University of Bonn
Prof. Dr. Christoph Thiele
Dr. Olli Saari

Due on Friday 8 May 2020. Hand in in groups of two or three.

Problem 1. Let H be a complex Hilbert space.

(a) Let $T : H \to H$ be a bounded linear operator. Prove that there exists an adjoint operator $T^* : H \to H$ so that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ with $\|T^*\|_{H\to H} = \|T\|_{H\to H}$. (We identify the Hilbert space with its dual)

(b) Show that $\|Ux\| = \|x\|$ for all x if and only if $U^* = U^{-1}$. In this case U is said to be unitary. Show that the inverse of a unitary operator is unitary.

(c) Let $\sigma(T) = \{\lambda \in \mathbb{C} | \exists x \in H \setminus \{0\} : Tx = \lambda x\}$. An operator T is self-adjoint if $T = T^*$. Show that if T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$ and if T is unitary, then $\sigma(T) \subset \partial\mathbb{D}$ (unit circle).

(d) The Caley transform C is defined as

$$CT = (T - i)(T + i)^{-1}.$$

Show that it maps every bounded self-adjoint operator to a unitary operator.

(e) Let $A = \{f \in C(\mathbb{R}) : |\lim_{|t|\to\infty} f(t)| < \infty\}$. Conclude that for any self-adjoint T and $f, g \in A$ there exists bounded linear operators $f(T)$ and $g(T)$ such that $f(T) \circ g(T) = (fg)(T)$ and $f(T) + g(T) = (f + g)(T)$.

Problem 2. Define *adjacent systems of dyadic intervals* by

$$\mathcal{D}^\alpha = \{2^{-k}([0, 1) + m + (-1)^k\alpha/3), m, k \in \mathbb{Z}\},$$

where $\alpha = 0, 1, 2$. Note that $\mathcal{D}^0$ is the usual system of dyadic intervals.

(a) Show that each $\mathcal{D}^\alpha$ is nested in the sense that for $I, J \in \mathcal{D}^\alpha$ we have $I \cap J \in \{I, J, \emptyset\}$.

(b) Show that for every interval $I = [a, b] \subset \mathbb{R}$ there exist $\alpha \in \{0, 1, 2\}$ and $J \in \mathcal{D}^\alpha$ such that $I \subset J$ and $|J| \le 4|I|$.

(c) Let $f \in L^1(\mathbb{R})$. The Hardy–Littlewood maximal functions are defined by

$$M_c f(x) := \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |f(y)|\, dy, \quad M_{d,\alpha} f(x) := \sup_{I \ni \mathcal{D}^\alpha} \frac{1_I(x)}{|I|} \int_I |f(y)|\, dy$$

Fix α. Show that there is no absolute constant $C > 0$ such that $M_c f(x) < C M_{d,\alpha} f(x)$ would hold for all x and all $f \in L^1(\mathbb{R})$.

(d) Show that

$$M_c f(x) \le C_1 \sum_{\alpha=0}^{2} M_{d,\alpha} f(x) \le C_2 M_c f(x)$$

where C_1 and C_2 are absolute constants.