle^{-1}\langle |x|-t\rangle^{-1/2}|T(u_{\leq N})| + A_1\varepsilon\langle |x|+t\rangle^{-2}\langle |x|-t\rangle^{1/2}|(u')_{\leq N}| + A_1^2\varepsilon^2\langle |x|+t\rangle^{-2}|(u')_{\leq N}| \\
&\lesssim A_1\varepsilon\langle |x|+t\rangle^{-1}\langle |x|-t\rangle^{-1/2}|T(u_{\leq N})| + A_1\varepsilon\langle t\rangle^{-3/2}|(u')_{\leq N}|.
\end{aligned}$$

In the last estimate, we notice that $\langle |x|-t\rangle \le \langle |x|+t\rangle$ and that $A_1\varepsilon \le 1$ if we choose $\varepsilon \ll_{A_1} 1$. Now we can apply Proposition 7.6 (with (w,u) replaced by $(u, Z^I u)$) to obtain

$$\begin{aligned}&A(t) + \left(\int_0^t \int_{\mathbb{R}^3} \langle |x|-t\rangle^{-1-\eta} |T(u_{\le N})(\tau,x)|^2 \, dx d\tau\right)^{1/2} \\ &\lesssim_\eta A(0) + \int_0^t A_1\varepsilon \left\| \langle |\cdot|+\tau\rangle^{-1}\langle |\cdot|-\tau\rangle^{-1/2} |T(u_{\le N})(\tau)|\right\|_{L^2} + A_1\varepsilon\langle\tau\rangle^{-3/2} A(\tau)\, d\tau.\end{aligned} \tag{7.18}$$

By choosing $\varepsilon \ll_{A_1} 1$, we can make $e^{CA_1\varepsilon} \le 2$, so we do not have $e^{CA_1\varepsilon}$ here. To continue, we note that

$$\begin{aligned}&\int_0^t A_1\varepsilon \left\| \langle |\cdot|+\tau\rangle^{-1}\langle |\cdot|-\tau\rangle^{-1/2} |T(u_{\le N})(\tau)|\right\|_{L^2} \, d\tau \\ &\lesssim \int_0^t A_1\varepsilon \langle\tau\rangle^{\eta/2-1} \left\| \langle |\cdot|-\tau\rangle^{-1/2-\eta/2} |T(u_{\le N})(\tau)|\right\|_{L^2} \, d\tau \\ &\lesssim A_1\varepsilon \left(\int_0^t \langle\tau\rangle^{\eta-2}\, d\tau\right)^{1/2} \left(\int_0^t \left\| \langle |\cdot|-\tau\rangle^{-1/2-\eta/2} |T(u_{\le N})(\tau)|\right\|_{L^2}^2 \, d\tau\right)^{1/2} \\ &\lesssim_\eta A_1\varepsilon \left(\int_0^t \left\| \langle |\cdot|-\tau\rangle^{-1/2-\eta/2} |T(u_{\le N})(\tau)|\right\|_{L^2}^2 \, d\tau\right)^{1/2}.\end{aligned}$$

To get the last estimate, we need to choose $0<\eta<1$. It is not hard to see that by choosing $\varepsilon \ll_{A_1} 1$, we can use the second term on the left hand side of (7.18) to absorb this integral. In summary, we have

$$A(t) \lesssim A(0) + \int_0^t A_1\varepsilon\langle\tau\rangle^{-3/2} A(\tau)\, d\tau.$$

It then follows from the Gronwall's inequality and $\varepsilon \ll_{A_1} 1$ that $A(t) \le CA(0) \le C\varepsilon$ where C is independent of A_1 and ε. By choosing $A_1 \gg 1$, we have $A(t) \le A_1\varepsilon/2$ for all $t\in[0,T]$. This finishes the proof of the continuity argument.

7.5 Alternative proof: I

Let me now present two different proofs. The first one is from [H97]. There Hörmander made use of the following energy estimate.

Proposition 7.7. *Fix $d\ge 2$. Let $u \in C^2([0,T]\times\mathbb{R}^d)$ vanish for large $|x|$ and satisfy*

$$g^{\alpha\beta}(w')\partial_\alpha\partial_\beta u(t,x) = F(t,x), \qquad \forall (t,x)\in[0,T)\times\mathbb{R}^d. \tag{7.19}$$

Suppose that $g^{\alpha\beta} = g^{\beta\alpha}$, that

$$g^{\alpha\beta}(w') = g_0^{\alpha\beta\gamma}\partial_\gamma w + g_c^{\alpha\beta}(w') \tag{7.20}$$

where the coefficients g_0^{***} are constants satisfying the null condition (7.2), and that for each sufficiently small constant $0<\delta\ll 1$ we have

$$\sum_{|J|\le 2} |Z^J w| \le \delta(1+t+|x|)^{-1}, \qquad \sum_{|J|\le 1} |Z^J(g_c^{\alpha\beta}(w'))| \le \delta(1+t+|x|)^{-2}. \tag{7.21}$$

Then, by setting

$$E_0(t) := \int_{\mathbb{R}^d} \sum_{|J|=1} |Z^J u(t,x)|^2 + (d-1)|u(t,x)|^2 \, dx,$$

we have

$$E_0(t)^{1/2} \lesssim (1+t)^{C\delta}(E_0(0)^{1/2} + \int_0^t (1+\tau)^{-C\delta} \left\| \langle t + |\cdot| \rangle F(\tau,\cdot) \right\|_{L^2(\mathbb{R}^d)} \, d\tau). \tag{7.22}$$

In the continuity argument, Hörmander considered the L^2 norm of not only $(u')_{\leq N}$ but also $u_{\leq N}$. This is why we cannot use the usual energy estimate (3.3) here.

Please read Proposition 6.6.6 and Lemma 6.6.7 in [H97] (note that $g^{\alpha\beta}(u') = -m^{\alpha\beta} + g_0^{\alpha\beta\gamma} u_\gamma + O(|u'|^2)$ in [H97]). The basic idea of proof is as follows. We apply the multiplier method to $Ku \cdot \Box_g u$ where

$$Ku := (1+t^2+|x|^2)u_t + 2tx\cdot\nabla_x u + (d-1)tu.$$

By tedious computations, we can again write

$$-Ku\Box_g u = \sum_{\beta=0}^d \partial_\beta e_\beta + R$$

where $0 \leq e_0 \sim \sum_{|J|=1} |Z^J u|^2 + (d-1)u^2$ (need $\delta \ll 1$ to get this estimate) and $|R| \lesssim \delta(1+t+|x|)^{-1}e_0$. By setting $E(t) := \int e_0(t,x)\, dx$ and noticing that $|Ku| \lesssim e_0$, we have

$$E'(t) \lesssim \delta(1+t)^{-1}E(t) + \left\| \langle t+|\cdot|\rangle F(t) \right\|_{L^2(\mathbb{R}^d)} E(t)^{1/2}$$

and thus

$$\frac{d}{dt}((1+t)^{-C\delta}E(t)^{1/2}) \lesssim (1+t)^{-C\delta} \left\| \langle t+|\cdot|\rangle F(t) \right\|_{L^2(\mathbb{R}^d)}.$$

We now integrate this inequality.

To finish the proof from [H97], we also need the following L^∞ estimate.

Lemma 7.8. *Let $F \in C^2(\mathbb{R}^{1+3}_+)$ and suppose that $w = w(t,x)$ solves $\Box w = F$ with $(w, w_t)|_{t=0} = 0$. Then,*

$$(1+t+|x|)|w(t,x)| \lesssim \int_0^t \int_{\mathbb{R}^3} \frac{\sum_{|I|\leq 2} |Z^I F(s,y)|}{1+s+|y|} \, dyds. \tag{7.23}$$

Proof. Let us first prove a homogeneous version of (7.23): we have

$$(t+|x|)|w(t,x)| \lesssim \int_0^t \int_{\mathbb{R}^3} \frac{\sum_{|I|\leq 2} |\widetilde{Z}^I F(s,y)|}{s+|y|} \, dyds. \tag{7.24}$$

Here $\widetilde{Z}$ is one of following commuting vector fields: scaling S, rotations Ω_{ij} and Lorentz boosts Ω_{0i}. Note that the coefficients of $\widetilde{Z}$ are homogeneous polynomials of (t,x) of degree 1, so we have

$(\widetilde{Z}^I w_\lambda)(t,x) = (\widetilde{Z}^I w)(\lambda t, \lambda x)$ where $w_\lambda := w(\lambda t, \lambda x)$. If we have proved (7.24) for w, then by replacing w with w_λ, we have $\Box w_\lambda = \lambda^2 F_\lambda$ and thus

$$\begin{aligned}(t+|x|)|w(\lambda t,\lambda x)| = (t+|x|)|w_\lambda(t,x)| &\lesssim \int_0^t \int_{\mathbb{R}^3} \frac{\sum_{|I|\le 2}(\widetilde{Z}^I(\lambda^2 F_\lambda))(s,y)}{s+|y|}\ dyds \\ &= \lambda^2 \int_0^t \int_{\mathbb{R}^3} \frac{\sum_{|I|\le 2}|(\widetilde{Z}^I F)(\lambda s,\lambda y)|}{s+|y|}\ dyds = \lambda^{-1}\int_0^{\lambda t}\int_{\mathbb{R}^3} \frac{\sum_{|I|\le 2}|(\widetilde{Z}^I F)(s,y)|}{s+|y|}\ dyds.\end{aligned}$$

It thus suffices to prove (7.24) with $t=1$.

The solution to $-\Box w = F$ with zero data can be written explicitly:

$$w(t,x) = \frac{1}{4\pi}\int_{|y|<t} \frac{F(t-|y|,x-y)}{|y|}\ dy. \tag{7.25}$$

This follows from the the Duhamel's principle: if $v(t,x;s)$ solves the linear wave equation with data $(v,v_t)|_{t=0} = (0,F(s,\cdot))$, then $w = \int_0^t v(t-s,x;s)\ ds$ solves $\Box w = F$ with zero data. You can also find this formula in I.1, [Sog08].

i) Let us first suppose that $\operatorname{supp} F \subset \{|y| < s/2\}$. If we go back to the proof of Lemma 5.2, we recall that

$$|\phi_t| + |\phi_r| \lesssim ||x|-t|^{-1}(|S\phi| + \sum_{i=1}^3 |\Omega_{0i}\phi|), \qquad |\phi_j| \lesssim |t+|x||^{-1}(\sum_{i=1}^3 |\Omega_{ij}\phi| + |\Omega_{0j}\phi| + |x||\phi_t - \phi_r|).$$

In summary, we have

$$|\partial\phi| \lesssim ||x|-t|^{-1}\sum_{|I|=1}|\widetilde{Z}^I\phi|.$$

And since $[\widetilde{Z},\widetilde{Z}'] = C\cdot\widetilde{Z}$, we have

$$|\partial^k\phi| \lesssim ||x|-t|^{-k}\sum_{|I|\le k}|\widetilde{Z}^I\phi|, \qquad k\ge 1.$$

In the support of F, we have $||y|-s| \sim s$, so

$$\int_0^1\int_{\mathbb{R}^3}\sum_{k\le 2}\frac{s^k|\partial^k F(s,y)|}{s}\ dyds \lesssim \int_0^1\int_{\mathbb{R}^3}\sum_{|I|\le 2}\frac{|\widetilde{Z}^I F(s,y)|}{s+|y|}\ dyds.$$

Since $\Box w = 0$ in $\{|y| > s/2\}$ and since $(w,w_t)|_{t=0} = 0$, we have $w(1,x) = 0$ whenever $|x| > 1$ by the finite speed of propagation. So to end the proof, we assume $|x| \le 1$.

Now, by the fundamental theorem of calculus, we have

$$|F(1-|y|,x-y)| = \int_0^1 |F(s,x-y) + \int_s^{1-|y|} F'_\tau(\tau,x-y)\ d\tau|\ ds \lesssim \int_0^1 |F(s,x-y)| + |F'_s(s,x-y)|\ ds.$$

If $|y|\le 1/2$, then similarly we have

$$|F(1-|y|,x-y)| = \int_{1/2}^1 |F(s,x-y) + \int_s^{1-|y|} F'_\tau(\tau,x-y)\ d\tau|\ ds \lesssim \int_{1/2}^1 |F(s,x-y)| + |F'_s(s,x-y)|\ ds.$$

It follows from (7.25) that

$$(1+|x|)|w(1,x)| \lesssim |w(1,x)| \lesssim \int_{|y|\le 1/2} \frac{|F(1-|y|,x-y)|}{|y|}\,dy + \int_{1/2\le|y|<1} \frac{|F(1-|y|,x-y)|}{|y|}\,dy$$
$$\lesssim \int_{|y|\le 1/2}\int_{1/2}^{1}(|F(s,x-y)|+|F_s'(s,x-y)|)\frac{dsdy}{|y|}$$
$$+\int_{1/2<|y|\le 1}\int_0^1 (|F(s,x-y)|+|F_s'(s,x-y)|)\,dsdy.$$

Note that in the second integral we have $1/|y|\le 2$. To end the proof, we notice that

$$\int_{\mathbb{R}^3}|g(y)|\frac{dy}{|y|}\lesssim \int_{\mathbb{R}^3}|g'(y)|\,dy,\qquad \forall g\in C_c^1(\mathbb{R}^3).$$

We introduce polar coordinates and use integration by parts to prove this estimate.

ii) Next we suppose $\operatorname{supp} F\subset\{|y|>s/3\}$. Set $G(t,r):=\sup_{|y|=r}|F(t,y)|$. By the Sobolev inequality on the sphere, we have

$$|G(t,r)|\lesssim \int_{\mathbb{S}^2}\sum_{|I|\le 2}|(\Omega^I F)(t,r\omega)|\,dS_\omega$$

where Ω^I denotes a product of rotations Ω_{ij}, $i,j>0$. Now we let U be the solution to $-\Box U = G(t,|x|)$ with zero data ($G(t,0)=0$ because of the support of F, no singularity). By (7.25), it follows that

$$|w(t,x)|\le U(t,x)=\frac{1}{4\pi}\int_{|y|<t}\frac{G(t-|y|,|x-y|)}{|y|}\,dy$$

which is rotationally symmetric in x (take substitution $z=Ly$ where L is a rotation). Writing $\Box$ in polar coordinates, we have

$$-\Box U = U_{tt}-U_{rr}-r^{-1}U_r = G \Longrightarrow (rU)_{tt}-(rU)_{rr}=rG.$$

Using the Duhamel's principle, we have

$$|x||w(1,x)|\le rU=\frac{1}{2}\int_0^1\int_{r-1+s}^{r+1-s}\rho G(s,\rho)\,d\rho ds\lesssim \int_0^1\int_0^\infty \rho G(s,\rho)\,d\rho ds$$
$$\lesssim \sum_{|I|\le 2}\int_0^1\int_0^\infty\int_{\mathbb{S}^2}\rho|(\Omega^I F)(s,\rho\omega)|\,dS_\omega d\rho ds\lesssim \sum_{|I|\le 2}\int_0^1\int_{\mathbb{R}^3}|(\Omega^I F)(s,y)||y|^{-1}\,dyds.$$

Note that $|y|^{-1}$ in the last estimate can be replaced by $(s+|y|)^{-1}$ because $F\equiv 0$ in $|y|<s/3$ and because $|y|+s\sim|y|$ whenever $|y|\ge s/3$. This finishes the proof when $|x|\ge 1/4$. When $|x|<1/4$, if $(1-|y|,x-y)\in\operatorname{supp}F$, we have $|x-y|\ge(1-|y|)/3$ and thus $|y|\ge|x-y|-|x|\ge(1-|y|/3)-|x|$ $\Longrightarrow 4|y|\ge 3(1-|x|)>1/4$. It follows from (7.25) that

$$|w(1,x)|\lesssim\int_{|y|<1}|F(1-|y|,x-y)|\,dy.$$

By viewing $F(1-|y|, x-y)$ as the value of $h(\tau) = F(\tau(1-|y|), \tau(x-y))$ at $\tau = 1$, we apply the fundamental theorem of calculus to obtain

$$\begin{aligned}|F(1-|y|, x-y)| &\lesssim \int_1^{16/15} |h(\tau)| + |h'(\tau)| \, d\tau \\ &\lesssim \int_1^{16/15} |F(\tau(1-|y|), \tau(x-y))| + |(\tau^{-1}SF)(\tau(1-|y|), \tau(x-y))| \, d\tau.\end{aligned}$$

The Jacobian of the the map $(\tau, y) \mapsto (\tau(1-|y|), \tau(x-y))$ is $\tau^3(1-(x\cdot y)/|y|) \geq 3/4$ whenever $\tau \in [1, 16/15]$ and $|y| \geq 1/16$. It follows that

$$\begin{aligned}|w(1,x)| &\lesssim \int_{|y|<1} \int_1^{16/15} |F(\tau(1-|y|), \tau(x-y))| + |(SF)(\tau(1-|y|), \tau(x-y))| \, d\tau dy \\ &\lesssim \int_0^1 \int_{|z|<2} |F(s,z)| + |(SF)(s,z)| \, dsdz.\end{aligned}$$

This finishes the proof of (7.24) in the case $\operatorname{supp} F \subset \{|y| > s/3\}$.

To end the proof of (7.24), we choose $\psi \in C_c^\infty(\mathbb{R}^3)$ such that $\psi|_{B(0,1/3)} = 1$ and $\psi|_{\mathbb{R}^3 \setminus B(0,1/2)} = 0$. Then, $\psi(y/s)F$ satisfies i) and $(1-\psi(y/s))F$ satisfies ii). To end the proof, we also note that $Z^I(\psi(y/s)) = O(1)$.

Finally, we return to (7.23). If $\operatorname{supp} F \subset \{s+|y| \geq 1\}$, then we have $\operatorname{supp} w \subset \{s+|y| \geq 1\}$ by the finite speed of propagation. In this case (7.23) follows from (7.24). Moreover, if $\operatorname{supp} F \subset \{s+|y| < 2\}$, then we make a translation $(s, y) \mapsto (s, y+(3,0,0))$ and apply the case already proved. The translations introduce constant vector fields. Combining these two cases by a partition of unity yields (7.23) in full generality. $\square$

Remark 7.8.1. This lemma and its proof are from Lemma 6.6.8, [H97]. There is another version of (7.23) proved in [Sog08] (Theorem II.1.5) where the author avoids using Lorentz boosts.

7.6 Alternative proof: II

Now let us discuss the proof in [Sog08]. In addition to the usual energy estimate, the author proved the following sharp weighted energy estimate.

Proposition 7.9. *Suppose v solves $\square v = G$ in $\mathbb{R}_+^{1+3}$. Then there is a uniform constant B such that*

$$(\ln(2+t))^{-1/2} \left\| \langle r \rangle^{-1/2} v' \right\|_{L^2([0,t]\times\mathbb{R}^3)} \leq B \left(\left\| v'(0) \right\|_{L^2} + \int_0^t \|G(\tau)\|_{L^2} \, d\tau \right).$$

We remark that this estimate is related to *local energy estimates* for wave equations.

Using this energy estimate and the estimate (7.23), we are able to finish the proof. You can check Section II.3 and II.5 in [Sog08] for more details.

8 Hörmander's asymptotic equations

In this section, we only consider three space dimensions ($\mathbb{R}^{1+3}$). For equations like $\Box u = u_t^2$ and $\Box u = u_t u_{tt}$, we know that any nontrivial global solutions with C_c^∞ data must blow up in finite time. For equations like (6.1) satisfying the null condition, we know that there exists a global solution as long as the initial data belong to C_c^∞ and are sufficiently small. The following question then arises naturally.

Question. Given an arbitrary quasilinear wave equation or an arbitrary system of quasilinear wave equations, how can we predict whether it has small data global existence or not?

To answer this question, we will introduce a type of asymptotic equations for quasilinear wave equations. This type of asymptotic equations was first introduced by Hörmander [H97, H87, H91], so I will also call it *Hörmander's asymptotic equations*.

8.1 Motivation

Let us consider the linear wave equation $\Box u = 0$ in $\mathbb{R}^{1+3}$ with C_c^∞ data (u^0, u^1) at time 0. For simplicity, we assume that $u^0 \equiv 0$ for all $x \in \mathbb{R}^3$ and $u^1 \equiv 0$ for all $|x| \geq 1$. Using the results in Section 2, we know that there exists a global C^∞ solution u. Now, can we say anything about the asymptotic behavior of u as $t \to \infty$?

Theorem 8.1. *There exists a smooth function $F = F(q, \omega) : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$, such that $F_0(q, \omega) = 0$ whenever $|q| \geq 1$, and*

$$|u(t,x) - r^{-1}F_0(r - t, \omega)| \lesssim r^{-2}, \qquad \forall t, r \gtrsim 1. \tag{8.1}$$

Proof. By the finite speed of propagation, there is nothing to prove when $|r - t| \geq 1$. From now on we shall assume $|r - t| < 2$ and $r, t \gtrsim 1$. Moreover, because of the rotation symmetry, we only need to prove (8.1) at $x = (r, 0, 0)$ where $r \geq 0$.

Now, by the Kirchoff's formula (2.2), we have

$$u(t,x) = \frac{t}{4\pi} \int_{\mathbb{S}^2} u^1(x + t\omega)\, dS_\omega = \frac{1}{4\pi t} \int_{\partial B(x,t)} u^1(y)\, dS_y.$$

We can replace $\partial B(x,t)$ here with $\partial B(x,t) \cap B(0,1)$ since $u^1 \equiv 0$ for all $|y| \geq 1$. In $\partial B(x,t) \cap B(0,1)$, if $r, t \gtrsim 1$ we must have $y_1 < x_1 = r$. This is because $y \in \partial B(x,t) \cap B(0,1)$ implies that $r = |x| \geq |x - y| - |y| \geq t - 1 \geq 1 \geq |y_1|$. So we only need to take the integral on the lower semisphere. This gives us

$$u(t,x) = \frac{1}{4\pi} \int_{y' \in \mathbb{R}^2,\ |y'| < t} \frac{u^1(r - \sqrt{t^2 - |y'|^2}, y')}{\sqrt{t^2 - |y'|^2}}\, dy' = \frac{1}{4\pi} \int_{y' \in \mathbb{R}^2,\ |y'| < 1} \frac{u^1(r - \sqrt{t^2 - |y'|^2}, y')}{\sqrt{t^2 - |y'|^2}}\, dy'.$$

Here we use $u^1 \equiv 0$ whenever $|y| \geq 1$. Note that by setting $q = r - t \in [-1, 1]$, we have

$$\sqrt{t^2 - |y'|^2} = r\sqrt{(1 - q/r)^2 - |y'|^2/r^2}, \qquad r - \sqrt{t^2 - |y'|^2} = \frac{2q + (|y'|^2 - q^2)/r}{1 + \sqrt{(1 - q/r)^2 - |y'|^2/r^2}}.$$

They are both smooth functions of $(q, 1/r, y')$ (it is easy to check that $(1 - q/r)^2 - |y'|^2/r^2 \geq C^{-1} > 0$ for some constant $C > 1$). In summary, we conclude that $u(t,x) = r^{-1}F(q, 1/r)$ where F is a smooth function. For other $x \in \mathbb{R}^3$, we can also show that $u(t,x) = r^{-1}F(q,\omega,1/r)$ for a certain smooth function F.

Finally, from our derivation, we notice that $F(q,\omega,0)$ is well-defined and that $(q,\omega,z) \mapsto F(q,\omega,z)$ is smooth in $\mathbb{R} \times \mathbb{S}^2 \times [0, C^{-1}]$. By setting $F_0(q,\omega) = F(q,\omega,0)$, we conclude (8.1) from the Taylor's theorem.

$\square$

Remark 8.1.1. We have a few remarks about this theorem.

- Such a smooth function F_0 exists for any initial data $(u^0, u^1) \in C_c^\infty$. It is called the *Friedlander radiation field.*
- In fact we can show

$$|Z^I(u(t,x) - r^{-1}F_0(r-t,\omega))| \lesssim_I r^{-2}, \qquad \forall t, r \gtrsim 1, \quad \forall I. \tag{8.2}$$

  Here Z^I is a product of commuting vector fields.
- One way to compute Friedlander radiation fields is to use the *Radon transform.* We refer to, for example, [Eva10].

Nonrigorously speaking, from Theorem 8.1 we have

$$u(t,x) \approx r^{-1}F_0(r-t,\omega), \qquad t \to \infty.$$

This motivates us to use the following ansatz for a general quasilinear wave equation:

$$u(t,x) \approx \varepsilon r^{-1} U(s,q,\omega). \tag{8.3}$$

Here U is a function of $(s,q,\omega) = (\varepsilon \ln t, r-t, \omega)$. We use the factor $\varepsilon \ll 1$ because only small solutions are considered. Later we will derive asymptotic equations for U in the coordinate set (s,q,ω). Note that $s \leq c$ is equivalent to $t \leq e^{c/\varepsilon}$, so we relate the local existence of U with the almost global existence of u together.

8.2 Derivation

Let us now derive the Hörmander's asymptotic equations. For simplicity, we only do the derivation for (6.1)

$$g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = F(u').$$

Moreover, we make the following assumptions.

- $t = r \to \infty$.
- $Z^I u = O(\varepsilon t^{-1})$, $\forall I$.
- $(\partial_s, \partial_q, \partial_\omega)^k U = O(1)$, $\forall k \geq 0$.
- $g^{\alpha\beta}(u') = m^{\alpha\beta} + g_0^{\alpha\beta\lambda} u_\lambda + O(|u'|^2)$.

- $F(u') = f_0^{\alpha\beta} u_\alpha u_\beta + O(|u'|^3)$.

In fact, an exact solution u to (6.1) might not satisfy these assumptions. For example, sometimes we might expect $Z^I u = O(\varepsilon t^{-1+C\varepsilon})$. This, however, does not matter, because those differences are usually negligible. I should also emphasize that the derivations below would not be rigorous.

Now we plug $u = \varepsilon r^{-1} U$ into the (6.1). We have

$$g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = \Box u + g_0^{\alpha\beta\lambda} u_\lambda u_{\alpha\beta} + O(\varepsilon^2 t^{-3}).$$

Here

$$\Box u = -\varepsilon r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)U + r^{-2}\Delta_\omega u$$

where $\Delta_\omega u = \sum_{i<j} \Omega_{ij}^2 u = O(\varepsilon t^{-1})$. Thus,

$$\begin{aligned}
\Box u &= -\varepsilon r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)U + O(\varepsilon t^{-3}) \\
&= -\varepsilon r^{-1}(\partial_t + \partial_r)(\varepsilon t^{-1} U_s - 2U_q) + O(\varepsilon t^{-3}) \\
&= -\varepsilon r^{-1}(-\varepsilon t^{-2} U_s + \varepsilon^2 t^{-2} U_{ss} - 2\varepsilon t^{-1} U_{sq}) + O(\varepsilon t^{-3}) \\
&= 2\varepsilon^2 (rt)^{-1} U_{sq} + O(\varepsilon t^{-3}).
\end{aligned}$$

Besides, we have

$$\begin{aligned}
u_t &= \varepsilon^2 (tr)^{-1} U_s - \varepsilon r^{-1} U_q = -\varepsilon r^{-1} U_q + O(\varepsilon^2 t^{-2}), \\
u_j &= -\varepsilon r^{-2}\omega_j U + \varepsilon r^{-1}(U_q \omega_j + U_\omega \cdot \partial\omega) = \varepsilon r^{-1}\omega_j U_q + O(\varepsilon t^{-2}), \\
u_{tt} &= -\varepsilon r^{-1}(\varepsilon t^{-1} U_{sq} - U_{qq}) + O(\varepsilon^2 t^{-2}) = \varepsilon r^{-1} U_{qq} + O(\varepsilon^2 t^{-2}), \\
u_{tj} &= \varepsilon r^{-1}\omega_j(\varepsilon t^{-1} U_{sq} - U_{qq}) + O(\varepsilon t^{-2}) = -\varepsilon r^{-1}\omega_j U_{qq} + O(\varepsilon t^{-2}), \\
u_{jk} &= \partial_k(\varepsilon r^{-1}\omega_j) U_q + \varepsilon r^{-1}\omega_j(U_{qq}\omega_k + U_{q\omega} \cdot \partial\omega) + O(\varepsilon t^{-2}) \\
&= \varepsilon r^{-1}\omega_j\omega_k U_{qq} + O(\varepsilon t^{-2}).
\end{aligned}$$

In summary, by writing $\widehat{\omega}_0 = -1$ and $\widehat{\omega}_j = \omega_j$, $j = 1, 2, 3$, we have

$$u_\alpha = \varepsilon r^{-1}\widehat{\omega}_\alpha U_q + O(\varepsilon t^{-2}), \qquad u_{\alpha\beta} = \varepsilon r^{-1}\widehat{\omega}_\alpha\widehat{\omega}_\beta U_q + O(\varepsilon t^{-2}).$$

It follows that

$$\begin{aligned}
g_0^{\alpha\beta\lambda} u_\lambda u_{\alpha\beta} &= G(\omega)\varepsilon^2 r^{-2} U_q U_{qq} + O(\varepsilon^2 t^{-3}), \\
f_0^{\alpha\beta} u_\alpha u_\beta &= F(\omega)\varepsilon^2 r^{-2} U_q^2 + O(\varepsilon^2 t^{-3})
\end{aligned}$$

where $F(\omega) = f_0^{\alpha\beta}\widehat{\omega}_\alpha\widehat{\omega}_\beta$, $G(\omega) = g_0^{\alpha\beta\lambda}\widehat{\omega}_\alpha\widehat{\omega}_\beta\widehat{\omega}_\lambda$. In summary, for $t = r \to \infty$, we have

$$\varepsilon^2 r^{-2}(2U_{sq} + G(\omega)U_q U_{qq} - F(\omega)U_q^2) = O(\varepsilon t^{-3}).$$

We thus obtain the asymptotic equation

$$2U_{sq} + G(\omega)U_q U_{qq} - F(\omega)U_q^2. \tag{8.4}$$

Using the same derivation, we can derive Hörmander's asymptotic equations for a general system of quasilinear wave equations.

Example 8.2. The asymptotic equation for $\Box u = u_t^2$ is $2U_{sq} = U_q^2$. Its solution is

$$\frac{1}{U_q(s,q,\omega)} = \frac{1}{U_q(0,q,\omega)} - \frac{s}{2}.$$

If $U_q(0,q,\omega) > 0$, then there is a blowup at $s = 2/U_q(0,q,\omega)$. But if $U|_{s=0} \in C_c^\infty$ is nonzero, then we must have $U_q|_{s=0} > 0$ for some (q,ω). So the we get a finite-time blowup result for $2U_{sq} = U_q^2$.

Example 8.3. The asymptotic equation for $\Box u = u_t u_{tt}$ is $2U_{sq} + U_q U_{qq} = 0$. This is the Burger's equation. Again, we have a finite-time blowup result for the Burger's equation with nonzero C_c^∞ data. See [H97].

Example 8.4. If (6.1) satisfies the null condition, then the associated asymptotic equation is $U_{sq} = 0$. Of course, we can find a global solution to this asymptotic equation.

Based on these examples, we notice that there seems to be a connection between the long time existence results for quasilinear wave equations and the long time existence results for associated Hörmander's asymptotic equations. In fact, there is a conjecture about this connection.

Definition 8.5. Consider a general system of quasilinear wave equations. Suppose that for at data at $s = 0$ decaying sufficiently fast in q, the corresponding system of Hörmander's asymptotic equations has a global solution U for all $s \geq 0$. Also suppose that U and all its derivatives grow at most exponentially ($\lesssim e^{Cs}$). Then we say the original system of quasilinear wave equations satisfies the *weak null condition*.

There is a conjecture by Lindblad and Rodnianski, which states that the weak null condition is sufficient for small data global existence. This conjecture is open up till today. But we have several examples supporting this conjecture.

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