University of Bonn
Prof. Dr. Christoph Thiele
Dr. Olli Saari

Due on Friday 3 July. On Monday 6 of July there will be no exercise session.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

- *A filtration* is sequence of sigma-algebras $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \mathcal{F}$ holding for all $k \geq 0$.
- *A stochastic process* adapted to a filtration $(\mathcal{F}_k)$ is a sequence of random variables (X_n) so that X_n is $\mathcal{F}_n$ measurable and $E|X_n| < \infty$.
- *Conditional expectation* w.r.t. the sigma algebra $\mathcal{F}_k$ of a random variable $X \in L^1(\Omega, \mathcal{F}, \mu)$ is another random variable $Y = E(X|\mathcal{F}_k)$ that satisfies $E(YZ) = E(XZ)$ for all bounded $\mathcal{F}_k$ measurable random variables.
- *A martingale* is a stochastic process (X_n) such that $E(X_n|\mathcal{F}_m) = X_m$ whenever $m \leq n$.
- *Stopping time* (relative to a filtration) is a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{T \leq m\} \in \mathcal{F}_m$.

Problem 1. Consider a non-negative Radon measure on $[0, 1)$ so that its martingale extension satisfies $L^1\ell^1 F < \infty$.

(a) Show that the martingales as defined in the lectures are an instance of the definition of martingale as written above, that is, identify a filtration, a stochastic process and a conditional expectation so that the martingale property from the lecture notes is equivalent with the process you defined being a martingale.

(b) Let $\lambda > 0$ and $A = \bigcup\{I : F(I) > \lambda\}$. Show that

$$1_A(x) = \sum_{k=0}^{\infty} \sum_{I \in \mathcal{D}_{-k}} 1_I(x) 1_{T(x)=k}(x)$$

where T is a stopping time (referring to the setting from part a).

Problem 2 (Doob's first martingale inequality)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and (X_n) a martingale.

(a) Let T be a bounded stopping time. Show that $EX_T = EX_0$.

(b) Let (X_n) be a non-negative martingale and $\lambda > 0$. Prove that

$$\lambda\mu\left(\max_{0\leq n\leq N} X_n > \lambda\right) \leq EX_N.$$