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Due on Friday 12 June.

Problem 1. Let m be an absolutely continuous Radon measure and M its martingale extension.

(a) Show that there exists a measurable function f on the unit interval so that for all continuous φ

$$\langle \varphi, m\rangle = \int_0^1 f(x)\varphi(x)\,dx.$$

Hence we can identify m with a measurable function f_m (hint: use the results from the section *Martingale average convergence*).

(b) Assume that m' is another Radon measure with martingale extension M' satisfying $\sup_I M'(I) < \infty$. Show that

$$\lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} |I| M(I) M'(I)$$

exists.

(c) Assume that there is a constant $c \in (0,1)$ such that $c \le M(I) \le c^{-1}$ for all I. Show that there exists a martingale G such that for almost all x

$$\lim_{\substack{|I|\to 0\\ x\in I}} F(I)G(I) = 1.$$

Problem 2. Recall that a Radon measure with martingale extension F is in BMO if $L^\infty\ell^2\Delta F < \infty$ and in Hardy space if $L^1\ell^2\Delta F < \infty$ and $F([0,1)) = 0$.

(a) Show that if F and G are martingale extensions of Radon measures in BMO and Hardy space, respectively, then

$$\lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} |I| F(I) G(I)$$

exists.

(b) Show that if F is martingale extension of a finite linear combination of characteristic functions m with $F([0,1)) = 0$, then

$$\|m\|_1 \le C L^1\ell^2\Delta F$$

for $C > 0$ independent of m. (hint: outer Hölder)

(c) Show that there exists a sequence of absolutely continuous Radon measures m_j with martingale extensions F_j satisfying $F_j([0,1)) = 0$ so that

$$\frac{L^1\ell^2\Delta F}{\|m_j\|_1} \to \infty \quad j \to \infty.$$

(d) Let m be a finite linear combination of characteristic functions of dyadic intervals and F its martingale extension. Show that

$$L^1\ell^\infty F \le C \int |m(x)| \log(e + |m(x)|)\,dx$$

(hint: use what we did with dyadic maximal function).

(e) Show that the embedding

$$L^1\ell^\infty F \le C\|m\|_{L^1}$$

cannot hold with C uniform in m.