STABILITY FOR GEOMETRIC AND FUNCTIONAL INEQUALITIES

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The celebrated isoperimetric inequality states that, for a measurable set $S \subset \mathbb{R}^n$, the inequality

$$\text{per}(S) \geq n\text{vol}(S)^{\frac{n-1}{n}}\text{vol}(B_1)^{\frac{1}{n}}$$

holds, where per(S) denotes the perimeter (or surface area) of S, and equality holds if and only if S is an euclidean ball. This result has many applications throughout analysis, but an interesting feature is that it can be obtained as a corollary of a more general inequality, the Brunn–Minkowski theorem: if $A, B \subset \mathbb{R}^n$, define $A + B = \{a + b, a \in A, b \in B\}$. Then

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

Here, equality holds if and only if A and B are homothetic and convex. A question pertaining to both these results, that aims to exploit deeper features of the geometry behind them, is that of stability: if S is close to being optimal for the isoperimetric inequality, can we say that A is close to being a ball? Analogously, if A, B are close to being optimal for Brunn–Minkowski, can we say they are close to being compact and convex?

These questions, as stand, have been answered only in very recent efforts by several mathematicians. In this talk, we shall outline these results, with focus on the following new result, obtained jointly with A. Figalli and K. Böröczky. If f, g are two non-negative measurable functions on $\mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is measurable such that

$$h(x + y) \geq f(2x)^{1/2}g(2y)^{1/2},\ \forall x, y \in \mathbb{R}^n,$$

then the Prekopa–Leindler inequality asserts that

$$\int h \geq \left(\int f\right)^{1/2} \left(\int g\right)^{1/2},$$

where equality holds if and only if h is log-concave, and f, g are 'homothetic' to h, in a suitable sense. We prove that, if $\int h \leq (1 + \varepsilon) \left(\int f\right)^{1/2} \left(\int g\right)^{1/2}$, then f, g, h are $\varepsilon^{\gamma_n}-$ L^1-close to being optimal. We will discuss the general idea for the proof and, time-allowing, discuss on a conjectured sharper version.