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Due on Friday 15 May. Hand in in groups of two or three.

Problem 1. Let $I_{i,j} = [3^{-j}i, 3^{-j}(i+1)]$ and define the standard Cantor set through

$$C_k = \bigcap_{j=1}^{k} \bigcup_{i=0}^{(3^j-1)/2} I_{2i,j}, \quad C = \bigcap_{j=1}^{\infty} \bigcup_{i=0}^{(3^j-1)/2} I_{2i,j}.$$

Define $f_k : [0,1] \to [0,1]$ as a continuous function with

$$f_k(0) = 0, \qquad f_k'(x) = (3/2)^k 1_{C_k}(x), \ x \notin \partial C_k.$$

(a) Show that C is closed and has Lebesgue measure zero.

(b) Show that f_k converge uniformly to a continuous and increasing function f with $f(0) = 0$ and $f(1) = 1$.

(c) Show that f is almost everywhere differentiable and $f'(x) = 0$ holds for almost every $x \in [0,1]$.

(d) Show that f_k' converge weakly* to a positive Radon measure μ but f_k' do not converge in L^1 norm.

(e) Show that for all $0 \le a \le b \le 1$ it holds

$$f(b) - f(a) = \int_a^b d\mu.$$

Problem 2. Let $\mathcal{D}$ be the collection of dyadic intervals on the real line. Define

$$M_d f(x) = \sup_{I \in \mathcal{D}} \frac{1_I(x)}{|I|} \int_I |f(y)|\, dy, \quad M_c f(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |f(y)|\, dy, \quad Mf(x) = \sup_{-\infty<a<x<b<\infty} \frac{1}{b-a} \int_a^b |f(y)|\, dy.$$

(a) Let $\mathcal{I}$ be any finite collection of open intervals (not necessarily dyadic). Show that there exists two subfamilies $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I}$ such that

$$\bigcup_{I \in \mathcal{I}} I = \bigcup_{i=1}^{2} \bigcup_{I \in \mathcal{I}_i} I, \quad \sum_{I \in \mathcal{I}_i} 1_I(x) \le 1, \quad x \in \mathbb{R}, i = 1, 2.$$

(b) Let $f \in L^1(\mathbb{R})$. Show that

$$|\{x \in \mathbb{R} : M_d f(x) > \lambda\}| \le \frac{1}{\lambda} \int_{\{x \in \mathbb{R} : M_d f(x) > \lambda\}} |f(y)|\, dy.$$

(c) Show that

$$|\{x \in \mathbb{R} : Mf(x) > \lambda\}| \le \frac{2}{\lambda} \int_{\{x \in \mathbb{R} : Mf(x) > \lambda\}} |f(y)|\, dy.$$

(d) What kind of inequality would you expect for M_c? What do you get by same arguments as in the cases above?