Harmonic Analysis
Lecture notes

Prof. Dr. Christoph Thiele*
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Universität Bonn

Contents

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*Notes by Marco Fraccaroli and João Pedro Ramos with supplements by Pavel Zorin-Kranich.

1 What is Harmonic in Harmonic Analysis: the Poisson extension

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The problem we want to start our course with is the following: what is a function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$?
The literal meaning of assignment of a value in the codomain to every point in the domain presents some problems:

1. the set of functions is too large. Already if the target set were $\{0, 1\}$ then the set of functions would be given by characteristic functions of subsets of $\mathbb{R}$, obtaining a cardinality strictly greater than the one of the domain;

2. there is no reasonable integration theory of such functions.

Mathematicians in different areas give different solutions: some Topologists restrict their attention to continuous functions (solving the firs problem, being enough to define values for a dense subset of the domain), some Differential Topologists to ∞-differentiable functions, etc.
We start with the following definition.

Definition 1.1 ($\mathcal{B}$)**.** Let $\mathcal{B}$ be the set of bounded Lipschitz functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, i.e. satisfying the following conditions:

- $\exists C < \infty \ \forall x \in \mathbb{R} \quad f(x) \leq C;$ (boundedness)
- $\exists L < \infty \ \forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq L|x - y|.$ (Lipschitz condition)

Note that the Lipschitz condition implies continuity in every point, so that $\mathcal{B}$ is a subset of the set of continuous functions. Moreover the elements of $\mathcal{B}$ are Riemann integrable in every compact interval $[a, b]$: given a partition $a = x_0, \dots, x_N = b$ we consider

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \sup_{x \in [x_{n-1}, x_n]} f(x)$$
$$\sum_{n=1}^{N} (x_n - x_{n-1}) \inf_{x \in [x_{n-1}, x_n]} f(x),$$

respectively the upper Riemann sum and the lower Riemann sum associated to the partition. Notice that they are both bounded by $C(b-a)$, hence finite. In particular their difference is bounded by

$$\sum_{n=1}^{N} (x_n - x_{n-1})^2 L \leq (b - a)L \max_{1 \leq n \leq N} (x_n - x_{n-1})$$

which goes to 0 as the biggest interval in the partition goes to 0. This allows us to define

$$\int_a^b f(x)dx$$

as the limit of both of these sums in the previous sense.[1] Moreover we define

$$\int_{-\infty}^{\infty} f(x)dx := \sup_{a<b} \int_a^b f(x)dx.$$

where the supremum may be ∞. Finally we denote with $\mathcal{B}_1$ the subset of $\mathcal{B}$ of all functions f such that $\int_{-\infty}^{\infty} f(x)dx < \infty$.

Example 1.2.

$$\frac{1}{1+x^2} \in \mathcal{B}_1, \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

This example is particularly useful in view of our starting problem. If we define the *Poisson kernel* $\mathcal{P}_t(y) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{P}_t(y) := \frac{1}{\pi} \frac{t}{t^2 + y^2},$$

then for every fixed $t > 0$ the function $\mathcal{P}_t(y) \in \mathcal{B}_1$, namely

$$\int_{-\infty}^{\infty} \mathcal{P}_t(y)dy = 1.$$

We "embed" a general $f \in \mathcal{B}$ into the upper half plane by defining for $(x, t) \in \mathbb{R} \times \mathbb{R}_{>0}$ the function

$$F(x,t) := \int_{-\infty}^{\infty} f(x-y)\mathcal{P}_t(y)dy = \int_{-\infty}^{\infty} f(y)\mathcal{P}_t(x-y)dy,$$

which we call the *Poisson extension* of f. We claim that F still contains the information of f but it is the "right" way to look at it.
Computing some derivatives of $\mathcal{P}_t(y)$ we obtain

$$\begin{aligned} \partial_y \mathcal{P}_t(y) &= \frac{1}{\pi} \frac{-2yt}{(t^2+y^2)^2} \\ \partial_y^2 \mathcal{P}_t(y) &= \frac{1}{\pi} \frac{6ty^2 - 2t^3}{(t^2+y^2)^3} \\ \partial_t \mathcal{P}_t(y) &= \frac{1}{\pi} \frac{y^2 - t^2}{(t^2+y^2)^2} \\ \partial_t^2 \mathcal{P}_t(y) &= \frac{1}{\pi} \frac{-6ty^2 + 2t^3}{(t^2+y^2)^3}, \end{aligned}$$

[1] One should show that this definition depends only on the property that the biggest interval in the partition goes to 0 and not on the choice of the points of the partition. This is straight-forward once observed that taking a refinement of a partition the lower Riemann sum associated increases and the upper one decreases. Therefore we have that for any two partitions the lower Riemann sum associated to the first one is less or equal to the upper Riemann sum associated to the second one (by passing to a common refinement).

so that

$$\Delta \mathcal{P}_t(y) := (\partial_y^2 + \partial_t^2)\mathcal{P}_t(y) = 0.$$

Therefore by good properties of integrability of partial derivatives of $\mathcal{P}_t(y)$ we have

$$\partial_x F(x,t) = \int_{-\infty}^{\infty} f(x-y)\partial_y \mathcal{P}_t(y)dy$$

etc.

In particular

$$\Delta F = 0$$

and being F a solution of the Laplace equation $\Delta u = 0$ we call it *harmonic*. Now we want to prove what we stated above, that we can recover f from F. We claim that

Claim 1.3. *The pointwise limit of F to $\mathbb{R} \times \{0\}$ is the function given by $(x,0) \mapsto f(x)$.*

Proof. It is enough to show it at $(0,0)$ so that the result follows by a simple translation argument.
For $0 < \varepsilon < 1$ set $|x|, t < \varepsilon^2$.

$$|F(x,t) - f(0)| \leq |\int_{|y|<\varepsilon} (f(x-y) - f(0))\mathcal{P}_t(y)dy| +$$

$$+ |\int_{|y|\geq\varepsilon} (f(x-y) - f(0))\mathcal{P}_t(y)dy| \leq$$

$$\leq 2L\varepsilon + 2C\int_{|y|\geq\varepsilon} \mathcal{P}_t(y)dy \leq 2L\varepsilon + 4C\varepsilon$$

where the last inequality is due to our restriction on t that gives

$$\mathcal{P}_t(y) \leq \frac{\varepsilon^2}{y^2},$$

which is an integrable function on the domain $\{|y| \geq \varepsilon\}$ with integral 2. □

Remark 1.4. In the proof the Lipschitz condition on f is not necessary, continuity of f is enough and the argument would be slightly more technical.

Summarizing the properties of F we have:

- F is harmonic in $\mathbb{R} \times \mathbb{R}_{>0}$, has a continuous extension to $\mathbb{R} \times \mathbb{R}_{\geq 0}$ and coincides with the function $(x,0) \mapsto f(x)$ on $\mathbb{R} \times \{0\}$;

- F is nonnegative and bounded by the same constant of f.

Claim 1.5. *F is the unique function with these properties.*

Remark 1.6. The idea is that we get a bijection of $\mathcal{B}$ with the set of functions with the properties stated above. Then we can forget about the starting definition of functions we used for f and consider this one as our definition. The extension of the idea of function will be given by not asking for the condition on F to have a continuous extension to $\mathbb{R} \times \mathbb{R}_{\geq 0}$.

Proof. We are going to use Complex Analysis machinery. Moreover we are not going to use the nonnegativity condition of f nor asking for the same property of F.
First of all we observe that it is enough to consider the case $f \equiv 0$. In fact if $f \neq 0$ had two such extensions to the upper half plane $F_1 \neq F_2$ then the function $f \equiv 0$ would have two too, the function constantly 0 and $F_1 - F_2 \neq 0$.
Suppose F is an extension of $f \equiv 0$ satisfying the wanted properties.
Through *Schwarz reflection principle* we extend F to a continuous and harmonic function[2] in $\mathbb{R}^2$ by setting for $t < 0$

$$F(x, t) = -F(x, -t).$$

Then there exists a holomorphic function h on $\mathbb{R}^2 = \mathbb{C}$ such that

$$F = \mathrm{Re}(h),$$

namely the primitive of $\partial_x F - i\partial_t F$.
Therefore the function e^h is holomorphic and bounded on $\mathbb{C}$ because

$$|e^h| = e^{\mathrm{Re}(h)} = e^F.$$

By Liouville's Theorem we get that e^h has to be a constant, thus also h is.
Since $\mathrm{Re}(h) = f \equiv 0$ on $\mathbb{R} \times \{0\}$ we get that

$$F = \mathrm{Re}(h) = 0.$$

□

The proof leads us to consider the following set.

Definition 1.7 (M)**.** Let M be the set of all functions $F : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- two times continuously differentiable and $\Delta F = 0$ on $\mathbb{R} \times \mathbb{R}_{>0}$, i.e. it is harmonic;
- $\forall t > 0 \quad \sup_{x \in \mathbb{R}, t' > t} F(x, t') < \infty$;

[2] While the continuity of the extension is trivial, the property of being harmonic in the points of $\{0\} \times \mathbb{R}$ is a more delicate issue. We refer to page 65 of the Lecture Notes of the Complex Analysis course taught by Prof. Thiele in the Sommersemester 2016.

- $\forall t > 0 \quad \int_{-\infty}^{\infty} F(x,t)dx < \infty$.

As we are going to prove, there is a bijection with the set of all nonnegative Borel measures on $\mathbb{R}$.
We turn now to the second of the problems with the literal definition of function we stated in the beginning: what does it mean for a function to be integrable in $\mathbb{R}$?
For a nonnegative integrable function f we should expect $\int fg$ to be defined in $\mathbb{R}_{\geq 0}$ for all $g \in \mathcal{B}$ and satisfy the linearity condition

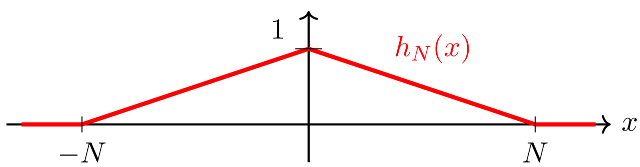
$$\int f(\lambda g_1 + g_2) = \lambda \int fg_1 + \int fg_2$$

for every $\lambda \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{B}$.
This gives us the motivation to introduce the following:

Definition 1.8 (Integral, $\mathcal{B}'$)**.** An *integral* is a map $\Lambda : \mathcal{B} \to \mathbb{R}_{\geq 0}$ such that

- (linearity)
$$\Lambda(\lambda g_1 + g_2) = \lambda\Lambda(g_1) + \Lambda(g_2)$$
for every $\lambda \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{B}$;

- (monotone convergence property)
$$\sup_{N>0} \Lambda(h_N) = \Lambda(1)$$
where h_N is the function defined by
$$h_N(x) := \begin{cases} \frac{N+x}{N} & \textit{for } -N < x \leq 0 \\ \frac{N-x}{N} & \textit{for } 0 < x \leq N \\ 0 & \textit{otherwise.} \end{cases}$$



We denote by $\mathcal{B}'$ the set of all these integrals.

The relation between $\mathcal{B}'$ and M is established by the following result:

Theorem 1.9 (Riesz-Herglotz Representation Thm)**.** *If Λ is an integral then the function defined in $\mathbb{R} \times \mathbb{R}_{>0}$ by*

$$F(x,t) := \Lambda(\mathcal{P}_t(\cdot - x))$$

is in M. Moreover this map provides a bijection from $\mathcal{B}'$ to M.

Proof. We postpone the proof to the next lecture. □

We conclude this lecture with the following example

Example 1.10. Let $\mathcal{H}$ be a Hilbert space, for example one of finite dimension is $\mathbb{C}^N$ with norm

$$\|x\| = \left(\sum_{n=1}^{N} |x_n|^2 \right)^{\frac{1}{2}}$$

and inner product

$$\langle x, y \rangle = \sum_{n=1}^{N} x_n \bar{y}_n.$$

However the result we are going to prove holds also in the case of infinite dimensional Hilbert spaces.
Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, i.e. there exists $C \in \mathbb{R}_{\geq 0}$ such that

$$\|Tx\| \leq C\|x\|,$$

and define the operator norm

$$\|Tx\|_{op} \coloneqq \sup_{\|x\|>0} \frac{\|Tx\|}{\|x\|},$$

and linearity means

$$T(\lambda g_1 + g_2) = \lambda T(g_1) + T(g_2)$$

for every $\lambda \in \mathbb{C}$ and $g_1, g_2 \in \mathcal{H}$.
We also assume T to be self-adjoint, i.e.

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for every $x, y \in \mathcal{H}$.
Then we can state the following version of a Spectral Theorem:

Theorem 1.11. *For $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$, $T + z$ is invertible and for all $x \in \mathcal{H}$ the function*

$$\{\mathrm{Im}(z) > 0\} \ni z \mapsto \mathrm{Im}(\langle x, (T+z)^{-1} x \rangle)$$

is in M.

Proof. Pick $\lambda > 0$, $\lambda > \|T\|_{op}$. We claim that $T + i\lambda$ is invertible. The idea is to use variants of the well know series for $\varepsilon < 1$

$$\frac{1}{1-\varepsilon} = \sum_{k=0}^{\infty} \varepsilon^k.$$

In the same fashion to exhibit the inverse we consider the formal series

$$(i\lambda)^{-1}\sum_{k=0}^{\infty}\left(\frac{iT}{\lambda}\right)^{k}.$$

The operator norm of the argument of the geometric series is strictly smaller than 1 by definition of λ. Therefore the series converges to an operator $\mathcal{H}\to\mathcal{H}$.

Exercise 1.12. *Prove that this formal definition gives the actual inverse, i.e.*

$$(T+i\lambda)(i\lambda)^{-1}\sum_{k=0}^{\infty}\left(\frac{iT}{\lambda}\right)^{k}x=x.$$

Formally

$$\begin{aligned}(T+z)^{-1}x &= (T+i\lambda+(z-i\lambda))^{-1}x= \\ &= (T+i\lambda)^{-1}\sum_{k=0}^{\infty}[-(z-i\lambda)(T+i\lambda)^{-1}]^{k}x \qquad (*)\end{aligned}$$

$$\begin{aligned}\|(T+i\lambda)x\|^2 &= \langle (T+i\lambda)x,(T+i\lambda)x\rangle = \\ &= \langle (T-i\lambda)(T+i\lambda)x,x\rangle = \\ &= \left\langle (T^2+\lambda^2)x,x\right\rangle \geq (\lambda^2-C^2)\langle x,x\rangle\end{aligned}$$

where $C=\|T\|_{op}$. This implies that $T+i\lambda$ is invertible and

$$\|(T+i\lambda)^{-1}\|_{op}\leq\frac{1}{\sqrt{\lambda^2-C^2}}.$$

A sufficient condition to ensure convergence of the sum in $(*)$ is

$$\frac{|z-i\lambda|}{\sqrt{\lambda^2-C^2}}=\frac{\sqrt{\mathrm{Re}(z)^2+(\lambda-\mathrm{Im}(z))^2}}{\sqrt{\lambda^2-C^2}}<1,$$

which is easily verified, once z with $\mathrm{Im}(z)>0$ is fixed, by choosing λ big enough. Therefore $z\mapsto\mathrm{Im}(\langle x,(T+z)^{-1}x\rangle)$ is harmonic because it is the imaginary part of a holomorphic function (the series expansion around $i\lambda$ is clear in $(*)$ and the series expansion around a generic z_0 can be foreseen). To prove it is nonnegative consider that for $\tilde{x}=(T+z)^{-1}x$ we have

$$\mathrm{Im}\left(\left\langle x,(T+z)^{-1}x\right\rangle\right)=\mathrm{Im}\left(\langle (T+z)\,\tilde{x},\tilde{x}\rangle\right).$$

Then

$$\begin{aligned}\frac{1}{2i}\left(\langle (T+z)\,\tilde{x},\tilde{x}\rangle-\langle \tilde{x},(T+z)\,\tilde{x}\rangle\right) &= \frac{1}{2i}\langle (z-\bar{z})\,\tilde{x},\tilde{x}\rangle = \\ &= \frac{z-\bar{z}}{2i}\langle \tilde{x},\tilde{x}\rangle=\mathrm{Im}(z)\langle \tilde{x},\tilde{x}\rangle\geq 0.\end{aligned}$$

Exercise 1.13. *Prove the two remaining properties of elements in M, namely:*

- $\forall t > 0 \quad \sup_{z\in\mathbb{C},\mathrm{Im}(z)>t} \mathrm{Im}\left(\left\langle x, (T+z)^{-1} x\right\rangle\right) < \infty$;
- $\forall t > 0 \quad \int_{-\infty}^{\infty} \mathrm{Im}\left(\left\langle x, (T+(s+it))^{-1} x\right\rangle\right) ds < \infty$.

□

2 Review

2016-10-20

Before beginning, it is useful to recall the definitions of M, $\mathcal{B}$ and $\mathcal{B}'$. First, we say that a function $F : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \in M$ if it satisfies the following conditions:

- F is twice continuously differentiable;
- $\Delta F(x,t) \equiv 0$, $\forall (x,t) \in \mathbb{R} \times \mathbb{R}_{>0}$;
- $\forall\, t > 0$; $\sup_{x\in\mathbb{R},t'>t} F(x,t') < +\infty$;
- $\forall\, t > 0$; $\int_{-\infty}^{\infty} F(x,t)dx < +\infty$.

We denote by $\mathcal{B}$ the set of bounded Lipschitz functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, i.e. satisfying the following conditions:

- $\exists C\ \forall x \in \mathbb{R} \quad f(x) \leq C$; (boundedness)
- $\exists L\ \forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq L|x-y|$. (Lipschitz condition)

Finally, we define the space $\mathcal{B}'$ as the space of $\Lambda : \mathcal{B} \to \mathbb{R}_{\geq 0}$ such that

- $\forall\, f, g \in \mathcal{B}, \forall\, \lambda > 0\ \Lambda(\lambda f + g) = \lambda\Lambda(f) + \Lambda(g)$.
- (weak monotone convergence)

$$\sup_{N>0} \Lambda(h_N) = \Lambda(1)$$

where h_N is the function defined by

$$h_N(x) := \begin{cases} \frac{N+x}{N} & \textit{for } -N < x \leq 0 \\ \frac{N-x}{N} & \textit{for } 0 < x \leq N \\ 0 & \textit{otherwise.} \end{cases}$$

In this lecture, we will prove in detail the following

Theorem 2.1 (Riesz-Herglotz Representation Theorem)**.** *If Λ is an integral then the function in $\mathbb{R} \times \mathbb{R}_{>0}$ by*

$$F(x,t) := \Lambda(\mathcal{P}_t(\cdot - x))$$

is in M. Moreover this map provides a bijection from the set of all integrals to M.

Let us work a bit more on those definitions, before we start the proof of the Theorem.

Example 2.2. We give an example of a function on the space M. Let

$$F(x,t) = \frac{1}{\pi}\frac{t}{t^2 + x^2}.$$

By what was done on the last lecture, we already know this is a twice continuously differentiable function on the upper half space $\mathbb{R} \times \mathbb{R}_{>0}$. Moreover, we also know it is harmonic. To prove it is in the space M, just two more properties are left:

- $\forall\, t > 0$; $\sup_{x\in\mathbb{R},t'>t} F(x,t') < +\infty$. In fact, for our function we have that $\sup_{x\in\mathbb{R},t'>t} F(x,t') < +\infty. = \frac{1}{\pi t}$.
- $\forall\, t > 0$; $\int_{-\infty}^{\infty} F(x,t)dx < +\infty$. In fact, one can show that $F(x,t) = \frac{1}{t}F(\frac{x}{t}, 1)$, and therefore all those numbers are equal to 1.

A more careful verification of those assertions will be left to the reader.

We now proceed to explore some of the properties of elements of the space $\mathcal{B}'$. Given a $\Lambda \in \mathcal{B}'$, we can extend it into the space $\mathcal{B} - \mathcal{B} = \{f = f_1 - f_2; f_1, f_2 \in \mathcal{B}\}$ by

$$\Lambda(f) := \Lambda(f_1) - \Lambda(f_2).$$

Of course, we must still verify that this extension is well defined. Indeed, $f_1 - f_2 = g_1 - g_2 \Rightarrow f_1 + g_2 = g_1 + f_2$. As both are positive functions, we have $\Lambda(f_1 + g_2) = \Lambda(g_1 + f_2) \Rightarrow \Lambda(f_1) - \Lambda(f_2) = \Lambda(g_1) - \Lambda(g_2)$. This concludes the verification.
For this extended Λ, we have the following continuity property:

Claim 2.3. *There is a positive constant $C = \Lambda(1) > 0$ such that, for all $f_1, f_2 \in \mathcal{B}$,*

$$|\Lambda(f_1 - f_2)| \le C \sup_{x\in\mathbb{R}} |f_1(x) - f_2(x)| =: C\|f_1 - f_2\|_\infty.$$

Proof. Define the functions $g_1 = \max\{f_1 - f_2, 0\}$, $g_2 = \max\{f_2 - f_1, 0\}$. g_1 is commonly called the *positive part* of $f_1 - f_2$, and g_2 its *negative part.* Notice

that $f_1 - f_2 = g_1 - g_2$.
Now suppose, without loss of generality, that $\Lambda(g_1 - g_2) \geq 0$. Then

$$\begin{aligned}\Lambda(f_1 - f_2) &= \Lambda(g_1 - g_2) = \Lambda(g_1) - \Lambda(g_2) \\ &\leq \Lambda(g_1) \\ &= \Lambda(\sup_{x\in\mathbb{R}} g_1) - \Lambda((\sup_{x\in\mathbb{R}} g_1) - g_1) \\ &\leq \Lambda(\sup_{x\in\mathbb{R}} g_1) = \Lambda(1)\cdot \sup_{x\in\mathbb{R}} g_1 \\ &\leq \Lambda(1) \sup_{x\in\mathbb{R}} |f_1(x) - f_2(x)|,\end{aligned}$$

where we used that $g_2 \geq 0$, $(\sup_{x\in\mathbb{R}} g_1) - g_1 \geq 0$. This proves the claim. ☐

Remark 2.4. As a corollary, if a sequence $f_n \to f$ on the norm $\|\cdot\|_\infty$, that is, if $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \to 0$ as $n \to \infty$, then $\Lambda(f_n) \to \Lambda(f)$. The details are left to the reader.

After having proved this, we will use its weak monotone convergence to prove the full, usual monotone convergence.

Claim 2.5 (Monotone Convergence Theorem)**.** *If $f_n, f \in \mathcal{B}$ are such that $f_n \nearrow f$, that is, f_n are a monotonically non-decreasing sequence, with* $\sup_n f_n(x) = f(x)$. *Then*

$$\Lambda(f_n) \nearrow \Lambda(f).$$

Sketch of proof. First of all, we assume, without loss of generality, that $\Lambda(1) = 1$. Fixing $\varepsilon > 0$, we take $N > 0$ large enough such that $\Lambda(1-h_N) < \varepsilon$. We then write

$$\Lambda(f) = \Lambda(fh_N) + \Lambda(f(1 - h_N)).$$

The second summand is controlled by $\sup_{x\in\mathbb{R}} f(x)\Lambda(1-h_N) \leq \varepsilon \sup_{x\in\mathbb{R}} f(x)$. But now fh_N has compact support, and, putting together the monotone convergence of f_n with an uniform continuity argument, we see that $f_n h_N \to fh_N$ *uniformly.* A formalization of this is left as an exercise to the reader. From the remark above, we get that, for sufficiently large n, $\Lambda(f_n h_N)$ is ε-close to $\Lambda(fh_N)$. Putting all together concludes the proof. ☐

To simplify notation, we will from now on call

$$\mathcal{P}_t(y - x) = \frac{1}{\pi}\frac{t}{t^2 + (y-x)^2} = P_{x,t}(y).$$

2.1 Proof of the Riesz-Herglotz Representation Theorem

We prove first that $F(x,t) = \Lambda(P_{x,t})$ is in M. For that, we must prove several assertions:

1. We will prove that F is continuously differentiable, and the reader can prove the existence and continuity of further derivatives by an analogous method. We claim that $\partial_x F(x,t) = \Lambda(\partial_x P_{x,t})$. In fact,

$$\left|\frac{F(x+h,t)-F(x,t)}{h} - \Lambda(\partial_x P_{x,t})\right| = \left|\Lambda\left(\frac{P_{x+h,t}-P_{x,t}}{h} - \partial_x P_{x,t}\right)\right|$$

   By uniform continuity, it suffices to estimate

$$\left|\frac{1}{h}[P_{x+h,t}-P_{x,t}](y) - \partial_x P_{x,t}(y)\right| \leq \sup_{x\in\mathbb{R}} |h\partial_x^2 P_{x,t}(y)| = h\sup_{x\in\mathbb{R}} |\partial_x^2 P_{x,t}(y)|,$$

   where we used a Taylor expansion to bound the difference above. Notice that the supremum ignores the action on y above, which concludes the proof of the differentiability of F.

2. Since $\Delta P_{x,t} \equiv 0$, and by the first item, we must have that

$$\Delta F = \Lambda(\Delta P_{x,t}) = \Lambda(0) = 0.$$

3. If $t' > t$, then we see that $F(x,t') = \Lambda(P_{x,t'}) \leq C\sup_{y\in\mathbb{R}} P_{x,t'}(y) \leq \frac{C}{\pi t'} \leq \frac{C}{\pi t}$.

4. Let $a > b \in \mathbb{R}$ be arbitrary, and let us evaluate

$$\int_a^b F(x,t)dx = \int_a^b \Lambda(P_{x,t})dx = \lim_{M\to\infty} \sum_{m=1}^{M} \Lambda(P_{a+\frac{b-a}{M}m,t}) \cdot \frac{b-a}{M}$$
$$= \lim_{M\to\infty} \Lambda\left(\sum_{m=1}^{M} P_{a+\frac{b-a}{M}m,t} \cdot \frac{b-a}{M}\right) = \Lambda\left(\int_a^b P_{x,t}dx\right) \leq \Lambda(1),$$

   where we used the uniform convergence of $\sum_{m=1}^{M} P_{a+\frac{b-a}{M}m,t} \cdot \frac{b-a}{M}$ to $\int_a^b P_{x,t}dx$ – which can be accomplished by a thorough calculation – to pass the limit to inside Λ. As $a > b$ were arbitrary, we conclude that

$$\int_{-\infty}^{\infty} F(x,t)dx \leq \Lambda(1),\ \forall t > 0.$$

This shows that $F(x,t) \in M$. Now we are only left with the task of showing that this map is, in fact, a bijection. Fortunately, there is an explicit expression for the inverse map: given $F \in M$, we define

$$\Lambda(f) := \lim_{t\to 0} \int_{\mathbb{R}} f(x)F(x,t)dx.$$

Of course, first we need to show the existence of the limit. We do so in some steps:

A If $f \equiv 1$, the limit exists, as we have that

$$\int_{\mathbb{R}} F(x,t_1)dx = \int_{\mathbb{R}} \int_{\mathbb{R}} F(y,t_1)P_{x,t_2-t_1}(y)dy\,dx = \int_{\mathbb{R}} F(y,t_2)dy.$$

Here we have used Fubini's theorem for the last equality, along with Poisson's representation formula for both.

B Assume $f(x) = \int_{\mathbb{R}} g(y)P_{x,s}(y)dy$, for some $g \in \mathcal{B}_\infty$ and some $s \in \mathbb{R}_{>0}$. In this case, we have

$$\int_{\mathbb{R}} f(x)F(x,t)dx = \int_{\mathbb{R}} \int_{\mathbb{R}} g(y)P_{x,s}(y)F(x,t)dy\,dx$$

$$= \int_{\mathbb{R}} g(y)F(y,t+s)dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(y)P_{x,t}(y)dy \right) F(x,s)dx.$$

In this calculation, we have used Fubini's theorem twice, along with Poisson's representation formula. To conclude, we must only see that $\int_{\mathbb{R}} g(y)P_{x,t}(y)dy \to g(x)$ uniformly as $t \to 0$, but this was already proved last time. This establishes this case.

C Let $f \in \mathcal{B}$ be arbitrary. This case should be handled by approximating uniformly f by $\int_{\mathbb{R}} f(y)P_{x,t}(y)dy$. The end of the argument – that is, the ϵ –δ part – is left as an exercise.

As we already know the limit exists, we shall advance into showing that the maps so defined are mutual inverses.

1. Let Λ be defined from every given F as above. Then we know that

$$\Lambda(P_{x,t}) = \lim_{s\to 0} \int_{\mathbb{R}} P_{x,t}(y)F(y,s)dy = \lim_{s\to 0} F(x,t+s) = F(x,t).$$

   Here we used the continuity of F on the last equality, and Poisson's representation formula on the second to last one.

2. Let F be defined from every given Λ as above, and let $f \in \mathcal{B}$. We write down

$$\lim_{t\to 0} \int_{\mathbb{R}} f(x)F(x,t)dx = \lim_{t\to 0} \lim_{N\to\infty} \int_{-N}^{N} f(x)F(x,t)dx$$

$$= \lim_{t\to 0} \lim_{N\to\infty} \lim_{M\to\infty} \Lambda \left(\frac{1}{M} \sum_{m=1}^{M} f(-N + \frac{2N}{M} m) P_{-N+\frac{2N}{M}m,t} \right)$$

$$= \lim_{t\to 0} \lim_{N\to\infty} \Lambda \left(\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} f(-N + \frac{2N}{M} m) P_{-N+\frac{2N}{M}m,t} \right)$$

$$= \lim_{t\to 0} \lim_{N\to\infty} \Lambda \left(\int_{-N}^{N} f(x)P_{x,t}(y)dx \right)$$

$$= \lim_{t\to 0} \Lambda \left(\int_{\mathbb{R}} f(x)P_{x,t}(y)dx \right) = \Lambda(f),$$

where, to justify the exchange of Λ with the various limits, we used the uniform continuity property we proved, a Riemann sum decomposition on the finite integrals on the second equality and the full monotone convergence on the penultimate inequality.

3 Primitives of martingales and measures

2016-10-25

We start recalling the definition of M, namely that we say a function $F : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \in M$ if it satisfies the following conditions:

- two times continuously differentiable and $\Delta F = 0$ on $\mathbb{R} \times \mathbb{R}_{>0}$, i.e. it is harmonic;
- $\forall t > 0 \quad \sup_{x \in \mathbb{R}, t' > t} F(x, t') < \infty$;
- $\forall t > 0 \quad \int_{-\infty}^{\infty} F(x, t) dx < \infty$.

We give the following

Definition 3.1 (PM)**.** Let PM be the set of all functions $F : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- two times continuously differentiable and $\Delta F = 0$ on $\mathbb{R} \times \mathbb{R}_{>0}$, i.e. it is harmonic;
- $\forall t > 0 \quad F(x, t)$ non-decreasing in x;
- it is bounded, i.e. $\exists C < \infty \; \forall x, t \quad F(x, t) \leq C$;
- $\forall t > 0 \quad \inf_{x \in \mathbb{R}} F(x, t) = 0$.

The P in PM stands for "primitive" and the meaning is explained by the following

Theorem 3.2. *Let $F \in PM$. Then $\partial_x F \in M$.*

Proof. 1. Since F is harmonic then it is infinitely differentiable. In particular $\partial_x F$ is twice differentiable and $\Delta \partial_x F = \partial_x \Delta F = 0$.

2.
$$\partial_x F(x, t + s) = \int_{-\infty}^{\infty} \partial_y \mathcal{P}_{x,t}(y) F(y, s) dy$$

because, as seen in previous lessons, F is its own Poisson's extension at every level. Boundedness of F implies the bound

$$C \int_{-\infty}^{\infty} \partial_y \mathcal{P}_{x,t}(y) dy \leq C C_t.$$

3.
$$\forall t \geq 0 \qquad \int_a^b \partial_x F(x,t)dx = F(b,t) - F(a,t) \leq F(b,t) \leq C.$$

□

We would like to exhibit an inverse of this map. The trouble with the naive idea of taking the integral of F in the first variable is due to the fact that $\mathbb{1}_{\mathbb{R}_{<x}}(z)$, the characteristic function of the set $\{z < x\}$, is not in $\mathcal{B}$. However we can approximate it with functions in $\mathcal{B}$ in the way described by the following

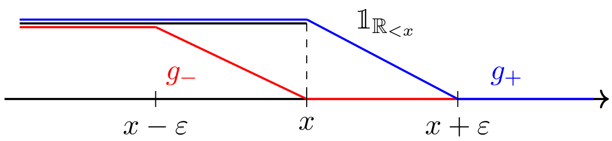
Lemma 3.3. *Let $F \in M$. Then*

$$\int_{-\infty}^{x} F(z,t)dz = \inf_{g\in\mathcal{B}, g\geq \mathbb{1}_{\mathbb{R}_{<x}}} \int g(z)F(z,t)dz$$
$$= \sup_{h\in\mathcal{B}, h\leq \mathbb{1}_{\mathbb{R}_{<x}}} \int h(z)F(z,t)dz$$

is a consistent definition, namely both the infimum and the supremum exist, are finite and equal.

Proof. We first observe that for $g, h \in \mathcal{B}$ such that $h \leq \mathbb{1}_{\mathbb{R}_{<x}} \leq g$ then by nonnegativity of F we have the trivial inequality, which is then preserved by taking the infimum over g and the supremum over h.
To prove the non trivial inequality we fix $\varepsilon > 0$ and consider the following functions

$$g_-(z) = \begin{cases} 1 & \text{for } z \leq x - \varepsilon \\ \frac{x-z}{\varepsilon} & \text{for } x - \varepsilon < z \leq x \\ 0 & \text{for } z > x \end{cases}, \quad g_+(z) = \begin{cases} 1 & \text{for } z \leq x \\ \frac{x-z}{\varepsilon} & \text{for } x < z \leq x + \varepsilon \\ 0 & \text{for } z > x + \varepsilon. \end{cases}$$



Then

$$\int g_+(z)F(z,t)dy - \int g_-(z)F(z,t)dz = \int (g_+(z) - g_-(z))F(z,t)dz \leq$$
$$\leq C_t \int (g_+(z) - g_-(z))dy \leq C_t\varepsilon.$$

Therefore the infimum over g is bounded by the supremum over h plus $C_t\varepsilon$. But ε is arbitrary, hence we get the claim. □

Remark 3.4. We could have "overkilled" the problem by using a result on monotone convergence. However we are trying to build a theory that fits in backgrounds we don't want to fix and therefore it is preferable to rely on naive computations as far as possible.

The obvious consequence is the following

Theorem 3.5. *Let $F \in M$. For $(x,t) \in \mathbb{R} \times \mathbb{R}_{>0}$ set*

$$G(x,t) = \int_{-\infty}^{x} F(z,t)dz.$$

Then $G \in PM$ and $\partial_x G = F$.

Proof. The proof is left as an exercise. □

Recalling the bijection between M and $\mathcal{B}'$ we are led to ask ourselves what objects on the real line correspond to elements of PM. We expect them to be the "primitives" of nonnegative Borel measures. The meaning will soon be clarified.
We want to understand the limiting object

$$\text{``}\lim_{t\to 0} \int_{-\infty}^{x} F(z,t)dz\text{''}.$$

We know that for $F \in M$ then

$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(z)F(z,t)dy$$

exists for all $g \in \mathcal{B}$. Since $\mathbb{1}_{\mathbb{R}_{<x}}$ is not in $\mathcal{B}$, as in previous Lemma we get around the problem defining the two functions

$$f_r(x) := \inf_{g\in\mathcal{B}, g\geq \mathbb{1}_{\mathbb{R}_{<x}}} \lim_{t\to 0} \int g(z)F(z,t)dz$$
$$f_l(x) := \sup_{h\in\mathcal{B}, h\leq \mathbb{1}_{\mathbb{R}_{<x}}} \lim_{t\to 0} \int h(z)F(z,t)dz.$$

In general these two functions are not equal, therefore the limiting object we wanted to study is not uniquely defined and there is no way to choose one above the other. However f_l and f_r are trying to be equal in the way enlightened by the following

Lemma 3.6.

$$\forall x \qquad f_l(x) \leq f_r(x),$$
$$\forall x < y \qquad f_r(x) \leq f_l(y).$$

Proof. For the first claim we trivially have that for $h \leq \mathbb{1}_{\mathbb{R}_{<x}} \leq g$ then

$$\int h(z)F(z,t)dz \leq \int g(z)F(z,t)dz$$

and the inequality is preserved when passing to the limit as t goes to 0. Therefore it holds when taking the supremum over all possible h on the left and the infimum over all possible g on the right.
For the second claim we observe that for the function

$$g(z) = \begin{cases} 1 & \text{for } z \leq x \\ \frac{y-z}{y-x} & \text{for } x < z \leq y \\ 0 & \text{for } z > y. \end{cases}$$

we have $\mathbb{1}_{\mathbb{R}_{<x}} \leq g \leq \mathbb{1}_{\mathbb{R}_{<y}}, g \in \mathcal{B}$. Therefore

$$f_r(x) \leq \lim_{t\to 0} \int g(z)F(z,t)dz \leq f_l(y).$$

□

f_l and f_r are easily both monotonic non-decreasing. Moreover

Lemma 3.7. *f_l is lower semicontinuous (and in this case it coincides with being left continuous), i.e.*

$$\lim_{x\nearrow y} f_l(x) = f_l(y).$$

f_r is upper semicontinuous (and in this case it coincides with being right continuous), i.e.

$$\lim_{x\searrow y} f_r(x) = f_r(y).$$

Proof. We prove the first claim, the second one being symmetrical.
Fix $\varepsilon > 0$. By definition of $f_l(y)$ there exists $g \leq \mathbb{1}_{\mathbb{R}_{<y}}, g \in \mathcal{B}$ such that

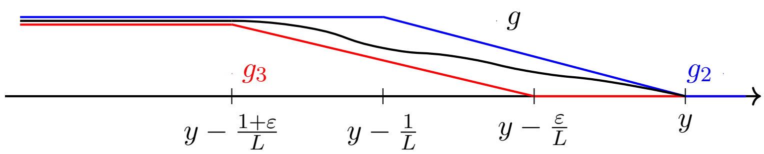
$$f_l(y) \leq \lim_{t\to 0} \int g(z)F(z,t)dz + \varepsilon.$$

Since g is Lipschitz (say L-Lipschitz, for $L < \infty$) then for

$$g_2(z) = \begin{cases} 1 & \text{for } z \leq y - \frac{1}{L} \\ Ly - Lz & \text{for } y - \frac{1}{L} < z \leq y \\ 0 & \text{for } z > y \end{cases}$$

we have

$$f_l(y) \leq \lim_{t\to 0} \int g_2(z)F(z,t)dz + \varepsilon.$$



Therefore for

$$g_3(z) = \begin{cases} 1 & \text{for } z \leq y - \frac{1+\varepsilon}{L} \\ Ly - \varepsilon - Lz & \text{for } y - \frac{1+\varepsilon}{L} < z \leq y - \frac{\varepsilon}{L} \\ 0 & \text{for } z > y - \frac{\varepsilon}{L}, \end{cases}$$

since $\sup_{z \in \mathbb{R}} |g_2(z) - g_3(z)| < \varepsilon$, we have

$$f_l(y) \leq \lim_{t \to 0} \int g_3(z) F(z,t) dz + C\varepsilon + \varepsilon$$

where $C = \sup_t \int F(z,t)dz$, which actually doesn't depend on t. By taking $x = y - \frac{\varepsilon}{L}$ and since f_l is monotonic non-decreasing we are done. □

We are finally ready to define the set of "primitives" of nonnegative Borel measures.

Definition 3.8 ($\mathcal{PB}'$)**.** Let $\mathcal{PB}'$ be the set of all pairs of monotonic non-decreasing functions $f_l, f_r : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- $\forall x \quad f_l(x) \leq f_r(x)$;
- $\forall x < y \quad f_r(x) \leq f_l(y)$;
- $\inf_{x \in \mathbb{R}} f_r(x) = 0 = \inf_{x \in \mathbb{R}} f_l(x)$;
- f_l is lower semicontinuous, f_r is upper semicontinuous.

Theorem 3.9. *There is a bijection $PM \to \mathcal{PB}'$ given by sending $F \in PM$ to*

$$f_l(x) = \sup_{h \in \mathcal{B}, h \leq \mathbb{1}_{\mathbb{R}_{<x}}} \lim_{t \to 0} \int h(z) \partial_z F(z,t) dz$$

$$f_r(x) = \inf_{g \in \mathcal{B}, g \geq \mathbb{1}_{\mathbb{R}_{<x}}} \lim_{t \to 0} \int g(z) \partial_z F(z,t) dz.$$

Proof. The proof is left as an exercise. □

The map $\mathcal{PB}' \to \mathcal{B}'$ given by

$$(f_l, f_r) \mapsto \left(\Lambda : \mathcal{B} \ni g \mapsto \int_{\mathbb{R}} g(x) df_l(x) \right),$$

where the last one is a Riemann-Stieltjes integral, is a bijection and close the following commutative diagram

$$
\begin{array}{ccc}
M & \longleftrightarrow & \mathcal{B}' \\
\updownarrow & & \uparrow \\
PM & \longrightarrow & \mathcal{P}\mathcal{B}'
\end{array}
$$

Before introducing our next tool we would like to comment on what we learned from extending harmonically functions on the real line to the upper half plane.
First of all we recall that there is no canonical way of defining a unique limiting object on $\mathbb{R}$ for $F \in PM$. If one between f_l, f_r (and therefore both) is continuous in a point then their values are equal and the limiting object is well defined in that point. The problem arises in points of discontinuity, where there is no way to assign canonically a value in the interval between ones of f_l and f_r. However the set of points of discontinuity for a monotonic bounded function is countable.
Secondly, for $\Lambda \in \mathcal{B}'$ we have a way to define it on the characteristic functions of intervals by setting

$$
\begin{aligned}
\Lambda(\mathbb{1}_{\mathbb{R}_{\leq x}}) &\coloneqq f_r(x), \\
\Lambda(\mathbb{1}_{\mathbb{R}_{< x}}) &\coloneqq f_l(x), \\
\Lambda(\mathbb{1}_{[a,b)}) &\coloneqq f_l(b) - f_l(a).
\end{aligned}
$$

This explain how we can interpret $\mathcal{B}'$ as the set of nonnegative Borel measure on $\mathbb{R}$.
Finally, to "embed" $f \in \mathcal{B}$ in the upper half plane through the harmonic function F we used translation and dilation of the function $\frac{1}{\pi}\frac{1}{1+y^2}$, namely

$$
\frac{1}{\pi}\frac{t}{t^2 + (y-x)^2},
$$

where the dilation parameter t varies in $\mathbb{R}_{>0}$ and the translation one x in $\mathbb{R}$. However we could have used another function, for example $\mathbb{1}_{[0,1)}$, and a countable set of dilations and translations. The extension will be no more harmonic but this new way allows us to use better tools to do some Analysis.

3.1 Discrete upper half plane

We start with some definitions to set our work enviroment.

Definition 3.10 (Dyadic interval, $\mathfrak{D}$)**.** A *dyadic interval* is an interval of the form

$$
[2^k n, 2^k(n+1))
$$

with $k, n \in \mathbb{Z}$, k is called the *scale parameter.*
We denote by $\mathfrak{D}$ the set of all these dyadic intervals.

Remark 3.11. $\forall x \in \mathbb{R}\ \forall k \in \mathbb{Z}$

$$\exists! n \in \mathbb{Z} : x \in [2^k n, 2^k(n+1)).$$

Remark 3.12. If $I \in \mathfrak{D}$ then there exist $I_l, I_r \in \mathfrak{D}$ such that

$$I = I_l \dot{\cup} I_r.$$

Remark 3.13. If $I, J \in \mathfrak{D}$ then one of the following is true:

$$I \cap J = \emptyset, I \subset J, J \subset I.$$

The way to prove is by observing that dyadic intervals are nested. The formalization of the argument is left as an exercise.

While big tool to do Analysis on the "harmonic upper half plane" was Complex Analysis machinery, on the "discrete upper half plane" it is Combinatorics. The two things go hand in hand, as we will see later on, and being able to translate between the two languages and reformulate problems may help in finding solutions. While the former is more elegant, the latter can be very useful when it comes to deal with actual computations and gets hands dirty.

Definition 3.14 (Martingale)**.** A function $F : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ is called *martingale* if for all $I \in \mathfrak{D}$ it satisfies

$$F(I) = \frac{1}{2}(F(I_l) + F(I_r)).$$

Remark 3.15. This property is a discrete version of the mean value property, therefore a discrete version of harmonicity.

Definition 3.16 ($M_{\mathfrak{D}}$)**.** Let $M_{\mathfrak{D}}$ be the set of nonnegative martingales satisfying the following properties:

- $\forall k \in \mathbb{Z}$
$$\sum_{n \in \mathbb{Z}} 2^k F([2^k n, 2^k(n+1))) < \infty;$$
- $\forall \{I_n\}_{n \in \mathbb{N}} \subset \mathfrak{D}$ such that $I_{n+1} = (I_n)_r$
$$\limsup_{n \to \infty} |I_n| F(I_n) = 0.$$

This definition matches with the one of M. "Harmonicity" condition is already required in the definition of martingales. The first condition translates the condition of integrability of F for every fixed value of the scale parameter $t < 0$, namely $\int F(x,t)dx < \infty$. Moreover, together with martingale property, it implies the condition of boundedness of F above every fixed value of the scale parameter $t < 0$, namely $\sup_{x \in \mathbb{R}, t' > t} F(x,t') < +\infty$

(as we will see in the next lecture). The second property is a technical one and is a consequence of having chosen intervals to be closed on the left and open on the right, therefore we choose the limiting object of $F \in M_{\mathfrak{D}}$ to be left continuous.

Remark 3.17. Notice that symmetrically we could have considered intervals open on the left and closed on the right and asked for the same condition with $\{J_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}$ such that $J_{n+1} = (J_n)_l$; we would have got the limiting object of F to be right continuous and obtained a theory completely symmetric. This double choice is due to the existence of f_l, f_r and the issue of non uniqueness of the limiting object.

Therefore we can state a dyadic version of Riesz-Herglotz Thm 1.9.

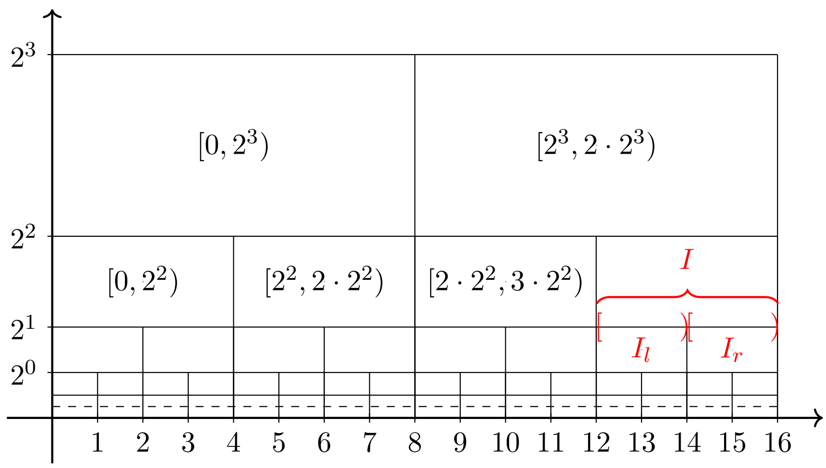
Theorem 3.18 (Dyadic Riesz-Herglotz Reperesentation Theorem)**.** *Given $F \in M_{\mathfrak{D}}$ then there exists a unique $\Lambda \in \mathcal{B}'$ such that*

- *if $x < 0$*
$$\sum_{I\in\mathfrak{D}, x\in I_r} |I_l| F(I_l) = \Lambda((-\infty, x));$$
- *if $x \geq 0$*
$$\sum_{I\in\mathfrak{D}, x\in I_r} |I_l| F(I_l) = \Lambda([0, x));$$
- $$\sup_{x<0} \sum_{I\in\mathfrak{D}, x\in I_r} |I_l| F(I_l) = \Lambda((-\infty, 0)).$$

The map sending F to Λ provides a bijection between $M_{\mathfrak{D}}$ and $\mathcal{B}'$.

Proof. We postpone the proof to the next lecture. □

Remark 3.19. We distinguish the cases $x < 0, x \geq 0$ because every dyadic interval with positive extremes is disjoint from every one with negative extremes.



4 Review

2016-10-27

Let us recall some definitions from the last time: let $\mathfrak{D}$ be the set of dyadic intervals, i.e., intervals of the form $I = [2^k n, 2^k(n+1))$, where $k, n \in \mathbb{Z}$. For every dyadic interval, we call $I_l = [2^k n, 2^k(n+\frac{1}{2}))$, $I_r = [2^k(n+\frac{1}{2}), 2^k(n+1))$, respectively, its *left* and *right* subintervals. Given these definitions, we say a function $F : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ is a (positive) *martingale* if, for every interval I, the following is satisfied:

$$|I|F(I) = |I_l|F(I_l) + |I_r|F(I_r).$$

Finally, we define the specific space of positive martingales we will be interested in:

Definition 4.1 ($M_{\mathfrak{D}}$)**.** Let $M_{\mathfrak{D}}$ be the set of nonnegative martingales satisfying the following properties:

- $\forall k \in \mathbb{Z}$
$$\sum_{n\in\mathbb{Z}} 2^k F([2^k n, 2^k(n+1))) < \infty;$$
- $\forall \{I_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}$ such that $I_{n+1} = (I_n)_r$
$$\limsup_{n\to\infty} |I_n| F(I_n) = 0.$$

With these in hands, we state our first

Lemma 4.2. *1. $\sum_{|I|=2^k} |I|F(I) =: \|F\|_1$ is independent of k. We will denote this quantity simply by $\|F\|_1$.*

2. If $\mathcal{I}$ is a collection of pairwise disjoint intervals in $\mathfrak{D}$, then

$$\sum_{I\in\mathcal{I}} |I|F(I) \leq \|F\|_1.$$

3. If, in addition, $I \subset J$, $\forall I \in \mathcal{I}$, then

$$\sum_{I\in\mathcal{I}} |I|F(I) \leq |J|F(J).$$

Sketch of the proof. 1.

$$\sum_{|I|=2^k} |I|F(I) \overset{\text{martingale property}}{=} \sum_{|I|=2^k} \{|I_l|F(I_l) + |I_r|F(I_r)\}$$
$$= \sum_{|I|=2^{k-1}} |I|F(I).$$

This proves the desired independence of the sum on the scales.

2.
$$\sum_{I\in\mathcal{I}}|I|F(I)=\sup_N \sum_{\substack{I\in\mathcal{I}\\ 2^{-N}<|I|<2^N}}|I|F(I)$$
$$\overset{\text{martingale property}}{=}\sup_N \sum_{\substack{|I|=2^{-N};\\ \exists J\in\mathcal{I};\\ 2^{-N}<|J|<2^N, I\subset J}}|I|F(I)\le\|F\|_1.$$

3. We skip this proof and leave it as an exercise, as it is an application of the techniques present in the two prior items.

□

4.1 The dyadic Riesz-Herglotz theorem

We advance to the main point of this lecture: the *dyadic Riesz-Herglotz theorem*

Theorem 4.3 (Dyadic Riesz-Herglotz Reperesentation Theorem)**.** *Given $F\in M_{\mathfrak{D}}$ then there exists a unique $\Lambda\in\mathcal{B}'$ such that*

- *if $x<0$*
$$\sum_{I\in\mathfrak{D},x\in I_r}|I_l|F(I_l)=\Lambda((-\infty,x));$$
- *if $x\ge 0$*
$$\sum_{I\in\mathfrak{D},x\in I_r}|I_l|F(I_l)=\Lambda([0,x));$$
- $$\sup_{x<0}\sum_{I\in\mathfrak{D},x\in I_r}|I_l|F(I_l)=\Lambda((-\infty,0)).$$

The map sending F to Λ provides a bijection between $M_{\mathfrak{D}}$ and $\mathcal{B}'$.

Proof. Define the following function:

$$f_l(x):=\begin{cases}\sum_{I\in\mathfrak{D};x\in I_r}|I_l|F(I_l), & \text{if } x<0,\\ \sup_{y<0}\sum_{I\in\mathfrak{D},y\in I_r}|I_l|F(I_l), & \text{if } x=0,\\ f_l(0)+\sum_{I\in\mathfrak{D},x\in I_r}|I_l|F(I_l), & \text{if } x>0.\end{cases}$$

We want to show that it is an element of $\mathcal{PB}'$, so that, from the fact that $\mathcal{PB}'$ and $\mathcal{B}'$ are in bijection, we indirectly prove the Theorem. Thus, we need to show:

1. f_l is non-decreasing.
2. f_l is lower semicontinuous.

3. $\limsup_{x\to-\infty} f_l(x) = 0$

4. f_l is bounded.

To simplify the technicalities, we present only a proof for $x < 0$, and leave to the reader the task of adapting the arguments here presented to the positive real numbers. We first prove (4): First, observe that $\{I_l;\, x \in I_r\}$ is a *pairwise disjoint* collection of intervals. Indeed, suppose not, that is, $\exists x' \in I_{1,l} \cap I_{2,l}$. As $I_{1,r}$ and $I_{2,r}$ are dyadic intervals, we may suppose that $I_{1,r} \supset I_{2,r}$. It is easy to see that, for every pair of points $x < y < 0$, there is one interval $J \in \mathfrak{D}$ of minimal length, and such that $x, y \in J$. It is also not hard to see that I_2 is this interval for x' and x. As $I_{2,r} \subsetneq I_{1,r}$, we conclude that $I_2 \subset I_{1,r}$, and, therefore, $x' \in I_{1,r}$. But this is a contradiction to the fact that $x' \in I_{1,l}$. Now, we simply apply Lemma 4.2 to the collection $\{I_l;\, x \in I_r\}$, and this implies directly that

$$\sum_{I\in\mathfrak{D}, x\in I_r} |I_l| F(I_l) \leq \|F\|_1.$$

Now we prove (1). Let $x < y < 0$ and select the interval J as above, such that $x, y \in J$, $x \in J_l$ and $y \in J_r$. Then

$$\begin{aligned} f_l(y) = \sum_{y\in I_r} |I_l|F(I_l) &\geq \sum_{x,y\in I_r} |I_l|F(I_l) + |J_l|F(J_l) \\ &\overset{\text{Lemma 4.2}}{\geq} \sum_{x,y\in I_r} |I_l|F(I_l) + \sum_{x\in I_r, y\notin I_r} |I_l|F(I_l). \end{aligned}$$

This ends the proof of (1).
To prove (2), we distinguish our situation in two cases:
Case (a): There are infinitely many $k < 0$ such that $|I| = 2^k$ and $y \in I_r$. In this case, there is a k such that

$$\sum_{\substack{y\in I_r \\ |I_r|\geq 2^k}} |I_l|F(I_l) + \varepsilon \geq f_l(y).$$

Pick $|J| < 2^k$, $y \in J_r$. If $x \in J_l$, then

$$f_l(x) \geq \sum_{\substack{y\in I_r \\ |I_r|\geq 2^k}} |I_l|F(I_l) \geq f_l(y) - \varepsilon.$$

This establishes the desired lower semicontinuity in this case.
Case (b): There are only finitely many $k < 0$ such that $|I| = 2^k$ and $y \in I_r$. This means that $y \in I_l$ for all intervals I with $|I| = 2^k, y \in I$ and $k < k_0$. Then y is the (left) endpoint of some $J \in \mathfrak{D}$. In this case, we now use the property that

$$2^k F([y - 2^k, y)) \to 0 \text{ as } k \to -\infty,$$

and writing the explicit formula for $f_l(y)$ and $f_l(x)$, $x < y$ once again will do. The details are left to the reader.

Finally, we come to prove (3). Notice that, from the martingale property and the definition of $\|F\|_1$,

$$\sum_{k\in\mathbb{Z}} 2^k F([-2^{k+1}, -2^k)) \leq \|F\|_1.$$

Pick then $k_0 > 0$ large enough so that

$$(F(-2^{k_0}) =) \sum_{k>k_0} 2^k F([-2^{k+1}, -2^k)) < \varepsilon.$$

This proves that $f_l \in \mathcal{PB}'$, and, therefore, there exists such a $\Lambda \in \mathcal{B}'$. To construct the inverse, we take, for every given $\Lambda \in \mathcal{B}'$, the function f_l naturally associated to it (see Lecture 3), and define $F([a,b)) = \frac{f_l(b)-f_l(a)}{b-a}$. Verifying that this is the inverse is a simple calculation, and we leave it as an exercise. □

4.2 The dyadic maximal function and existence of limits

Now we pass to a topic which is related, yet not that closely, to the last one. Explicitly, let first $F \in M_{\mathfrak{D}}$, and define, for $x \in \mathbb{R}$, $k \in \mathbb{Z}$, the interval $I_{x,k}$ as the unique dyadic interval such that $x \in I_{x,k}$ and $|I_{x,k}| = 2^k$.

Question 4.4. Does the limit

$$\lim_{k\to-\infty} F(I_{x,k})$$

exist? For which x? For how many x?

The fact that the limit does not always exist can be shown by an explicit example: define

$$F(I) = \begin{cases} \frac{1}{|I|}, & \text{if } 0 \in I, \\ 0, & \text{if } 0 \notin I. \end{cases}$$

Of course, this example does not allow a limit at 0.
As every convergent sequence of real numbers is bounded, we can relax the convergence assumptions, and go to a yet more general question:

Question 4.5. When is $\sup_k F(I_{x,k}) < +\infty$? For which and/or how many x does it exist?

This leads to the following definition:

Definition 4.6 (Hardy-Littlewood maximal function of F)**.** Let $F : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ be a martingale. Define, then, its *Hardy-Littlewood maximal function* as the real function

$$\mathcal{M}_{\mathfrak{D}} F(x) := \sup_k F(I_{x,k}).$$

This definition might seem artificial at first sight, but let us have a look at a more natural way to build maximal functions.

Example 4.7. Let dx be the Lebesgue measure on $\mathbb{R}$. Define, for some integrable function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$,

$$F_L([a, b)) := \frac{1}{b-a} \int_a^b f(t)dt.$$

It is easy to check that this definition gives rise to a martingale (although it is not on the space $M_{\mathfrak{D}}$, but we will not bother this for now). Then, from the definition above, we can just define the *Hardy-Littlewood maximal function* of f – or, more specifically, of F_L – as

$$\sup_{x \in I \in \mathfrak{D}} \frac{1}{|I|} \int_I f(t)dt = \mathcal{M}_{\mathfrak{D}} F_L(x) (=: \mathcal{M}f(x)).$$

Let now $F \in M_{\mathfrak{D}}$. Define, for every $N \in \mathbb{N}$, the sets

$$\mathcal{I}_N = \{I \in \mathfrak{D}; F(I) > N \text{ and } \nexists J; F(J) > N \text{ and } J \supset I.\}.$$

From the definition, we have that

$$N \sum_{I \in \mathcal{I}_N} |I| \leq \sum_{I \in \mathcal{I}_N} |I| F(I) \leq \|F\|_1.$$

$$\Rightarrow \sum_{I \in \mathcal{I}_N} |I| \leq \frac{\|F\|_1}{N}.$$

Also, if $\mathcal{M}_{\mathfrak{D}} F(x) = +\infty$, then, for all N, there is $I \in \mathcal{I}_N$ with $x \in I$.

Definition 4.8 (Vanishing outer Lebesgue measure)**.** A set $E \subset \mathbb{R}$ is said to have *vanishing outer Lebesgue measure* if

$$\inf_{\mathcal{I}} \sum_{I \in \mathcal{I}} |I| = 0,$$

where the infimum is taken over all possible coverings of E with elements in $\mathfrak{D}$, i.e.,

$$E \subset \bigcup_{I \in \mathcal{I}} I.$$

With this definition, we can already define the following Theorem, to be proved on the next lecture:

Theorem 4.9. *If $F \in M_{\mathfrak{D}}$, then the set $\{x \in \mathbb{R}; \mathcal{M}_{\mathfrak{D}} F(x) = +\infty\}$ has vanishing outer Lebesgue measure*

The popular way to state this would be: “If $F \in L^1$, then $\mathcal{M}_{\mathfrak{D}} F < +\infty$ almost everywhere in $\mathbb{R}$.”

5 A dyadic proof of the decomposition theorem for nonnegative Borel measures

2016-11-03

Before starting we would like to comment on the difference between $F(I)$ and $|I|F(I)$ hoping this will clarify their meaning and make it easier to understand when we need one or the other. $F(I)$ can be thought of as an average of a function over the dyadic interval I. Therefore $|I|F(I)$ corresponds to the value of the integral of the same function over I.

We recall some definitions:

- $\mathfrak{D}$ is the set of dyadic intervals $[n2^k, (n+1)2^k)$, where $n, k \in \mathbb{Z}$;
- $M_{\mathfrak{D}}$ is the set of functions $F : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ such that:
    1. $\forall I \in \mathfrak{D} \qquad |I|F(I) = |I_l|F(I_l) + |I_r|F(I_r);$
    2. $\exists k \in \mathbb{Z}$ such that
    $$\sum_{|I|=2^k} |I|F(I) < \infty.$$
    The sum is independent on k (once assumed first condition), we call it *total mass* of F and denote it by $\|F\|_1$;
    3. $\forall \{I_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}$ such that $I_{n+1} = (I_n)_r$
    $$(\limsup_{n\to\infty} =) \inf_n |I_n|F(I_n) = 0;$$
- given $F \in M_{\mathfrak{D}}$ we define $\mathcal{I}_N = \mathcal{I}_{N,F}$ the set of maximal (with respect to the inclusion) dyadic intervals I such that $F(I) \geq N$.

We want to use the theory and language we developed for the discrete upper half plane extension to prove the result on canonical decomposition of nonnegative Borel measures into absolutely continuous and singular part. We start observing that

$$N \sum_{I \in \mathcal{I}_{N,F}} |I| \leq \sum_{I \in \mathcal{I}_{N,F}} |I|F(I) \leq \|F\|_1,$$

by the very definition of $\mathcal{I}_{N,F}$.

Definition 5.1 (Absolutely continuous)**.** $F \in M_{\mathfrak{D}}$ is called *absolutely continuous* if

$$\inf_N \sum_{I \in \mathcal{I}_{N,F}} |I|F(I) = 0.$$

This condition is not verified by every $F \in M_{\mathfrak{D}}$.

Example 5.2. $F \in M_{\mathfrak{D}}$ defined by

$$F(I) = \begin{cases} \frac{1}{|I|} & \text{if } 0 \in I \\ 0 & \text{if } 0 \notin I \end{cases}$$

is not absolutely continuous. It corresponds to the Dirac delta δ;

Example 5.3. $F \in M_{\mathfrak{D}}$ defined by

$$F(I) = \begin{cases} 1 & \text{if } I \subseteq [0,1) \\ 0 & \text{if } I \nsubseteq [0,1) \end{cases}$$

instead is absolutely continuous. It gives the measure defined by $\mathbb{1}_{[0,1)}(x)$.

Definition 5.4 (Singular part)**.** Let $F \in M_{\mathfrak{D}}$. For $J \in \mathfrak{D}$ we define

$$G_N(J) := \begin{cases} \dfrac{1}{|J|} \displaystyle\sum_{I \in I_{N,F}, I \subseteq J} |I| F(I) & \text{if } J \not\subset I \text{ for all } I \in \mathcal{I}_{N,F} \\ F(I) & \text{if } J \subseteq I \text{ for some } I \in \mathcal{I}_{N,F}, \end{cases}$$

$$G(J) := \inf_N G_N(J).$$

G is called the *singular part* of F.

Notice that the two definitions of G_N coincide when $J \in \mathcal{I}_{N,F}$.

Theorem 5.5. $G \in M_{\mathfrak{D}}$.

Proof. We first claim that $G_N \in M_{\mathfrak{D}}$:

1. if $G_N(J), G_N(J_l), G_N(J_r)$ all fall in the first case of the definition of G_N then the martingale property follows by the one of F. Similarly if they all fall in the second case. Since the definition of $G_N(J)$ for $J \in \mathcal{I}_{N,F}$ coincides we are always in one of the two conditions;

2. & 3. follows from the observation that $G_N(J) \leq F(J)$ for all $J \in \mathfrak{D}$, which is easily verified in both cases of the definition of $G_N(J)$.

We observe that, once fixed $J \in \mathfrak{D}$, $G_N(J)$ is nonincreasing in N because for $M \geq N$, for every $I \in \mathcal{I}_{M,F}$ there exists $I' \in \mathcal{I}_{N,F}$ such that $I \subseteq I'$. Therefore

$$G(J) = \inf_N G_N(J) = \lim_{N \to \infty} G_N(J). \qquad (*)$$

We are ready to prove the statement:

1. for every fixed N the martingale property is satisfied for $G_N(J)$. Therefore it is preserved when taking the limit and holds for $G(J)$;

2. & 3. follows from the observation that $G(J) \leq G_N(J) \leq F(J)$ for all $J \in \mathfrak{D}$.

□

Since $G \leq F$ there exists $H : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ such that $F = G + H$. Again $H \in M_{\mathfrak{D}}$ easily: the martingale property comes from the same one for F, G and for the other two properties is enough to observe $H \leq F$. Moreover

Theorem 5.6. *H is absolutely continuous.*

Proof. We have to show that

$$\inf_N \sum_{I \in \mathcal{I}_{N,H}} |I| H(I) = 0.$$

We claim that it is enough to show

$$\inf_N \sum_{I \in \mathcal{I}_{N,F}} |I| H(I) = 0.$$

This is due to the fact that every $I \in \mathcal{I}_{N,H}$ has to be contained in a $J \in \mathcal{I}_{N,F}$ by definition of $\mathcal{I}_{N,\cdot}$ and $H \leq F$.
Pick $\varepsilon > 0$. Then there exists N large enough such that

$$\forall M > N \qquad \sum_{I \in \mathcal{I}_{N,F}} |I| F(I) \leq \sum_{I \in \mathcal{I}_{M,F}} |I| F(I) + \varepsilon,$$

because the sum over the elements of the set $\mathcal{I}_{N,F}$ is decreasing in N. Now

$$\varepsilon + \sum_{I \in \mathcal{I}_{N,F}} |I| G(I) = \varepsilon + \inf_{M > N} \sum_{I \in \mathcal{I}_{M,F}} |I| F(I) \geq \sum_{I \in \mathcal{I}_{N,F}} |I| F(I),$$

where for the first equality we used the very definition of G and for the second inequality we used the previous estimate. Moreover notice that we don't have to worry about interchanging sums and limits since the summands are positive.
To conclude we observe that by additivity

$$\sum_{I \in \mathcal{I}_{N,F}} |I| G(I) + \sum_{I \in \mathcal{I}_{N,F}} |I| H(I) = \sum_{I \in \mathcal{I}_{N,F}} |I| F(I),$$

which implies

$$\varepsilon \geq \sum_{I \in \mathcal{I}_{N,F}} |I| H(I).$$

The claim follows by taking ε arbitrarily small. □

Therefore we call H the *absolutely continuous part* of F and $F = G + H$ gives the wanted decomposition. It satisfies the following

Theorem 5.7. *If $F = \tilde{G} + \tilde{H}$ where $\tilde{G}, \tilde{H} \in M_{\mathfrak{D}}$ and $\tilde{H}$ is absolutely continuous then $\tilde{G} \geq G$.*

Proof. Pick $J \in \mathfrak{D}$. Then

$$\sum_{I \in \mathcal{I}_{N,F}, I \subseteq J} |I|\tilde{G}(I) + \sum_{I \in \mathcal{I}_{N,F}, I \subseteq J} |I|\tilde{H}(I) = \sum_{I \in \mathcal{I}_{N,F}, I \subseteq J} |I|F(I).$$

The first sum is bounded by $|J|\tilde{G}(J)$ independently on N. The third sum goes to $|J|G(J)$ as $N \to \infty$ by the observation $(*)$ and the fact that for $I \supseteq J$ then $F(I) \leq \frac{\|F\|_1}{|J|} < \infty$.
Therefore it is enough to prove that the second sum goes to 0 as $N \to \infty$. For $\varepsilon > 0$, by almost continuity of $\tilde{H}$, there exists M large enough so that

$$\sum_{I \in \mathcal{I}_{M,\tilde{H}}} |I|\tilde{H}(I) \leq \varepsilon.$$

Then pick N large enough so that

$$M \sum_{I \in \mathcal{I}_{N,F}} |I| \leq \varepsilon.$$

To conclude we observe

$$\sum_{I \in \mathcal{I}_{N,F}} |I|\tilde{H}(I) \leq \sum_{I \in \mathcal{I}_{N,F}, \tilde{H}(I) > M} |I|\tilde{H}(I) + \sum_{I \in \mathcal{I}_{N,F}, \tilde{H}(I) \leq M} |I|\tilde{H}(I) \leq$$
$$\leq \sum_{I \in \mathcal{I}_{M,\tilde{H}}} |I|\tilde{H}(I) + M \sum_{I \in \mathcal{I}_{N,F}} |I| \leq 2\varepsilon.$$

The claim follows by taking ε arbitrarily small. □

Therefore the decomposition $F = G + H$ is canonical, i.e. uniquely determined by the property of minimizing $\tilde{G}$ over all decomposition $F = \tilde{G} + \tilde{H}$ such that $\tilde{H}$ is absolutely continuous. It is also interesting here to observe that we gave an explicit description of the singular part and then proved that the difference is absolutely continuous instead of doing the opposite.

5.1 On convergence of $F(I_{x,k})$ for $k \to -\infty$.

We recall some definitions and results:

- $I_{x,k}$ is the unique $I \in \mathfrak{D}$ such that $|I_{x,k}| = 2^k$ and $x \in I_{x,k}$;
- for $F \in M_{\mathfrak{D}}$ we define $\mathcal{M}_{\mathfrak{D}}F(x) := \sup_k F(I_{x,k})$;

- for $F \in M_{\mathfrak{D}}$ then $\mathcal{M}_{\mathfrak{D}}F(x)$ is finite for almost every x, i.e. for every $\varepsilon > 0$ there exists $\mathcal{I} \subset \mathfrak{D}$ such that

$$\sum_{I\in\mathcal{I}}|I| \leq \varepsilon,\ \mathcal{M}_{\mathfrak{D}}F(x) < \infty \text{ for all } x \notin \bigcup_{I\in\mathcal{I}} I.$$

We studied the dyadic Hardy-Littlewood maximal function as an introduction for the question on convergence of $F(I_{x,k})$ as $k \to -\infty$. The next step in this direction is the following

Lemma 5.8. *Let $F \in M_{\mathfrak{D}}$. Fix $\varepsilon > 0, n \in \mathbb{N}_{>0}, \delta > 0$. Then there exists $\mathcal{I} \subset \mathfrak{D}$ such that*

$$\sum_{I\in\mathcal{I}}|I| \leq \delta,$$

and for all $x \notin \cup_{I\in\mathcal{I}} I$ we have

$$\begin{aligned} &F(I_{x,k}) < \varepsilon(n+1) && \textit{for all but finitely many } k < 0 \\ \textit{or } &F(I_{x,k}) \geq \varepsilon n && \textit{for all but finitely many } k < 0. \end{aligned}$$

Proof. Let $\mathcal{I}_0$ be the set of maximal dyadic intervals I such that $F(I) \geq (n+1)\varepsilon$. Then

$$\sum_{I\in\mathcal{I}_0}|I| \leq \frac{\|F\|_1}{(n+1)\varepsilon} < \infty.$$

For $m \in \mathbb{N}$ let $\mathcal{J}_{m+1}$ be the set of maximal dyadic intervals I such that $I \subset J$ for some $J \in \mathcal{I}_m$ and $F(I) < \varepsilon n$.
Similarly let $\mathcal{I}_{m+1}$ be the set of maximal dyadic intervals I such that $I \subset J$ for some $J \in \mathcal{J}_{m+1}$ and $F(I) \geq \varepsilon(n+1)$.
Trivially

$$\sum_{I\in\mathcal{J}_{m+1}}|I| \leq \sum_{I\in\mathcal{I}_m}|I|.$$

We claim

$$\sum_{I\in\mathcal{I}_{m+1}}|I| < \frac{n}{n+1}\sum_{I\in\mathcal{J}_{m+1}}|I|,$$

from which the Lemma follows using recursively this argument in order to find m large enough so that $\sum_{I\in\mathcal{I}_m}|I| \leq \delta$; one of the two property can be proven to hold distinguishing the cases

$$x \notin \bigcup_{I\in\mathcal{I}_0} I,\ x \in \bigcup_{I\in\mathcal{I}_m} I \setminus \bigcup_{J\in\mathcal{J}_{m+1}} J,\ x \in \bigcup_{J\in\mathcal{J}_m} J \setminus \bigcup_{I\in\mathcal{I}_m} I.$$

To prove the claim we notice

$$\varepsilon(n+1)\sum_{I\in\mathcal{I}_{m+1}}|I| \leq \sum_{I\in\mathcal{I}_{m+1}}F(I)|I| \leq \sum_{I\in\mathcal{J}_{m+1}}F(I)|I| < \varepsilon n\sum_{I\in\mathcal{J}_{m+1}}|I|.$$

□

Remark 5.9. The meaning of the Lemma is that for every $\varepsilon > 0, n \in \mathbb{N}_{>0}$ the interval $[\varepsilon n, \varepsilon(n+1))$ is not an obstruction to almost everywhere convergence: for x outside a set of measure arbitrarily small the sequence $\{F(I_{x,k})\}_{k<0}$ is definitely either above εn or below $\varepsilon(n+1)$.

The following step in studying the convergence is

Theorem 5.10. *Let $F \in M_{\mathfrak{D}}, \varepsilon > 0$. Then there exist $\mathcal{I} \subset \mathfrak{D}$ and $k_0 \in \mathbb{Z}$ such that*

$$\sum_{I \in \mathcal{I}} |I| \leq \varepsilon,$$

and for all $k, k' < k_0, x \notin \cup_{I \in \mathcal{I}} I$ we have

$$|F(I_{x,k}) - F(I_{x,k'})| \leq \varepsilon.$$

Proof. We postpone the proof to the next lecture. □

For every fixed ε it establishes a uniform Cauchy property outside a set of measure smaller or equal to ε. Applying it with $\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{4}, \ldots$ and by choosing ε arbitrarily small we get pointwise convergence almost everywhere. Actually the statement describes an almost uniform convergence for the sequence of functions $\{F(I_{\cdot,k})\}_{k<0}$, which is stronger.

6 The $L^1 - L^\infty$ pairing in the dyadic setting

2016-11-08

In order to prove last Theorem we stated in the previous lecture we need a quantitative stronger version of the Lemma we want to use. Its statement was only qualitative, it means that for x outside a set of measure arbitrarily small the sequence $\{F(I_{x,k})\}_{k<0}$ definitely stops "jumping" between values above and below $[a, b)$. However a priori the value of $k_{0,x}$ below which we are definitely below b or above a may depend on x. Instead, since we are aiming at an uniform result, we need uniform conditions at every step.

Lemma 6.1. *Let $F \in M_{\mathfrak{D}}$. Fix $0 < a < b, \delta > 0$. Then there exists $\mathcal{I} \subset \mathfrak{D}$ and $k_0 \in \mathbb{Z}$ such that*

$$\sum_{I \in \mathcal{I}} |I| \leq \delta,$$

and for all $x \notin \cup_{I \in \mathcal{I}} I$ we have

$$\begin{aligned} F(I_{x,k}) &< b && \text{for all } k \leq k_0 \\ \text{or } F(I_{x,k}) &\geq a && \text{for all } k \leq k_0. \end{aligned}$$

Proof. Let $\mathcal{I}_0$ be the set of maximal dyadic intervals I such that $F(I) \geq b$. Then

$$\sum_{I \in \mathcal{I}_0} |I| \leq \frac{\|F\|_1}{b} < \infty.$$

For $m \in \mathbb{N}$ let $\mathcal{J}_{m+1}$ be the set of maximal dyadic intervals I such that $I \subset J$ for some $J \in \mathcal{I}_m$ and $F(I) < a$.
Similarly let $\mathcal{I}_{m+1}$ be the set of maximal dyadic intervals I such that $I \subset J$ for some $J \in \mathcal{J}_{m+1}$ and $F(I) \geq b$.
We have

$$\sum_{I \in \mathcal{J}_{m+1}} |I| \leq \sum_{I \in \mathcal{I}_m} |I|, \quad \sum_{I \in \mathcal{I}_{m+1}} |I| \leq \frac{a}{b} \sum_{I \in \mathcal{J}_{m+1}} |I|,$$

the first being trivial, the second following from

$$b \sum_{I \in \mathcal{I}_{m+1}} |I| \leq \sum_{I \in \mathcal{I}_{m+1}} F(I)|I| \leq \sum_{I \in \mathcal{J}_{m+1}} F(I)|I| < a \sum_{I \in \mathcal{J}_{m+1}} |I|.$$

By recursion

$$\sum_{I \in \mathcal{I}_m} |I| \leq \left(\frac{a}{b}\right)^m \sum_{I \in \mathcal{I}_0} |I| \xrightarrow{m \to \infty} 0,$$

and for M large enough

$$\sum_{I \in \mathcal{I}_M} |I| \leq \frac{\delta}{2}.$$

Now consider

$$\lim_{k \to -\infty} \sum_{m=0}^{M} \sum_{\substack{I \in \mathcal{I}_m \cup \mathcal{J}_m \\ |I| \geq 2^k}} |I| = \sum_{m=0}^{M} \sum_{I \in \mathcal{I}_m \cup \mathcal{J}_m} |I| < \infty,$$

where the limit makes sense since the sum is a countable sum of positive summands. Therefore there exists k_0 such that

$$\sum_{m=0}^{M} \sum_{\substack{I \in \mathcal{I}_m \cup \mathcal{J}_m \\ |I| < 2^{k_0}}} |I| \leq \frac{\delta}{2}.$$

We define

$$\mathcal{I} = \mathcal{I}_M \cup \bigcup_{m=0}^{M} \{I \in \mathcal{I}_m \cup \mathcal{J}_m : |I| < 2^{k_0}\}.$$

For $x \notin \cup_{I \in \mathcal{I}} I$ we have both less than $2M$ "jumps" (taken care by the first collection of dyadic intervals) and all of them are at scale grater or equal to 2^{k_0} (taken care by the second one). $\square$

We are ready to prove the following

Theorem 6.2. *Let $F \in M_{\mathfrak{D}}, \varepsilon > 0$. Then there exist $\mathcal{I} \subset \mathfrak{D}$ and $k \in \mathbb{Z}$ such that*

1. $\sum_{I \in \mathcal{I}} |I| \leq \varepsilon$;

2. for all $k', k'' < k, x \notin \cup_{I\in\mathcal{I}} I$ we have

$$|F(I_{x,k'}) - F(I_{x,k''})| \leq 2\varepsilon.$$

Proof. Pick N large enough such that $\mathcal{I}_{N,F}$, the set of maximal dyadic intervals I such that $F(I) > N$, satisfies

$$\sum_{I\in\mathcal{I}_{N,F}} |I| \leq \frac{\|F\|_1}{N} \leq \frac{\varepsilon}{2}.$$

For every $n \geq 0$ such that $n\varepsilon < N$ we pick $\mathcal{J}_n, k_n$ defined by applying previous Lemma to the interval $[n\varepsilon, (n+1)\varepsilon]$ and $\delta = \frac{\varepsilon^2}{2(N+1)}$. Therefore

$$\sum_{n=0}^{\lfloor \frac{N}{\varepsilon} \rfloor} \sum_{I\in\mathcal{J}_n} |I| \leq \frac{\varepsilon}{2},$$

and for all $x \notin \bigcup_{I\in\mathcal{J}_n} I$

$$\begin{aligned} F(I_{x,k}) &< \varepsilon(n+1) && \text{for all } k \leq k_n \\ \text{or } F(I_{x,k}) &\geq \varepsilon n && \text{for all } k \leq k_n. \end{aligned}$$

Set

$$k = \min_{n\in\{0,\dots,\lfloor \frac{N}{\varepsilon} \rfloor\}} k_n, \; \mathcal{I} = \mathcal{I}_{N,F} \cup \bigcup_{n=0}^{\lfloor \frac{N}{\varepsilon} \rfloor} \mathcal{J}_n.$$

By construction

$$\sum_{I\in\mathcal{I}} |I| \leq \varepsilon.$$

Once fixed $x \notin \cup_{I\in\mathcal{I}} I$ we pick the minimal n such that $F(I_{x,\tilde{k}}) < \varepsilon(n+1)$ for all $\tilde{k} < k_n$. By minimality of n we have also that $F(I_{x,\tilde{k}}) \geq \varepsilon(n-1)$ for all $\tilde{k} < k_{n-1}$. This ends the proof because of the definition of k. □

If we apply the Theorem with $\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{4}, \dots$ we get that outside the union of exceptional sets we have uniform convergence. However the k in the statement depends on the measure ε of the exceptional set and we cannot get rid of this dependence, i.e. in general the sequence of functions $\{F_{x,k}\}_k$ doesn't converge uniformly almost everywhere.
Anyway our interest in studying the convergence of this sequence of functions was a warm-up for the following topic, dyadic paraproducts. We turn to products of martingales: for F, G martingales define

$$H(I) := F(I)G(I),$$

which may not be a martingale. In the same way in general the product of harmonic functions is not harmonic: for example if F is a harmonic real

valued function that vanishes at (x, y), then $F \cdot F$ is nonnegative and vanishes at (x, y), which will be a local minimum.
We could worry about the possibility that this is a limit of the model of discrete upper half plane. However the underlying question is whether or when we can multiply Borel nonnegative measures. In general this is not possible, for example we don't expect to give meaning to the product of two copies of Dirac delta. Therefore this means that it isn't a lack of our model but instead it portrays a proper situation for measures.
We want to understand when and how it is possible to make sense of

$$\text{``}\lim_{k\to-\infty} \sum_{|I|=2^k} |I|F(I)G(I)\text{''}.$$

It is trying to be the product on the real line but actually we are not after the product, more after multilinear forms of the type $\int fg$, $\int f_1 f_2 f_3$.

Theorem 6.3. *Let $F \in M_{\mathfrak{D}}$ absolutely continuous and let $G : \mathfrak{D} \to \mathbb{R}_{\geq 0}$ be a bounded martingale, i.e.*

1. $\forall I \in \mathfrak{D} \qquad |I|G(I) = |I_l|G(I_l) + |I_r|G(I_r)$;
2. $\|G\|_\infty := \sup_{I\in\mathfrak{D}} G(I) < \infty$.

Then

$$\lim_{k\to-\infty} \sum_{|I|=2^k} |I|F(I)G(I)$$

exists.

Remark 6.4. This is the classical result about the pairing $\int fg$ of $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ in disguise. However in the language of the discrete upper half plane there exists no "almost everywhere" and therefore no classes of equivalence for functions. Moreover we would like to stress the fact that absolute continuity of F is an essential condition (it means that the limiting object will be a proper function and not a general measure), for example we don't expect to give meaning to $\int \delta g$ when g is defined only almost everywhere.

Proof. Without loss of generality we can assume that $\|G\|_\infty = 1$. Therefore

$$\sum_{|I|=2^k} |I|F(I)G(I) \leq \sum_{|I|=2^k} |I|F(I) = \|F\|_1,$$

and the supremum over $k \in \mathbb{Z}$ of the first sum exists finite.
By dominated convergence theorem for series it suffices to show that for every $J_0 \in \mathfrak{D}$

$$\lim_{k\to-\infty} \sum_{\substack{|I|=2^k \\ I\subseteq J_0}} |I|F(I)G(I),$$

exists, because

$$\sum_{|I|=2^k} |I|F(I)G(I) = \sum_{|J|=2^{k_0}} \sum_{\substack{|I|=2^k \\ I\subseteq J}} |I|F(I)G(I),$$

and domination is given by the first observation of the proof.
Let J_0 and $\varepsilon > 0$ be fixed.
Since F is absolutely continuous we can choose $\delta \leq \varepsilon$ such that whenever $\tilde{\mathcal{I}} \subset \mathfrak{D}$ and $\sum_{I\in\tilde{\mathcal{I}}}|I| \leq \delta$ then

$$\sum_{I\in\tilde{\mathcal{I}}}|I|F(I) \leq \varepsilon. \tag{1}$$

We claim that there exists a constant C and $k \in \mathbb{Z}$ such that for all $k' \leq k$

$$|\sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0}} |I|F(I)G(I) - \sum_{\substack{|I|=2^k \\ I\subseteq J_0}} |I|F(I)G(I)| \leq C\varepsilon.$$

We first use Theorem (6.2) with $\frac{\delta}{2}$ to obtain $\mathcal{I}, k$. Now let $\mathcal{I}'$ be the set of "parents" (i.e. the dyadic intervals I such that either $I_l \in \mathcal{I}'$ or $I_r \in \mathcal{I}'$) of elements of $\mathcal{I}$ and $\mathcal{J}$ be the set of maximal elements in $\mathcal{I}'$.
We split the first sum in the following way

$$\sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0}} |I|F(I)G(I) = \sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0 \\ I\subset\cup_{J\in\mathcal{J}}J}} |I|F(I)G(I) + \sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0 \\ I\not\subset\cup_{J\in\mathcal{J}}J}} |I|F(I)G(I).$$

The first one is bounded by ε by boundedness of G and (1).
By Theorem (6.2) and boundedness of G the second one is within $\delta|J_0|$ of

$$\sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0 \\ I\not\subset\cup_{J\in\mathcal{J}}J}} |I|F(I_{(k)})G(I),$$

where $I_{(k)}$ is the dyadic interval of length 2^k containing I. Now we would like to control

$$\sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0 \\ I\subset\cup_{J\in\mathcal{J}}J}} |I|F(I_{(k)})G(I),$$

by ε. This would tell us that the previous sum is close to

$$\sum_{\substack{|I|=2^{k'} \\ I\subseteq J_0}} |I|F(I_{(k)})G(I) = \sum_{\substack{|I|=2^k \\ I\subseteq J_0}} |I|F(I)G(I),$$

and the equality is due to the martingale property of G, ending the proof. The job is done in the following way. First of all we use the martingale property of G to rise the scale of I in the sum to the maximal one staying inside the exceptional set, obtaining

$$\sum_{\substack{2^{k'}\leq|I|\leq 2^k \\ I\subseteq J_0 \\ I\in\mathcal{J}}} |I|F(I_{(k)})G(I).$$

Then we get rid of $G(I)$ by boundedness of G. After that we observe that since I is a maximal double of an element $L \in \mathcal{I}$ then the twin of L is not contained completely in $\cup_{J\in\mathcal{I}} J$. Therefore we can apply Theorem (6.2) to go from $F(I_k)$ to $F(I)$ with an error controlled by $\delta|J_0|$. We conclude by using absolute continuity for the family $\mathcal{J}$, which is made of disjoint intervals and defines a set of measure less or equal than δ by construction. $\square$

However in other situations the study of

$$\lim_{k\to-\infty} \sum_{|I|=2^k} |I|F(I)G(I),$$

is more complicated. For many reasons we would prefer to translate the limit problem into . To obtain it we consider the parallelogram law

$$\forall l, r \in \mathbb{R} \qquad \left(\frac{l-r}{2}\right)^2 = \frac{1}{2}l^2 + \frac{1}{2}r^2 - \left(\frac{l+r}{2}\right)^2.$$

We translate it into the martingale language through the definition

$$\Delta F(I) := \left(\frac{F(I_l) - F(I_r)}{2}\right),$$

$$|I|\Delta F(I)^2 = |I_l|F(I_l)^2 + |I_r|F(I_r)^2 - F(I)^2.$$

Then

$$\sum_{|I|=2^{k-1}} |I|F(I)^2 - \sum_{|I|=2^k} |I|F(I)^2 = \sum_{|I|=2^k} \Delta F(I)^2.$$

Assuming the condition on F

$$\lim_{k\to\infty} |I|F(I)^2 = 0,$$

which is not hard to be satisfied, we translate the problem of existence of a limit into the absolute summability of a series, namely

$$\lim_{k\to-\infty} \sum_{|I|=2^k} |I|F(I)^2 = \sum_{I\in\mathfrak{D}} |I|\Delta F(I)^2,$$

by effect of a telescoping sum. The same algebra works for a general G, leading to

$$\sum_{I\in\mathfrak{D}}|I|\Delta F(I)\Delta G(I).$$

Once we have absolute summability, the following object

$$\sum_{I\in\mathfrak{D}}|I|E(I)\Delta F(I)\Delta G(I),$$

where $E(I)\in[-1,1]$ for every $I\in\mathfrak{D}$, is well defined and we can investigate instances of it.

7 Dyadic paraproducts

2016-11-10

Following up last lecture, we would like to *multiply* martingales, in order to see what happens with the limit

$$\lim_{k\to-\infty}\sum_{|I|=2^k}|I|\prod_{i=1}^{n}F_i(I).$$

In order to do that, we define the auxiliary *difference martingale* as

$$\Delta F(I)=\frac{1}{2}(F(I_l)-F(I_r)).$$

With respect to those functions, we have the following

Lemma 7.1 (Telescoping identity)**.**

$$\sum_{|I|=2^k}\left(|I_l|\prod_{i=1}^{n}F_i(I_l)+|I_r|\prod_{i=1}^{n}F_i(I_r)-|I|\prod_{i=1}^{n}F_i(I)\right)=$$

$$=\sum_{\substack{A\subset\{1,\ldots,n\}\\ |A|\ even\\ A\neq\emptyset}}\sum_{|I|=2^k}|I|\prod_{i\in A}\Delta F_i(I)\times\prod_{i\notin A}F_i(I).$$

Sketch of the proof. Roughly, we do a distributive law. Explicitly, we write

$$\prod_{i=1}^{n}F_i(I_l)=\prod_{i=1}^{n}(F_i(I)+\Delta F_i(I))=\sum_{A\subset\{1,\ldots,n\}}\prod_{i\in A}\Delta F_i(I)\prod_{i\notin A}F_i(I).$$

An analogous formula for $F_i(I_r)$, on the other hand, implies

$$\prod_{i=1}^{n}(F_i(I)+\Delta F_i(I))+\prod_{i=1}^{n}(F_i(I)-\Delta F_i(I))=2\sum_{\substack{A\subset\{1,\ldots,n\}\\ |A| even}}\prod_{i\in A}\Delta F_i(I)\prod_{i\notin A}F_i(I).$$

After summing in I; $|I| = 2^k$ and accounting for the fact that the null set contributes with a factor of $|I|\prod_{i=1}^n F_i(I)$ on the sum above, the lemma is proved. $\square$

Now, if we assume that

1. $$\lim_{k\to\infty} \sum_{|I|=2^k} |I| \prod_{i=1}^n F_i(I) = 0;$$

2. $$\sum_{I\in\mathfrak{D}} ||I| \prod_{i\in A} \Delta F_i(I) \prod_{i\notin A} F_i(I)| < \infty, \forall\, A \subset \{1, ..., n\};$$

and use the previous Lemma, we get that

$$\lim_{k\to-\infty} |I| \prod_{i=1}^n F_i(I) = \sum_{\substack{A\subset\{1,...,n\}\\ |A| \text{ even}\\ A\neq\emptyset}} \sum_{I\in\mathfrak{D}} |I| \prod_{i\in A} \Delta F_i(I) \prod_{i\notin A} F_i(I).$$

We call the inner expression in the last sum, that is,

$$\sum_{I\in\mathfrak{D}} |I| \prod_{i\in A} \Delta F_i(I) \prod_{i\notin A} F_i(I),$$

is usually called the *dyadic paraproduct of type* A, and will appear soon again in our course.

7.1 Hölder's inequality

We start with the following innocent-looking real analysis Lemma:

Lemma 7.2. *Let* $a_i > 0$, $i = 1, ..., n$, $1 \le p_1 < +\infty$ *and* $\sum_{i=1}^n \frac{1}{p_i} = 1$, *then*

$$\prod_{i=1}^n a_i \le \sum_{i=1}^n \frac{a_i^{p_i}}{p_i}.$$

Proof. Let $a_i = e^{t_i/p_i}$. We want them to prove

$$\exp\left(\sum_{i=1}^n \frac{t_i}{p_i}\right) \le \sum_{i=1}^n \frac{e^{t_i}}{p_i}.$$

After defining $\tilde{t_i} = t_i - \sum_{i=1}^n \frac{t_i}{p_i}$, it boils down to prove that

$$1 \le \sum_{i=1}^n \frac{1}{p_i} e^{\tilde{t_i}}.$$

It suffices then to show that, if $g(t) = \sum_{i=1}^n \frac{1}{p_i} e^{t\cdot\tilde{t_i}}$, then $g(t) \ge 1$, $\forall t \ge 0$. But this itself follows if we show that:

1. $g(0) = 1$, which is obvious from the definition.

2. $g'(0) = 0$, which follows from the fact that $g'(0) = \sum_{i=1}^{n} \frac{\tilde{t_i}}{p_i} = 0$.

3. $g''(t) \geq 0$, which follows from the fact that $g''(t) = \sum_{i=1}^{n} \frac{\tilde{t_i^2}}{p_i} e^{t \cdot \tilde{t_i}}$.

This finishes the proof. □

We suppose then that we have a collection of functions $f_i : \mathbb{Z} \to \mathbb{R}$, $i = 1, ..., n$, and we want to estimate

$$\sum_{l \in \mathbb{Z}} \prod_{i=1}^{n} f_i(l).$$

By using the previous Lemma with each of the terms $\prod_{i=1}^{n} |f_i(l)|$ and changing the order of summation, we have that that quantity is bounded by

$$\sum_{i=1}^{n} \frac{1}{p_i} \sum_{l \in \mathbb{Z}} |f_i(l)|^{p_i}.$$

Our next assumption is then that, for each $i = 1, ..., n$, it holds that

$$\sum_{l \in \mathbb{Z}} |f_i(l)|^{p_i} = 1.$$

With that, we get then that

$$\sum_{l \in \mathbb{Z}} \prod_{i=1}^{n} f_i(l) \leq 1 = \prod_{i=1}^{n} \left(\sum_{l \in \mathbb{Z}} |f_i(l)|^{p_i} \right)^{1/p_i}.$$

But both sides in the last expression above are invariant under scaling, i.e., the inequality remains the same if we change f_i by $\lambda_i f_i$, where $\lambda_i > 0$. By picking such λ_i such that $\sum_{l \in \mathbb{Z}} \lambda_i^{p_i} |f_i(l)|^{p_i} = 1$, and using the last inequality above for $\lambda_i f_i$, we conclude that it must always hold that

$$\sum_{l \in \mathbb{Z}} \prod_{i=1}^{n} f_i(l) \leq \prod_{i=1}^{n} \left(\sum_{l \in \mathbb{Z}} |f_i(l)|^{p_i} \right)^{1/p_i},$$

and this is what we call *Hölder's inequality.* From that we define the spaces

$$L^p(\mathbb{R}) = \{F : \mathfrak{D} \to \mathbb{R} \text{ martingale such that}$$
$$\|F\|_p^p := \sup_{k \in \mathbb{Z}} \sum_{|I| = 2^k} |I| F(I)^p < +\infty\}.$$

At least for such martingales and $p_i \in [1, +\infty)$, by using Hölder's inequality, we have that

$$\sup_{k \in \mathbb{Z}} \sum_{|I|=2^k} |I| \prod_{i=1}^{n} F_i(I) \leq \prod_{i=1}^{n} \|F_i\|_{p_i}.$$

7.2 Outer measures

In this section, we will devote ourselves to defining an outer structure on various model sets. Basically, an *outer measure* is going to be a set function μ, defined on subsets of a set S, that arises from a previously defined set function σ on a smaller class of subsets $\mathcal{E}$ of S. The name 'outer measure' is simple to understand: once we have our σ, our outer measure μ is then defined as

$$\mu(A) = \inf_{\substack{\mathcal{E}' \subset \mathcal{E} \\ A \subset \cup_{E \in \mathcal{E}'} E.}} \sum_{E \in \mathcal{E}'} \sigma(E).$$

Why 'outer'? Well, covering a set has encompasses always the notion of 'containing' it, and some object that contains another is generally regarded as being on the outside. A good metaphor is: when you cover yourself with a blanket, the blanket it outside you.
Now we pass to our examples:

Example 7.3. *The set of integers* $\mathbb{Z}$. In this set, we take our collection of subsets $\mathcal{E}$ to be the set of all singletons, i.e.,

$$\mathcal{E} = \{\{n\}, n \in \mathbb{Z}.\},$$

and define therein the set function $\sigma(\{n\}) = 1$. For this definition, as every subset of $\mathbb{Z}$ is a disjoint union of elements in $\mathcal{E}$, the outer measure so defined will be

$$\mu(A) = \#A,$$

or, as it is commonly called, it is going to be then the *counting measure* on $\mathbb{Z}$. In order to go on, we need a class of functions $\mathcal{B}$ with the property that, if $f \in \mathcal{B}$, $\mathcal{E}' \subset \mathcal{E}$, then $f \mathbb{1}_{S \setminus \cup_{E \in \mathcal{E}'} E} \in \mathcal{B}$. In this case, it will be enough to take

$$\mathcal{B} = \{ \text{ all functions } f : \mathbb{Z} \to \mathbb{R}.\}$$

We are also going to need a way to measure the 'size' of functions. This is achieved by a functional $S : \mathcal{B} \times \mathcal{E} \to \mathbb{R}_{\geq 0}$, which satisfies the following properties:

1. *Monotonicity.* If $f \leq g$, then we must have, for all $E \in \mathcal{E}$, $Sf(E) \leq Sg(E)$.
2. *Scaling.* If $f \in \mathcal{B}, \lambda > 0$, then $S(\lambda f)(E) = \lambda Sf(E)$, for all $E \in \mathcal{E}$.
3. *Quasi-subadditivity.* There is $C > 0$ such that, if $f, g \in \mathcal{B}, E \in \mathcal{E}$, then $S(f+g)(E) \leq C(Sf(E) + Sg(E))$.

In the case of the integers, it is going to suffice to take $Sf(\{n\}) = |f(n)|$.
Given those tools, we are ready to define the outer L^p spaces. In the right endpoint case, the definition boils down to the following:

$$L^\infty(S) = \{f \in \mathcal{B}; \|f\|_{L^\infty(S)} := \sup_{E \in \mathcal{E}} Sf(E) < +\infty\}.$$

In the present case, we will recover the space of bounded functions on $\mathbb{Z}$, $\ell^\infty(\mathbb{Z})$. In the other cases, we define it in another way, which at first glance might seem artificial, but which also whenever we translate into our measurable language will make complete sense:

$$L^p(S) = \{f \in \mathcal{B};$$

$$\|f\|^p_{L^p(S)} := \int_0^\infty p\lambda^{p-1} \left(\inf_{\substack{\mathcal{E}' \subset \mathcal{E} \\ \|f\mathbb{1}_{S\setminus \cup_{E\in\mathcal{E}'}E}\|_{L^\infty(S)} < \lambda}} \sum_{E\in\mathcal{E}'} \sigma(E) \right) d\lambda < +\infty.\}$$

This frighteningly big expression makes complete sense in our case, though: let $F_\lambda = \{n \in \mathbb{Z}; |f(n)| \geq \lambda.\}$. Then for sure $f\mathbb{1}_{S\setminus F_\lambda} < \lambda$, and obviously any other subcollection $\mathcal{E}'$ that fulfills this property must also contain F_λ. Therefore, in this case we have the explicit expression

$$\inf_{\substack{\mathcal{E}' \subset \mathcal{E} \\ \|f\mathbb{1}_{S\setminus \cup_{E\in\mathcal{E}'}E}\|_{L^\infty(S)} < \lambda}} \sum_{E\in\mathcal{E}'} \sigma(E) = \#\{n; |f(n)| > \lambda\}.$$

The outer L^p norm becomes, then,

$$\int_0^\infty p\lambda^{p-1}\#\{n \in \mathbb{Z}; |f(n)| > \lambda\} d\lambda = \int_0^\infty p\lambda^{p-1} \left(\sum_{n\in\mathbb{Z}} \mathbb{1}_{\{|f|>\lambda\}} \right) d\lambda$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{|f(n)|} p\lambda^{p-1} d\lambda = \sum_{n\in\mathbb{Z}} |f(n)|^p,$$

which is the usual L^p norm on $\mathbb{Z}$.

Example 7.4. *The real line* $\mathbb{R}$. In this setting, we have to be a little more careful while choosing our subsets and subfamilies. Explicitly, in this case we will take as our model subset $\mathcal{E} = \mathfrak{D}$, the set of dyadic intervals. Moreover, obviously every interval on the real line is endowed with its measure, and this defines σ by $\sigma(I) = |I|$. The (outer) measure μ we therefore obtain from this procedure is the so-called *outer Lebesgue measure.*

Now we need to choose our prototypical space $\mathcal{B}$. In this case, our objects of study will not be precisely functions, but, as we have been seeing lately, martingales. The space $M_\mathfrak{D}$ defined previously is going to be our "function space", and we must show that it satisfies the property of elements in $\mathcal{B}$, namely, if $\mathcal{E}'$ is a collection of dyadic intervals, then we define

$$F\mathbb{1}_{\mathbb{R}\setminus\cup_{E\in\mathcal{I}}E}(J) = \begin{cases} 0, & \text{if } J \subset I \in \mathcal{I}, \\ F(J) - \sum_{I\in\mathcal{I}; I\subset J} F(I), & \text{otherwise.} \end{cases}$$

It is easy then to verify that this gives rise to another element in $M_\mathfrak{D}$. Alternatively, we could have used any equivalent class of functions, like the class

of Borel measures, to define such function space.
The size functional, in this case, is given explicitly by $SF(I) = |F|(I)$. It is easy from the definition to check that the outer $L^\infty(S)$ space in this case coincides with the space $L^\infty \cap L^1(\mathbb{R})$, as the definition of elements in $M_{\mathfrak{D}}$ already supposes $F \in L^1(\mathbb{R})$. In the same way as it was done to the case of $\mathbb{Z}$, one can show that the outer L^p spaces in this case coincide with $L^p \cap L^1(\mathbb{R})$.

Example 7.5. *The set of dyadic intervals* $\mathfrak{D}$. This is perhaps the most important case, for it gives rise to non-standard spaces that can be used for the control of paraproducts. Explicitly, our set $S = \mathfrak{D}$, and our collection of subsets will be

$$\mathcal{T} = \{T_J = \{I \in \mathfrak{D}; I \subset J\},\ J \in \mathcal{D}\}.$$

Those subsets are often called "trees", as they are collections of dyadic intervals associated to a common "root", which is the interval J. Instinctively, we endow such a tree with a measure of its top, that is, we define our $\sigma(T_J) = |J|$. Surprisingly, the arising outer measure is somewhat nonstandard, in the sense that it has not been extensively studied since recently, and therefore has no standardized name.
We can also here define our space $\mathcal{B}$ as the space of "all functions", that is, all functions from $\mathcal{D}$ to $\mathbb{R}$, and with that define not only one, but rather than a *family* of size functionals by

$$S_r F(T_J) = \left(\frac{1}{|J|} \sum_{I \subset J} |I| |F(I)|^r \right)^{1/r}.$$

Now there are no "trivial" reductions to be made on the definition of the outer L^p spaces. On the case of $p = \infty$, those elements are a little more famous than usual: $L^\infty(S_r)$, for when $r < +\infty$, were spaces studied by Lennart Carleson, and $L^\infty(S_\infty)$ is just – by definition – the usual $L^\infty(\mathbb{R})$. Due to this intrinsic difference, we will denote from now on these outer L^p spaces as $\mathbb{L}^p(S_r)$.

As promised in this last example, we are going to put together our three themes of today's lecture: we wish to use outer measure spaces, along with Hölder inequalities, to bound paraproducts! Explicitly, we hope that there exists some sort of "outer Hölder inequality", that should give us something like

$$\sum_{I \in \mathfrak{D}} \left| |I| \prod_{i \in A} \Delta F_i(I) \prod_{i \notin A} F_i(I) \right| \leq C \prod_{i \in A} \|\Delta F_i\|_{\mathbb{L}^{p_i}(S_2)} \prod_{i \notin A} \|F_i\|_{\mathbb{L}^{p_i}(S_\infty)}.$$

Next, we hope we can find suitable embedding theorems for outer measure spaces, in order to bound an outer norm by an usual L^p one. More explicitly, we also hope that an inequality like

$$\prod_{i \in A} \|\Delta F_i\|_{\mathbb{L}^{p_i}(S_2)} \prod_{i \notin A} \|F_i\|_{\mathbb{L}^{p_i}(S_\infty)} \leq C \prod_{i \in A} \|F_i\|_{p_i} \prod_{i \notin A} \|F_i\|_{p_i}$$

could hold. This would help us analyze paraproducts, and, therefore, make sense of the limits of the products as defined in the previous lecture, as we desired.

Below, the reader may find a table that summarizes all we have discussed in terms of outer measures and our examples:

	$\mathbb{Z}$	$\mathbb{R}$	$\mathfrak{D}$
$\mathcal{E}$	$\{\{n\}, n \in \mathbb{Z}\}$	$\mathfrak{D}$	$\mathcal{T} = \{T_J = \{I \in \mathfrak{D}, I \subset J\}, J \in \mathfrak{D}\}$.
σ	$\sigma(\{n\}) = 1$	$\sigma(I) = \|I\|$	$\sigma(T_J) = \|J\|$.
μ	Counting measure	Lebesgue outer measure	Not named yet.
$\mathcal{B}$	All functions	Martingales in $M_{\mathfrak{D}}$	All functions.
S	$Sf(\{n\}) = \|f(n)\|$	$SF(I) = \|F(I)\|$	$S_r F(T_J) = \left(\frac{1}{\|J\|} \sum_{I \subset J} \|I\| \|F(I)\|^r\right)^{1/r}$
$L^\infty(S)$	$\ell^\infty(\mathbb{Z})$	$L^\infty \cap L^1(\mathbb{R})$	$L^\infty(\mathbb{R})$, if $r = \infty$, $r -$ Carleson const. , $r \in (1, \infty)$.
$L^p(S)$	$\ell^p(\mathbb{Z})$	$L^p \cap L^1(\mathbb{R})$	$\mathcal{L}^p(S_r)$.

8 $\mathcal{L}^p$-theory for outer measures

2016-11-15

We start recalling some notation: X is a set, $\mathcal{E}$ is a collection of subsets of X and $\sigma\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$.
We want to define an *outer measure* μ on the set of subsets of X, which is in general too big. The data $(X, \mathcal{E}, \sigma)$ specify the values of a pre-measure σ on a small collection of subsets of X. The following step is to define μ on an arbitrary subset by means of covering it with elements in $\mathcal{E}$. Namely for every $A \subset X$ we define

$$\mu(A) := \inf_{\substack{\mathcal{E}' \subset \mathcal{E} \\ \mathcal{E}' \text{ covers } A}} \sum_{E \in \mathcal{E}'} \sigma(E),$$

where "$\mathcal{E}'$ covers A" means that $A \subset \cup_{E \in \mathcal{E}'} E$.
This construction satisfies the following properties, which sometimes are used as a definition of outer measures:

Theorem 8.1. *1. if $A \subset A'$ for $A, A' \subset X$, then $\mu(A) \leq \mu(A')$;*

2. $\mu(\emptyset) = 0$;

3. (Subadditivity) for $\{A_i\}_{i \in \mathbb{N}}$ countable collection of subsets of X, then

$$\mu(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mu(A_i).$$

Proof. 1. It is enough to observe that every cover of A' is a cover of A.
2. $\emptyset$ is a cover of $\emptyset$.
3. Without loss of generality we can assume that $\mu(A_i) < \infty$ for every $i \in \mathbb{N}$, otherwise the inequality is trivial. Once fixed $\delta > 0$, let $\mathcal{E}_i$ covers A_i and satisfies $\mu(A_i) + 2^{-i}\delta \geq \sum_{E\in\mathcal{E}_i} \sigma(E)$. Then

$$\sum_{i=1}^{\infty} \mu(A_i) + \delta \geq \sum_{i=1}^{\infty} \sum_{E\in\mathcal{E}_i} \sigma(E) \geq \sum_{E\in\cup_{i=1}^{\infty}\mathcal{E}_i} \sigma(E) \geq \mu(\bigcup_{i=1}^{\infty} A_i),$$

where the last inequality is due to the fact that $\cup_{i=1}^{\infty}\mathcal{E}_i$ covers $\cup_{i=1}^{\infty}A_i$ □

Remark 8.2. $A \subset X$ is called *Carathéodory measurable* if

$$\forall E \in \mathcal{E} \qquad \mu(E) = \mu(E \cap A) + \mu(E \cap A^c). \tag{8.3}$$

Lemma 8.4 (Ex. sheet 4, Pb. 2)**.** *For $\{A_i\}_{i\in\mathbb{N}}$ countable collection of pairwise disjoint subsets of X that are Carathéodory measurable, then*

$$\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

This lemma allows us to go from a subadditive theory for an outer measure to an additive theory for the associated measure, provided that we restrict to Carathéodory measurable sets.
In the case of Lebesgue outer measure (second example from previous lecture) there are plenty of Carathéodory measurable sets, which are called Lebesgue measurable sets, and the measure is interesting. However in general the set of Carathéodory measurable sets can be very poor, even containing only the two trivial ones $\emptyset, X$ (as in the third example from previous lecture, see again Ex. sheet 4, Pb. 2). Therefore the measure associated to such an outer measure would be trivial.
In particular we are interested in developing an integration theory for outer measures. First of all we observe that the "splitting property" (8.3) of measurable sets doesn't hold for outer measures on a general subset of X. Therefore there is no hope to obtain additive integral objects. For example if we give the following meaning to $\int$

$$\text{“}\mu(A) = \text{”} \int \mathbb{1}_A,$$

then for $E \in \mathcal{E}$ we have the subadditivity

$$\int \mathbb{1}_E \leq \int \mathbb{1}_{E\cap A} + \int \mathbb{1}_{E\cap A^c},$$

and for a general $A \subset X$ there exists E for which the inequality is strict.
Therefore in the case of outer measures we consider integral objects which

are already subadditive in the classical case. The standard example of this kind of integral objects is

$$\left(\int |f|^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty.$$

Before moving on in the construction of this integration theory we need to request further structure:

- let $\mathcal{B}$ be a collection of function on X closed under linear operation, i.e. for $f, g \in \mathcal{B}$ then $\lambda f, f+g \in \mathcal{B}$, and such that $\mathbb{1}_X \in \mathcal{B}$. Moreover if $f \in \mathcal{B}, \mathcal{E} \subset \mathcal{E}'$, then
$$f \mathbb{1}_{(\cup_{E \in \mathcal{E}'} E)^c} \in \mathcal{B};$$
- a *size*, i.e. a map $S \colon \mathcal{B} \times \mathcal{E} \to \mathbb{R}_{\ge 0}$ satisfying the following properties for every $f, g \in \mathcal{B}, E \in \mathcal{E}$:
    1. (monotone) if $|f| \le |g|$, then $Sf(E) \le Sg(E)$;
    2. (scaling) for $\lambda \ge 0$, $S(\lambda f)(E) = \lambda Sf(E)$;
    3. (quasi subadditive) $S(f+g)(E) \le C(Sf(E) + Sg(E))$ for some $C < \infty$ independent of f, g, E.

  The meaning of S is of an already given integration theory for specific functions on specific subsets, namely averages on sets in $\mathcal{E}$. Usually we can get around the unease of having these data by means of the very way in which elements in $\mathcal{B}$ are defined. For example in the case of martingale the functions are defined morally through their averages on dyadic intervals.

For example in the case of Lebesgue outer measure $\mathcal{B}$ is the set of functions in L^1_{loc} and the size map S given by taking averages of the absolute value of such a function on dyadic intervals.
We are ready to start the construction of an integration theory for outer measure spaces.

Definition 8.5 (Outer essential supremum $\|\cdot\|_{\mathcal{L}^\infty(S)}$, $\mathcal{L}^\infty(S)$)**.** For $f \in \mathcal{B}$ we define
$$\|f\|_{\mathcal{L}^\infty(S)} \coloneqq \sup_{E \in \mathcal{E}} Sf(E),$$
and we call it the *outer essential supremum* of f.
We denote by $\mathcal{L}^\infty(S)$ the set of functions $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{L}^\infty(S)} < \infty$.

This definition conceives the idea that in our theory $Sf(E)$ takes the moral role of point evaluation in the classical case. The outer essential supremum satisfies properties analogous to the ones that hold for S. We use it to define

Definition 8.6 (Super level measure)**.** For $f \in \mathcal{B}, \lambda > 0$ we define

$$\mu(Sf > \lambda) := \inf_{\mathcal{E}' : \|f \mathbb{1}_{(\cup_{E \in \mathcal{E}'} E)^c}\|_{\mathcal{L}^\infty(S)} \leq \lambda} \sum_{E \in \mathcal{E}'} \sigma(E),$$

and we call it the *super level measure* associated to f, λ.

Remark 8.7. We would like to draw attention to the notation and the fact that it has not to be intended as "the value of the outer measure on the subset of X where $|f|$ is larger than λ". In many cases it will be precisely this, but in general it is not even clear that it corresponds to an outer measure of a subset of X.

Again, it satisfies properties analogous to the ones that hold for S, namely for every $f, g \in \mathcal{B}, E \in \mathcal{E}$:

Lemma 8.8. *1. (monotone) if $|f| \leq |g|$, then $\mu(Sf > \lambda) \leq \mu(Sg > \lambda)$;*

2. (scaling) for $\lambda,\ \lambda' \geq 0,\ \mu(S(\lambda f) > \lambda\lambda') = \mu(Sf > \lambda')$;

3. (quasi subadditive) $\mu(S(f+g) > C\lambda) \leq \mu(Sf > \lambda) + \mu(Sg > \lambda)$, for some $C < \infty$ independent of f, g.

Proof. The proof is left as an exercise. □

The definition of $\mathcal{L}^p(S)$-spaces follow the ones of L^p-spaces in the classical context, once we have substituted our definitions of outer essential supremum and super level measure for their classical counterparts.

Definition 8.9 ($\|\cdot\|_{\mathcal{L}^p(S)}, \mathcal{L}^p(S)$)**.** For $0 < p < \infty, f \in \mathcal{B}$ we define

$$\|f\|_{\mathcal{L}^p(S)} := \left(\int_0^\infty p\lambda^{p-1} \mu(Sf > \lambda) d\lambda \right)^{\frac{1}{p}}.$$

We denote by $\mathcal{L}^p(S)$ the set of functions $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{L}^p(S)} < \infty$.

Definition 8.10 ($\|\cdot\|_{\mathcal{L}^{p,\infty}(S)}, \mathcal{L}^{p,\infty}(S)$)**.** For $0 < p < \infty, f \in \mathcal{B}$ we define

$$\|f\|_{\mathcal{L}^{p,\infty}(S)} := \left(\sup_{\lambda > 0} \lambda^p \mu(Sf > \lambda) \right)^{\frac{1}{p}}.$$

We denote by $\mathcal{L}^{p,\infty}(S)$ the set of functions $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{L}^{p,\infty}(S)} < \infty$.
For $p = \infty$

$$\|f\|_{\mathcal{L}^{\infty,\infty}(S)} := \|f\|_{\mathcal{L}^\infty(S)}.$$

and therefore $\mathcal{L}^{\infty,\infty}(S) = \mathcal{L}^\infty(S)$.

The spaces in this last definition sometimes are referred to also as *weak* $\mathcal{L}^p(S)$ *spaces* or *Lorentz spaces* $\mathcal{L}^{p,\infty}(S)$.
The function $\|\cdot\|_{\mathcal{L}^p(S)}$ satisfies properties analogous to the ones that hold for super level measure, namely for every $f, g \in \mathcal{B}$:

Lemma 8.11. *1. (monotone) if $|f| \leq |g|$, then $\|f\|_{\mathcal{L}^p(S)} \leq \|g\|_{\mathcal{L}^p(S)}$;*

2. (scaling) for $\lambda \geq 0, \|\lambda f\|_{\mathcal{L}^p(S)} = \lambda \|f\|_{\mathcal{L}^p(S)}$;

3. (quasi subadditive) $\|f+g\|_{\mathcal{L}^p(S)} \leq C\left(\|f\|_{\mathcal{L}^p(S)} + \|g\|_{\mathcal{L}^p(S)}\right)$, for some $C < \infty$ independent of f, g.

Proof. The proof is left as an exercise. □

Remark 8.12. The proof of (3.) based on the corresponding property of the super level measure yields a constant C different from 1. Even in the case of the outer measure leading to the Lebesgue measure, for which we have

$$\mu(|(f+g)(x)| > 2\lambda) \leq \mu(|f(x)| > \lambda) + \mu(|g(x)| > \lambda),$$

we cannot recover in this way a better quasi subadditivity than

$$\|f+g\|_{L^p} \leq 2(\|f\|_{L^p} + \|g\|_{L^p}).$$

Therefore it is rather "miraculous" that in this case we actually have

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Lemma 8.13 (Chebysheff's inequality). $\|f\|_{\mathcal{L}^{p,\infty}(S)} \leq \|f\|_{\mathcal{L}^p(S)}$.

Proof. Assume first that the left hand side is finite and let $\delta > 0$. Pick λ_0 such that

$$\|f\|^p_{\mathcal{L}^{p,\infty}(S)} \leq \delta + \lambda_0^p \mu(Sf > \lambda_0).$$

Then

$$\begin{aligned}\|f\|^p_{\mathcal{L}^p(S)} &= \int_0^\infty p\lambda^{p-1}\mu(Sf > \lambda)d\lambda \geq \int_0^{\lambda_0} p\lambda^{p-1}\mu(Sf > \lambda)d\lambda \geq \\ &\geq \int_0^{\lambda_0} p\lambda^{p-1}\mu(Sf > \lambda_0)d\lambda = \lambda_0^p\mu(Sf > \lambda_0) \geq \|f\|^p_{\mathcal{L}^{p,\infty}(S)} - \delta.\end{aligned}$$

By choosing δ arbitrarily small we get the claim.
The proof of the inequality if the left hand side is infinite, namely that also the right hand one is infinite, is already contained in the last chain of inequalities. □

In general the converse doesn't hold. However a "weaker version" is provided by the following

Lemma 8.14 (Logarithmic convexity). *Let $0 < p_1 < p < p_2 \leq \infty$ and $\alpha_1, \alpha_2 \in (0,1)$ such that*

$$\alpha_1 + \alpha_2 = 1, \ \frac{1}{p} = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2}.$$

Then for $f \in \mathcal{B}$

$$\|f\|_{\mathcal{L}^p(S)} \leq C_{p_1,p_2,p} \|f\|^{\alpha_1}_{\mathcal{L}^{p_1,\infty}(S)} \|f\|^{\alpha_2}_{\mathcal{L}^{p_2,\infty}(S)},$$

for some $C_{p_1,p_2,p} < \infty$ independent of f.

Proof. If either of the factors on the right hand side of the inequality vanishes then $\mu(Sf > \lambda) = 0$ for all $\lambda > 0$. Therefore also the left hand side is 0 and the inequality trivially holds.
Thus without loss of generality we can assume $\|f\|_{\mathcal{L}^{p_i,\infty}} \neq 0, i = 1, 2$.
$\underline{p_2 < \infty}$: by a scaling argument we may assume

$$A := \|f\|^{p_1}_{\mathcal{L}^{p_1,\infty}(S)} = \|f\|^{p_2}_{\mathcal{L}^{p_2,\infty}(S)}.$$

If not consider $\lambda_0 f$ which produces a relative factor $\lambda_0^{p_1-p_2}$.
Then

$$\mu(Sf > \lambda) \leq A \min\{\lambda^{-p_1}, \lambda^{-p_2}\},$$

$$\begin{aligned} \|f\|^p_{\mathcal{L}^p(S)} &= \int_0^\infty p\lambda^{p-1}\mu(Sf > \lambda) d\lambda \leq \\ &\leq Ap \left(\int_0^1 \lambda^{p-p_1-1} d\lambda + \int_1^\infty \lambda^{p-p_2-1} d\lambda \right) \leq \tilde{C} A. \end{aligned}$$

This yields

$$\|f\|_{\mathcal{L}^p(S)} \leq CA^{\frac{1}{p}} = C(A^{\frac{\alpha_1}{p_1}} A^{\frac{\alpha_2}{p_2}}) = C\|f\|^{\alpha_1}_{\mathcal{L}^{p_1,\infty}(S)} \|f\|^{\alpha_2}_{\mathcal{L}^{p_2,\infty}(S)}.$$

$\underline{p_2 = \infty}$: the proof is left as an exercise. □

Theorem 8.15 (Hölder's inequality)**.** *Let $S, S_1, \ldots, S_n$ be sizes. Assume that for every $E \in \mathcal{E}$ there exist $E_1, \ldots, E_n \in \mathcal{E}$ such that whenever for a fixed $f \in \mathcal{B}$ we have $f = \prod f_i$ with $f_i \in \mathcal{B}$, then*

$$Sf(E) \leq \prod_{i=1}^n S_i f_i(E_i). \tag{8.16}$$

Let $p, p_1, \ldots, p_n \in (0, \infty]$ such that $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$. Then

$$\|f\|_{\mathcal{L}^p(S)} \leq n \prod_{i=1}^n \|f_i\|_{\mathcal{L}^{p_i}(S_i)}.$$

Remark 8.17. The hypothesis in (8.16) asks for a Hölder inequality for the sizes $S, S_1, \ldots, S_n$.

Proof. $\underline{\forall i, p_i \neq \infty}$: by a scaling argument we may assume

$$\forall i \in \{1, \dots, n\} \qquad \|f\|_{\mathcal{L}^{p_i}(S_i)} = 1.$$

Fix $\lambda > 0, \delta > 0$ and pick covers $\mathcal{E}_i$ such that

$$\sum_{E \in \mathcal{E}_i} \sigma(E) \leq \mu(S_i f_i > \lambda^{\frac{p}{p_i}}) + \delta,$$

$$\sup_{E \in \mathcal{E}} S_i \left(f_i \mathbb{1}_{(\cup_{E \in \mathcal{E}_i})^c} \right)(E) \leq \lambda^{\frac{p}{p_i}}.$$

Define $\tilde{\mathcal{E}} = \bigcup_{i=1}^n \mathcal{E}_i$. Let $E \in \mathcal{E}$ be given, then there exist $\{E_i\}_{i=1}^n$ such that

$$S\left(f \mathbb{1}_{(\cup_{E \in \tilde{\mathcal{E}}})^c}\right)(E) \leq \prod_{i=1}^n S_i \left(f_i \mathbb{1}_{(\cup_{E \in \tilde{\mathcal{E}}})^c}\right)(E_i) \leq \prod_{i=1}^n S_i \left(f_i \mathbb{1}_{(\cup_{E \in \mathcal{E}_i})^c}\right)(E_i) \leq$$
$$\leq \prod_{i=1}^n \lambda^{\frac{p}{p_i}} = \lambda,$$

$$\mu(Sf > \lambda) \leq \sum_{i=1}^n \sum_{E \in \mathcal{E}_i} \sigma(E) \leq n\delta + \sum_{i=1}^n \mu(S_i f_i > \lambda^{\frac{p}{p_i}}).$$

By choosing δ arbitrarily small we can forget about the first summand. To conclude we observe

$$\|f\|^p_{\mathcal{L}^p(S)} = \int_0^\infty p\lambda^{p-1} \mu(Sf > \lambda) d\lambda \leq \int_0^\infty p\lambda^{p-1} \sum_{i=1}^n \mu(S_i f_i > \lambda^{\frac{p}{p_i}}) d\lambda =$$
$$\overset{\nu = \lambda^{p/p_i}}{=} \sum_{i=1}^n \int_0^\infty p_i \nu^{p_i - 1} \mu(S_i f_i > \nu) d\nu = \sum_{i=1}^n \|f\|_{\mathcal{L}_i^p(S_i)} =$$
$$= n = n \prod_{i=1}^n \|f\|_{\mathcal{L}_i^p(S_i)}.$$

$\underline{\max\{p_i\} = \infty}$: the proof is left as an exercise. □

9 The Marcinkiewicz interpolation theorem

2016-11-17

We keep extending the basic result of L^p-spaces' theory to $\mathcal{L}^p(S)$-spaces presenting the *Marcinkiewicz Interpolation Theorem.*

Theorem 9.1 (Marcinkiewicz interpolation theorem)**.** *Let* $X, \sigma, \mathcal{E}, \mathcal{B}, S$ *and* $X', \sigma', \mathcal{E}', \mathcal{B}', S'$ *be two outer measure structures. Assume then that* $0 < p_1 < p < p_2 \leq \infty$, *and that there is an operator* $T : \mathcal{B}' \to \mathcal{B}$ *such that*

$$\begin{cases} T(\lambda f) = \lambda T f, & \text{for } \lambda > 0, \\ T(f+g) \leq C(Tf + Tg), & \text{for all } f, g \in \mathcal{B}', \\ \|Tf\|_{\mathcal{L}^{p_i,\infty}(S)} \leq C_i \|f\|_{\mathcal{L}^{p_i}(S')}, & \text{for all } f \in \mathcal{L}^p(S') \text{ and } i = 1, 2. \end{cases}$$

Then it holds that

$$\|Tf\|_{\mathcal{L}^p(S)} \leq C_{p_1,p_2,p,C_1,C_2}\|f\|_{\mathcal{L}^p(S')}.$$

Proof. Pick a function f, $\|f\|_{\mathcal{L}^p(S')} < +\infty$ and $\lambda \in (0, +\infty)$. Let also $\mathcal{E}'$ be a collection of subsets such that $\|f\mathbb{1}_{(\cup_{E\in\mathcal{E}'}E)^c}\|_{L^\infty(S')} < \lambda$ and $\sum_{E\in\mathcal{E}'}\sigma'(E) \leq 2\mu'(S'f > \lambda)$. Let then $f = f_1 + f_2$, where $f_2 = f\mathbb{1}_{(\cup_{E\in\mathcal{E}'}E)^c}$. Then we have that

$$\mu(S(Tf) > C\lambda) \leq \mu(S(Tf_1) > \lambda) + \mu(S(Tf_2) > \lambda),$$

for some constant $C > 0$. This is just a consequence of the quasi-subadditivity properties of both S and T, and we leave the details to the reader. By using the boundedness properties of T, we have that

$$\mu(S(Tf) > C\lambda) \leq \lambda^{-p_1}\|f_1\|^{p_1}_{\mathcal{L}^{p_1}(S')} + \lambda^{-p_2}\|f_2\|^{p_2}_{\mathcal{L}^{p_2}(S')}.$$

On the other hand, it is easy to see from the definitions of f_1 and f_2 that the right hand side of the inequality above is bounded by

$$C\lambda^{-p_1}\int_0^\infty \nu^{p_1-1}\min(\mu'(S'f > \lambda), \mu'(S'f > \nu))d\nu +$$
$$C\lambda^{-p_2}\int_0^\lambda \nu^{p_2-1}\mu'(S'f > \nu)d\nu.$$

Therefore, we may bound then

$$\begin{aligned}
&\int_0^\infty \lambda^{p-1}\mu(S(Tf) > \lambda)d\lambda \\
&\quad \leq C\int_0^\infty\int_0^\lambda \lambda^{-p_1}\lambda^{p-1}\nu^{p_1-1}\mu'(S'f > \lambda)d\nu d\lambda \\
&\quad + C\int_0^\infty\int_\lambda^\infty \lambda^{-p_1}\lambda^{p-1}\nu^{p_1-1}\mu'(S'f > \nu)d\nu d\lambda + \\
&\quad + C\int_0^\infty\int_0^\lambda \lambda^{-p_2}\lambda^{p-1}\nu^{p_2-1}\mu'(S'f > \nu)d\nu d\lambda.
\end{aligned}$$

Finally, a use of Fubini's theorem - which is mechanical and therefore must be ommited here - shows that each of the terms above is controlled by

$$C\int_0^\infty \nu^{p-1}\mu'(S'f > \nu)d\nu \leq C'\|f\|^p_{\mathcal{L}^p(S')}.$$

This finishes the proof, except for the case $p_2 = \infty$, where a simple modification of the proof above gives the result. The details are left to the reader. □

9.1 Back to the upper half plane

Let, as we defined in a previous lecture,

$L^p(\mathbb{R}) = \{$ set of functions $F : \mathfrak{D} \to \mathbb{R}$ such that

1. $|I|F(I) = |I_l|F(I_l) + |I_r|F(I_r)$,
2. $\sup_k \sum_{|I|=2^k} |I||F(I)|^p (=: \|F\|_p^p) < +\infty$.
3. $p = 1 \Rightarrow \lim_{k\to\infty} |I_k|F(I_k) = 0,\ I_{k+1} = (I_k)_r.\}$

Remark 9.2. If $1 < p < \infty$, then the third condition above is a consequence of the second one, and the proof of this fact is left as an exercise to the reader.

Of course, we must also include and exception: if $p = \infty$, we define the space $L^\infty(\mathbb{R})$ as the set of martingales $f : \mathfrak{D} \to \mathbb{R}$ such that $\|F\|_\infty := \sup_I |F(I)| < +\infty$. We are, from this moment on, going to be specified on the example of outer measure spaces on dyadic intervals. More specifically, we let $X = \mathfrak{D}$, $\mathcal{E} = \{T_J, J \in \mathfrak{D}\}$, where $T_J = \{I \in \mathfrak{D} : I \subset J\}$, and $\sigma(T_J) = |J|$. Moreover, our set $\mathcal{B}$ will then be the set of all function from $\mathfrak{D} \to \mathbb{R}$. We are, though, going to consider two different size functionals. The first is going to be the 2-size

$$S_2F(T_J) = \left(\frac{1}{|J|} \sum_{I\in T_J} |I|F(I)^2 \right)^{1/2},$$

while the second is then the ∞-functional given by

$$S_\infty F(T_J) = \sup_{I\in T_J} |F(I)|.$$

Theorem 9.3 (Embedding theorem). *If $f \in L^p(\mathbb{R})$, then, for all $1 < p \leq +\infty$, the following two inequalities hold:*

1. $\|F\|_{\mathcal{L}^p(S_\infty)} \leq C\|F\|_{L^p(\mathbb{R})}$,
2. $\|\Delta F\|_{\mathcal{L}^p(S_2)} \leq C\|F\|_{L^p(\mathbb{R})}$,

where we remember the definition of $\Delta F = \frac{1}{2}(F(I_l) - F(I_r))$.

Remark 9.4. As a corollary of such a theorem, it is possible to show that $\|\Delta F\|_{\mathcal{L}^p(S_\infty)} \leq C\|F\|_{L^p(\mathbb{R})}$.

Proof of theorem 25.2. We will use as a crucial weapon the outer measure version of Marcinkiewicz interpolation theorem. Explicitly, we will prove that the assertions above hold if $p = 1$ and $p = \infty$, and the result for intermediate p is then going to follow for free by this theorem. So let us begin:

Proof of item (1): $\underline{p = \infty}$. This is simple, as

$$\|F\|_{\mathcal{L}^\infty(S_\infty)} = \sup_{T_J} \sup_{I \subset J} |F(I)| \leq \sup_I |F(I)| = \|F\|_{L^\infty(\mathbb{R})}.$$

$\underline{p = 1}$. Let $\lambda > 0$, and $\mathcal{E}'$ be the set of maximal dyadic intervals J such that $F(J) > \lambda$. Then, from definition, we have that

$$\|f \mathbb{1}_{(\cup_{E \in \mathcal{E}'} E)^c}\|_{L^\infty(S)} \leq \lambda.$$

Because of that, we have that

$$\lambda \sum_{J \in \mathcal{E}'} |J| \leq \sum_{J \in \mathcal{E}'} |J| F(J) \leq \|F\|_1. \Rightarrow \mu(S_\infty F > \lambda) \leq \frac{\|F\|_1}{\lambda}.$$

Finally, this implies that

$$\|F\|_{\mathcal{L}^{1,\infty}(S_\infty)} \leq \|F\|_1.$$

This case is, therefore, finished.

Proof of item (2): $\underline{p = \infty}$. Let $F \in L^\infty(\mathbb{R})$. By the corollary of the telescoping identity in Lecture 7, used with $n = 2$, $F_1 = F_2$, then we see that

$$\frac{1}{|J|} \sum_{I \subset J} |I| \Delta F(I)^2 = \left(\lim_{k \to -\infty} \frac{1}{|J|} \sum_{\substack{I \subset J, \\ |I| = 2^k}} |I| F(I)^2 \right) - F(J)^2$$
$$\leq \lim_{k \to -\infty} \frac{1}{|J|} |J| \|F\|_\infty^2 = \|F\|_\infty^2.$$

This finishes the first part of the second case.

$\underline{p = 1}$. Let, as before, $\mathcal{E}'$ be the set of maximal J with $F(J) > \lambda$, but now define also the set $\mathcal{E}'$ of maximal *parents* of elements in $\mathcal{E}'$. What does that mean? For all elements $J \in \mathcal{E}'$, let $\tilde{J}$ be its parent interval. Then subextract from this collection of parents a collection of *maximal* intervals with respect to set inclusion. This collection is our $\mathcal{E}''$.
From this collection, we will perform a so called *Calderón-Zygmund decomposition.* Explicitly, let

$$G(I) = \begin{cases} F(I), & \text{if } \forall J \in \mathcal{E}'', I \not\subset J, \\ F(J), & \text{if } \exists J \in \mathcal{E}'', I \subset J. \end{cases}$$

We let then $F = G + B$. It is not difficult to see that G is a martingale, and moreover, $\|G\|_\infty \leq \lambda$. This already implies that

$$\sum_{J \in \mathcal{E}''} |J| \leq \frac{2\|F\|_1}{\lambda}.$$

Also notice that, as $G \equiv F$ on the complementary of $\cup_{J\in\mathcal{E}''} J$, then

$$\Delta F \mathbb{1}_{(\cup_{J\in\mathcal{E}''} T_J)^c}(I) = \Delta G \mathbb{1}_{(\cup_{J\in\mathcal{E}''} T_J)^c}(I).$$

We are a couple of steps from finishing. The last inequality already tells us that

$$\|\Delta F \mathbb{1}_{(\cup_{J\in\mathcal{E}''} T_J)^c}\|_{\mathcal{L}^\infty(S_2)} = \|\Delta G \mathbb{1}_{(\cup_{J\in\mathcal{E}''} T_J)^c}\|_{\mathcal{L}^\infty(S_2)}.$$

But, from the definition,

$$\|\Delta G \mathbb{1}_{(\cup_{J\in\mathcal{E}''} T_J)^c}\|_{\mathcal{L}^\infty(S_2)} = \sup_{J\notin\mathcal{E}''} \left(\frac{1}{|J|} \sum_{I\subset J} |I| \Delta G(I)^2 \right)^{1/2}.$$

From the case $\underline{p=\infty}$ in this same item, we know that this last expression – as $G \in L^\infty$, with $\|G\|_\infty \le \lambda$ – is bounded by $\|G\|_\infty \le \lambda$. Therefore, the collection $\mathcal{E}''$ is an allowed collection for the set above, and thus we must have that

$$\mu(S_2(\Delta F) > \lambda) \le \sum_{J\in\mathcal{E}''} \sigma(T_J) = \sum_{J\in\mathcal{E}''} |J| \le \frac{2\|F\|_1}{\lambda}.$$

This is exactly what we wanted to prove, and therefore our result is completed. □

10 Atomicity in the dyadic setting

2016-11-22

We consider the following simple observation. Let $\Lambda\colon \ell^1(\mathbb{Z}) \to \mathbb{R}$ be linear, possibly defined only on a dense subset of $\ell^1(\mathbb{Z})$, for example on functions supported on finite sets. Define

$$e_n(m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise,} \end{cases},$$

which we call *atoms*. Assume that we have a uniform bound for Λ on them, namely for all $n \in \mathbb{Z}, \lambda \in \mathbb{R}$

$$|\Lambda(\lambda e_n)| \le |\lambda| 1.$$

This implies a bound for Λ on $f \in \ell^1(\mathbb{Z})$

$$|\Lambda(f)| \le \sum_{n\in\mathbb{Z}} |f(n)| = \|f\|_{\ell^1}.$$

A version of this criterion for outer measures is given by the following

Theorem 10.1. *Let $X, \mathcal{E}, \sigma, \mathcal{B}, S$ give the outer measure structure as in previous lectures and assume $\mathcal{E}$ to be countable.*
Let $\Lambda\colon \mathcal{B} \to \mathbb{R}$ be countably subadditive, i.e. for $f_n \in \mathcal{B}, \lambda_n \in \mathbb{R}$ then

$$|\Lambda(\sum_{n\in\mathbb{N}} \lambda_n f_n)| \leq \sum_{n\in\mathbb{N}} |\lambda_n||\Lambda(f_n)|.$$

Moreover assume that for all $f \in \mathcal{L}^\infty(S)$ supported on $\cup_{E\in\mathcal{E}} E$ and $E \in \mathcal{E}$

$$|\Lambda(f\mathbb{1}_E)| \leq S(f\mathbb{1}_E)(E)\sigma(E). \tag{10.2}$$

Then there exists $C \in \mathbb{R}$ such that for all $f \in \mathcal{L}^\infty(S)$ supported on $\cup_{E\in\mathcal{E}} E$

$$|\Lambda(f)| \leq C\|f\|_{\mathcal{L}^1(S)}.$$

Remark 10.3. The assumption in (10.2) is inspired by the analogous one for atoms in $\ell^1(\mathbb{Z})$: $\sigma(E)$ plays the role of 1, which is the counting measure of the set $\{n\}$;$S(f\mathbb{1}_E)(E)$ has the role analogous to $|\lambda|$.

Proof. Let $\mathcal{E}_k \subset \mathcal{E}$ such that

$$\|f\mathbb{1}_{(\cup_{E\in\mathcal{E}_k} E)^c}\|_{\mathcal{L}^\infty(S)} \leq 2^k, \tag{10.4}$$

$$2\mu(Sf > 2^k) \geq \sum_{E\in\mathcal{E}_k} \sigma(E), \tag{10.5}$$

and enumerate $\mathcal{E}_k = \{E_{k,1}, E_{k,2}, \dots\}, \mathcal{E} = \{E_1, E_2, \dots\}$. These collections can be countably infinite or finite, however we are not going to take care of this in the following notation. We claim that

$$f = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}} f\mathbb{1}_{(E_{k,n} \setminus \cup_{m<n} E_{k,m} \setminus \cup_{l>k} \cup_{m\in\mathbb{N}} E_{l,m})} + \\ + \sum_{i\in\mathbb{N}} f\mathbb{1}_{(E_i \setminus \cup_{j<i} E_j \setminus \cup_{k\in\mathbb{Z}} \cup_{m\in\mathbb{N}} E_{k,m})}.$$

In fact if $x \notin \cup_{E\in\mathcal{E}} E$ then both sides are 0 because of the condition on the support of f.
If $x \in \cup_{E\in\mathcal{E}} E$ then either $x \in \cup_{k,n} E_{k,n}$ or $x \notin \cup_{k,n} E_{k,n}$. In the first case since $f \in \mathcal{L}^\infty(S)$ there exists a maximal k_0 such that $x \in E_{k_0,n}$ for some n ($2^{k_0} \simeq \|f\|_{\mathcal{L}^\infty}$) and there is a minimal n_0 such that $x \in E_{k_0,n_0}$ and

$$x \in E_{k_0,n_0} \setminus \cup_{m<n_0} E_{k_0,m} \setminus \cup_m \cup_{l>k} E_{l,m}.$$

In the second case there exists a minimal i_0 such that $x \in E_{i_0}$ and

$$x \in E_{i_0} \setminus \cup_{j<i_0} E_j \setminus \cup_{k\in\mathbb{Z}} \cup_{m\in\mathbb{N}} E_{k,m}$$

Then

$$
\begin{aligned}
|\Lambda(f)| \leq & \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}|\Lambda(f\mathbb{1}_{(E_{k,n}\setminus\cup_{m<n}E_{k,m}\setminus\cup_{l>k}\cup_{m\in\mathbb{N}}E_{l,m})})|+\\
&+\sum_{i\in\mathbb{N}}|\Lambda(f\mathbb{1}_{(E_i\setminus\cup_{j<i}E_j\setminus\cup_{k\in\mathbb{Z}}\cup_{m\in\mathbb{N}}E_{k,m})})| \leq\\
\leq & \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}|\Lambda(f\mathbb{1}_{(E_{k,n}\setminus\cup_{m<n}E_{k,m}\setminus\cup_{l>k}\cup_{m\in\mathbb{N}}E_{l,m})}\mathbb{1}_{(E_{k,n})})|+\\
&+\sum_{i\in\mathbb{N}}|\Lambda(f\mathbb{1}_{(E_i\setminus\cup_{j<i}E_j\setminus\cup_{k\in\mathbb{Z}}\cup_{m\in\mathbb{N}}E_{k,m})}\mathbb{1}_{(E_i)})| \leq\\
\leq & \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}S(f\mathbb{1}_{(E_{k,n}\setminus\cup_{m<n}E_{k,m}\setminus\cup_{l>k}\cup_{m\in\mathbb{N}}E_{l,m})})(E_{k,n})\sigma(E_{k,n})+\\
&+\sum_{i\in\mathbb{N}}S(f\mathbb{1}_{(E_i\setminus\cup_{j<i}E_j\setminus\cup_{k\in\mathbb{Z}}\cup_{m\in\mathbb{N}}E_{k,m})})(E_i)\sigma(E_i) \leq\\
\leq & \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}2^{k+1}\sigma(E_{k,n})+\sum_{i\in\mathbb{N}}0\sigma(E_i) \leq\\
\leq & \sum_{k\in\mathbb{Z}}2^{k+2}\mu(Sf>2^k) \leq 8\int_0^\infty \mu(Sf>\lambda)d\lambda = 8\|f\|_{\mathcal{L}^1(S)}.
\end{aligned}
$$

In the first inequality we used the countably subadditivity of Λ together with the claim. In the second we introduced a multiplication for a proper characteristic function which is always 1 in the support of $f\mathbb{1}_{(E_{k,n}\setminus\cdots)}$ (or $f\mathbb{1}_{(E_i\setminus\cdots)}$). In the third we used the hypothesis of boundedness on atoms (10.2). In the fourth we used (10.4) and in the fifth (10.5). In the last inequality we used the fact that $\mu(SF>\lambda)$ is nonincreasing in λ, therefore the discrete sum and the integral are related. □

We are interested in using this result in the estimate of Paraproducts. We start recalling their general form: for $A\subset\{1,\dots,n\}, |A|\geq 2, |A|$ even and $a\colon\mathfrak{D}\to\mathbb{R}, \sup_{I\in\mathfrak{D}}|a(I)|\leq 1$ we consider

$$
P(F_1,\dots,F_n)=\sum_{I\in\mathfrak{D}}|I|a(I)\prod_{i\in A}\Delta F_i(I)\prod_{i\notin A}F_i(I). \tag{10.6}
$$

We want to estimate it in terms of outer measure $\|F_i\|_{\mathcal{L}^{p_i}(S_i)}$ and then use the Embedding Theorem we proved in the last lecture to recover a bound by means of classical L^{p_i}-norms.
Our setting is the following

- $X=\mathfrak{D}$, $\mathcal{E}=\{T_J|J\in\mathfrak{D}\}$, where $T_J=\{I\in\mathfrak{D}|I\subset J\}$, $\sigma(T_J)=|J|$;
- $\mathcal{B}$ is the set of all functions $F\colon\mathcal{B}\to\mathbb{R}$;
- $S_1F(T_J)=\frac{1}{|J|}\sum_{I\in T_J}|I||F(I)|$.

The definition

$$\Lambda\colon \mathcal{B} \to \mathbb{R}, \qquad \Lambda(F) := \sum_{I\in\mathfrak{D}} |I||F(I)|,$$

yields

$$\Lambda(F\mathbb{1}_{T_J}) = \sum_{I\in T_J} |I||F(I)| = |J|S_1F(T_J).$$

Applying last Theorem we obtain that for $F \in \mathcal{L}^\infty(S_1)$ supported on $\cup_{I\in\mathfrak{D}} I$

$$|\Lambda(F)| \le C\|F\|_{\mathcal{L}^1(S_1)}. \tag{$*$}$$

Corollary 10.7. *Let* $p_1, \dots, p_n \in [1,\infty]$ *such that* $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$. *Moreover let* P, a *as in* (10.6) *and* $A_0 \subset A, |A_0| = 2$. *If* $(*)$ *applies then*

$$|P(F_1, \dots, F_n)| \le C \prod_{i=1}^n \|F_i\|_{L^{p_i}}.$$

Sketch of the proof. We use in a row boundedness of a, $(*)$ to prove

$$|P(F_1, \dots, F_n)| \le C\|\prod_{i\in A} \Delta F_i \prod_{i\notin A} F_i\|_{\mathcal{L}^1(S_1)}.$$

Since $\frac{1}{1} = \frac{1}{2} + \frac{1}{2} + \frac{1}{\infty} + \dots$ then (8.16) holds and we can apply Hölder inequality to obtain

$$\|\prod_{i\in A} \Delta F_i \prod_{i\notin A} F_i\|_{\mathcal{L}^1(S_1)} \le C \prod_{i\in A_0} \|\Delta F_i\|_{\mathcal{L}^{p_i}(S_2)} \prod_{i\in A\setminus A_0} \|\Delta F_i\|_{\mathcal{L}^{p_i}(S_\infty)} \prod_{i\notin A} \|F_i\|_{\mathcal{L}^{p_i}(S_\infty)}.$$

To conclude we apply the Embedding Theorem from last lecture

$$\prod_{i\in A_0} \|\Delta F_i\|_{\mathcal{L}^{p_i}(S_2)} \prod_{i\in A\setminus A_0} \|\Delta F_i\|_{\mathcal{L}^{p_i}(S_\infty)} \prod_{i\notin A} \|F_i\|_{\mathcal{L}^{p_i}(S_\infty)} \le C \prod_{i=1}^n \|F_i\|_{L^{p_i}}.$$

□

Consider the following two transformations:

- for $\lambda > 0$ the dilation D_λ

$$D_\lambda f(x) := f(\lambda^{-1}x);$$

- for $y \in \mathbb{R}$ the translation T_y

$$T_y f(x) := f(x - y).$$

Then

$$\int \prod_{i=1}^{n} D_\lambda f_i(x)dx = \lambda \int \prod_{i=1}^{n} f_i(x)dx, \quad \int \prod_{i=1}^{n} T_y f_i(x)dx = \int \prod_{i=1}^{n} f_i(x)dx.$$

All the paraproducts share a similar behaviour under translations and dilations (in the dyadic setting dilations are restricted to $\lambda = 2^k$).
For $F\colon \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ harmonic there exists a unique $G\colon \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ harmonic with $\lim_{y\to\infty} G(x,y) = 0$ such that $F + iG$ is holomorphic. We call it the *Hilbert Transform* of F and denote it by $H(F)$. We observe that $H(\cdot)$ is linear and invariant under dilations and translations, i.e.

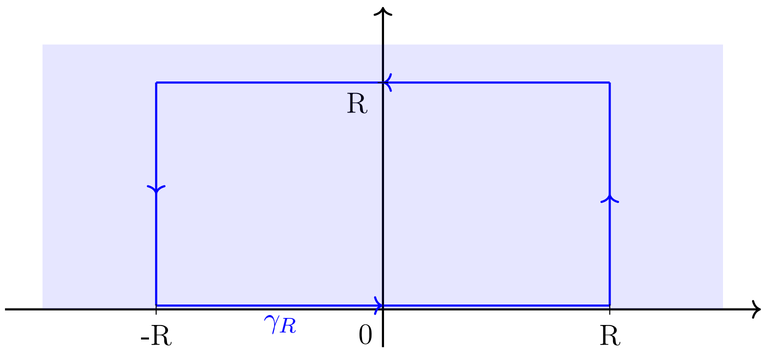
$$H(D_\lambda T_y F) = D_\lambda T_y(H(F)).$$

Therefore the following is a good candidate for a paraproduct

$$\int H(F_1)F_2 dx.$$

Now assume F harmonic in $\mathbb{R}\times\mathbb{R}_{>0}$ and continuous in $\mathbb{R}\times\mathbb{R}_{\geq 0}$ Under other suitable conditions on F which provide that the following integral makes sense, the Cauchy integral

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} F(\xi) \frac{1}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\mathbb{R}} F(t) \frac{1}{t-z} dt,$$



gives a holomorphic function in z.
For $z = x + iy$

$$\frac{1}{t-z} = \frac{1}{t-x-iy} = \frac{t-x+iy}{(t-x)^2+y^2} + i\frac{y}{(t-x)^2+y^2},$$

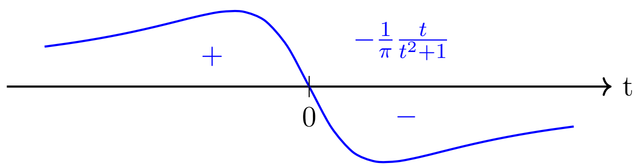
and the Cauchy integral becomes

$$\frac{1}{2\pi} \int_{\mathbb{R}} F(t) \frac{y}{(t-x)^2+y^2} dt + i\frac{-1}{2\pi} \int_{\mathbb{R}} F(t) \frac{t-x}{(t-x)^2+y^2} dt.$$

The real part is $\frac{F}{2}$ because F is harmonic and by properties of Poisson kernel. The imaginary one is $\frac{H(F)}{2}$. Therefore we consider the kernel of the second summand

$$-\frac{1}{2\pi}\frac{t-x}{(t-x)^2+y^2} = D_y T_x\left(-\frac{1}{\pi}\frac{t}{t^2+1}\right),$$

and we call it the *conjugate Poisson kernel.* We can draw a parallel between the Poisson kernel and martingale averages on one hand and the conjugate Poisson kernel and martingale differences on the other.



11 Classical operators and paraproducts

2016-11-24

In this lecture, we start with a little digression with respect to our previous theorem. More specifically, we wish to illustate a little how to apply some of our techniques to bound some classical operators in harmonic analysis. Our first main operator of the day will then be the *Hilbert transform.*

Before going on, we start by redefinind a few basic concepts that appeared on our first lectures. For example, we define the *Poisson kernel* as the function

$$\varphi(z) = \frac{1}{\pi}\frac{1}{1+t^2};$$

Moreover, we parametrize a family of functions accordingly by $\varphi_t(z) = \frac{1}{t}\varphi\left(\frac{x}{t}\right)$. This definition is useful as, as we have already seen, we can extend a (reasonably well behaved) function on the real line with the help of this kernel to the upper half plane, in order that the extension is *harmonic* on $\mathbb{R}\times\mathbb{R}_{>0}$. Explicitly, we define

$$F(x,y) := \int_{\mathbb{R}} f(z)\frac{1}{\pi}\frac{y}{(x-z)^2+y^2}dz = \int_{\mathbb{R}} f(z)\varphi_y(x-z)dz = f*\varphi_y(x).$$

Closely related to this kernel is the *conjugate Poisson kernel*, which is defined by

$$\psi(z) = \frac{1}{\pi}\frac{z}{1+z^2}.$$

Just like before, we define $\psi_t(z)$ and $f*\psi_y(x) = G(x,y)$, as long as the function f is sufficiently well behaved. The most noteseworthy property of this conjugate function is the fact that, if F, G are defined as above, then $F+iG$ is a *holomorphic* function in $\mathbb{R}\times\mathbb{R}_{>0}$. There are many ways to see that, one of them being simply and directly using the Cauchy-Riemann equations. We omit the details.

The next question we want to pose is: as we know that we can make sense of $\lim_{t\to 0} f*\varphi_t(x)$, and this converges almost everywhere to the original function f, do we have necessarily a similar phenomenom happening for G? That is, when does $\lim_{t\to 0} f*\psi_t(x)$ exist? One possible first way to start investigating that is by simply looking at the limit of $\psi_t(x)$ as $t\to 0$. If $z\neq 0$:

$$\lim_{t\to 0}\psi_t(x)=\lim_{t\to 0}\frac{1}{t}\frac{1}{\pi}\frac{z/t}{1+(z/t)^2}=\frac{1}{\pi}\frac{1}{z}.$$

We have, however, a singularity problem around the origin. The way to remmediate this problem is by ignoring the pointwise limits for ψ_t and focusing on the definition of G directly, for a suitable class of functions f. In a clearer way: let f be a function that belongs to some $L^p(\mathbb{R})$, for $1<p<+\infty$. Moreover, assume that f is continuously differentiable around every point on the real line. We have, then:

$$\begin{aligned}
\lim_{t\to 0} f*\psi_t(x) &= \lim_{t\to 0}\int f(x-z)\frac{1}{\pi}\frac{z}{t^2+z^2}dz\\
&\overset{\text{odd function}}{=} \lim_{t\to 0}\int \left(f(x-z)-f(x)\mathbb{1}_{|z|<\varepsilon}\right)\frac{1}{\pi}\frac{z}{t^2+z^2}dz\\
&\overset{f\in C^1\cap L^p}{=} \int \left(f(x-z)-f(x)\mathbb{1}_{|z|<\varepsilon}\right)\frac{1}{\pi}\frac{dz}{z}\\
&= \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f(x-z)\frac{1}{\pi}\frac{dz}{z}+\int_{[-\varepsilon,\varepsilon]}(f(x-z)-f(x))\frac{1}{\pi}\frac{dz}{z}\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} f(x-z)\frac{1}{\pi}\frac{dz}{z} =: \text{p.v.}\int_{\mathbb{R}} f(x-z)\frac{dz}{\pi z}.
\end{aligned}$$

Alternatively, one can also show that the expression above also equals

$$\frac{1}{2}\int_{\mathbb{R}}(f(x-z)-f(x+z))\frac{dz}{\pi z} =: Hf(x).$$

This last object is what we call the *Hilbert transform.* We want to use the techniques and the general philosophy we have so far to prove L^p bounds for this object. With this intent, our main idea is the follwoing:

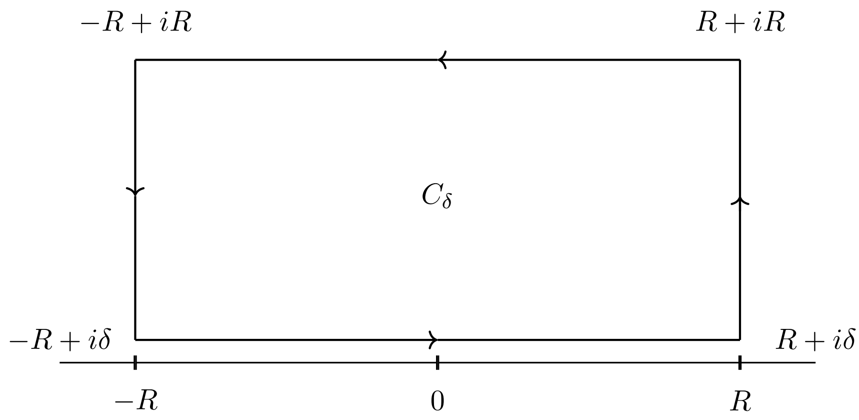
Idea: $\int Hf(x)g(x)dx$ can be decomposed into paraproducts.

In the end, we wish to establish:

Goal: $\left|\int Hf(x)g(x)dx\right|\leq C\|f\|_{p_1}\|g\|_{p_2}$, $\frac{1}{p_1}+\frac{1}{p_2}=1.$

Note that this last goal is equivalent to the wished L^p bounds for the Hilbert transform. One of the implications is proved by simply setting

$g = \overline{Hf}|Hf|^{p-2}$ and calculating explictly. The other direction is established by Hölder's ineqaulity. The first observation we make is that for $p = 2$ the proof is actually simple, and resorts to complex analysis techniques: as the function $F + iG$ is analytic on $\mathbb{R} \times \mathbb{R}_{>0}$, we integrate its square over the following rectangle:



We have then

$$\int_{C_\delta} (F + iG)^2 dz = 0.$$

By taking the real part, employing $\lim_{R\to\infty}$ and then taking $\delta \to 0$, we see that

$$0 = \int_{\mathbb{R}} (F^2 - G^2) dx = \int_{\mathbb{R}} (f^2 - (Hf)^2) dx.$$

This implies that $\|f\|_2 = \|Hf\|_2$. Of course, the details in the proof above are far from being completed, but the reader is invited to take them as an exercise.

Remark 11.1. In higher dimensions, there are various ways to define analogous objects to the Hilbert transform. For example, one can then define the so-called *Riesz transforms* as

$$R_i f = \text{p.v.} f * c \frac{x_i}{|x|^{d+1}} = \lim_{\varepsilon \to 0} \int_{|z| > \varepsilon} f(x - z) \frac{z_i}{|z|^{d+1}} dz_1 \cdots dz_d.$$

We remark that this is the critical exponent case of a homogeneous function in dimension d. This means that

$$\frac{\lambda z_i}{|\lambda z|^{d+1}} = \lambda^{-d} \frac{z_i}{|z|^{d+1}}.$$

While this is a triviality, there is more to it: it means essentially that we have, in this only critical case, bad integration both near the origin and near infinity. Of course, we also need, as we are going to see soon, some sort of cancellation. In this example, as in the case of the Hilbert transform, we have that the integral of the kernel we are integrating on every annulus of $\mathbb{R}^d$ is actually *zero*. Inspired by that, we can also define the *second order Riesz transforms* as the kernels

$$R_{i,j}f(x) = \begin{cases} \text{p.v.} \int_{\mathbb{R}^d} f(x-z) \frac{z_i z_j}{|z|^{d+1}} dz, & i \neq j; \\ \text{p.v.} \int_{\mathbb{R}^d} f(x-z) \frac{z_i^2 - c|z|^2}{|z|^{d+1}} dz, & i = j; \end{cases}$$

where c is taken so that the inner kernel in the formula above has integral zero on the unit sphere $\mathbb{S}^{d-1}$.

11.1 A PDE motivation

In the study of partial differential equations – or simply often called 'PDEs' –, inequalities involving the original function and its derivatives are crucial for many of the regularity theorems, as well as for most of the existence and uniqueness results. Particularly, one important question to be raised was the following:

Question 11.2. Suppose we have control on $\|\Delta f\|_p$, for some sufficiently regular f. Can we say anything about $\|\partial_i \partial_j f\|_p$, for arbitrary i, j? In other terms: would it be true that

$$\|\partial_i \partial_j f\|_p \leq C_{p,n} \|\Delta f\|_p,$$

for some constant $C_{p,n}$?

The question can be rephrased in the following way: can we find, for a general $u \in L^p$, bounds of the type

$$\|\partial_i \partial_j \Delta^{-1} u\|_p \leq C_{n,p} \|u\|_p?$$

Of course, we need to make sense of what Δ^{-1} here means. For this, we let the function N be defined as follows:

$$N(x) = \begin{cases} \frac{c_d}{|x|^{d-2}}, & d > 2; \\ c \log |x|, & d = 2. \end{cases}$$

Here c, c_d are dimensional constants. The claim is then the following:

Claim 11.3. *The function $u = f * N$ satisfies $\Delta u = f$, whenever f is bounded and sufficiently fast decaying.*

Sketch of proof. We prove this statement in dimensions $d > 2$, as for $d = 2$ a separate argument, but in the same spirit as this one, is needed. Define, then, the function

$$N_\varepsilon(x) = \begin{cases} N(x), & \text{if } |x| \geq \varepsilon; \\ a|x| + b, & \text{if } |x| \leq \varepsilon. \end{cases}$$

Moreover, choose a and b such that N_ε is still of class C^1. A simple calculation then shows that $f*N = \lim_{\varepsilon\to 0} f*N_\varepsilon$, and that the same holds for derivatives. Finally, a direct computation shows that

$$\Delta N_\varepsilon(x) = \begin{cases} 0, & |x| \geq \varepsilon \\ 2da(= c\varepsilon^{-d}), & |x| \leq \varepsilon. \end{cases}$$

The last line is justified with either a direct calculation or a scaling argument. Therefore, by Lebesgue's differentiation theorem,

$$\lim_{\varepsilon\to 0} \Delta(f * N_\varepsilon) = \lim_{\varepsilon\to 0} \frac{c}{\varepsilon^{-d}} \int_{|x-y|\leq\varepsilon} f(x)dx = cf(y).$$

Multiplying by the constant in the definition of N then gives the result. The details are left to the reader. □

One particular property that could be derived directly from the definition of N is that it satisfies

$$\partial_i\partial_j N(x) = \frac{cx_ix_j}{|x|^{d+2}} \text{ and } \partial_i\partial_i N(x) = c\frac{x_i^2 - \frac{1}{d}|x|^2}{|x|^{d+2}}.$$

Notice that these are *exactly* the kernels defining the second order Riesz transforms. This observation, together with the fact we have proved above, gives a proof of the following

Theorem 11.4. *If* $\|R_{ij}f\|_p \leq C_p\|f\|_p, \forall f \in L^p(\mathbb{R})$, *then*

$$\|\partial_j\partial_i f\|_p \leq C_p\|\Delta f\|_p,$$

for each $f \in W^{2,p} = \{f : \|f\|_p + \|\Delta f\|_p < +\infty\}$.

Remark 11.5. Notice that, as a consequence of those formulas, we have that the Riesz transforms satisfy

$$R_i = \partial_i \Delta^{-1/2},$$

where $\Delta^{-1/2}$ is defined as $\Delta^{-1/2}f = c(f * |\cdot|^{1-d})$. With respect to this operator, we have that $\Delta^{-1/2}(\Delta^{-1/2}f) = \Delta^{-1}f$ as defined above, and a proof of this goes roughly as follows: first, one proves that the convolution $|\cdot|^{1-d} * |\cdot|^{1-d}$ is well-defined and radial. Then it can also be shown that it is homogeneous of degree $2-d$, and therefore we must have that it equals $c|\cdot|^{2-d}$. To find the constant c explicitly, a direct computation will suffice. The details are then left to the reader.

12 The S^∞ embedding in the continuous setting

2016-11-29

We get finally to the touching point of our last two topics: singular operators and outer measure theory. We wish, as promised, to perform a paraproduct decomposition in our operators, so that we reach, in the end, the promised L^p bounds for such classes of operators.

The goal of this lecture is to establish "continuous versions" of the embedding theorem (25.2) for outer $\mathcal{L}^p$-spaces we discussed in the dyadic environment.

First of all, we set our outer measure space structure:

- $X = \mathbb{R}^d \times \mathbb{R}_{>0}$;
- $\mathcal{E} = \{T(x,s) = \{(y,t) \in \mathbb{R}^d \times \mathbb{R}_{>0} \colon \|x-y\| < s-t\}\}$.
  The elements of this set are called *tents*;
- $\sigma(T(x,s)) = s^d$;
- $\mathcal{B}$ is the set of Borel measurable functions in $\mathbb{R}^d \times \mathbb{R}_{>0}$;
- for $F \in \mathcal{B}$, we define the sizes

$$S^p(F)(T(X,s)) \coloneqq \left(\frac{1}{s^d}\int_{T(x,s)} |F(y,t)|^p \mathrm{d}y\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$
$$S^\infty(F)(T(X,s)) \coloneqq \sup_{(y,t)\in T(x,s)} |F(y,t)|.$$

Remark 12.1. The tents $T(x,s)$ substitute for the dyadic ones T_J, where $J = [n2^k, (n+1)2^k)$: x plays the role of n, s of the scaling parameter k. In particular for $(y,t) \in T(x,s)$ we have $T(y,t) \subset T(x,s)$, as well as for $I \subset J$ then $T_I \subset T_J$.

For $f \colon \mathbb{R}^d \to \mathbb{R}$ measurable, locally integrable and such that

$$\int |f(z)|(1+\|z\|)^{-d-\epsilon} dz < \infty,$$

the first embedding map is defined by

$$Af(y,t) \coloneqq \sup_\phi |t^{-d}\int f(z)\varphi(t^{-1}(y-z))dz|,$$

where the supremum is taken over the set of functions $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ such that $|\varphi(z)| \le (1+|z|)^{-d-\epsilon}$, where $\epsilon > 0$ is a fixed number (in applications we can usually take $\epsilon = 1$).

Equivalently,

$$Af(y,t) = t^{-d}\int |f|(z)(1+t^{-1}|y-z|)^{-d-\epsilon}\mathrm{d}z.$$

Remark 12.2. In the lecture we have written $F = Af$.

The adjective "embedding" is explained by the following result, where $\|\cdot\|_p$ stands for the classical L^p-norm.

Theorem 12.3. *For every* $1 < p \leq \infty$

$$\|Af\|_{\mathcal{L}^p(S^\infty)} \leq C_p \|f\|_p.$$

Moreover in the endpoint a weak type inequality holds

$$\|Af\|_{\mathcal{L}^{1,\infty}(S^\infty)} \leq C\|f\|_1.$$

Proof. By Marcinkiewicz interpolation theorem it is enough to prove the cases $p = 1, p = \infty$.
$\underline{p = \infty}$. For all y, t we have

$$Af(y,t) \leq \|f\|_\infty t^{-d} \int (1 + t^{-1}|y - z|)^{-d-\epsilon} \mathrm{d}z \lesssim \|f\|_\infty.$$

Therefore

$$S^\infty(Af)(T(x,s)) = \sup_{(y,t)\in T(x,s)} |F(y,t)| \lesssim \|f\|_\infty,$$

$$\|Af\|_{\mathcal{L}^\infty(S^\infty)} = \sup_{x,s} S^\infty(Af)(T(x,s)) \lesssim \|f\|_\infty.$$

$\underline{p = 1}$. We define the *Hardy-Littlewood maximal function* by

$$Mf(x) := \sup_{r>0} \frac{1}{r^d} \int_{B(x,r)} |f(z)| dz,$$

where $B(x,r) := \{z : \|z - x\| < r\}$. To be consistent with the standard definition one should put also a constant taking care of the d-volume of the unitary ball, but for our purpose we can forget about it. It is a continuous version of the dyadic maximal function we defined in the discrete environment for martingales.
We want to prove a weak type (1,1) bound for it, i.e.

$$|\{x : Mf(x) > \lambda\}| \leq C\frac{\|f\|_1}{\lambda}.$$

In the discrete case this was done straightforward by considering maximal dyadic intervals. In the continuous case we have to be more careful.
The first step in order to prove it is a so-called *Vitali covering argument.* For $i = 1, 2, \ldots$ pick $B(x_i, r_i)$ such that

$$\frac{1}{r_i^d} \int_{B(x_i,r_i)} |f(z)| dz > \lambda,$$

$B(x_i, r_i)$ is disjoint from all $B(x_j, r_j), j < i$ and there is no such ball $B(\tilde{x}_i, \tilde{r}_i)$ with $\tilde{r}_i > 2r_i$. In the continuous setting this condition plays the same role of restricting to maximal dyadic intervals in the discrete environment.
The condition $f \in L^1$ ensures that this choice can be made.
Now we claim that

$$\{x\colon Mf(x) > \lambda\} \subset \bigcup_i B(x_i, 6r_i).$$

In fact suppose that for x there exists r such that

$$\frac{1}{r^d}\int_{B(x,r)} |f(z)|dz > \lambda.$$

Since

$$\lambda \sum_i r_i^d \leq \bigcup_i \int_{B(x_i,r_i)} |f(z)|dz \leq \|f\|_1,$$

then there exist finitely many $r_i > r$. By construction of our sequence of balls there exists i with $B(x, r) \cap B(x_i, r_i) \neq \emptyset$ and $2r_i > r$. Therefore $B(x, r) \subset B(x_i, 6r_i)$ which proves the claim. Thus

$$|\{x\colon Mf(x) > \lambda\}| \leq \sum_i C_d(6r_i)^d \leq \tilde{C}_d \frac{\|f\|_1}{\lambda}.$$

Now pick the tents $T(x_i, 6r_i)$ for (x_i, r_i) as above, for which we have

$$\sum_i \sigma(T(x_i, 6r_i)) \leq C\frac{\|f\|_1}{\lambda}.$$

We want to show that

$$\sup_{x,s} S(Af\mathbb{1}_{(\cup_i T(x_i,6r_i))^c})(T(x, s)) \leq C\lambda.$$

We claim

$$Af(y, t) \lesssim \inf_{x\in B(y,t)} Mf(x).$$

Assuming the validity of this claim we can conclude that for $(y, t) \notin \bigcup_i T(x_i, 6r_i)$ we have

$$Af(y, t) \lesssim \lambda$$

since

$$\begin{aligned}(y, t) \notin \bigcup_i T(x_i, 6r_i) &\iff B(y, t) \not\subset \bigcup_i B(x_i, 6r_i)\\ &\implies \exists x \in B(y, t), x \notin \{Mf > \lambda\} \subset \bigcup_i B(x_i, 6r_i).\end{aligned}$$

Now we verify the claim. By scaling and translation we may assume $t = 1$ and $y = 0$. Note

$$\frac{1}{(1+\|z\|)^{d+\epsilon}} \lesssim \sum_{k\geq 0} 2^{-\epsilon k} 2^{-dk} \mathbb{1}_{B(0,2^k)}(z).$$

In fact if $\|z\| < 1$ then

$$\frac{1}{(1+\|z\|)^{d+\epsilon}} \leq 1 = 1 2^{-\epsilon 0} 2^{-d0} \mathbb{1}_{B(0,2^0)}(z),$$

and if $2^k \leq \|z\| < 2^{k+1}$ as well

$$\frac{1}{(1+\|z\|)^{d+\epsilon}} \leq 4^{d+\epsilon} 2^{-\epsilon k} 2^{-dk} \mathbb{1}_{B(0,2^{k+1})}(z).$$

Therefore

$$\begin{aligned} Af(0,1) &\lesssim \sum_{k\geq 0} 2^{-\epsilon k} \frac{1}{2^{kd}} \int_{B(0,2^k)} |f(z)| dz \leq \\ &\lesssim \sum_{k\geq 0} 2^{-\epsilon k} \sup_{r\geq 1} \frac{1}{r^d} \int_{B(0,r)} |f(z)| dz \leq \\ &\lesssim \sup_{r\geq 1} \frac{1}{r^d} \int_{B(0,r)} |f(z)| \mathrm{d}z, \end{aligned}$$

and the claim follows. □

Remark 12.4. In the dyadic argument we had no dependence on dimension, namely no multiplicative constant depending on it. Therefore we should expect the same to happen also in the continuous setting, meaning that our argument is not optimal.

12.1 S^2 embeddings

For $f\colon \mathbb{R}^d \to \mathbb{R}$ measurable, locally integrable and such that

$$\int |f(z)|(1+|z|)^{-d-\epsilon} dz < \infty,$$

the second embedding map is defined by

$$Df(y,t) := \sup_{\phi\in\mathcal{C}} \Big| t^{-d} \int f(z)\varphi(t^{-1}(y-z)) dz \Big|,$$

where the supremum is taken over the set $\mathcal{C}$ of the functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ such that for all $z, z' \in \mathbb{R}^d$ we have

$$\textstyle\int \varphi(z) dz = 0, \tag{12.5}$$

$$|\varphi(z)| \leq (1+|z|)^{-d-\epsilon}, \tag{12.6}$$

$$|\phi(z) - \phi(z')| \leq |z-z'|^{\epsilon}((1+|z|)^{-d-\epsilon} + (1+|z'|)^{-d-\epsilon} \tag{12.7}$$

for some fixed $\epsilon > 0$.

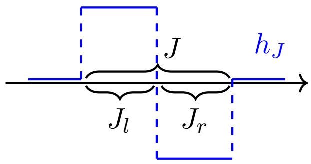
Remark 12.8. The above conditions become more restrictive as ϵ grows. In most applications it suffices to consider $\epsilon = 1$, in that case (12.7) is a local form of Lipschitz continuity and follows for instance from a derivative estimate of the form $\|\nabla\phi(z)\| \lesssim (1+|z|)^{-d-2}$.

However, it seems convenient to use the larger family of wave packets with small ϵ since it restricts the kind of estimates that can be made and leads to cleaner proofs.

Remark 12.9. In the lecture we have used the notation $Df = \Delta F$.

Remark 12.10. The smoothness condition (12.7) prevents us from considering the *Haar functions* with integral 0. Namely for $J \in \mathfrak{D}$ the function

$$h_J(x) := \begin{cases} 1 & \text{if } x \in J_l, \\ -1 & \text{if } x \in J_r, \\ 0 & \text{if } x \notin J. \end{cases}$$



Theorem 12.11. *For every* $1 < p \leq \infty$

$$\|Df\|_{\mathcal{L}^p(S^2)} \leq C_p \|f\|_p.$$

Moreover in the endpoint a weak type inequality holds

$$\|Df\|_{\mathcal{L}^{1,\infty}(S^2)} \leq C\|f\|_1.$$

In the dyadic setting we used a telescoping identity, basically relying on the orthogonality of the Haar functions.
However this property no longer holds for the functions used in defining Df, even if it is not completely lost. Therefore the struggle in this case will be to recover enough information for our purpose from this only quasi orthogonality.

We linearize the supremum in the definition of Df by choosing for each pair (y,t) a function $\phi \in \mathcal{C}$ for which the supremum is almost attained. Denote then $\phi_{y,t}(z) = t^{-d}\phi(t^{-1}(y-z))$. This is an L^1 normalized wave packet at scale t. The almost orthogonality of these wave packets is captured by the following estimate.

Lemma 12.12.

$$|\langle \varphi_{y,t}, \varphi_{y',t'} \rangle| \lesssim \frac{(tt')^{\epsilon/2}}{(\max(t,t') + |y-y'|)^{d+\epsilon}}.$$

Proof. Without loss of generality assume $t \le t'$. Using the cancellation condition (12.5) we split the integral

$$\int_{\mathbb{R}^d} \varphi_{y,t}(z)\varphi_{y',t'}(z)\mathrm{d}z = \int_{\mathbb{R}^d} \varphi_{y,t}(z)(\varphi_{y',t'}(z) - \varphi_{y',t'}(y))\mathrm{d}z$$
$$= \int_B \varphi_{y,t}(z)(\varphi_{y',t'}(z)-\varphi_{y',t'}(y))\mathrm{d}z+\int_{B^c} \varphi_{y,t}(z)\varphi_{y',t'}(z)\mathrm{d}z-\int_{B^c} \varphi_{y,t}(z)\varphi_{y',t'}(y)\mathrm{d}z$$
$$= I_{\mathrm{local}} + I_{\mathrm{tail},1} - I_{\mathrm{tail},2},$$

where $B = B(y, R)$ and $t \le R$ will be chosen later. For future reference note

$$\int_{B^c} |\varphi_{y,t}(z)|\mathrm{d}z \lesssim \int_R^\infty \frac{t^\epsilon}{(t+r)^{d+\epsilon}} r^{d-1}\mathrm{d}r \lesssim (t/R)^\epsilon.$$

Using the Hölder continuity condition (12.7) we estimate

$$|I_{\mathrm{local}}| \le \int_{\mathbb{R}^d} |\varphi_{y,t}(z)|\mathrm{d}z \cdot (R/t')^\epsilon \frac{(t')^\epsilon}{(t' + \mathrm{dist}(y', B))^{d+\epsilon}}$$
$$\lesssim \frac{R^\epsilon}{(t' + \mathrm{dist}(y', B))^{d+\epsilon}}.$$

Using the decay condition (12.6) we estimate

$$|I_{\mathrm{tail},2}| \le \frac{(t')^\epsilon}{(t' + |y-y'|)^{d+\epsilon}} \int_{B^c} |\varphi_{y,t}(z)|\mathrm{d}z$$
$$\lesssim \frac{(t't/R)^\epsilon}{(t' + |y-y'|)^{d+\epsilon}}$$

In the estimate for $I_{\mathrm{tail},1}$ we distinguish the spatially separated case $|y - y'| \ge t$ and the non-separated case $|y - y'| \le t$.

Case $|y - y'| \ge t'$ Using the decay condition (12.6) we obtain

$$|I_{\mathrm{tail},1}| \le \int_{B^c \cap B(y', |y-y'|/2)} \frac{t^\epsilon}{(t + |y-y'|/2)^{d+\epsilon}} |\varphi_{y',t'}(z)|\mathrm{d}z$$
$$+ \int_{B^c \cap B(y', |y-y'|/2)^c} |\varphi_{y,t}(z)| \frac{(t')^\epsilon}{(t' + |y-y'|/2)^{d+\epsilon}}\mathrm{d}z$$
$$\lesssim \frac{t^\epsilon}{(t + |y-y'|)^{d+\epsilon}} + \frac{(t')^\epsilon}{(t' + |y-y'|/2)^{d+\epsilon}} \int_{B^c} |\varphi_{y,t}(z)|\mathrm{d}z$$
$$\lesssim \frac{t^\epsilon}{(t + |y-y'|)^{d+\epsilon}} + \frac{(t't/R)^\epsilon}{(t' + |y-y'|/2)^{d+\epsilon}}.$$

Case $|y - y'| \le t'$ This case is easier because we do not have to exploit spatial separation between y and y':

$$I_2 \lesssim (t')^{-d} \int_{B^c} |\phi_{y,t}| \lesssim (tt'/R)^\epsilon/(t')^{d+\epsilon}.$$

In both cases we obtain the claim upon choosing $R = (tt')^{1/2}$. □

We use the almost orthogonality statement in Lemma 12.12 to deduce a warm up claim for $p = 2$.

Lemma 12.13.

$$\int_{\mathbb{R}^d \times \mathbb{R}_{>0}} |Df(y,t)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t} \lesssim \|f\|_2^2.$$

Proof. We intend to prove it by a Hilbert space argument.
Denoting by

$$\langle f, \varphi \rangle = \int f(z)\varphi(z) dz,$$

we recall that for every (y,t) there exists $\varphi_{y,t}$ such that $|Df(y,t)| \leq 2\langle f, \varphi_{y,t}\rangle$, then

$$\begin{aligned}
\left(\int_{\mathbb{R}^d \times \mathbb{R}_{>0}} |Df(y,t)|^2 \mathrm{d}y \frac{\mathrm{d}t}{t}\right)^2 &\leq 16 \left(\int \langle f, \varphi_{y,t}\rangle \langle \varphi_{y,t}, f\rangle \mathrm{d}y \frac{\mathrm{d}t}{t}\right)^2 \\
&\lesssim \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \times \mathbb{R}_{>0}} \langle f, \varphi_{y,t}\rangle \varphi_{y,t}(z) \mathrm{d}y \frac{\mathrm{d}t}{t}\right) f(z) \mathrm{d}z\right)^2 \\
&\leq \|\int_{\mathbb{R}^d \times \mathbb{R}_{>0}} \langle f, \varphi_{y,t}\rangle \varphi_{y,t}(z) \mathrm{d}y \frac{\mathrm{d}t}{t}\|_2^2 \|f\|_2^2 \\
&= \iint \langle f, \varphi_{y,t}\rangle \langle \varphi_{y,t}, \varphi_{y',t'}\rangle \langle \varphi_{y',t'}, f\rangle \mathrm{d}y \frac{\mathrm{d}t}{t} \mathrm{d}y' \frac{\mathrm{d}t'}{t'} \|f\|_2^2 \\
&\leq 2C \int |\langle f, \varphi_{y,t}\rangle|^2 \mathrm{d}y \frac{\mathrm{d}t}{t} \|f\|_2^2.
\end{aligned}$$

In the last inequality we used

$$|\langle f, \varphi_{y,t}\rangle \langle \varphi_{y',t'}, f\rangle| \leq |\langle f, \varphi_{y,t}\rangle|^2 + |\langle f, \varphi_{y',t'}\rangle|^2,$$

together with the definition of the constant

$$C = \sup_{y,t} \int |\langle \varphi_{y,t}, \varphi_{y',t'}\rangle| \mathrm{d}y' \frac{\mathrm{d}t'}{t'}.$$

Since the integral over (y,t) in the last estimate is bounded by the left-hand side of the conclusion, it remains to show $C < \infty$. By Lemma 12.12 we have

$$\begin{aligned}
\int |\langle \varphi_{y,t}, \varphi_{y',t'}\rangle| \mathrm{d}y' \frac{\mathrm{d}t'}{t'} &\lesssim \int_{t \leq t'} \int_{\mathbb{R}^d} \frac{(tt')^{\epsilon/2}}{(t' + \|y - y'\|)^{d+\epsilon}} \mathrm{d}y' \frac{\mathrm{d}t'}{t'} + \\
&\quad + \int_{t > t'} \int_{\mathbb{R}^d} \frac{(tt')^{\epsilon/2}}{(t + \|y - y'\|)^{d+\epsilon}} \mathrm{d}y' \frac{\mathrm{d}t'}{t'} \\
&\lesssim \int_{t \leq t'} \left(\frac{t}{t'}\right)^{\varepsilon/2} \frac{\mathrm{d}t'}{t'} + \int_{t > t'} \left(\frac{t'}{t}\right)^{\varepsilon/2} \frac{\mathrm{d}t'}{t'} \\
&\lesssim 1.
\end{aligned}$$

$\square$

13 The S^2 embedding in the continuous setting

2016-12-01

Proof of Theorem 12.11 with $p = \infty$. We need to show that

$$S^2(Df)(T(x,s)) \lesssim \|f\|_\infty.$$

But this is equivalent to

$$\left(\frac{1}{s^d}\int_{T(x,s)} |Df(y,t)|^2 dy\frac{dt}{t}\right)^{1/2} \lesssim \|f\|_\infty.$$

Split then $f = f\mathbb{1}_{B(x,3s)} + f\mathbb{1}_{B(x,3s)^c}$. By subadditivity of D we can consider these summands separately. For the first summand we use Lemma 12.13 together with the estimate

$$\|f\mathbb{1}_{B(x,3s)}\|_2 \lesssim s^{d/2}\|f\|_\infty.$$

For the second summand we linearize the supremum in the definition of D and estimate

$$D(f\mathbb{1}_{B(x,3s)^c})(y,t) \lesssim |\langle f\mathbb{1}_{B(x,3s)^c}, \varphi_{y,t}\rangle| \lesssim \|f\|_\infty \int_s^\infty \frac{t^\epsilon}{(t+r)^{d+\epsilon}} r^{d-1} dr \lesssim \|f\|_\infty \frac{t^\epsilon}{s^\epsilon}.$$

This implies

$$\begin{aligned}
&\frac{1}{s^d}\int_{T(x,s)} |D(f\mathbb{1}_{B(x,3s)^c})(y,t)|^2 dy\frac{dt}{t} \\
&\quad\lesssim \|f\|_\infty^2 \frac{1}{s^d}\int_0^s \int_{|y-y'|\le s} \left(\frac{t}{s}\right)^{2\epsilon} dy\frac{dt}{t} \\
&\quad\lesssim \|f\|_\infty^2 s^{-2\epsilon}\int_0^s t^{2\epsilon-1}\mathrm{d}t \\
&\quad\lesssim \|f\|_\infty^2.
\end{aligned}$$

□

Proof of Theorem 12.11 with $p = 1$. The proof relies on a smart and classical trick in harmonic analysis. Explicitly, we want to truncate f in such a way that the truncation is going to be a good function, in the sense that it is bounded by a fixed parameter we will choose. On the other hand, there is still going to be a reminiscent term, which we will call our "bad function", which could possibly cause us trouble. Luckily, this "bad function" is not as bad as it sounds, as the measure of it supporting set is finite – and well bounded in terms of our fixed parameter –, and it possesses some (extremely) important cancellation properties, that will allow us to perform the desired bounds. A such decomposition is generally called a *Calderón-Zygmund decomposition*, and can be described as follows:
First, fix a $f \in L^1$. Recall we have a sequence of balls $B(x_i, r_i)$ such that

1. $\frac{1}{r_i^d}\int_{B(x_i,r_i)}|f(z)|dz > \lambda$.

2. $B(x_i, r_i)$ is *disjoint* from all the other balls in the collection;

3. There is no $B(\tilde{x_i}, \tilde{r_i})$ satisfying the two conditions above and $2\tilde{r_i} > r_i$.

In the last lecture we have included the set

$$\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\} \subset \bigcup_i B(x_i, 6r_i).$$

Define then

$$Q_i = B(x_i, 6r_i) \cap \left(\bigcup_{j<i} Q_j\right)^c \cap \left(\bigcup_{j>i} B(x_j, r_j)\right)^c.$$

Those cubes here are playing the role of dyadic intervals, as they contain the ball $B(x_i, r_i)$, are contained in $B(x_i, 6r_i)$ and are pairwise disjoint. Define then b_i be function such that

- $\text{supp} b_i \subset Q_i$;
- $\int b_i(z)dz = 0$;
- $f - b_i$ is *constant* on Q_i (and equal to $\frac{1}{|Q_i|}\int_{Q_i} f$.)

Those are the main ingredients to our desired decomposition. In fact, we call $b = \sum_i b_i$, and this is supposed to be our "bad part". On the other hand, if we write $f = g + b$, then g is supposed to be the good part. We are going to work more thoroughly on this decomposition on the next lectures.

Our aim is to show that

$$\mu(S^2(Df) > C\lambda) \lesssim |\{\mathcal{M}f > \lambda\}|$$

for some sufficiently large C. Since the good part in the CZ decomposition satisfies $\|g\|_\infty \lesssim \lambda$ and by the $p = \infty$ case of the theorem we may replace f by the bad part b in the claim.

From the proof of the $L^{1,\infty}(S^\infty)$ embedding we know that

$$\|Db\mathbb{1}_{(\cup_i T(x_i, 6r_i))^c}\|_{L^\infty(S^\infty)} \lesssim \lambda.$$

By logarithmic convexity of the S^p sizes it therefore suffices to show

$$\|Db\mathbb{1}_{(\cup_i T(x_i, 10r_i))^c}\|_{L^\infty(S^1)} \lesssim \lambda.$$

By scaling we may assume $\lambda = 1$.

We are going to estimate Db_i for each fixed i. For notational simplicity assume $x_i = 0$ and let $b = b_i$, $r = r_i$. Let $(x,t) \notin T(0,10r)$. Consider first the case $t \leq 3r$, so that $|x| \geq 7r$. Then

$$Db(x,t) \leq t^{-d} \int_{B(0,6r)} |b|(y)(1+|(x-y)/t|)^{-d-\epsilon} \mathrm{d}y$$

$$\lesssim t^{-d} \int_{B(0,3r)} |b|(y)|x/t|^{-d-\epsilon} \mathrm{d}y \lesssim (r/t)^d |x/t|^{-d-\epsilon}.$$

Consider now the case $t > 3r$. Then for every $x \in \mathbb{R}^d$ we have

$$\begin{aligned} Db(x,t) &\leq \sup_{\phi \in \mathcal{C}} t^{-d} \Big| \int b(y)(\phi((x-y)/t) - \phi(x/t)) \mathrm{d}y \Big| \\ &\leq (r/t)^d \sup_{\phi \in \mathcal{C}, |y| \leq 3r} |\phi((x-y)/t) - \phi(x/t)| \\ &\lesssim (r/t)^{d+\epsilon} \sup_{|y|\leq 3r} ((1+|x-y|/t)^{-d-\epsilon} + (1+|x|/t)^{-d-\epsilon}) \\ &\lesssim (r/t)^{d+\epsilon}(1+|x|/t)^{-d-\epsilon}. \end{aligned}$$

Next we estimate the integral of Db over the vertical line $\{x\} \times (0,s) \setminus T(0,10r)$. For $|x| \leq 7r$ we have

$$\int_{3r}^{s} Db(x,t) \frac{\mathrm{d}t}{t} \lesssim \int_{3r}^{\infty} (r/t)^{d+\epsilon}(1+|x|/t)^{-d-\epsilon} \frac{\mathrm{d}t}{t} \lesssim 1$$

for every $s > 0$. For $|x| > 7r$ we distinguish several cases in the estimate for

$$\int_0^s Db(x,t) \frac{\mathrm{d}t}{t}.$$

Case $s \leq 3r$

$$\int_0^s Db(x,t)\frac{\mathrm{d}t}{t} \lesssim \int_0^s (r/t)^d |x/t|^{-d-\epsilon} \frac{\mathrm{d}t}{t} \lesssim r^d |x|^{-d-\epsilon} \int_0^s t^{\epsilon} \frac{\mathrm{d}t}{t} \lesssim r^d |x|^{-d-\epsilon} s^{\epsilon}.$$

Case $3r < s \leq |x|$ We split the integral at $t = 3r$. For the first part we obtain the estimate $\lesssim r^{d+\epsilon}|x|^{-d-\epsilon}$ by the previous case. The second part we estimate by

$$\int_{3r}^s Db(x,t)\frac{\mathrm{d}t}{t} \lesssim \int_{3r}^s (r/t)^{d+\epsilon}(1+|x|/t)^{-d-\epsilon}\frac{\mathrm{d}t}{t} \lesssim (r/|x|)^{d+\epsilon} \int_{3r}^s \frac{\mathrm{d}t}{t} \lesssim (r/|x|)^{d+\epsilon} \log s/(3r).$$

Case $|x| < s$ We split the integral at $t = |x|$. For the first part we obtain the estimate $(r/|x|)^{d+\epsilon}(1+\log s/(3r))$ by the previous case. The last part is estimated by

$$\int_{|x|}^s Db(x,t)\frac{\mathrm{d}t}{t} \lesssim \int_{|x|}^s (r/t)^{d+\epsilon}(1+|x|/t)^{-d-\epsilon}\frac{\mathrm{d}t}{t} \lesssim \int_{|x|}^{\infty} (r/t)^{d+\epsilon}\frac{\mathrm{d}t}{t} \lesssim (r/|x|)^{d+\epsilon}.$$

Overall the integral of Db restricted to $T(0,10r)^c$ over the vertical line $\{x\}\times(0,s)\setminus T(0,10r)$ is bounded by

$$V(r,s,x)=\begin{cases}0, & |x|\le 7r, s\le 3r,\\ 1, & |x|\le 7r, s>3r,\\ r^d s^\epsilon |x|^{-d-\epsilon}, & |x|>7r, s\le 3r,\\ r^{d+\epsilon'}|x|^{-d-\epsilon'}, & |x|>7r, s>3r,\end{cases}$$

where $0<\epsilon'<\epsilon$ is arbitrary.

We claim that for an arbitrary family of disjoint balls $B(x_i,r_i)$, every $c\in\mathbb{R}^d$, and every $0<s<\infty$ we have

$$s^{-d}\int_{B(c,s)}\sum_i V(r_i,s,x-c_i)\mathrm{d}x \lesssim 1.$$

By scaling and translation we may assume $s=1$ and $c=0$.

Observe first that for $3r<1$ we have $\int_{\mathbb{R}^d} V(r,1,x)\mathrm{d}x\lesssim r^d$, so the contributions of the small balls inside $B(0,10)$ can be summed up using disjointness. The contributions of the small balls outside of $B(0,9)$ and of the large balls decay exponentially with scale and distance from the origin. This ends the proof of the claim.

Applying the claim to the bad cubes in the CZ decomposition we obtain the required S^1 estimate. □

14 Supplement: tent spaces, square functions, shearings

2016-12-06

This lecture actually continued the proof of the $L^{1,\infty}(S^2)$ embedding from the previous lecture. Here the proof is presented in one section for improved readability. This section contains solutions to several homework problems and Lemma 14.4 that is important in the sequel.

The outer L^p spaces can be embedded into the "tent spaces" introduced in [CMS85].

Definition 14.1. The cone in the upper half-space with vertex $x\in\mathbb{R}^d$ is the set $\Gamma(x):=\{(y,t)\in\mathbb{R}^d\times(0,\infty),|x-y|<t\}$. For a function $G:\mathbb{R}^d\times(0,\infty)\to\mathbb{R}$ we define

$$A_qG(x):=\Big(\int_{\Gamma(x)}|G(y,t)|^q\frac{\mathrm{d}t\mathrm{d}y}{t^{d+1}}\Big)^{1/q},\quad A_\infty G(x):=\sup_{(y,t)\in\Gamma(x)}|G(y,t)|,$$

where the supremum is taken in the almost everywhere sense. The *tent spaces* T^p_q are the spaces of functions on the upper half-space $\mathbb{R}^d\times(0,\infty)$ defined by the norms

1. $\|A_q G\|_{L^p}$ if $1 \leq p, q < \infty$,
2. same for $1 \leq p < q = \infty$, but with an additional continuity assumption on the functions,
3. $\|G\|_{L^\infty(S^q)}$ if $p = \infty$ and $q = 2$

Lemma 14.2. *For every $1 < p, q < \infty$ we have*

$$\|A_q G\|_{L^p} \lesssim \|G\|_{L^p(S^q)}$$

and

$$\|A_q G\|_{L^{1,\infty}} \lesssim \|G\|_{L^{1,\infty}(S^q)}.$$

Proof. We begin with the case $p = q$. The estimate

$$\|A_q G\|_{L^q} \lesssim \|G\|_{L^q(S_q)}, \quad 0 < q < \infty.$$

follows easily from the case $q = 1$, and in that case we note

$$\begin{aligned}\|A_1 G\|_{L^1} &= \int_{\mathbb{R}^d} \int_{\Gamma(x)} |G(y,t)| \frac{\mathrm{d}t\mathrm{d}y}{t^{d+1}} \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_0^\infty \int_{|x-y| \leq t} |G(y,t)| \mathrm{d}x \frac{\mathrm{d}t}{t^{d+1}} \mathrm{d}y \sim \int_{\mathbb{R}^d} \int_0^\infty |G(y,t)| \frac{\mathrm{d}t}{t^1} \mathrm{d}y,\end{aligned}$$

and this is bounded by $\|G\|_{L^1(S^1)}$ by atomicity.

Next we will prove the weak type $(1,1)$ estimate. Let $\lambda > 0$ and let $\mathcal{E}'$ be a collection of tents such that

$$\sum_{T \in \mathcal{E}'} \sigma(T) \leq 2\mu(S^q G > \lambda), \quad \|G'\|_{L^\infty(S^q)} \leq \lambda,$$

where $G' = G \mathbb{1}_{(\cup \mathcal{E}')^c}$. Then

$$\begin{aligned}\|G'\|_{L^q(S^q)}^q &= \int_0^\infty q(\lambda')^{q-1} \mu(S^q G' > \lambda') \mathrm{d}\lambda' \leq \int_0^\lambda q(\lambda')^{q-1} \mu(S^q G > \lambda') \mathrm{d}\lambda' \\ &\leq \|G\|_{L^{1,\infty}(S^q)} \int_0^\lambda q(\lambda')^{q-2} \mathrm{d}\lambda' = \frac{q}{q-1} \lambda^{q-1} \|G\|_{L^{1,\infty}(S^q)}.\end{aligned}$$

It follows that

$$|\{A_q G' > \lambda\}| \leq \lambda^{-q} \|A_q G'\|_{L^q}^q \lesssim \lambda^{-q} \|G'\|_{L^q(S^q)}^q \lesssim \lambda^{-1} \|G\|_{L^{1,\infty}(S^q)}.$$

For a tent $T = T(x,r)$ let $B(T) = B(x,r)$. Note that $A_q G = A_q G'$ on $\mathbb{R}^d \setminus \cup_{T \in \mathcal{E}'} B(T)$. Therefore

$$|\{A_q G > \lambda\}| \leq |\{A_q G' > \lambda\}| + \sum_{T \in \mathcal{E}'} |B(T)|,$$

and the conclusion follows. By interpolation we obtain the cases $1 < p < q$.

It remains to consider $p > q$. In this case we have

$$\|A_q G\|_{L^p}^q = \|(A_q G)^q\|_{L^{p/q}} = \int f(A_q G)^q$$

for some $f \in L^{(p/q)'}$ with $\|f\|_{(p/q)'} = 1$. The right-hand side can be written as

$$\begin{aligned} \int f(x) \int_{\Gamma(x)} |G(y,t)|^q \frac{\mathrm{d}t\mathrm{d}y}{t^{d+1}} \mathrm{d}x &= \int_{\mathbb{R}^d} \int_0^\infty |G(y,t)|^q \int_{|x-y|\le t} f(x) \mathrm{d}x \frac{\mathrm{d}t}{t^{d+1}} \mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^d} \int_0^\infty |G(y,t)|^q A_t f(x) \frac{\mathrm{d}t}{t} \mathrm{d}y \lesssim \|G^q A_t f\|_{L^1(S^1)} \\ &\lesssim \|G^q\|_{L^{p/q}(S^1)} \|A_t f\|_{L^{(p/q)'}(S^\infty)} \lesssim \|G\|_{L^p(S^q)}^q \|f\|_{L^{(p/q)'}} \lesssim \|G\|_{L^p(S^q)}^q \end{aligned}$$

as required. □

In particular, the S^2 embeddings allow us to recover standard estimates for the Littlewood–Paley square function, and in fact also for the "intrinsic" square function [Wil07].

Corollary 14.3. *For every $1 < p < \infty$ we have*

$$\|A_2(Df)\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Moreover,

$$\|A_2(Df)\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}.$$

Proof. Use Theorem 12.11 and Lemma 14.2. □

We will need certain uniform estimates on sheared and rescaled functions in the upper half-space.

Lemma 14.4. *Let $\alpha \in B(0,1)$ and $0 < \beta \le 1$. For a function $F : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ define $F_{\alpha,\beta}(x,t) := F(x + \alpha t, \beta t)$. Then*

$$\mu(S^2 F_{\alpha,\beta} > \lambda) \lesssim \beta^{-d} \mu(S^2 F > 2^{-d}\lambda)$$

and

$$\|F_{\alpha,\beta}\|_{L^{2,\infty}(S^2)} \lesssim \Big(\int_0^\infty \int_{\mathbb{R}^d} F(x,t)^2 \mathrm{d}x \frac{\mathrm{d}t}{t} \Big)^{1/2}.$$

Recall from Lemma 14.2 that

$$\int_0^\infty \int_{\mathbb{R}^d} F(x,t)^2 \mathrm{d}x \frac{\mathrm{d}t}{t} \lesssim \|F\|_{L^2(S^2)}^2.$$

Therefore, by interpolation we obtain

$$\|F_{\alpha,\beta}\|_{L^p(S^2)} \lesssim \beta^{-d\alpha_p} \|F\|_{L^p(S^2)},$$

where

$$\alpha_p = \begin{cases} 2/p - 1, & 1 < p < 2, \\ \text{any number } > 0, & p = 2, \\ 0, & 2 < p < \infty. \end{cases}$$

Proof. We begin with the estimate for the superlevel measure. Note that

$$\begin{aligned} \int_{T(x,r)} F_{\alpha,\beta}(y,t)^2 \mathrm{d}y \frac{\mathrm{d}t}{t} &= \int_{|x-y|+t \le r} F(y+\alpha t, \beta t)^2 \mathrm{d}y \frac{\mathrm{d}t}{t} \\ &\le \int_{|x-z|+t\le 2r} F(z,\beta t)^2 \mathrm{d}z \frac{\mathrm{d}t}{t} \le \int_{|x-z|+t\le 2r} F(z,t)^2 \mathrm{d}z \frac{\mathrm{d}t}{t}, \quad (14.5) \end{aligned}$$

so that $\|F_{\alpha,\beta}\|_{L^\infty(S^2)} \le 2^d \|F\|_{L^\infty(S^2)}$. Let $\mathcal{E}$ be a collection of tents such that $\|F 1_{(\cup \mathcal{E})^c}\|_{L^\infty(S^2)} \le 2^{-d}\lambda$. Then $F_{\alpha,\beta} 1_{(\cup_{T\in\mathcal{E}} \frac{2}{\beta} T)^c} \le (F 1_{(\cup_{T\in\mathcal{E}} T)^c})_{\alpha,\beta}$, and the superlevel measure estimate follows.

It remains to prove the $L^{2,\infty}(S^2)$ estimate. Let $\mathcal{E}$ be the collection of all tents T such that $S^2 F_{\alpha,\beta}(T) > \lambda$. If the right-hand of the claimed estimate is finite, then the radii of the tents in $\mathcal{E}$ are bounded from above.

By Vitali's covering lemma applied to the space $\mathbb{R}^d \times [0,\infty)$ with the metric $d((x,t),(x',t')) = |x-x'| + |t-t'|$ we can cover $\mathcal{E}$ by $\cup_{T\in\mathcal{E}'} 10T$, where $\mathcal{E}' \subset \mathcal{E}$ is a collection of tents T such that the expanded tents $2T$ are pairwise disjoint. It follows from (14.5) that

$$\begin{aligned} \sum_{T\in\mathcal{E}'} \sigma(10T) &\lesssim \lambda^{-2} \sum_{T\in\mathcal{E}'} \sigma(T) S^2(F_{\alpha,\beta})(T)^2 \\ &\le \lambda^{-2} \sum_{T\in\mathcal{E}'} \int_{2T} F^2(x,t) \mathrm{d}x \frac{\mathrm{d}t}{t} \le \lambda^{-2} \int_0^\infty \int_{\mathbb{R}^d} F^2(x,t) \mathrm{d}x \frac{\mathrm{d}t}{t}. \end{aligned}$$

On the other hand, the restriction of F to $(\cup_{T\in\mathcal{E}'} 10T)^c$ vanishes identically on every tent from $\mathcal{E}$, so it certainly has $L^\infty(S^2)$ norm bounded by λ. □

15 Paraproduct decomposition for classical operators

2016-12-08

Aiming to use the tools we have developped so far, we define a prototype of a function φ that we have been used before. Explicitly, take φ a function such that

- $\int_{\mathbb{R}^d} \varphi(z) dz = 1$;
- $|\varphi(z)| \le (1+|z|)^{-d-1}$;
- $\varphi_{y,t}(z) = \frac{1}{t^d} \varphi\left(\frac{z-y}{t}\right)$.

Notice that the famly $\varphi_{y,t}$ has always the same L^1-norm, and therefore we call it L^1*-normalized.*
To exemplify our method, let f_i be continuous, compactly supported real functions. Then we have that

$$\lim_{t\to 0}\langle f, \varphi_{y,t}\rangle = f_i(y).$$

As all functions are compactly supported, we can even state a *uniform convergence* for the equality above. This translates into the fact that

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\prod_{i=1}^{n}\langle f_i, \varphi_{y,t}\rangle dy = \int_{\mathbb{R}^d}\prod_{i=1}^{n} f_i(y)dy.$$

On the other hand, by the fact that each of the f_i are compactly supported, for $n \geq 2$ we have

$$\lim_{t\to\infty}\int_{\mathbb{R}^d}\prod_{i=1}^{n}\langle f_i, \varphi_{y,t}\rangle dy = 0.$$

Let us calculate then:

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^d}\prod_{i=1}^{n}\langle f_i, \varphi_{y,t}\rangle dy\right) = \frac{-nd}{t}\int_{\mathbb{R}^d}\prod_{i=1}^{n}\langle f_i, \varphi_{y,t}\rangle dy \\ -\frac{1}{t^d}\sum_{i=1}^{n}\sum_{k=1}^{d}\int\left(\int f_j(z)\frac{z_k-y_k}{t^2}\partial_k\varphi\left(\frac{z-y}{t}\right)dz\right)\prod_{i\neq j}\langle f_i, \varphi_{y,t}\rangle dy.\end{aligned} \tag{15.1}$$

Assume that φ is even in all arguments. This implies immediately that both $z_k\varphi$ and $\partial_k\varphi$ are odd in z_k. Therefore, by a partial integration we have that the expression in 15.1 is equal to

$$\begin{aligned}&= \frac{-nd}{t}\int_{\mathbb{R}^d}\prod_{i=1}^{n}\langle f_i, \varphi_{y,t}\rangle dy + \\ &\frac{1}{t}\sum_{j=1}^{n}\sum_{k=1}^{d}\int\left(\int f_j(z)\frac{1}{t^d}\varphi\left(\frac{z-y}{t}\right)dz\right)\prod_{i\neq j}\langle f_i, \varphi_{y,t}\rangle dy \\ &+\frac{1}{t}\sum_{j=1}^{n}\sum_{k=1}^{d}\sum_{l\neq j}\int \\ &\left(\int f_j(z)\frac{1}{t^d}\frac{z_k-y_k}{t}\varphi\left(\frac{z-y}{t}\right)dz\int f_l(\tilde{z})\frac{1}{t^d}\partial_k\varphi\left(\frac{\tilde{z}-y}{t}\right)d\tilde{z}\right) \\ &\prod_{i\neq j,l}\langle f_i, \varphi_{y,t}\rangle dy.\end{aligned} \tag{15.2}$$

It is though easy to see that the two summands above are in fact the same. Now we also take two extra assumptions, namely, that $|\partial_k\varphi(z)| \leq (1+$

$|z|)^{-(d+2)}$ and $|\nabla\partial_k\varphi(z)| \leq (1+|z|)^{-(d+2)}$. This makes automatically the functions $\varphi_{y,t}$, $y_k\varphi_{y,t}$, $\partial_k\varphi_{y,t}$ are all allowed for our embedding theorem. We call then

$$\psi_{k,y,t} := y_k\varphi_{y,t} \text{ and } \tilde{\psi}_{k,y,t} = \partial_k\varphi_{y,t}.$$

With these definitions, the calculations above and an use of the fundamental theorem of calculus, we see that

$$\begin{aligned}
\int_{\mathbb{R}^d}\prod_{i=1}^n f_i(y)dy &= \sum_{j=1}^n\sum_{k=1}^d\sum_{l\neq j}\int_0^\infty\int_{\mathbb{R}^d}\langle f_j,\psi_{k,y,t}\rangle\langle f_l,\tilde{\psi}_{k,y,t}\rangle\prod_{i\neq j,l}\langle f,\varphi_{y,t}\rangle dy\frac{dt}{t}\\
&\overset{\text{atomicity}}{\leq} \|\langle f_i,\psi_{k,y,t}\rangle\langle f_l,\tilde{\psi}_{k,y,t}\rangle\prod_{i\neq l,j}\langle f_j,\varphi_{y,t}\rangle\|_{\mathcal{L}^1(S^1)}\\
&\overset{\text{outer Hölder}}{\leq} \|\langle f_i,\psi_{k,y,t}\rangle\|_{\mathcal{L}^{p_j}(S^2)}\|\langle f_l,\tilde{\psi}_{k,y,t}\rangle\|_{\mathcal{L}^{p_l}(S^2)}\prod_{i\neq j,l}\|\langle f_i,\varphi_{y,t}\rangle\|_{\mathcal{L}^{p_i}(S^\infty)}\\
&\overset{\text{emb. theorems}}{\leq} C\prod_{i=1}^n\|f_i\|_{p_i}.
\end{aligned}$$

Remark 15.3. We emphasize that, if we take our model function $\varphi(z) = Ce^{-z^2}$, then our functions $\psi,\tilde{\psi}$ above can be taken such that $\psi = c'\tilde{\psi}$.

15.1 Classical operators and paraproduct decomposition

We want to do the same kind of procedure as above to more general, classical operators in harmonic analysis. Explicitly, if we take $n = 2$ above, then our toy example was

$$\int_{\mathbb{R}^d} f_1(y)f_2(y)dy.$$

We want to transform and adapt some of our techniques, so that we can estimate also the following more general operators

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)K(y,z)g(z)dy\,dz,$$

where we demand that our operator $K(y,z)$ has some special properties. Of course, by taking such an operator to be $K(y,z) = \delta_{y=z}$, we recover our toy case above. Let us assume then that we have a more general operator $K(x,s,y,t)$ such that

$$|K(x,s,y,t)| \leq \frac{\min(t,s)}{\max(t,s,\|y-x\|)^{d+1}}.$$

Then we can state the following:

Theorem 15.4. *If* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $d\max(\alpha_p,\alpha_{p'})<\epsilon$*, then*

$$\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\min(t,s)^\epsilon|F(x,s)G(y,t)|}{\max(t,s,\|y-x\|)^{d+\epsilon}}dx\,dy\,\frac{ds}{s}\,\frac{dt}{t}\lesssim\|F\|_{L^p(S^2)}\|G\|_{L^{p'}(S^2)}.$$

Proof. We set first $r=\max(t,s,\|y-x\|)$.
Domain 1: $r\neq\|x-y\|$. Assume, without loss of generality – as the symmetries of K allow us to do so – that $t\le s$. First we notice that our kernel can, in this domain, be bounded by

$$\frac{t^\epsilon}{s^{d+\epsilon}}.$$

Letting $y=x+\alpha s, t=\beta s$ and computing the change of variables explicitly, we see that

$$\int_0^\infty\int_{\mathbb{R}^d}\int_0^s\int_{B(x,s)}|F\cdot K\cdot G|\,dy\,\frac{dt}{t}\,dx\,\frac{ds}{s}$$

is bounded by

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}\int_0^1\int_{B(0,1)}F(x,s)G(x+\alpha s,\beta s)\,d\alpha\,\frac{d\beta}{\beta^{1-\epsilon}}\,dx\,\frac{ds}{s}\\
&\overset{\text{atomicity + Hölder}}{\le}C\int_0^1\int_{B(0,1)}\|F\|_{\mathcal{L}^p(S^2)}\cdot\|G_{\alpha,\beta}\|_{\mathcal{L}^{p'}(S^2)}\,d\alpha\,\frac{d\beta}{\beta^{1-\epsilon}}\\
&\overset{\text{Lemma 14.4}}{\le}C\int_0^1\int_{B(0,1)}\|F\|_{L^p(S^2)}\|G\|_{L^{p'}(S^2)}\cdot\beta^{-1+\epsilon-d\alpha_{p'}}\,d\alpha\,d\beta\\
&\lesssim\|F\|_{L^p(S^2)}\|G\|_{L^{p'}(S^2)}.
\end{aligned}$$

This shows us that this term is suitably controlled.
Domain 2: $r=\|x-y\|$. We assume one more time that $s\ge t$, as the other case is entirely analogous. In this case, we wish to estimate

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^{\|x-y\|}\int_0^s F(x,s)\frac{t^\epsilon}{\|x-y\|^{d+\epsilon}}G(y,t)\,\frac{dt}{t}\,\frac{ds}{s}\,dx\,dy.$$

But, by a change of variables, we have that

$$\begin{aligned}
&\overset{\text{Change of variables}}{=}\int_0^1\int_0^\alpha\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F(x,\alpha\|x-y\|)\\
&\times\frac{1}{\|x-y\|^d}G(y,\beta\|x-y\|)\,dx\,dy\,\frac{d\beta}{\beta^{1-\epsilon}}\,\frac{d\alpha}{\alpha}\\
&\overset{\text{polar coordinates}}{=}\int_0^1\int_0^\alpha\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_{\mathbb{R}^d}F(x,\alpha r)G(x+ur,\beta r)\,dx\,\frac{dr}{r}\,du\,\frac{d\beta}{\beta^{1-\epsilon}}\,\frac{d\alpha}{\alpha}\\
&\overset{\text{atomicity + Hölder}}{\le}C\int_0^1\int_0^\alpha\int_{\mathbb{S}^{d-1}}\|F_{0,\alpha}\|_{\mathcal{L}^p(S^2)}\|G_{u,\beta}\|_{\mathcal{L}^{p'}(S^2)}\,du\,\frac{d\beta}{\beta^{1-\epsilon}}\,\frac{d\alpha}{\alpha}\\
&\overset{\text{Lemma 14.4}}{\le}C\|F\|_{L^p(S^2)}\|G\|_{L^{p'}(S^2)}\int_0^1\int_0^\alpha\alpha^{-d\alpha_p}\beta^{-d\alpha_{p'}}\,\frac{d\beta}{\beta^{1-\epsilon}}\,\frac{d\alpha}{\alpha}\\
&\lesssim\|F\|_{L^p(S^2)}\|G\|_{L^{p'}(S^2)}
\end{aligned}$$

This clearly finishes the proof of the theorem. $\square$

15.2 Example: the Hilbert transform

From the calculations above, we see that, for two sufficiently regular functions f, g, we have that

$$\int_{\mathbb{R}^d} g(y)f(y)\,dy = \sum_{k=1}^{d}\sum_{j=1}^{2}\int_0^\infty \int_{\mathbb{R}^d} \langle f, \psi_{k,y,t,j}\rangle \langle g, \tilde{\psi}_{k,y,t,j}\rangle\,dy\,\frac{dt}{t}.$$

Now we may use the fact that, if two function f_1 and f_2 satisfy that, for all g smooth and compactly supported,

$$\int f_1 g = \int f_2 g,$$

then $f_1 = f_2$. This implies promptly that

$$f = \sum_{k=1}^{d}\sum_{j=1}^{2}\int_0^\infty \int_{\mathbb{R}^d} \langle f, \psi_{k,y,t,j}\rangle \tilde{\psi}_{k,y,t,j}\,dy\,\frac{dt}{t},$$

at least formally. Therefore, we can try to apply the same sort of "formal" reasoning, so that, for a bilinear form Λ, we have that

$$\Lambda(f,g) = \sum_{k,k'=1}^{d}\sum_{j,j'=1}^{2}\int_0^\infty \int_{\mathbb{R}^d}\int_0^\infty \int_{\mathbb{R}^d} \langle f, \psi_{k,y,t,j}\rangle$$

$$\times \Lambda(\psi_{k,y,t,j}, \tilde{\psi}_{k,y,t,j})\langle g, \tilde{\psi}_{k,y,t,j}\rangle\,dy\,\frac{dt}{t}\,dy'\,\frac{dt'}{t'}.$$

Therefore, we can wonder whether we can use a similar method to deal with, for example, the Hilbert transform. This is, nevertheless, the aim of the next lecture.

16 $T(1)$ theorem

2016-12-13

Definition 16.1. We start by defining a *Calderón-Zygmund kernel* K as a function

$$K\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x)\colon x \in \mathbb{R}^d\} \to \mathbb{R}$$

such that:

1. $\forall x, x_0, y\colon 2\|x - x_0\| \le \|y - x_0\|$, then

$$|K(x,y) - K(x_0,y)| \le \frac{\|x - x_0\|}{\|y - x_0\|^{d+1}};$$

2. $\forall x, y, y_0 \colon 2\|y - y_0\| \le \|x - y_0\|$, then

$$|K(x,y) - K(x,y_0)| \le \frac{\|y - y_0\|}{\|x - y_0\|^{d+1}};$$

3. $\forall \varepsilon > 0 \quad \exists N \colon \|x - y\| > N$ then $|K(x,y)| \le \varepsilon$.

The first two properties provide local conditions on K, while the third requests a qualitative decay away from the diagonal. For example, it prevents from considering the constant functions.
Observe that these three properties imply a bound for K, namely

$$\begin{aligned} |K(x,y)| &= |\sum_{n=1}^{\infty} K(x, x + n(y-x)) - K(x, x + (n+1)(y-x))| \le \\ &\le \sum_{n=1}^{\infty} \frac{\|y - x\|}{((n+1)\|y - x\|)^{d+1}} = C \frac{1}{\|y - x\|^d}, \end{aligned}$$

where we can telescope because of the third property.

Example 16.2. While Definition 16.1 might seem a little bit abstract, we emphasize that those conditions naturally appear on classical operators in Harmonic Analysis. For example, if $d = 1$, then taking

$$K(x,y) = \frac{1}{x - y}$$

returns us the classical example of the Hilbert transform, and if $d > 1$, by taking

$$K(x,y) = \frac{x_k - y_x}{\|x - y\|^{d+1}},$$

gives us the $k-$Riesz transform.

Definition 16.3. The *space of Schwartz functions* or the *Schwartz class* is

$$\mathcal{S}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} \colon \forall N, \sup_{|\alpha|,|\beta| < N} \|x^\alpha \partial^\beta f\|_\infty < +\infty\}.$$

Definition 16.4 (Λ associated to CZ kernel)**.** A bilinear form $\Lambda \colon \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ is *associated with the CZ kernel K* if for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi) = \emptyset$, then

$$\Lambda(\varphi, \psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) K(x,y) \psi(y) \mathrm{d}x \mathrm{d}y. \tag{16.5}$$

Example 16.6. The identity operator

$$\Lambda(\varphi, \psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y)\, dx\, dy$$

is associated to $K \equiv 0$. In fact, any multiple of the identity has such property. In particular, there is no claim about uniqueness of a form associated to a CZ kernel.

We can finally state our main theorem:

Theorem 16.7 ($T(1)$ Theorem)**.** *Let $\Lambda\colon \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ be associated with the CZ kernel K. Assume that there exists N such that*

$$\Lambda(\varphi_{y,t}, \psi) \leq t^{-\frac{d}{2}} \|\psi\|_2,$$
$$\Lambda(\psi, \varphi_{y,t}) \leq t^{-\frac{d}{2}} \|\psi\|_2,$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, where φ satisfies

- $\operatorname{supp}(\varphi) \subset B(0,1)$*;*
- $\sup_{\alpha < N} \|\partial^\alpha \varphi\|_\infty \leq 1$*;*
- $\varphi_{y,t}(z) := \frac{1}{t^d} \varphi\left(\frac{z-y}{t}\right)$.

Then

$$\Lambda(\varphi, \psi) \leq C \|\varphi\|_2 \|\psi\|_2.$$

In this lecture, we will focus our work on the first part of the proof, which will consist of adding an additional assumption to the theorem and proving it.

Example 16.8. *The Hilbert kernel.* Let us prove that the Hilbert kernel satisfies the properties required in the $T(1)$ theorem. We start by noticing that

$$\begin{aligned}
&\text{p.v.} \int_{\mathbb{R}} \varphi_{y,t}(x-z) \frac{dz}{z} = \\
&\frac{1}{2} \int_{\mathbb{R}} (\varphi_{y,t}(x-z) - \varphi_{y,t}(x+z)) \frac{dz}{z} \leq \\
&\frac{1}{2} \int_{\{|z| \leq 2t\}} |\varphi_{y,t}(x-z) - \varphi_{y,t}(x+z)| \frac{dz}{|z|} + \\
&\int_{\{|z| \geq 2t\}} |\varphi_{y,t}(x-z) - \varphi_{y,t}(x+z)| \frac{dz}{|z|} \\
&= I_1 + I_2.
\end{aligned}$$

On I_2, we use the hypotheses on φ in the $T(1)$ theorem, and it is easy to see that it is bounded by $\frac{C}{t+|x-y|}$. On I_1, on the other hand, we use the fundamental theorem of calculus to bound

$$I_1 \leq Ct \|\varphi'_{y,t}\|_\infty.$$

Finally, the conditions on φ show one more time that this part is also controlled by $\frac{C}{t+|x-y|}$, and thus we have that

$$|\Lambda(\varphi_{y,t},\psi)|+|\Lambda(\psi,\varphi_{y,t})|\leq C'\left\|\frac{1}{t+|x-y|}\right\|_2\cdot\|\psi\|_2.$$

By scalling, it is easy to see that $\left\|\frac{1}{t+|x-y|}\right\|_2\leq\frac{C}{t^{-d/2}}$. This proves the desired, and therefore the Hilbert kernel is included in our class of Calderón-Zygmund kernels.

Proof of the $T(1)$ theorem. *Part (a)*: Let $\rho\in\mathcal{S}(\mathbb{R}^d)$ be a smooth, compactly supported, and positive function with

$$\rho\equiv 1 \text{ on } B(0,1);\ \ \rho\equiv 0 \text{ on } B(0,2)^c.$$

Define $\rho_r(z)=\rho\left(\frac{x}{r}\right)$. Assume that for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$ such that

1. φ has compact support;
2. $\int_{\mathbb{R}^d}\varphi(z)\,dz=0$;

we have $\lim_{r\to+\infty}\Lambda(\varphi,\rho_r)=\lim_{r\to+\infty}\Lambda(\rho_r,\varphi)=0$.

Remark 16.9. We can see the assumption as, formally,

$$\lim_{r\to+\infty}\Lambda(\varphi,\rho_r)\text{“}=\text{”}\Lambda(\varphi,1)\text{“}=\text{”}\langle\varphi,T(1)\rangle.$$

That is the condition which gives its name to the theorem. Notice also that all cases which we have seen so far satisfy this assumption.

Under this assumption, we will prove that

$$\Lambda(\varphi_{y,t},\psi_{x,t})\leq C\frac{\min(t,s)^{1-\varepsilon}}{\max(t,s,\|x-y\|)^{d+1-\varepsilon}},$$

and then the L^2-boundedness will follow from Theorem 15.4. We assume the same hypotheses on ψ as on φ.

First domain: $\|x-y\|\geq 4\max(t,s)$. Without loss of generality, we also suppose that $t\leq s$. In this case we want to get an upper bound of the form

$$\frac{t^{1-\varepsilon}}{\|x-y\|^{d+1-\varepsilon}}.$$

The hypotheses on x, y, t, s imply that the supports of $\varphi_{y,t}$ and $\psi_{x,s}$ are actually *disjoint*. This gives us directly that

$$\begin{aligned}
\Lambda(\varphi_{y,t}, \psi_{x,s}) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi_{y,t}(u)K(u,v)\psi_{x,s}(v)\,du\,dv \\
&\overset{\int\psi=0}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi_{y,t}(u)(K(u,v)-K(y,v))\psi_{x,s}(v)\,du\,dv \\
&\overset{\text{assumptions}}{\leq} C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi_{y,t}(u)\frac{t}{\|x-y\|^{d+1}}\psi_{x,s}(v)\,du\,dv \\
&\overset{\text{assumptions}}{\leq} \frac{Ct}{\|x-y\|^{d+1}},
\end{aligned}$$

in which case we get something actually *better* than promised.

Second domain: $\|x-y\| < 4\max(t,s)$. We again assume $t \leq s$. In this case, we write

$$\begin{aligned}
\Lambda(\varphi_{y,t}, \psi_{x,s}) &\overset{\text{additional assumption}}{=} \Lambda(\varphi_{y,t}, \psi_{x,s} - \psi_{x,s}(y)) \\
&= \Lambda(\varphi_{y,t}, (\psi_{x,s} - \psi_{x,s}(y))\rho_{y,t}) \ \ (= I_1) \\
&+ \Lambda(\varphi_{y,t}, (\psi_{x,s} - \psi_{x,s}(y))(1-\rho_{y,t})) \ \ (= I_2).
\end{aligned}$$

Of course, we understand I_2 above as a limit. We begin then: I_1 can be estimated, by assumption, by

$$I_1 \leq t^{-d/2}\|(\psi_{x,s} - \psi_{x,s}(y))\rho_{y,t}\|_2 \leq Ct^{-d/2}t^{d/2}\frac{t}{s^{d+1}} = C\frac{t}{s^{d+1}}.$$

By assumption, this case is clear. We split now

$$I_2 = \int_{\mathbb{R}^d}\int_{B(y,3s)} + \int_{\mathbb{R}^d}\int_{B(y,3s)^c} = I_{2,1} + I_{2,2}.$$

But the functions involved in I_2 have disjoint supports, and then we have that

$$
\begin{aligned}
I_{2,1} &= \sum_{k=1}^{\lfloor\log(\frac{s}{t})\rfloor+C} \int_{\mathbb{R}^d}\int_{B(y,2^k t)\setminus B(y,2^{k-1}t)} \varphi_{y,t}(u)K(u,v)(\psi_{x,s}(v)-\psi_{x,s}(y))(1-\rho_{y,t}(v))\,du\,dv \\
&\le C' \sum_{k=1}^{\lfloor\log(\frac{s}{t})\rfloor+C} \int_{\mathbb{R}^d}\int_{B(y,2^k t)\setminus B(y,2^{k-1}t)} \varphi_{y,t}(u)(K(u,v)-K(y,v))(\psi_{x,s}(v)-\psi_{x,s}(y))(1-\rho_{y,t}(v))\,du\,dv \\
&\overset{\text{assumptions on}\psi\text{and}K}{\le} C \sum_{k=1}^{\lfloor\log(\frac{s}{t})\rfloor+C} \frac{t}{(2^k t)^{d+1}}\frac{2^k t}{s^{d+1}}(2^k t)^d \le C' \sum_{k=1}^{\lfloor\log(\frac{s}{t})\rfloor+C} \frac{t}{s^{d+1}} \\
&\le C\log(\frac{s}{t})\frac{t}{s^{d+1}} \le C''\frac{t^{1-\varepsilon}}{s^{d+1-\varepsilon}}.
\end{aligned}
$$

Finally, the estimate for the last part should be easier: as $s \ge t$, then $1-\rho_{y,t} \equiv 1$. Moreover, by the fact that $\int \varphi = 0$, we have that

$$
\begin{aligned}
I_{2,2} &\le \int_{\mathbb{R}^d}\int_{B(y,3s)^c} |\varphi_{y,t}(u)||K(u,v)-K(y,v)||\psi_{x,s}(v)-\psi_{x,s}(y)|\,dv\,du \\
&\overset{\text{hypothesis on}\psi}{\le} \int_{\mathbb{R}^d}\int_{B(y,3s)^c} |\varphi_{y,t}(u)|\frac{t}{\|v-y\|^{d+1}}\cdot\frac{1}{s^d}\,dv\,du \\
&\overset{\text{integrating}}{\le} C\cdot\frac{t}{s}\frac{1}{s^d} = C\frac{t}{s^{d+1}}.
\end{aligned}
$$

This proves part (a) of the $T(1)$ theorem. □

17 Conclusion of the proof of $T(1)$ theorem

2016-12-15

Proof of Theorem 16.7, continued. We have already proved half of the Theorem.
Pick $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that $\rho\restriction_{B(0,1)} = 1$, $\rho\restriction_{B(0,2)^c} = 0$ and define

$$\rho_r(z) = \rho\left(\frac{z}{r}\right).$$

We claimed that

$$\lim_{r\to\infty} \Lambda(\varphi,\rho_r)$$

exists for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ compactly supported and such that $\int \varphi(z)\mathrm{d}z = 0$. Notice that the limit of ρ_r is the constant function 1 (it eventually coincides with this function in the compact support of φ, and this explains the notation $T(1)$). Fix r_0 so that $\text{supp}(\varphi) \subset B(0, \frac{r_0}{4})$. We want to show that

$$\lim_{r\to\infty} \Lambda(\varphi, \rho_r - \rho_{r_0})$$

exists. The supports of φ and $\rho_r - \rho_{r_0}$ are disjoint, so we can use (16.5), namely

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)(K(x,y) - K(0,y))(\rho_r(y) - \rho_{r_0}(y))\mathrm{d}x\mathrm{d}y,$$

where we also used the property $\int \varphi(x)dx = 0$. By the Lebesgue Dominated Convergence Theorem we can conclude that the limit exists and is 0, upon proving that the integral is bounded independently on r. In fact

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)(K(x,y) - K(0,y))(\rho_r(y) - \rho_{r_0}(y))\mathrm{d}x\mathrm{d}y\Big| \le \\ &\le \int_{\text{supp}(\varphi)}\int_{\|y\|\ge\frac{r_0}{2}} |\varphi(x)| \underbrace{|K(x,y) - K(0,y)|}_{\le \frac{\|x\|}{\|y\|^{d+1}}} \underbrace{|\rho_r(y) - \rho_{r_0}(y)|}_{\le 2\|\rho\|_\infty} \mathrm{d}x\mathrm{d}y \le C,\end{aligned}$$

where we used the fact that for $\|y\| < \frac{r_0}{2}$ then $\rho_r(y) - \rho_{r_0}(y) = 0$, while for $\|y\| \ge \frac{r_0}{2} \ge 2\|x\|$ we can apply the first property of K.
Last time we proved the Theorem assuming that for every $\psi \in \mathcal{S}(\mathbb{R}^d)$ compactly supported such that $\int \psi(z)dz = 0$ then

$$\lim_{r\to\infty} \Lambda(\psi, \rho_r) = \lim_{r\to\infty} \Lambda(\rho_r, \psi) = 0.$$

To complete the proof we sketched, we need to find another bilinear form $\Pi\colon \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ associated to a (possibly different) CZ kernel that satisfies:

- $|\Pi(\varphi, \psi)| \le C_p\|\varphi\|_p\|\psi\|_{p'}$;
- for all ψ as above

$$\lim_{r\to\infty} \Pi(\psi, \rho_r) = \lim_{r\to\infty} \Lambda(\psi, \rho_r), \tag{17.1}$$

$$\lim_{r\to\infty} \Pi(\rho_r, \psi) = 0. \tag{17.2}$$

The proof will be completed by considering $\Lambda - \Pi - \widetilde{\Pi}$, where $\widetilde{\Pi}$ is the symmetric of Π.
For $j \in J$ a finite set, pick ψ^j, $\widetilde{\psi}^j$ supported in $B(0,1)$, $\int \psi^j = \int \widetilde{\psi}^j = 0$ such that for every $g, h \in L^2(\mathbb{R}^d)$

$$\langle g, h\rangle = \sum_{j\in J}\int_{\mathbb{R}^d}\int_0^\infty \left\langle g, \psi^j_{y,t}\right\rangle \left\langle \widetilde{\psi}^j_{y,t}, h\right\rangle \frac{\mathrm{d}t}{t}\mathrm{d}y,$$

which gives the paraproduct decomposition.
Now pick $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $B(0,1)$, $\int \varphi = 1$ and define

$$\Pi(g,h) := \sum_{j\in J} \int_{\mathbb{R}^d} \int_0^\infty \left\langle g, \psi^j_{y,t} \right\rangle \lim_{r\to\infty} \Lambda(\widetilde{\psi}^j_{y,t}, \rho_r) \left\langle \varphi_{y,t}, h \right\rangle \frac{\mathrm{d}t}{t} \mathrm{d}y \qquad (17.3)$$

We observe that:

- $\left\langle g, \psi^j_{y,t} \right\rangle \in \mathcal{L}^p(S^2)$, by S^2 embedding theorem, since $\int \psi^j_{y,t} = 0$;
- $\left\langle \varphi_{y,t}, h \right\rangle \in \mathcal{L}^{p'}(S^\infty)$, by S^∞ embedding theorem, since $\int \varphi_{y,t} = 1$.

We would like to apply Hölder's inequality, upon proving the following

Claim 17.4. $\|\sup_{r>\|y\|+t} |\Lambda(\widetilde{\psi}^j_{y,t}, \rho_r)|\|_{\mathcal{L}^\infty(S^2)} \leq C$.

Proof. We need to show that for every tent $T(x,s)$

$$\int_{T(x,s)} \sup_{r>\|y\|+t} |\Lambda(\tilde{\psi}^j_{y,t}, \rho_r)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq Cs^d.$$

Pick ρ such that $\rho\restriction_{B(0,2)} = 1$, $\rho\restriction_{B(0,4)^c} = 0$ and define $\rho_{x,s}$ as before. Since $\Lambda(g, \rho_{x,s}) \leq C\|g\|_2 s^{\frac{d}{2}}$ for all $g \in \mathcal{S}(\mathbb{R}^d)$, then there exists $f \in L^2(\mathbb{R}^d)$ such that

$$\Lambda(g, \rho_{x,s}) = \langle g, f \rangle, \qquad \text{for all } g,$$
$$\|f\|_2 \leq Cs^{\frac{d}{2}}.$$

Therefore

$$\int_{T(x,s)} |\Lambda(\tilde{\psi}^j_{y,t}, \rho_{x,s})|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq \int_{\mathbb{R}^d} \int_0^\infty \left| \left\langle \tilde{\psi}^j_{y,t}, f \right\rangle \right|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq C\|f\|_2^2 \leq Cs^d,$$

where the second inequality is given by the S^2 embedding theorem.
The restriction $r > \|y\| + t$ ensures that the functions $\tilde{\psi}^j_{y,t}(u)$ and $\rho_r(v) - \rho_{x,s}(v)$ have disjoint supports, so that we can use the kernel representation of the bilinear form Λ. Using the fact that $\tilde{\psi}^j_{y,t}(u)$ has integral zero to subtract $K(y,v)$ below, we are left to estimate

$$\int_{T(x,s)} \sup_{r>\|y\|+t} \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \tilde{\psi}^j_{y,t}(u)(K(u,v) - K(y,v))(\rho_r(v) - \rho_{x,s}(v)) \mathrm{d}u\mathrm{d}v \right|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq$$
$$\leq C \int_{\|y-x\|<s} \int_0^s \left| \iint_{\|u-v\|>s} |\tilde{\psi}^j_{y,t}(u)| \frac{t}{\|u-v\|^{d+1}} \mathrm{d}u\mathrm{d}v \right|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq$$
$$\leq C \int_{\|y-x\|<s} \int_0^s \left(\frac{t}{s}\right)^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq Cs^d.$$

□

Now we claim that

$$\limsup_{r\to\infty}|\Lambda(\tilde{\psi}^j_{y,t},\rho_r)| \le C,$$

and it can be proved very similarly to the previous claim.
The following step is to construct the CZ kernel associated to Π.

$$K(u,v) = \sum_{j\in J}\int_{\mathbb{R}^d}\int_0^\infty \psi^j_{y,t}(u)\lim_{r\to\infty}\Lambda(\tilde{\psi}^j_{y,t},\rho_r)\varphi_{y,t}(v)\frac{\mathrm{d}t}{t}\mathrm{d}y.$$

We want to prove it is a CZ kernel. The first property is verified in this way: assuming $u, u_0, v\colon 2\|u-u_0\| \le \|u-v\|$, then

$$|K(u,v)-K(u_0,v)| = \Big|\sum_{j\in J}\int_{\mathbb{R}^d}\int_0^\infty (\psi^j_{y,t}(u)-\psi^j_{y,t}(u_0))\lim_{r\to\infty}\Lambda(\tilde{\psi}^j_{y,t},\rho_r)\varphi_{y,t}(v)\frac{\mathrm{d}t}{t}\mathrm{d}y\Big| \le$$

First of all, we may restrict the domain of integration to $t \ge \frac{\|u-v\|}{2}$, since $B(y,t)$ needs to contain u, v, obtaining

$$\begin{aligned}&\le \sum_{j\in J}\int_{\frac{\|u-v\|}{2}}^\infty\int_{B(u,3t)}\|\nabla\psi^j_{y,t}\|_\infty\|u-u_0\|C\|\varphi_{y,t}\|_\infty \mathrm{d}y\frac{\mathrm{d}t}{t} \le \\ &\le C\frac{\|u-u_0\|}{\|u-v\|^{d+1}}.\end{aligned}$$

The other properties can be checked, so that Π is associated with a multiple of a CZ kernel.

Claim 17.5. *If $\varphi \in \mathcal{S}(\mathbb{R}^d)$ compactly supported and such that $\int\varphi = 0$ then*

$$\left\langle \varphi, \psi^j_{y,t}\right\rangle \in \mathcal{L}^1(S^2). \tag{17.6}$$

Moreover

$$\langle\varphi,\varphi_{y,t}\rangle \in \mathcal{L}^1(S^\infty).$$

Remark 17.7. A priori we only have $\left\langle \varphi, \psi^j_{y,t}\right\rangle \in \mathcal{L}^{1,\infty}(S^2)$.

Proof. The proofs are left as an exercise. □

The claim is needed in order to use Hölder's inequality in (17.3) with $p = 1, p' = \infty$

$$\begin{aligned}\lim_{r\to\infty}\Pi(\varphi,\rho_r) &= \lim_{r\to\infty}\sum_{j\in J}\int_{\mathbb{R}^d}\int_0^\infty \left\langle \varphi, \psi^j_{y,t}\right\rangle \lim_{\tilde{r}\to\infty}\Lambda(\tilde{\psi}^j_{y,t},\rho_{\tilde{r}})\left\langle\varphi_{y,t},\rho_r\right\rangle\frac{\mathrm{d}t}{t}\mathrm{d}y = \\ &= \sum_{j\in J}\int_{\mathbb{R}^d}\int_0^\infty \left\langle \varphi, \psi^j_{y,t}\right\rangle \lim_{\tilde{r}\to\infty}\Lambda(\tilde{\psi}^j_{y,t},\rho_{\tilde{r}})1\frac{\mathrm{d}t}{t}\mathrm{d}y,\end{aligned}$$

where we moved the limit in r inside the integrals using the Dominated Convergence Theorem and used the fact that

$$\lim_{r\to\infty} \langle \varphi_{y,t}, \rho_r \rangle = \langle \varphi_{y,t}, 1 \rangle = 1.$$

The application of the dominated convergence theorem is justified by (17.6), Claim 17.4, and the fact that

$$\sup_r |\langle \varphi_{y,t}, \rho_r \rangle| \in \mathcal{L}^\infty(S^\infty).$$

These three facts together with outer Hölder inequality and atomicity allow us to conclude that the integral converges absolutely with $\lim_r$ replaced by $\sup_r$ and moved inside the integral.

By the monotone convergence theorem we have

$$\lim_{r\to\infty} \Pi(\varphi, \rho_r) = \sum_{j\in J} \int_{\mathbb{R}^d} \int_0^{t_0} \left\langle \varphi, \psi^j_{y,t} \right\rangle \lim_{\tilde{r}\to\infty} \Lambda(\tilde{\psi}^j_{y,t}, \rho_{\tilde{r}}) \frac{\mathrm{d}t}{t} \mathrm{d}y + o_{t_0\to\infty}(1).$$

We concentrate on the first term on the right-hand side. If $0 < t_0 < \infty$ is so large that $\operatorname{supp}\phi \subset B(0, t_0)$ and $t \leq t_0$, then $\left\langle \varphi, \psi^j_{y,t} \right\rangle \neq 0 \implies \|y\| \leq 2t_0$. Let $T_0 := T(0, t_0)$. We claim

$$\|1_{10T_0} \sup_{\tilde{r}>r_0} \Lambda(\tilde{\psi}^j_{y,t}, \rho_{r_0} - \rho_{\tilde{r}})\|_{L^\infty(S^2)} \lesssim t_0/r_0$$

for $r_0 > 100t_0$.

Proof of the claim. It suffices to consider tents $T(x, s) \subset 20T_0$. For such tents we have

$$\begin{aligned}
&\int_{T(x,s)} \sup_{r>r_0} \Big| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \tilde{\psi}^j_{y,t}(u)(K(u,v) - K(y,v))(\rho_r(v) - \rho_{r_0}(v)) \mathrm{d}u\mathrm{d}v \Big|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq \\
&\quad \leq C \int_{\|y-x\|<s} \int_0^s \Big| \iint_{\|u-v\|>r_0/2} |\tilde{\psi}^j_{y,t}(u)| \frac{t}{\|u-v\|^{d+1}} \mathrm{d}u\mathrm{d}v \Big|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq \\
&\quad \leq C \int_{\|y-x\|<s} \int_0^s \left(\frac{t}{r_0}\right)^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \leq C s^{d+2}/r_0^2.
\end{aligned}$$

□

Using the claim and outer Hölder inequality we obtain

$$\lim_{r\to\infty} \Pi(\varphi, \rho_r) = \sum_{j\in J} \int_{\mathbb{R}^d} \int_0^{t_0} \left\langle \varphi, \psi^j_{y,t} \right\rangle \Lambda(\tilde{\psi}^j_{y,t}, \rho_{t_0^{1+\epsilon}}) \frac{\mathrm{d}t}{t} \mathrm{d}y + o_{t_0\to\infty}(1)$$

for any $\epsilon > 0$. By the restricted boundedness property of Λ there exist functions $f_{\tilde{r}} \in L^2(\mathbb{R}^d)$ with $\|f_{\tilde{r}}\|_2 \leq \|\rho_{\tilde{r}}\| \leq C\tilde{r}^{\frac{d}{2}}$ such that $\Lambda(\cdot, \rho_{\tilde{r}}) = \langle \cdot, f_{\tilde{r}} \rangle$. Then by the Calderón reproducing formula

$$\begin{aligned}\lim_{r\to\infty} \Pi(\varphi, \rho_r) &= \sum_{j\in J} \int_{\mathbb{R}^d} \int_0^{t_0} \left\langle \varphi, \psi^j_{y,t} \right\rangle \left\langle \tilde{\psi}^j_{y,t}, f_{t_0^{1+\epsilon}} \right\rangle \frac{\mathrm{d}t}{t} \mathrm{d}y + o_{t_0\to\infty}(1) \\ &= -\sum_{j\in J} \int_{\mathbb{R}^d} \int_{t_0}^{\infty} \left\langle \varphi, \psi^j_{y,t} \right\rangle \left\langle \tilde{\psi}^j_{y,t}, f_{t_0^{1+\epsilon}} \right\rangle \frac{\mathrm{d}t}{t} \mathrm{d}y \\ &+ \left\langle \varphi, f_{t_0^{1+\epsilon}} \right\rangle + o_{t_0\to\infty}(1).\end{aligned}$$

The second term on the right-hand side equals $\Lambda(\varphi, \rho_{t_0^{1+\epsilon}})$, so it remains to show that the first term is $o_{t_0\to\infty}(1)$. To this end we use Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_{t_0}^{\infty} |\left\langle \varphi, \psi^j_{y,t} \right\rangle \left\langle \tilde{\psi}^j_{y,t}, f_{t_0^{1+\epsilon}} \right\rangle| \frac{\mathrm{d}t}{t} \mathrm{d}y \\ &\leq \Big(\int_{\mathbb{R}^d} \int_{t_0}^{\infty} |\left\langle \varphi, \psi^j_{y,t} \right\rangle|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \Big)^{1/2} \cdot \Big(\int_{\mathbb{R}^d} \int_0^{\infty} |\left\langle \tilde{\psi}^j_{y,t}, f_{t_0^{1+\epsilon}} \right\rangle|^2 \frac{\mathrm{d}t}{t} \mathrm{d}y \Big)^{1/2} \\ &\qquad \lesssim \Big(\int_{t_0}^{\infty} \int_{\|y\|\leq 2t} |\left\langle \varphi, \psi^j_{y,t} \right\rangle|^2 \mathrm{d}y \frac{\mathrm{d}t}{t} \Big)^{1/2} \cdot \|f_{t_0^{1+\epsilon}}\|_2 \\ &\qquad \lesssim \Big(\int_{t_0}^{\infty} \int_{\|y\|\leq 2t} t^{-2(d+1)} \mathrm{d}y \frac{\mathrm{d}t}{t} \Big)^{1/2} \cdot t_0^{(1+\epsilon)d/2} \\ &\qquad \lesssim \Big(\int_{t_0}^{\infty} t^{-d-2} \frac{\mathrm{d}t}{t} \Big)^{1/2} \cdot t_0^{(1+\epsilon)d/2} \\ &\qquad \lesssim t_0^{-d/2-1} \cdot t_0^{(1+\epsilon)d/2} \\ &\qquad = t_0^{-1+\epsilon d/2}.\end{aligned}$$

This finishes the proof of (17.1). To prove (17.2) we observe

$$\begin{aligned}\lim_{r\to\infty} \Pi(\rho_r, \varphi) &= \lim_{r\to\infty} \sum_{j\in J} \int_{\mathbb{R}^d} \int_0^{\infty} \left\langle \rho_r, \psi^j_{y,t} \right\rangle \lim_{\tilde{r}\to\infty} \Lambda(\tilde{\psi}^j_{y,t}, \rho_{\tilde{r}}) \left\langle \varphi_{y,t}, \varphi \right\rangle \frac{\mathrm{d}t}{t} \mathrm{d}y \\ &= 0,\end{aligned}$$

where we used the same argument as before to take the limit inside the integrals, and we observed

$$\lim_{r\to\infty} \left\langle \rho_r, \psi^j_{y,t} \right\rangle = \left\langle 1, \psi^j_{y,t} \right\rangle = 0.$$

□

18 Coifman, Jones, Semmes: complex analysis proof

2016-12-20

In the next two lectures we are going to present the paper *"Two Elementary Proofs of the L^2 Boundedness of Cauchy Integrals on Lipschitz Curves"* by Coifman, Jones on Semmes. First, we will exhibit the proof in the complex setting.

For a function $\gamma\colon \mathbb{R} \to \mathbb{C}, \gamma(x) = x + iA(x)$, where $\|A'\|_\infty < \infty$, we have the Lipschitz curve Γ

$$\Gamma := \{\gamma(x)\colon x \in \mathbb{R}\}.$$

It will play the role of a horizontal axis. We denote with $\Omega_\pm$ the half planes in which $\mathbb{C}$ is divided

$$\Omega_+ := \{x + iA(x) + iy, y > 0\},$$
$$\Omega_- := \{x + iA(x) + iy, y < 0\}.$$

In order to make the proof easier, we assume that A is compactly supported and smooth and that g is compactly supported and smooth in Γ. Our estimates depend only on $\|A'\|_\infty$, so the general case descends from an approximation argument.
In the case $A \equiv 0$, when Γ is the x-axis, the relation between the Cauchy integral and the Hilbert Transform is well known. Therefore the former is a proper way to extend the latter in the general case of a Lipschitz curve. For $z \in \Omega_+$, we define the Cauchy integral

$$Cg(z) := \int_\Gamma \frac{g(\xi)}{z-\xi}\mathrm{d}\xi = \int_\mathbb{R} \frac{g(x+iA(x))(1+A'(x))}{z-(x+iA(x))}\mathrm{d}x. \tag{18.1}$$

We extend the definition to $\partial\Omega_+$ setting, for $z \in \Gamma$,

$$Cg(z) = \lim_{y \searrow 0} Cg(z+iy),$$

and the limit exists in the assumptions on A, g.
We want to recover a L^2-boundedness of this operator, in the same fashion of the one for the Hilbert transform, with respect to the norm

$$\|g\|^2_{L^2(\Gamma)} := \int_\mathbb{R} |g(x+iA(x))|^2 \sqrt{1+(A'(x))^2}\mathrm{d}x =: \int_\Gamma |g(s)|^2 \mathrm{d}s,$$

and the last one is the arc length integral.

Theorem 18.2. $\|Cg\|_{L^2(\Gamma)} \le c(\|A'\|_\infty)\|g\|_{L^2(\Gamma)}$.

Remark 18.3. The L^2-boundedness problem with this definition of the L^2 norm is equivalent to the same result for

$$\|g\|^2_{\tilde{L}^2} = \int_\mathbb{R} |g(x+iA(x)|^2 \mathrm{d}x.$$

In fact

$$\|g\|_{L^2(\Gamma)} \approx \|g\|_{\tilde{L}^2},$$

since, due to $\|A'\|_\infty < \infty$,

$$1 \le \sqrt{1 + (A'(x))^2} \le C$$

The proof of the Theorem relies on two Lemmata. To state them, we fix the notation $\mathrm{d}(z) = \mathrm{dist}(z, \Gamma)$ for $z \in \mathbb{C}$ and $\mathcal{H}_\pm$

$$\begin{aligned} \mathcal{H}_+ &= \{f \colon \Omega_+ \to \mathbb{C} \text{ measurable}\}, \\ \mathcal{H}_- &= \{f \colon \Omega_- \to \mathbb{C} \text{ measurable}\}. \end{aligned}$$

For $f \in \mathcal{H}_+$ we define the norm

$$\|f\|_{\mathcal{H}_+} = \left(\iint_{\Omega_+} |f(z)|^2 \mathrm{d}(z) \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{2}},$$

and for $g \in \mathcal{H}_-$, in an analogous way, $\|g\|_{\mathcal{H}_-}$.

Lemma 18.4. *Let F be holomorphic in Ω_+, smooth on $\overline{\Omega_+}$, $|F(z)| \le \frac{C}{|z|}$, $|F'(z)| \le \frac{C}{|z|^2}$, where F' is the complex derivative. Then*

$$\|F\|_{L^2(\Gamma)} \le c(\|A'\|_\infty) \|F'\|_{\mathcal{H}_+}.$$

Lemma 18.5. *Let $f \in \mathcal{H}_+$ be compactly supported in Ω_+, and for $\zeta \in \Gamma$ define*

$$Tf(\zeta) = \iint_{\Omega_+} \frac{f(z)}{z - \zeta} \mathrm{d}(z) \mathrm{d}x \mathrm{d}y.$$

Then

$$\|Tf\|_{L^2(\Gamma)} \le c(\|A'\|_\infty) \|f\|_{\mathcal{H}_+}.$$

The proof of the Theorem is straight forward once these two results are settled.

Proof of Thm.

$$\begin{aligned} \|Cg\|_{L^2(\Gamma)} &\overset{\text{first Lemma}}{\le} c(\|A'\|_\infty) \|(Cg)'\|_{\mathcal{H}_+} = \\ &\overset{\text{duality}}{=} c(\|A'\|_\infty) \sup_{\substack{\|f\|_{\mathcal{H}_+} \le 1 \\ f \text{ cmpct. supp.}}} \left| \langle (Cg)', f \rangle_{\mathcal{H}_+} \right| = \\ &= c(\|A'\|_\infty) \sup_f \left| \iint_{\Omega_+} \left(- \int_\Gamma \frac{g(\xi)}{(z-\xi)^2} \mathrm{d}\xi \right) \overline{f(z)} \mathrm{d}(z) \mathrm{d}x \mathrm{d}y \right| = \\ &\overset{\text{Fubini}}{=} c(\|A'\|_\infty) \sup_f \left| \int_\Gamma g(\zeta) T(\overline{f})(\zeta) \mathrm{d}\zeta \right| \le \\ &\overset{\text{Cauchy-Schwarz}}{\le} c(\|A'\|_\infty) \|g\|_{L^2(\Gamma)} \sup_f \|T(\overline{f})\|_{L^2(\Gamma)} \le \\ &\overset{\text{second Lemma}}{\le} c(\|A'\|_\infty) \|g\|_{L^2(\Gamma)}. \end{aligned}$$

□

Before proving the first Lemma, we investigate its statement in the trivial case of $A \equiv 0$, when $\Omega_+ = \mathbb{R} \times \mathbb{R}_{>0}$. Recalling our conditions, we consider H holomorphic in $\mathbb{R} \times \mathbb{R}_{>0}$, smooth on $\mathbb{R} \times \mathbb{R}_{\geq 0}$, $|H(z)| \leq \frac{C}{|z|}$, $|H'(z)| \leq \frac{C}{|z|^2}$. We can prove a stronger equality result, namely

$$\int_{\mathbb{R}} |H(x)|^2 \mathrm{d}x = 4 \int_{\mathbb{R} \times \mathbb{R}_{>0}} |H'(z)|^2 y \mathrm{d}x \mathrm{d}y,$$

by a partial integration argument. We use the *Green's Theorem*, i.e. for a compact set Ω with piecewise smooth boundary $\partial\Omega$, and for f, g smooth in a neighbourhood of Ω, then

$$\int_{\Omega} (\Delta f g - f \Delta g) \mathrm{d}x \mathrm{d}y = \int_{\partial\Omega} ((\nabla f \cdot \vec{n}) g - f(\nabla g \cdot \vec{n})) \mathrm{d}s, \tag{18.6}$$

where $\vec{n}$ is the outward pointing normal vector with respect to $\partial\Omega$.
In particular, we apply this result with $f = H\overline{H}$, $g = y$, $\Omega = \mathbb{R} \times \mathbb{R}_{\geq 0}$[3]. Since $\Delta = 4\partial_z \partial_{\overline{z}}$, then it becomes

$$4 \int_{\Omega} (H' \overline{H'} y - 0) \mathrm{d}x \mathrm{d}y = \int_{\partial\Omega} (0 - H\overline{H}) \mathrm{d}s. \tag{18.7}$$

If in addition we have D holomorphic in $\mathbb{R} \times \mathbb{R}_{>0}$, smooth on $\mathbb{R} \times \mathbb{R}_{\geq 0}$, $|D(z)| \leq 1$, $|D'(z)| \leq \frac{C}{|z|}$, then

$$\iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H(z) D'(z)|^2 y \mathrm{d}x \mathrm{d}y \leq \int_{\mathbb{R}} |H(x)|^2 \mathrm{d}x. \tag{18.8}$$

In fact, upon observing

$$\int_{\mathbb{R}} |H(x)|^2 \mathrm{d}x \geq \int_{\mathbb{R}} |H(x) D(x)|^2 \mathrm{d}x = 4 \iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H'(z) D(z) + H(z) D'(z)|^2 y \mathrm{d}x \mathrm{d}y,$$

we have

$$\iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H(z) D'(z)|^2 y \mathrm{d}x \mathrm{d}y \leq C \int_{\mathbb{R}} |H(x)|^2 \mathrm{d}x + \iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H'(z) D(z)|^2 y \mathrm{d}x \mathrm{d}y.$$

To conclude we notice that for the second summand

$$\iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H'(z) D(z)|^2 y \mathrm{d}x \mathrm{d}y \leq \iint_{\mathbb{R} \times \mathbb{R}_{>0}} |H'(z)|^2 y \mathrm{d}x \mathrm{d}y \leq \int_{\mathbb{R}} |H(x)|^2 \mathrm{d}x,$$

because of (18.7).

[3] Even if in this case Ω is not compact, we can recover the result through Green's Theorem on sets getting bigger and the good decay properties of H.

Proof of first Lemma. The example provides enough preparation to tackle the case of a general A Lipschitz function.
Because of the definition of Γ, there exists a Riemann map $\Phi\colon \mathbb{R}\times\mathbb{R}_{\geq 0} \to \Omega_+$ bijective, holomorphic in $\mathbb{R}\times\mathbb{R}_{>0}$, smooth on $\mathbb{R}\times\mathbb{R}_{\geq 0}$, mapping $\mathbb{R}\times\{0\}$ to Γ and such that $\Phi(z) = z + o\left(\frac{1}{z}\right)$. The *Köbe's $\frac{1}{4}$ Theorem* yields a useful estimate.

Theorem 18.9 (Köbe's $\frac{1}{4}$ Theorem)**.** *If $f\colon B(0,1) \to \mathbb{C}$ is holomorphic and injective then*

$$B\left(f(0), \frac{1}{4}f'(0)\right) \subset f(B(0,1)).$$

In particular translating and dilating f, one obtains $f(0) = 0$, $f'(0) = 1$, so that the claim becomes $B(0, \frac{1}{4}) \subset f(B(0,1))$.

The application of this result to our case implies

$$|\Phi'(z)y| \lesssim \mathrm{d}(\Phi(z)).$$

The other direction $\gtrsim$ is given by considering the inverse map Φ^{-1}.
We want to prove

$$\begin{aligned} A &:= \int_{\mathbb{R}} |G(x)|^2 |\Phi'(x)| \mathrm{d}x \leq \qquad\qquad \text{where } F\circ\Phi = G, \\ &\leq C \underbrace{\iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G'(z)|^2 |\Phi'(z)| y \mathrm{d}x\mathrm{d}y.}_{B} \end{aligned}$$

To obtain $A \leq CB$, we will show the equivalent statement $A \leq C(B + \sqrt{AB})$. Since the curve is Lipschitz, there exists $\varepsilon < 1$ such that

$$|\arg(\Phi')| \leq \frac{\pi}{2}\varepsilon.$$

In other words, $|\Phi'| \leq C\,\mathrm{Re}(\Phi')$. Thus

$$\begin{aligned} A &= \int_{\mathbb{R}} |G(x)|^2 |\Phi'(x)| \mathrm{d}x \leq C \Big| \int_{\mathbb{R}} G(x)\overline{G}(x)\Phi'(x) \mathrm{d}x \Big| = \\ &= C \Big| \iint_{\mathbb{R}\times\mathbb{R}_{>0}} \Delta(G\overline{G}\Phi')(z) y \mathrm{d}x\mathrm{d}y \Big| = \\ &= 4C \Big| \iint_{\mathbb{R}\times\mathbb{R}_{>0}} \partial_z\partial_{\overline{z}}(G\overline{G}\Phi')(z) y \mathrm{d}x\mathrm{d}y \Big| \leq \\ &\leq C \left(\Big| \iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G'(z)|^2 \Phi'(z) y \mathrm{d}x\mathrm{d}y \Big| + \Big| \iint_{\mathbb{R}\times\mathbb{R}_{>0}} G(z)\overline{G'}(z)\Phi''(z) y \mathrm{d}x\mathrm{d}y \Big| \right). \end{aligned}$$

To bound the first summand we have the trivial estimate by CB. To bound the second one we use Cauchy-Schwarz. Since Φ' is defined on a simply connected domain and does not vanish in it, we can write it as an exponential, namely $\Phi' = e^v$, thus $\Phi'' = v'e^v = v'\Phi'$. Recalling that $\mathrm{Im}(v) < \frac{\pi}{2}$, we get

$$|\Phi''| < e^{\frac{\pi}{2}} + |v'e^{iv}\Phi'| = e^{\frac{\pi}{2}}|D'\Phi'|, \qquad \text{where } D = e^{iv}.$$

Therefore Cauchy-Schwarz applied to the integral above with the splitting $G\overline{G'}\Phi'' = Gv'|\Phi'|^{\frac{1}{2}}y^{\frac{1}{2}}\ \overline{G'}|\Phi'|^{\frac{1}{2}}y^{\frac{1}{2}}$ yields

$$\Big|\iint_{\mathbb{R}\times\mathbb{R}_{>0}} G(z)\overline{G'}(z)\Phi''(z)y\mathrm{d}x\mathrm{d}y\Big| \leq \Big(\iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G'(z)|^2|\Phi'(z)|y\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}} \cdot$$
$$\cdot \Big(\iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G(z)|^2|v'|^2|\Phi'(z)|y\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}.$$

The first factor is $B^{\frac{1}{2}}$. To estimate the second we recall $D = e^{iv}$, then $D' = iv'e^{iv}$ and

$$\iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G|^2|v'|^2|\Phi'|y\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G|^2|D'|^2|\Phi'|y\mathrm{d}x\mathrm{d}y.$$

To conclude we use (18.8) with $H = G|\Phi'|^{\frac{1}{2}}$

$$\iint_{\mathbb{R}\times\mathbb{R}_{>0}} |G(z)|^2|D'|^2|\Phi'(z)|y\mathrm{d}x\mathrm{d}y \leq C\int_{\mathbb{R}} |G(x)|^2|\Phi'(x)|\mathrm{d}x = CA.$$

□

In the following proof we denote by $\mathcal{L}_+$ the space of functions on Ω_+ satisfying

$$\|f\|_{\mathcal{L}_+} = \iint_{\Omega_+} |f|^2\mathrm{d}x\mathrm{d}y < \infty,$$

and in an analogous way we define $\mathcal{L}_-$.

Proof of second Lemma. We extend Tf to Ω_-. By the first Lemma we have

$$\|Tf\|_{L^2(\Gamma)} \leq \|(Tf)'\|_{\mathcal{H}_-}.$$

Now for $\omega \in \Omega_-$

$$|Tf'(\omega)| \leq 2\iint_{\Omega_+} \frac{|f(z)|}{|z-\omega|^3}\mathrm{d}(z)\mathrm{d}x\mathrm{d}y,$$

so we want to estimate the operator $\mathcal{H}_+ \to \mathcal{H}_-$ given by

$$f \mapsto \Big(\Omega_- \ni \omega \mapsto \iint_{\Omega_+} \frac{f(z)}{|z-\omega|^3}\mathrm{d}(z)\mathrm{d}x\mathrm{d}y\Big).$$

It is enough to show that $S\colon \mathcal{L}_+ \to \mathcal{L}_-$ is bounded, where

$$Sf(\omega) = \mathrm{d}(\omega)^{\frac{1}{2}} \iint_{\Omega_+} \frac{f(z)}{|z-\omega|^3} (\mathrm{d}(z))^{\frac{1}{2}} \mathrm{d}x\mathrm{d}y.$$

By means of the so-called *Schur's Lemma*, it is enough to prove that for every $\omega \in \Omega_-$

$$\mathrm{d}(\omega)^{\frac{1}{2}} \iint_{\Omega_+} \frac{1}{|z-\omega|^3} \mathrm{d}(z)\mathrm{d}x\mathrm{d}y \leq C.$$

The analogous statement

$$\mathrm{d}(z)^{\frac{1}{2}} \iint_{\Omega_-} \frac{1}{|z-\omega|^3} \mathrm{d}(\omega)\mathrm{d}x'\mathrm{d}y' \leq C,$$

for a fixed $z \in \Omega_+$ is completely symmetric.
The wanted bound is trivially given by the estimate

$$\mathrm{d}(\omega)^{\frac{1}{2}} \iint_{\Omega_+} \frac{1}{|z-\omega|^3} \mathrm{d}(z)\mathrm{d}x\mathrm{d}y \leq \mathrm{d}(\omega)^{\frac{1}{2}} \iint_{|z-\omega|>\mathrm{d}(\omega)} \frac{1}{|z-\omega|^{2.5}} \mathrm{d}x\mathrm{d}y \leq$$
$$\leq C \frac{\mathrm{d}(\omega)^{\frac{1}{2}}}{\mathrm{d}(\omega)^{\frac{1}{2}}} \leq C,$$

where we used the fact that if $z \in \Omega_+$ then $|z-\omega| > \mathrm{d}(\omega), \mathrm{d}(z)$. □

19 Coifman, Jones, Semmes: alternative proof

2016-12-22

As we already did in our last lecture, we are going to present the paper *"Two Elementary Proofs of the L^2 Boundedness of Cauchy Integrals on Lipschitz Curves"*, by Coifman, Jones and Semmes. Today we will, however, present the second proof.

Let therefore $(z =)\Gamma : \mathbb{R} \to \mathbb{C}$ be a curve that satisfies the following properties:

1. *Rectifiability,* that is, $|\Gamma(t_1) - \Gamma(t_2)| \leq C|t_1 - t_2|$.
2. *Chord-arc,* that is, $|\Gamma(t_1) - \Gamma(t_2)| \geq c|t_1 - t_2|$.
3. *Jordan curve (through ∞),* that is, it is non self-intersecting.

Notice that:

1. Condition 1 implies we can define a function

$$S(t) = \sup_{n; 0=t_0<\cdots<t_n=t} \sum_{k=1}^{n} |\Gamma(t_k) - \Gamma(t_{k-1})|,$$

if $t > 0$, and analogously if $t < 0$. By reparametrizing Γ by s, we obtain the so-called *arc-length parametrization of* Γ. If Γ is arc-length parametrized, then it satisfies that $|\Gamma'(t)| = 1$, $\forall t$.

2. Condition 3 is directly implied by condition 2, as it already forbids two different values of t_1, t_2 to have same values of Γ.

Example 19.1 (Graph of a Lipschitz curve). $\Gamma(x) = x + iA(x)$, where A is a Lipschitz function with $\|A'\|_\infty \leq C$ clearly satisfies all conditions above.

Remark 19.2. It is easy to see that graphs are not the general type of such curves.

Define then, for f a linear combination of characteristic functions of dyadic intervals, the operator

$$Tf(x) = \lim_{\delta \to 0_+} \int_{-\infty}^{\infty} \frac{z'(y)}{z(y) - z(x) + i\delta z'(x)} f(y) dy.$$

Let $f = \mathbb{1}_{(a,b)}$. Suppose also that $\min(|x-a|, |x-b|) > \varepsilon$. Then one is able to prove that (if $(a,b) = I$)

$$\left| \int_{-\infty}^{\infty} \frac{z'(y)f(y)}{z(y) - z(x) - i\delta z'(x)} dy \right| < C \log\left(\frac{|I|}{\varepsilon}\right). \tag{19.3}$$

We will use this fact to prove the following

Theorem 19.4.

$$\|Tf\|_2 \leq C\|f\|_2,$$

where C is a constant that depends only *on the chord-arc constant in 2.*

In order to prove this theorem, we still need to do some work. First, define the pseudo-inner product

$$\langle f, g \rangle_\Gamma = \int_{\mathbb{R}} f(x) g(x) z'(x) dx.$$

In particular, by the Lipschitz condition, if Γ is arc-length parametrized, then $|\langle f, \overline{f} \rangle_\Gamma| \leq \|f\|_2^2$. Let next

$$m(I) = \frac{1}{|I|} \int_I z'(x) dx = \frac{1}{|I|}(z(b) - z(a)),$$

where $I = (a,b)$. By using both Lipschitz and chord-arc conditions, we conclude that $1 \geq |m(I)| \geq c$ for all intervals $I \subset \mathbb{R}$. This leads us to our last definition, namely, the one of the *adapted Haar functions to* $\langle \cdot, \cdot \rangle_\gamma$. Explicitly, those are

$$\beta_I(x) = \frac{1}{|I|^{1/2}} \left(\frac{m(I_l) m(I_r)}{m(I)} \right)^{\frac{1}{2}} \cdot (m(I_l)^{-1} \mathbb{1}_{I_l} - m(I_r)^{-1} \mathbb{1}_{I_r}).$$

We notice that this function is defined up to a sign, whose choice is going to be irrelevant for us.

Claim 19.5.

$$\langle \beta_I, \beta_J \rangle_\Gamma = \begin{cases} 0, & \textit{if } I \neq J; \\ 1, & \textit{if } I = J. \end{cases}$$

Proof. From the definition, if $J \cap I = \emptyset$, we have nothing to show, as $\text{supp}\beta_I \subset I$. As both I, J are dyadic intervals, we can suppose, without loss of generality, that $I \subset J$.

1. *If* $I \neq J$, then we have to show that

$$\int_{\mathbb{R}} (m(I_l)^{-1} \mathbb{1}_{I_l} - m(I_r)^{-1} \mathbb{1}_{I_r}) z'(x) dx = 0,$$

   by a simple of the fundamental theorem of calculus.

2. *If* $I = J$, then

$$\begin{aligned} \langle \beta_I, \beta_I \rangle_\Gamma &= \frac{m(I_l) m(I_r)}{|I| m(I)} \left(\int \left(\frac{\mathbb{1}_{I_l}}{m(I_l)^2} + \frac{\mathbb{1}_{I_r}}{m(I_r)^2} \right) z'(x) dx \right) \\ &= \frac{m(I_l) m(I_r)}{|I| m(I)} \left(\frac{m(I_l)|I_l|}{m(I_l)^2} + \frac{m(I_r)|I_r|}{m(I_r)^2} \right) \\ &= \frac{|I_l| m(I_l) + |I_r| m(I_r)}{|I| m(I)} = 1. \end{aligned}$$

□

Lemma 19.6. *If* $f \in L^2(\mathbb{R})$, *then*

$$f = \sum_{I \in \mathfrak{D}} \langle f, \beta_I \rangle_\Gamma \beta_I,$$

where the convergence of the sum on the left-hand side is taken in the L^2-*sense, and*

$$\frac{1}{C} \|f\|_2^2 \leq \sum_{I \in \mathfrak{D}} |\langle f, \beta_I \rangle_\Gamma|^2 \leq C \|f\|_2^2.$$

Proof. Let $\mathfrak{D}_k$ be the set of dyadic intervals of length 2^{-k}. Define then an *modified martingale operator* as

$$E_k f(x) = m(I)^{-1} |I|^{-1} \int_I f(x) z'(x) dx.$$

Observe that this dyadic martingale operator is bounded by $C \cdot \mathcal{M}f(x)$, where $\mathcal{M}f(x)$ is the Hardy-Littlewood maximal function. Also observe that

- $\lim_{k \to -\infty} E_k f(x) = 0$, as

$$\frac{1}{m(I)} \frac{1}{|I|} \int_I f(x) dx \leq \frac{1}{m(I)} \frac{\|f\|_2}{|I|^{1/2}}.$$

- $\lim_{k\to+\infty} E_k f(x) = f(x)$, as it is true if f is, for example, constant on a sufficiently small dyadic scale, and for general f by approximation and by the Hardy-Littlewood maximal theorem.

We now define the difference operators

$$\Delta_k f(x) = E_{k+1} f(x) - E_k f(x).$$

The last expression is, however, equal to $\sum_{I\in\mathfrak{D}_k}\langle f,\beta_I\rangle_\Gamma \beta_I$. Indeed, this holds by a simple calculation for $f = \beta_I$, therefore it does also hold for linear combinations of those, and, thus, for all $f \in L^2(\mathbb{R})$, by approximation. Then we obtain that

$$\begin{aligned} f - \lim_{k\to\infty}(E_k f(x) - E_{-k} f(x)) &= \lim_{k\to+\infty}\sum_{n=-k}^{k}\sum_{I\in\mathfrak{D}_n}\langle f,\beta_I\rangle_\Gamma \beta_I \\ &\overset{L^2-\text{sense}}{=} \sum_{I\in\mathfrak{D}}\langle f,\beta_I\rangle_\Gamma \beta_I. \end{aligned}$$

This proves the first part.

For the second one, we are going to resort to the usual martingale and difference operators. Let then P_k be this martingale operator, and $Q_k = P_{k+1} - P_k$ be the respective difference operator, with respect to the standard inner product on the real line.

Claim 19.7.

$$\Delta_k f = \frac{Q_k(z'f)}{P_k(\overline{z'})} - \frac{Q_k(\overline{z'})}{P_k(\overline{z'})P_{k+1}(\overline{z'})} P_k(z'f).$$

Idea of the proof of the claim 19.7. It is enough to prove, multiplying out, that

$$P_{k+1}(z')P_k(z')\Delta_k f = Q_k(z'f)P_k(z') + Q_k(z')P_k(z'f).$$

The left hand side is equal to an expression of the form

$$\sum_{I\in\mathfrak{D}_k} c_I\langle f,\beta_I\rangle_\Gamma(\mathbb{1}_{I_l} - \mathbb{1}_{I_r}).$$

Proving the claim amounts then to decomposing $\beta_I = A_I h_I + B_I \mathbb{1}_I$, identifying A_I, B_I, plugging into the right hand side and verifying that they match the ones on the left hand side. The details are left. □

We will use claim 19.7 to finish the proof of the lemma. Indeed, we check directly that $1 \geq |P_k(\overline{z'})| \geq c$ for all k. We must therefore only show that:

$$\int \sum_k |Q_k(z'f)|^2 dx \leq C\|f\|_2^2,$$

which is due directly to the standard Haar orthogonality, and

$$\begin{aligned}\int \sum_k |Q_k(\overline{z'})|^2 |P_k(z'f)|^2 dx &\overset{\text{Paraproduct}}{\leq} \|\overline{z'}\|^2_{\mathcal{L}^\infty(S^2)} \|z'f\|^2_{\mathcal{L}^2(S^\infty)} \\ &\overset{\text{Emb. theorem}}{\leq} \|z'\|^2_\infty \|z'f\|^2_2 \leq C\|f\|^2_2,\end{aligned}$$

which proves the first inequality in the statement of the lemma. By the paraproduct estimate, we have that

$$\sum_{I\in\mathfrak{D}_k} |\langle f, \beta_I\rangle_\Gamma|^2 \leq C\|f\|_2^2.$$

Suppose, without loss of generality, that $\|f\|_2 = 1$, and let $g = \overline{z'f}$. Then:

$$\begin{aligned}1 = \|f\|_2^2 = \langle f, g\rangle_\Gamma &= \sum_{I\in\mathfrak{D}} \langle f, \beta_I\rangle_\Gamma \langle g, \beta_I\rangle_\Gamma \\ &\leq \left(\sum_{I\in\mathfrak{D}} |\langle f, \beta_I\rangle_\Gamma|^2\right)^{1/2} \left(\sum_{I\in\mathfrak{D}} |\langle g, \beta_I\rangle_\Gamma|^2\right)^{1/2} \\ &\leq C\|g\|_2^2 \left(\sum_{I\in\mathfrak{D}} |\langle f, \beta_I\rangle_\Gamma|^2\right)^{1/2}.\end{aligned}$$

This ends the proof of the lemma. □

Finally, to achieve another proof of our theorem, we do the following estimate:

$$\begin{aligned}\langle Tf, g\rangle_\Gamma &\leq \sum_{I,J\in\mathfrak{D}} |\langle f, \beta_I\rangle_\Gamma \langle T\beta_I, \beta_J\rangle_\Gamma \langle g, \beta_J\rangle_\Gamma| \\ &\leq \sum_{I\in\mathfrak{D}} |\langle f, \beta_I\rangle_\Gamma|^2 \left(\sum_{J\in\mathfrak{D}} |\langle T\beta_I, \beta_J\rangle_\Gamma|\right) \\ &\quad + \sum_{I\in\mathfrak{D}} |\langle g, \beta_I\rangle_\Gamma|^2 \left(\sum_{J\in\mathfrak{D}} |\langle T\beta_I, \beta_J\rangle_\Gamma|\right).\end{aligned}$$

On the other hand, we can estimate this last expression from the following

Lemma 19.8.

$$\sup_I \sum_{J\in\mathfrak{D}} |\langle T\beta_I, \beta_J\rangle_\Gamma| \leq C.$$

The details of the proof are going to be ommitted, and we mention only the main ideas. Namely, we can rewrite

$$|T\beta_I(x)| = \left|\int \left(\frac{1}{z(x)-z(y)} - \frac{1}{z(x)-z(c(I))}\right) z'(y)\beta_I(y)dy\right|.$$

From this expression, by a careful analysis we can get that

$$|T\beta_I(x)| \leq \begin{cases} c|x-c(I)|^{-2}|I|^{3/2}, & \text{if } x \in 2I; \\ c|I|^{-1/2}\log\left(\frac{10|I|}{\min(|x-a|,|x-b|,|x-c(I)|)}\right), & \text{otherwise.} \end{cases}$$

The second case already follows from inequality 19.3, and the first one by the analysis mentioned. Putting all those estimates together gives us the result.

20 Wolff's proof of the Corona Theorem

2017-01-10

In this lecture we are going to present Wolff's proof (1980) of the Corona Theorem, a statement conjectured by Kakutani in 1941 and first proved by Carleson in 1962. The interest in this proof is motivated by the application of the outer measures and paraproducts theory in the proof in a non naive way. This context, far from being the most useful application of it, historically represents one of its starting point.
Before stating the Theorem we study two significant examples.

Example 20.1. For $D = B(0,1) \subset \mathbb{C}$ the open unitary ball in $\mathbb{C}$, let $B = H^\infty(D)$ be the set of bounded analytic functions on D. It is a commutative Banach algebra, i.e.

- it has a structure of normed vector space with the norm $\|f\|_\infty$. Moreover it is complete with respect to this norm;
- it has a structure of commutative algebra with the pointwise product, that satisfies the Banach inequality

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty;$$

- it has a unit, the constant function 1.

The homomorphisms from B to $\mathbb{C}$ are the linear, multiplicative, bounded functionals. We denote by S, the *spectrum* of B, the set of these homomorphisms. For example, for $z \in D$, the definition

$$\lambda(f) := f(z), \qquad \text{for } f \in B,$$

gives an element of S. However, they are not all of the form of evaluation in a point.
For an arbitrary Banach algebra B, we let B^* be the set of bounded linear functionals (without any condition on the product) equipped with the weak-* topology. Therefore we say that $\lambda_n \to \lambda$ if for every $f \in B$ we have $\lambda_n(f) \to \lambda(f)$. The topology is generated by the sets

$$\{\lambda \in B^* \colon |\lambda(f) - c| < \delta\}, \qquad \text{for } c \in \mathbb{C}, \delta > 0, f \in B,$$

in the sense that an arbitrary open set is given by the union of finite intersection of those. On B^* we also have the operatorial norm

$$\|\lambda\| := \sup_{\|f\|_B \le 1} |\lambda(f)|.$$

A well known result is the following

Theorem 20.2 (Banach–Alaoglu Theorem). *The unit ball in B^* is weak-* compact.*

If λ is multiplicative and nonzero, B is unital, then for every $f \in B$

$$\lambda(f) = \lambda(1f) = \lambda(1)\lambda(f).$$

Thus, by existence of f such that $\lambda(f) \neq 0$, we obtain $\lambda(1) = 1$, which implies $\|\lambda\| \ge 1$. Now suppose $\|\lambda\| > 1$, hence there exists f such that $\lambda(f) > 1, \|f\| < 1$. Then

$$\|\lambda\| > \|\lambda\| \|f\|^n \ge |\lambda(f^n)| = |\lambda(f)|^n \to \infty,$$

giving a contradiction. Therefore $\|\lambda\| = 1$, and S is a subset of the unit ball in B^*.
Moreover, S is closed, thus weak-* compact. Upon identifying $z \in D$ with the point evaluation homomorphism, D"$\subset$"B^* is not weak-* compact (the proof requires the Axiom of Choice), hence D"$\subsetneq$"S.

Example 20.3. In this second example let $B = \ell^\infty(\mathbb{N})$ the unital commutative Banach algebra of bounded sequence. Once again, for every $n \in \mathbb{N}$, $\lambda_n(f) := f(n)$ defines an element of S, but not all the elements of S are point evaluation homomorphisms.
In particular for $A \subset \mathbb{N}$,

$$\begin{aligned} 0 &= \lambda(0) = \lambda(\mathbb{1}_A \cdot \mathbb{1}_{A^c}) = \lambda(\mathbb{1}_A)\lambda(\mathbb{1}_{A^c}), \\ 1 &= \lambda(1) = \lambda(\mathbb{1}_A + \mathbb{1}_{A^c}) = \lambda(\mathbb{1}_A) + \lambda(\mathbb{1}_{A^c}). \end{aligned}$$

This yields

$$\lambda(\mathbb{1}_A) \in \{0, 1\}.$$

Therefore every element of S produces a so called *ultrafilter*. We distinguish two cases:

- $\exists A \colon |A| < \infty, \lambda(\mathbb{1}_A) = 1$.
  Then there exists $n \in \mathbb{N}$ such that $\lambda(\mathbb{1}_{\{n\}}) = 1$, and λ is given by the evaluation in n;
- $\forall A, |A| < \infty \colon \lambda(\mathbb{1}_A) = 0$.

As above, Banach-Alaoglu Theorem implies that this second case happens. A Heine-Borel argument implies that given $f \in \ell^\infty(\mathbb{N}), \lambda \in S$, then $\forall \delta > 0\ \exists A \subset \mathbb{N}, c_\delta$ such that $\lambda(\mathbb{1}_A) = 1$ and $\forall n \in A$

$$|f(n) - c_\delta| < \delta.$$

For $\delta \to 0$ we have $c_\delta \to c$. We claim $\lambda(f) = c$. In particular knowing the ultrafilter one can reconstruct the homomorphism, so that S is in bijection with the set of ultrafilters.

Remark 20.4. One might think that S is given by the compactification of $\mathbb{N}$ obtained by $\mathbb{N} \cup \{\infty\}$ with a proper topology, so that one has only to make sense of the homomorphism associated to $\{\infty\}$. Actually the right compactification to consider is the *Stone-Čech compactification.* In the same way, for the first example, S is not given by $\mathbb{S}^2$ or $\overline{D}$, a richer compactification is needed.

After this introduction we can state the following

Theorem 20.5 (Corona Theorem)**.** *D is dense in the spectrum of $H^\infty(D)$.*

Remark 20.6. The name comes from the solar corona, the aura of plasma that surrounds the sun that is most easily seen during a total solar eclipse. Let S be the sun and $\overline{D}^S$ be the moon. The theorem asserts that the corona is empty.

Proof. It is enough to show that for all $n \in \mathbb{N}, \delta > 0, f_1, \dots, f_n \in H^\infty(D)$ with $\|f_j\|_\infty \le 1$ and such that $\forall z \in D\ \exists i$: $|f_i(z)| > \delta$, then there exist $g_1, \dots, g_n \in H^\infty(D)$ with $\sum_{i=1}^n f_i g_i \equiv 1$.
Why? Assume $\lambda \in S$ to be not in the closure of D. Then there exists an open neighbourhood of λ not intersecting D. Equivalently, there exist $f_1, \dots, f_n \in H^\infty(D), \delta > 0$ such that for all $z \in D\ \exists i \in \{1, \dots, n\}$ so that $|\lambda(f_i) - \lambda_z(f_i)| > \delta$. Subtracting a constant from f_i we have $\lambda(f_i) = 0$, so the last condition becomes $|f_i(z)| = |\lambda_z(f_i)| > \delta$, implying $f_i \neq 0$, and for all $g_1, \dots, g_n$

$$\lambda(\sum_{i=1}^n f_i g_i) = \sum_{i=1}^n \lambda(f_i)\lambda(g_i) = 0 \neq \lambda(1),$$

so that $\sum f_i g_i \neq 1$ (this is the contrapositive of the claim).
We will actually show the claim with $\|g_i\|_\infty \le C(n, \delta)$. It allows us to assume by approximation that the functions f_i extend to holomorphic functions on some $B(0, 1 + \varepsilon)$. We start with f_i and define $f_{i,r_k}(z) = f_i(r_k z)$, $r_k \nearrow 1$. Now suppose we find g_{i,r_k} that do the job. Since $\|g_{i,r_k}\|_\infty \le C(n, \delta)$, then there exists a subsequence $g_{i,r_k} \to g_i$ as $k \to \infty$, with g_i holomorphic and $\|g_i\|_\infty \le C(n, \delta)$. Moreover,

$$\sum_{i=1}^n f_i g_i = \lim_{k \to \infty} \sum_{i=1}^n f_{i,r_k} g_{i,r_k} = 1.$$

The first try is to define

$$h_i = \frac{\bar{f}_i}{\sum_{j=1}^n |f_j|^2}.$$

Then $\|h_i\|_\infty \leq C(\delta), \sum_{i=1}^n f_i h_i = 1$, but the functions h_i are not analytic in general (this first guess controlled the algebra part of the problem but not the analysis one). The second ansatz is

$$g_i = h_i + \sum_{j=1}^n A_{ij} f_j.$$

If A_{ij} is antisymmetric, then

$$\sum_{i=1}^n (\sum_{j=1}^n A_{ij} f_j) f_i = 0,$$

and hence, to keep the good algebraic property of the first guess, we have

$$g_i = h_i + \sum_{j=1}^n (w_{ij} - w_{ji}) f_j.$$

If $\partial_{\bar{z}} w_{ij} = h_i \partial_{\bar{z}} h_j$, then g_i is analytic, namely

$$\begin{aligned} \partial_{\bar{z}} g_i &= \partial_{\bar{z}} h_i + \sum_{j=1}^n (\partial_{\bar{z}} w_{ij} - \partial_{\bar{z}} w_{ji}) f_j = \\ &= \partial_{\bar{z}} h_i + \sum_{j=1}^n h_i \partial_{\bar{z}} h_j f_j - \sum_{j=1}^n \partial_{\bar{z}} h_i h_j f_j = 0, \end{aligned}$$

since the second summand is $\partial_{\bar{z}}(\sum h_j f_j) = \partial_{\bar{z}} 1 = 0$, the third one is $\partial_{\bar{z}} h_i$.
To complete the proof it suffices to find w such that $\partial_{\bar{z}} w = u, u = h_i \partial_{\bar{z}} h_j$, satisfying also $\|w\|_\infty \leq C(n, \delta)$. The function

$$w_0(\zeta) = c \iint_{B(0,1+\varepsilon)} \frac{u(z)}{z - \zeta} \mathrm{d}x \mathrm{d}y,$$

where c is a universal constant, defines a solution. In fact, $\partial_{\bar{z}} w_0(\zeta) = u(\zeta)$ on the domain of integration, so that every solution is given by adding a holomorphic function to w_0.
We sketch the proof: for a fixed ζ_0, consider a bump function φ_{ζ_0} in the neighbourhood of ζ_0 contained in the disc $B(\zeta_0, \varepsilon')$. Use it to split $u(z) = u(z)\varphi_{\zeta_0}(z) + u(z)(1 - \varphi_{\zeta_0}(z))$.

$$\iint_{B(0,1+\varepsilon)} \frac{u(z)(1 - \varphi_{\zeta_0}(z))}{z - \zeta} \mathrm{d}x \mathrm{d}y$$

is analytic in ζ near ζ_0. If in addition $u(\zeta_0) = 0$ (we can assume it by adding a proper constant function), then

$$\lim_{\varepsilon' \to 0} \iint_{B(0,1+\varepsilon)} \frac{u(z)}{z - \zeta_0} \varphi_{\zeta_0}(z) \mathrm{d}x\mathrm{d}y = 0.$$

It remains to show the claim for the case of u constant, but this is done through explicit computations.
Now a priori we don't have a bound of the form $\|w_0\|_\infty \leq C(n, \delta)$.
However we observe that for $w \in L^\infty(\partial D)$, it defines a linear functional Λ_1 on $L^1(\partial D)$ with norm $\|w\|_\infty$. The space

$$H^1(D) := \{f \text{ analytic in } D \colon \forall r < 1 \int_0^{2\pi} |f(re^{i\theta})| \mathrm{d}\theta \leq C < \infty\}$$

is a closed subspace of $L^1(\partial D)$.
The linear functional Λ_1 restricted to $H^1(D)$ has norm $\|\Lambda_2\| \leq \|w\|_\infty$. By Hahn-Banach, Λ_2 extends to Λ_3 on $L^1(\partial D)$ with $\|\Lambda_3\| = \|\Lambda_2\|$. Therefore, by Riesz representation Theorem, there is $w_3 \in L^\infty(\partial D)$ realizing Λ_3, and $\|w_3\|_\infty = \|\Lambda_3\|$. By construction,

$$\int_0^{2\pi} (w - w_3) f \mathrm{d}\theta = 0, \qquad \text{for all } f \in H^1(D),$$

where we have to make sense of the integral, since a priori f is not defined on ∂D (as stated above, one observes $H^1(D) \subset L^1(\partial D)$). In particular

$$\int_0^{2\pi} (w - w_3) \underbrace{e^{2\pi i n\theta}}_{z^n} \mathrm{d}\theta = 0, \qquad \forall n \geq 0,$$

and the Fourier series of $w - w_3$ has only positive frequencies. Hence

$$w - w_3 \in H^\infty(D),\ \partial_{\bar{z}}(w - w_3) = 0.$$

Therefore if $w = w_0$ then w_3 solves the equation $\partial_{\bar{z}} w_3 = u$. To estimate $\|w_3\|_\infty$ it is enough to find an upper bound on $\|w_0 \restriction_{H^1(D)}\|_\infty$, i.e. we need a bound of the form

$$\sup_{\substack{F \in H^1(D) \\ \|F\|_1 \leq 1}} \int_0^{2\pi} w_0(e^{2\pi i\theta}) F(e^{2\pi i\theta}) \mathrm{d}\theta \leq C(n, \delta).$$

Using the fact that without loss of generality $w_0(0) = 0$, applying the Green's formula we obtain

$$\iint_D \underbrace{\Delta(w_0 F)}_{4\partial_z \partial_{\bar{z}}(w_0 F) = 4(u\partial_z F + F\partial_z u)} \log \frac{1}{|z|} \mathrm{d}x\mathrm{d}y.$$

To estimate this integral we want use the theory of paraproducts.
We have that $\log\frac{1}{|z|} \cong |I|$ plays the role of the measure of the dyadic interval, and

$$F \in L^1(S^\infty),\ \log\frac{1}{|z|}\partial_z F \in L^1(S^2).$$

For $u = h_j \partial_{\bar{z}} h_k$

$$u = \frac{\bar{f}_j}{\sum|f_m|^2}\frac{\partial_z \bar{f}_k}{\sum|f_m|^2} - \frac{\bar{f}_j}{\sum|f_m|^2}\bar{f}_k \sum_l \frac{f_l \partial_z f_l}{(\sum|f_m|^2)^2},$$

and

$$f_j \in L^\infty(S^\infty),\ \sum|f_m|^2 > \delta^2,\ \log\frac{1}{|z|}\partial_z f_j \in L^\infty(S^2),$$

so that

$$\log\frac{1}{|z|}u \in L^\infty(S^2).$$

In the same way, since

$$\partial_z u = \partial_z \bar{f}_k \ldots \partial_z f_m \ldots,$$

we have

$$\left(\log\frac{1}{|z|}\right)^2 \partial_z u \in L^\infty(S^\infty).$$

□

21 An introduction to the Carleson's Theorem

2017-01-12

The purpose of this lecture is to given an introduction to another theme of Lennart Carleson, namely, Carleson's theorem on almost everywhere convergence of Fourier series. Let then $f : L^2[0,1] \to \mathbb{C}$ and define for it the *partial sums*

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}_n e^{2\pi i n x},$$

where we define

$$\widehat{f}_N = \int_0^1 f(y) e^{-2\pi i N y} dy.$$

These are generally called *partial Fourier sums* of the function f. We may prove the following properties about these partial sums:
Cauchy as $N \to \infty$: In $L^2[0,1]$, this is due directly to orthogonality of $e^{2\pi i n x}$ and Hilbert space theory: We have that

$$\|S_N f\|_2^2 = \sum_{m,n=-N}^{N} \widehat{f}_m \widehat{f}_n \langle e^{2\pi i n x}, e^{2\pi i m x}\rangle = \sum_{n=-N}^{N} |\widehat{f}_N|^2.$$

In particular, we have, if $1 \ll N < M$,

$$\|S_N f - S_M f\|_2^2 = \sum_{N<|n|<M} |\widehat{f}_n|^2 < \varepsilon,$$

whenever $\sum_n |\widehat{f}_n|^2 < +\infty$.

<u>*Existence of the limit:*</u> This follows from the fact that $L^2[0,1]$ is complete, which in turn follows from its definitions via the completion of the space (under L^2 norm) spanned by Haar functions and $\mathbb{1}_{[0,1]}$.

<u>*Equality of the limit with f:*</u> We must only prove that the Fourier basis $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is complete in $L^2[0,1]$. For it, it suffices to prove that *every* indicator function $\mathbb{1}_{[a,b]}$ is on the span of this basis. In order to do so, we notice that, as periodic functions, it holds that

$$\mathbb{1}_{[a,b)} + \mathbb{1}_{[b,a+1)} \equiv 1. \tag{21.1}$$

Let us then analyse $f = \mathbb{1}_{[a,b]}$. If $x \notin [a,b]$, then we write

$$\begin{aligned}
S_N f(y) &= \sum_{n=-N}^{N} \left(\int f(x) e^{-2\pi inx} dx \right) e^{2\pi iny} \\
&= \int f(y-x) \left(\sum_{n=-N}^{N} e^{2\pi inx} \right) dx \\
&= \int f(y-x) \frac{e^{2\pi i(N+\frac{1}{2})x} - e^{-2\pi i(N+\frac{1}{2})x}}{e^{2\pi i\frac{x}{2}} - e^{-2\pi i\frac{x}{2}}} dx \\
&= \int f(y-x) D_N(x) dx.
\end{aligned}$$

By analyzing D_N carefully, we get that

$$|S_N f(x)| \le \min(C, \frac{C}{N}) \xrightarrow{L^2} 0,$$

as $N \to \infty$. If, on the other hand, $x \in [a,b]$, we use equation (21.1) and the same argument as before. This proves the last of our claims, which amounts to the following

Theorem 21.2 (Plancherel-Parseval)**.** *Let $f \in L^2[0,1]$. Then*

$$S_N f \overset{L^2,\ as\ N\to\infty}{\to} f.$$

Therefore, a natural question that arises from this analysis is: When do we have actually a *pointwise* convergence? More specifically, if we fix $x \in \mathbb{R}$, then can we establish that

$$S_N f(x) \to f(x)?$$

This question is surprisingly more difficult than it seems, but we can at least give a first attempt at it: if$f$ is twice continuously differentiable, we may write

$$\begin{aligned}\widehat{f}_n &= \int f(x)e^{2\pi inx}dx \\ &\overset{\text{partial integration}}{=} \int f'(x)\frac{1}{2\pi in}e^{-2\pi inx}dx \\ &\overset{\text{partial integration}}{=} \int f''(x)\frac{1}{(2\pi in)^2}e^{-2\pi inx}dx,\end{aligned}$$

which implies that

$$|\widehat{f}_n| \leq \frac{C}{n^2}.$$

This makes the partial sums $S_N f$ *absolutely convergent*, and, of course, that $S_N f \to f$ for *all* $x \in [0,1]$. Nevertheless, Lennart Carleson (1966) was able to prove much, much more:

Theorem 21.3 (Carleson)**.** *Let f be continuous and periodic on $\mathbb{R}$, with period one. Alternatively, let $f \in L^2[0,1]$. Then*

$$S_N f(x) \to f(x) \textit{ for almost every } x \in [0,1].$$

We will not prove this theorem today. However, we may still make some enlightening comments about it: first, define the *Carleson maximal operator* as

$$Cf(x) = \sup_{N\geq 1} |S_N f(x)|.$$

Although Lennart Carleson has not defined himself this operator, it has been hidden in his proof, and was definitively unveiled in the clarified proof given by Fefferman in 1973. About this operator, we can state the following:

Theorem 21.4 (Carleson-Hunt)**.** *With the same conditions as in Theorem 21.3, we have that there exists a constant $C > 0$ such that, for all $f \in L^2[0,1]$,*

$$\|Cf\|_2 \leq C\|f\|_2.$$

One may see trivially that, by an application of Theorem 21.4, we must have

$$|\{x\colon Cf(x) > \lambda\}| \leq \frac{C\|f\|_2^2}{\lambda^2}.$$

We are going to prove now that Theorem 21.4 implies Theorem 21.3. Indeed, let N_0 be a positive integer such that for all $N > N_0$, we have that

$$\|S_N f - f\|_2 < \varepsilon^2.$$

Then we have that, as $S_N f - f = S_N(f - S_{N_0} f) - (f - S_N f)$,

$$|\{ \sup_{N>N_0} |S_N f - f| > \varepsilon^2\}| \leq |\{C(f - S_N f) + |f - S_N f| > \varepsilon^2\}| \overset{\text{assumption on} N_0}{\leq} C\varepsilon^2.$$

Apply this with $\varepsilon = \varepsilon_0 2^{-k}$, $k \geq 1$, let $E_\varepsilon = \{\sup_{N>N_0} |S_N f - f| > \varepsilon^2\}$, and define

$$E = \cup_{k\geq 0} E_{\varepsilon_0 2^{-k}}.$$

This implies that $|E| \leq C\varepsilon_0$. If then $x \notin E$, then for every $\delta > 0$, there is N' such that if $N'' > N'$, then

$$|S_{N''} f(x) - f(x)| < \varepsilon.$$

We have thus proved that the set where the Fourier series of f converges has measure $\geq 1 - C\varepsilon_0$. As ε_0 was arbitrary, we conclude it has actually measure 1, as desired.

We will now move to a related question, that is going to help us in the task of proving Theorem 21.4. As we did for the periodic setting, fix $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and We define then the *partial Fourier integral* of f as

$$\tilde{S}Nf(y) = \int_{-N}^{N} \widehat{f}(\xi) e^{2\pi i \xi y} d\xi,$$

where $\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \xi} dx$ is the Fourier transform on the real line. As we calculated before, we can show that

$$\tilde{S}_N f(y) = \int_{\mathbb{R}} f(y-x) \frac{e^{2\pi i N x} - e^{-2\pi i N x}}{2\pi i x} dx = \int_{\mathbb{R}} f(y-x) \frac{\sin(2\pi N x)}{\pi x} dx,$$

where $\frac{\sin(2\pi N x)}{\pi x} \in L^p(\mathbb{R})$, $1 < p \leq +\infty$. Finally, we define also the (continous) *Carleson maximal operator* as

$$\mathcal{C}f(x) = \sup_N |\tilde{S}_N f(x)|. \tag{21.5}$$

For this operator, we will want an inequality of the form

$$\|\mathcal{C}f\|_2 \leq C\|f\|_{L^2(\mathbb{R})}. \tag{21.6}$$

We can prove then:

Theorem 21.7. *Let Cf be the Fourier series maximal operator, and $\mathcal{C}f$ be the continuous version of it defined in 21.5. Then, if (21.6) holds, automatically also Theorem 21.4 holds.*

Proof. Let F be a 1−periodic function in $L^2[0,1]$, and define $f = F \cdot \varphi$, where φ is bounded from above and below on $[0,1]$ and $\widehat{\varphi}$ is smooth, nonnegative, compactly supported in $[-1/2, 1/2]$ and symmetric function. The existence of such a function is simple to prove: take first a function $\widehat{\varphi_1}$ that has all the desired properties, with the additional one that it is supported on $[-1/10, 1/10]$. Define then φ_1 to be its Fourier transform, and let $\varphi_2 = (\varphi_1)^2$. It is easy to show that, from the properties we have set, φ_1 is analytic. Therefore, φ_2 is a nonnegative, analytic function. Finally, let $\varphi = \varphi_2 * \varphi_2$. As then $\widehat{\varphi} = (\widehat{\varphi_2})^2$, then φ is *also* analytic. By the fact that φ is analytic, we see that $\varphi(x) > 0$, $\forall x \in \mathbb{R}$. This is our desired function.

Now we estimate:

$$
\begin{aligned}
\| \sup_N |S_N F| \|_{L^2[0,1]} &\lesssim \| \sup_N (S_N F)\varphi \|_{L^2(\mathbb{R})} \\
&\lesssim \| \sup_N \tilde{S}_N (F\varphi) \|_{L^2(\mathbb{R})} \\
&\overset{(21.6)}{\lesssim} \| F \cdot \varphi \|_{L^2(\mathbb{R})} \\
&\lesssim \| F \|_{L^2[0,1]}.
\end{aligned}
$$

This ends the proof. □

On the next week, we are going to analyze a little bit more the embedding and invariation properties of the Carleson maximal operator, in order to establish inequality (21.6).

22 The Walsh model

2017-01-17

Consider the symmetry group given by:

- translations, for $y \in \mathbb{R}$

$$T_y \varphi(x) = \varphi(x - y);$$

- modulations, for $\eta \in \mathbb{R}$

$$M_\eta \varphi(x) = \varphi(x) e^{2\pi i x \eta};$$

- (L^1-norm preserving) dilations, for $t \in \mathbb{R}_+$

$$D_t \varphi(x) = \frac{1}{t} \varphi \left(\frac{1}{t} \right);$$

- (multiplication by a scalar of modulus one).

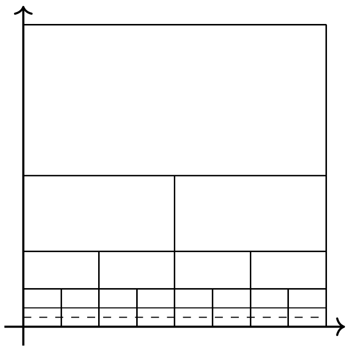
The theory of paraproducts is invariant under these symmetries.
We observe

$$(M_\eta(T_y\varphi))(x) = (T_y\varphi)(x)e^{2\pi i x\eta} = \phi(x-y)e^{2\pi i\eta x},$$
$$(T_y(M_\eta\varphi))(x) = (M_\eta\varphi)(x-y) = M_\eta(T_y\varphi)(x)e^{-2\pi i\eta y};$$

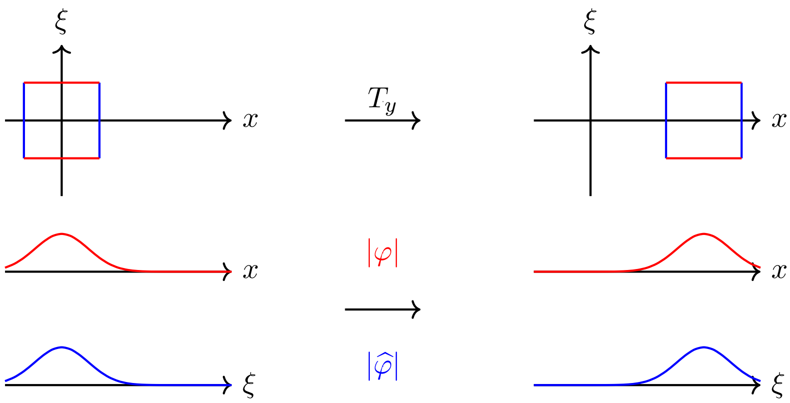
For the Fourier transform $\widehat{\varphi}(\xi) = \int \varphi(x)e^{-2\pi i x\xi}\mathrm{d}x$, we have

$$(T_y\varphi)\widehat{\ }(\xi) = \int \varphi(x-y)e^{-2\pi i x\xi}\mathrm{d}x = \widehat{\varphi}(\xi)e^{-2\pi i y\xi} = M_{-y}(\widehat{\varphi})(\xi),$$
$$(M_\eta\varphi)\widehat{\ }(\xi) = T_\eta(\widehat{\varphi})(\xi).$$

The old picture of the dyadic model of the upper half plane encoded translations and dilations.



We want to add also modulations. We restrict to the (x,ξ) space/frequency plane and we represent translations in space as horizontal translations and modulations, which are translations in frequency as seen, as vertical translations.

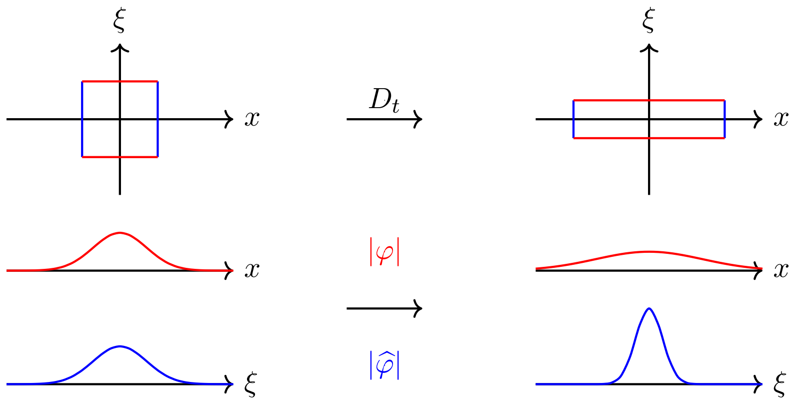


Remark 22.1. The intervals defining the sides of the rectangle should be thought as the region in which the relevant part of φ (resp. $\widehat{\varphi}$) is localized rather than the proper support of the function. In fact there cannot be a function such that both it and its Fourier transform have compact support.

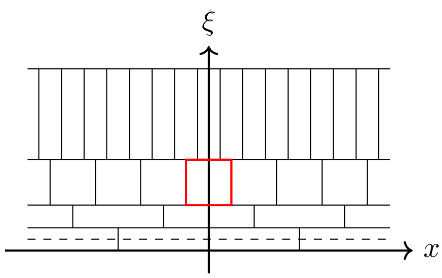
A similar picture can be drawn for modulations.
What is instead the effect of dilations?

$$\widehat{D_t \varphi}(\xi) = \int \frac{1}{t}\varphi\left(\frac{1}{t}\right) e^{-2\pi i x \xi} \mathrm{d}x = \widehat{\varphi}(\xi t) = t^{-1} D_{t^{-1}} \varphi(\xi).$$

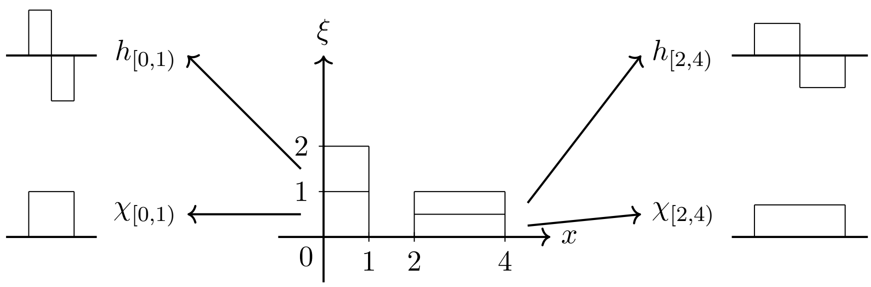
We observe that the area of the rectangle with blue and red sides is preserved. This effect encodes the so called "Heisenberg uncertainty principle".



Now assume $\widehat{\varphi}$ has compact support. Therefore $M_\eta \varphi$ has integral zero for η large enough. Adding translations and dilations of the rectangle associated to $M_\eta \varphi$ we obtain



This is the same old picture where $t \mapsto \frac{1}{t}$. We recall that the effect of modulation gives vertical translations of this structure.
The dyadic model in this case is also called the *Walsh model*. We work with the conditions $(x, \xi) \in \mathbb{R}_+ \times \mathbb{R}_+, t \in \mathbb{R}_+$.



We use the L^2-normalized functions

$$\chi_I = \frac{1}{|I|}\mathbb{1}_I = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{|I_l|}}\mathbb{1}_{I_l} + \frac{1}{\sqrt{|I_r|}}\mathbb{1}_{I_r}\right),$$
$$h_I = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{|I_l|}}\mathbb{1}_{I_l} - \frac{1}{\sqrt{|I_r|}}\mathbb{1}_{I_r}\right).$$

For these functions it holds

$$\begin{pmatrix}\chi_I \\ h_I\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}\chi_{I_l} \\ \chi_{I_r}\end{pmatrix}.$$

Let $\mathfrak{p}$ be the set of dyadic rectangles $I \times \omega$ such that $|I||\omega| = 1$.

Claim 22.2. *There is a unique map* $w\colon \mathfrak{p} \to L^2(\mathbb{R}_+)$ *such that*

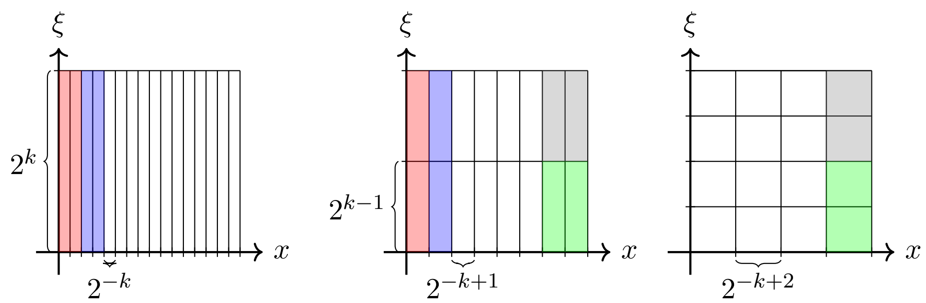
- $w(I \times [0, |I|^{-1})) = \chi_I$;
- *if* $|I||\omega| = 2$, *then*

$$\begin{pmatrix}w(I \times \omega_l) \\ w(I \times \omega_r)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}w(I_l \times \omega) \\ w(I_r \times \omega)\end{pmatrix}. \tag{22.3}$$

Proof. To prove both uniqueness and existence we consider the following argument. For $k \geq 0$ we restrict to the square $[0, 2^k) \times [0, 2^k)$ and we divide it into vertical tiles $I \times [0, 2^k)$ with $|I| = 2^{-k}$, $I \subset [0, 2^k)$. By the first property of w we know the image of these elements of $\mathfrak{p}$. Using (22.3) we determine uniquely the images under w of the elements of $\mathfrak{p}$ of the form $I' \times [0, 2^{k-1})$, $I' \times [2^{k-1}, 2^k)$ with $|I'| = 2^{-k+1}$. By recursion w is uniquely determined on all the elements of $\mathfrak{p}$ contained in $[0, 2^k) \times [0, 2^k)$. To conclude existence (and uniqueness) it is enough to observe that for χ_I we have

$$\chi_I = \frac{1}{\sqrt{2}}(\chi_{I_l} + \chi_{I_r}),$$

i.e. the relation in (22.3) is verified for $w(I \times [0, |I|^{-1}) = \chi_I$. All instances of the property (22.3) appear in the construction of w we described.



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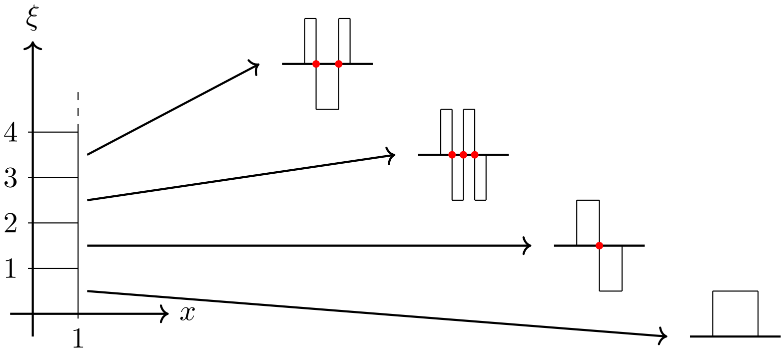


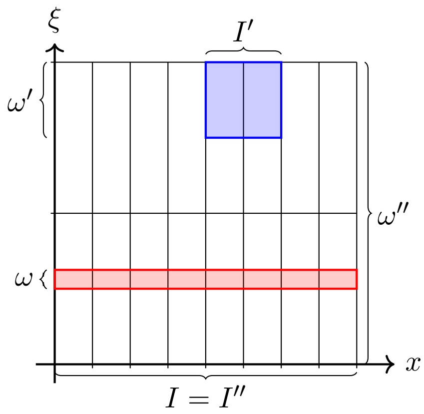
Observe that for every $n \in \mathbb{N}$ there is a unique $n' \in \mathbb{N}$ such that $w([0,1) \times [n', n'+1))$ has n "zero crossing", just like $\cos(2\pi n x)$ and $\sin(2\pi n x)$. The functions we described above are really the counterpart of these two functions. In fact they are the characters of $(\mathbb{Z}/2\mathbb{Z})^\infty$, and there is an algebraic way of defining them, as the main ingredients of a "Fourier Transform" for functions defined on this group, which we are not going to treat here. A fundamental property of w is stated by the following

Lemma 22.4 (Orthogonality). *Let $p, p' \in \mathfrak{p}$, $p \cap p' = \emptyset$. Then*

1. $\|w(p)\|_{L^2} = 1$;
2. $w(p) \perp w(p')$.

Proof. 1. The claim is clear for $p = I \times [0, |I|^{-1})$. The statement for an arbitrary p follows by recursive use of (22.3).

2. Since $p \cap p' = \emptyset$, either $\omega \cap \omega' = \emptyset$ or $I \cap I' = \emptyset$. Without loss of generality we can restrict to the first.
Let ω'' be the smallest dyadic interval containing ω, ω', and I'' be defined in the same way for I, I'. We can assume $\omega \subset \omega''_l, \omega' \subset \omega''_r$, and we subdivide $I'' \times \omega''_l$ and $I'' \times \omega_r$ vertically into elements of $\mathfrak{p}$. Because of the property (22.3), it is enough to prove the orthogonality of the image through w of these rectangles. They are of the form $\tilde{I} \times \omega''_l$ and $\tilde{I}' \times \omega''_r$, where $\tilde{I}, \tilde{I}' \subset I''$. If $\tilde{I} \neq \tilde{I}$ the orthogonality is clear, since the functions $w(\tilde{I} \times \omega''_l), w(\tilde{I}' \times \omega''_r)$ have disjoint supports. If $\tilde{I} = \tilde{I}'$ the orthogonality follows from (22.3).



□

Even if historically the dyadic model was studied after, a posteriori it is interesting to start from it rather than attack directly the problem in the continuous case. In particular, the dyadic model provides a simpler setting where to study the analogous problem of pointwise convergence almost everywhere of the Fourier series, and develop the necessary techniques. However it doesn't mean that the passage from the proof in the dyadic case to that in the continuous one is always naive.

22.1 Walsh-Fourier series for $f \in L^2([0,1))$

Define

$$S_N f(x) = \sum_{n=0}^{N-1} \langle f, w([0,1) \times [n, n+1)) \rangle \, w([0,1) \times [n, n+1))(x).$$

It is a Cauchy sequence in $L^2([0,1))$, therefore it converges to f in $L^2([0,1))$. In fact, $S_{2^k} f$ are given by the dyadic martingale averages $\mathbb{E}_k f$, which are known to converge to f in $L^2([0,1))$.
We can consider the problem of pointwise convergence almost everywhere in this case. The Carleson's Theorem in this setting is the following

Theorem 22.5 (Billard-Carleson-Hunt Theorem for Walsh-Fourier series)**.**

$$\|\sup_N S_N f\|_p \le C_p \|f\|_p, \qquad \textit{for } 1 < p < \infty.$$

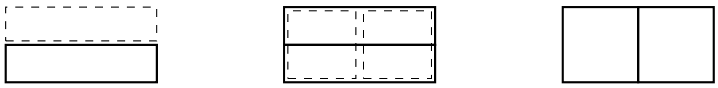
Remark 22.6. We fairly trivially have

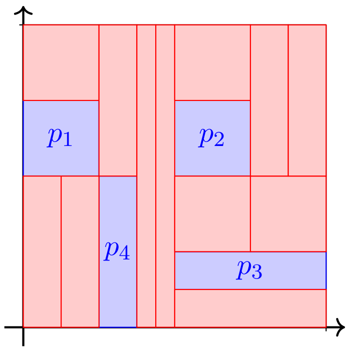
$$\|\sup_k S_{2^k} f\|_p \le C_p \|f\|_p,$$

by boundedness of the Hardy-Littlewood maximal operator, since $S_{2^k} f$ is given by a martingale average. The boundedness result is trivial also in the continuous setting.

Lemma 22.7. *Let* $p_1, \ldots, p_n$ *pairwise disjoint in* $\mathfrak{p}$ *and contained in* $[0, 2^k) \times [0, 2^k)$. *Then we can find* $p_{n+1}, \ldots, p_{2^k}$ *such that* $p_1, \ldots, p_{2^k}$ *is an orthonormal basis of* $H = \operatorname{span}\{\chi_{I\times[0,2^k)} \colon I \subset [0, 2^k)\}$.

Proof. Without loss of generality we can assume that for every p_i with $i \in \{1, \ldots, n\}$, its vertical sibling is not in $p_1, \ldots, p_n$. Otherwise suppose it to be p_j. Then replace, through (22.3), p_i, p_j with p_i', p_j', the horizontal siblings giving the same union.
The claim follows by induction on $\max_j |I_j|$. If $\max_j |I_j| = 2^{-k}$ it is clear, it means we have to add enough tiles of the form $I \times [0, 2^k)$. In general for i such that $|I_i| = \max_j |I_j| > 2^{-k}$ we can add the vertical sibling of p_i to the collection. Then apply (22.3) to change the generators from $w(I \times \omega_l), w(I \times \omega_r)$ to $w(I_l \times \omega), w(I_r \times \omega)$. Repeating these steps we can recollect to the case $\max_j |I_j| = 2^{-k}$.





□

Remark 22.8. We are basically using Hilbert space techniques, namely orthogonality is the tool we apply with (22.3). That is another reason why the L^2-normalization for the functions is the right one to choose.

Corollary 22.9. *If* $p_1, \ldots, p_n$ *are pairwise disjoint in* $\mathfrak{p}$ *and* $\tilde{p}_1, \ldots, \tilde{p}_n$ *are pairwise disjoint in* $\mathfrak{p}$ *such that*

$$\bigcup_{i=1}^{n} p_i = \bigcup_{i=1}^{n} \tilde{p}_i.$$

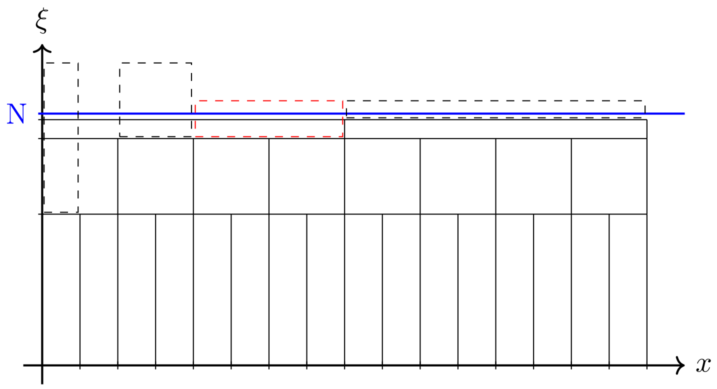
Then both are orthonormal bases of some subspace of $L^2(\mathbb{R}_+)$.

Proof. Complete both by the same vectors to a basis of H as before. □

22.2 Walsh-Fourier integral for $f \in L^2(\mathbb{R}_+)$

For $N \in \mathbb{R}_+$ define

$$S_N f := \sum_{\underbrace{p = I \times \omega_l \in \mathfrak{p}\colon N \in \omega_r}_{\mathfrak{p}_N}} \langle f, w(p)\rangle\, w(p).$$



We observe that

- the elements of $\mathfrak{p}_N$ are pairwise disjoint: suppose $(x,\xi) \in p, p' \in \mathfrak{p}_N$, i.e. $p = I \times \omega_l$, $p' = I' \times \omega_l'$, and $p \neq p'$. Since $x \in I \cap I'$ then without loss of generality $I \subsetneq I'$, hence $|\omega_l| > |\omega_l'|$. But $\xi \in \omega_l \cap \omega_l' \neq \emptyset$, that yields $\omega_r \cap \omega_r' = \emptyset$, giving a contradiction with $N \in \omega_r \cap \omega_r'$;
- the elements of $\mathfrak{p}_N$ cover $\mathbb{R}_+ \times [0, N)$.

If $N \in \mathbb{N}$ and $x \in [0,1)$, then partial Walsh-Fourier series and partial Walsh-Fourier integral coincide.
Now pick $N\colon \mathbb{R}_+ \to \mathbb{R}$ measurable (you should think of $N(x)$ picking a N for which $|S_N f(x)|$ "almost" attains the supremum) and define

$$S_{N(x)} f := \sum_{|I||\omega|=2} \langle f, w(I \times \omega_l)\rangle\, w(I \times \omega_l) \mathbb{1}_{N(x) \in \omega_r}.$$

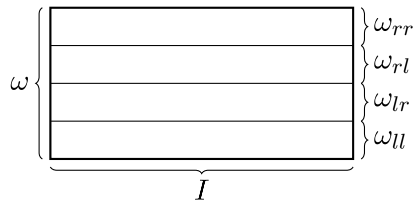
Then

$$\left\langle S_{N(\cdot)} f, g\right\rangle = \sum_{|I||\omega|=2} \langle f, w(I \times \omega_l)\rangle \left\langle w(I \times \omega_l) \mathbb{1}_{N(x)\in\omega_r}, g\right\rangle.$$

22.3 Quartile operator

We introduce the *quartile operator*, which plays the role of paraproducts in this context

$$Q(f_1, f_2, f_3) := \sum_{|I||\omega|=4} |I| \langle f_1, w(I \times \omega_{ll})\rangle \langle f_2, w(I \times \omega_{lr})\rangle \langle f_3, w(I \times \omega_{rl})\rangle.$$



Theorem 22.10. *The quartile operator is bounded, i.e. there exists* $C \in \mathbb{R}$ *independent on* f_1, f_2, f_3 *such that*

$$Q(f_1, f_2, f_3) < C\|f_1\|_{p_1}\|f_2\|_{p_2}\|f_3\|_{p_3},$$

where $2 < p_1, p_2, p_3 < \infty$ *and* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$.

23 Quartile operator

2017-01-19

Recall that we define a *tile* to be a rectangle $I \times \omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that I, ω are both dyadic and $|I||\omega| = 1$. We define also the functions $w_{I\times\omega}$ satisfying the following properties:

1. If $\omega = [0, |\omega|)$, then
$$w_{I\times\omega} = \frac{1}{|I|}\mathbb{1}_I.$$

2. If $|I||\omega| = 2$, then the functions $w_{I\times\omega_l}, w_{I\times\omega_r}, w_{I_l\times\omega}$ and $w_{I_r\times\omega}$ satisfy
$$\begin{pmatrix} w_{I\times\omega_l} \\ w_{I\times\omega_r} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_{I_l\times\omega} \\ w_{I_r\times\omega} \end{pmatrix}.$$

As we have seen in the previous lecture, this determines uniquely these functions. Moreover, it is also easy to verify that all those functions satisfy

$$|w_{I\times\omega}| = w_{I\times[0,|\omega|)}.$$

Therefore, we define a *quartile* to be a rectangle $P = I \times \omega$ such that $|I||\omega| = 4$, where both I, ω are dyadic intervals. From this definition, we let then

$$\omega_0 = (\omega_l)_l,\ \omega_1 = (\omega_l)_r,\ \omega_2 = (\omega_r)_l,\ \omega_3 = (\omega_r)_r.$$

From this definition, we define then the *quartile form* as

$$\Lambda(f_1, f_2, f_3) = \sum_{I\times\omega \text{ quartile}} |I| \prod_{i=1}^{3} \langle f_i, w_{I\times\omega_i}\rangle. \tag{23.1}$$

Our main aim of this lecture will be to prove the following

Theorem 23.2. *For $2 < p_1, p_2, p_2 \leq \infty$, such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, there exists a $C > 0$ such that*

$$|\Lambda(f_1, f_2, f_3)| \leq C \prod_{i=1}^{3} \|f_i\|_{p_i}.$$

The analysis to prove this theorem will be based, of course, on the outer measure spaces we have been studying throughout the course. Indeed, let $X = \mathcal{P}$ be the space of all quartiles, and define the set of generating sets – or tents – as

$$\mathcal{E} = \mathcal{T} = \text{ set of all trees in } \mathcal{P},$$

where a set T is a *tree* if there exists $I_T \in \mathfrak{D}$ and $\xi_T \in \mathbb{R}_+$ such that, for all $I \times \omega \in T$, then $I \subset I_T$ and $\xi_T \in \omega_1 \cup \omega_2 \cup \omega_3$. We also define our set function $\sigma : \mathcal{T} \to \mathbb{R}_+$ as $\sigma(T) = |I_T|$.
Therefore, we only need to make sense of the *sizes*: we define first the S^1 size as

$$S^1(F)(T) = \frac{1}{|I_T|} \sum_{I \times \omega \in T} |I||F(I \times \omega)|,$$

where $F : \mathcal{P} \to \mathbb{R}$ and $T \in \mathcal{T}$. With this definition, we have the following:

Lemma 23.3 (atomicity)**.**

$$\sum_{I \times \omega \in \mathcal{P}} |I||F(I \times \omega)| \leq c\|F\|_{\mathcal{L}^1(S^1)}.$$

For the proof of this lemma, we need to use that the left hand side is additive, and that, for $F \in \mathcal{L}^\infty(S^1)$, by considering $F = F\mathbb{1}_{\cup_{P \in \mathcal{P}} P}$, then $|\Lambda(F\mathbb{1}_T)| \leq S_1(F\mathbb{1}_T)|I_T|$. Verifying these conditions and putting them together to prove the lemma is left as an exercise.
From that, we must yet define further sizes (or "energies"), as follows: for $i = 0, 1, 2, 3$, then

$$S_i(F)(T) = \sup_{P \in T, \xi_T \in \omega_i} |F(I \times \omega)| + \sum_{i \neq j} \left(\frac{1}{|I_T|} \sum_{I \times \omega \in T, \xi_T \in \omega_j} |I||F(P)|^2 \right)^{1/2}.$$

Note that, from this definition, we must have that

$$S^1(F_1 \times F_2 \times F_3)(T) \leq CS_1(F_1)S_2(F_2)S_3(F_3).$$

The justification of this fact follows from the definition of a tree and a Cauchy-Schwarz inequality:

$$S^1(F_1 \times F_2 \times F_3)(T) = \sum_{i=0}^{3} \frac{1}{|I_T|} \sum_{I \times \omega \in T, \xi_T \in \omega_i} |I||F_1F_2F_3(I \times \omega)|$$
$$\overset{\ell^\infty \times \ell^2 \times \ell^2 \text{ Hölder}}{\leq} CS_i(F_i)(T).$$

This implies, by Hölder's inequality, that

$$\|F_1F_2F_3\|_{\mathcal{L}^1(S^1)} \le C\prod_{i=1}^{3}\|F_i\|_{\mathcal{L}^{p_i}(S_i)}.$$

Therefore, to prove the theorem we have to show the following Embedding theorem:

Theorem 23.4. *There exists $C > 0$ depending on $2 < p \le \infty$ such that*

$$\|\langle f, w_{I\times\omega_i}\rangle\|_{\mathcal{L}^p(S_i)} \le C\|f\|_{L^p(\mathbb{R})}.$$

Proof. By interpolation, it suffices to show for $p = \infty$ and $p = 2$.
Case $p = \infty$: In this case, we need to show that, for all $T \in \mathcal{T}$,

$$S_i(\langle f, w_{I\times\omega_i}\rangle)(T) \le C\|f\|_\infty.$$

For the first summand in the definition of S_i, we estimate

$$\sup_{I\times\omega\in T,\xi_T\in\omega_i}|\langle f, w_{I\times\omega_i}\rangle| \le \sup_{P\in\mathcal{P}}\|w_{\mathcal{P}}\|_1\|f\|_\infty = \|f\|_\infty,$$

where we used the fact that $|w_{I\times\omega}| = w_{I\times[0,|\omega|)}$. For the other part, we use that, for $j \ne i$,

$$\sum_{I\times\omega\in T,\xi_T\in\omega_j}|I||\langle f, w_{I\times\omega_{I\times\omega_i}}\rangle|^2 = \sum_{I\times\omega\in T,\xi_T\in\omega_j}|I||\langle f\mathbb{1}_{I_T}, w_{I\times\omega_{I\times\omega_i}}\rangle|^2$$
$$\overset{I\times\omega_i\text{p.w. disjoint}}{\le} \frac{1}{|I_T|}\|f\mathbb{1}_{I_T}\|_2^2 \le C\|f\|_\infty^2,$$

where the pairwise disjointness of those intervals can be justified as follows: let $P, P' \in T$ be $P = I\times\omega$ and $P' = I'\times\omega'$, and suppose that $I\times\omega_i \cap I'\times\omega_i' \ne \emptyset$. Then, without loss of generality, we might suppose that $I \subset I'$, and this implies automatically that $\omega_i' \subset \omega_i$. But we also have that $\omega_j' \cap \omega_j \ne \emptyset$. Therefore, we must also have that $\omega_j' \subset \omega_j$. If $4|I| \le |I'|$, then we reach to a contradiction automatically. If not, then a case analysis will do it. The details of this last part are left as an exercise.
Case $p = 2$: We need, in this case, a weak type bound. Let $\lambda > 0$. We need to find a collection of trees $\mathcal{T}'$ such that

$$\sum_{T\in\mathcal{T}'}|I_T| \le C\frac{\|f\|_2^2}{\lambda^2}$$

and

$$S_i(|\langle f, w_{I\times\omega_i}\rangle|\mathbb{1}_{(\cup\mathcal{T}')^c})(T) \le C\lambda.$$

To this intent, pick $P_1 = I_1 \times \omega_1$ a quartile such that $|\langle f, w_{I\times\omega_i}\rangle| > \frac{\lambda}{10}$ and $|I_1|$ is maximal. The maximality of the length of this interval can be

assured due to the fact that $|\langle f, w_{I\times\omega_i}\rangle| \leq \|f\|_2 |I|^{-1/2}$. Then we iterate this process: pick P_{n+1} such that $|\langle f, w_{I\times\omega_i}\rangle| \geq \frac{\lambda}{10}$, with $|I_{n+1}|$ maximal and $I_{n+1}\times(\omega_{n+1})_i$ *disjoint* from all the other previously selected $I_m \times (\omega_m)_i$, for $m = 1, ..., n$.
For each of those n, we pick then a $\xi_{T_n} \in (\omega_n)_i$, and let $I_{T_n} = I_n$. This defines already a first collection of trees $\{T_n\}$, and we see that it satisfies

$$\sum_n |I_n| \leq 100\lambda^{-2} \sum_n |I_n| |\langle f, w_{I_n \times (\omega_n)_i}\rangle|^2 \overset{\text{p.w. disjoint}}{\leq} 100\lambda^{-2}\|f\|_2^2.$$

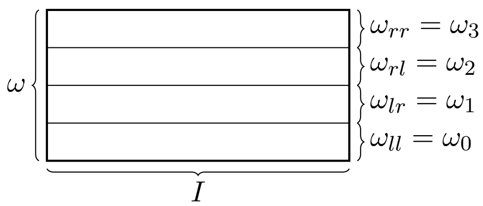
This is one of the inequalities we want. On the set $\mathcal{P} \setminus \cup_n T_n$ we have also that $|\langle f, w_{I\times\omega_i}\rangle| \leq \frac{\lambda}{10}$. Indeed, if this were not the case, pick one quartile $P \in \mathcal{P} \setminus \cup_n T_n$ with $|\langle f, w_{I_P\times(\omega_P)_i}\rangle| > \frac{\lambda}{10}$. Then, as it is in none of the trees $\{T_n\}$, it must *not* intersect $I' \times (\omega')_i$, if $I' \times \omega' \in \cup_n T_n$. As each time we select a new tree T_n, we go down one scale, then $\lim_{n\to\infty} |I_n| = 0$. This is a contradiction, as then P would have to be selected in this procedure, because, for some $k > 0$, P is a quartile with all the selection properties above – because of its disjointness properties– and such that $|I_P| \geq |I_n|$, $\forall n > k$. This contradiction finishes this selection.
On the next class we will do one more selection procedure, which will serve for us to finish the proof of this theorem. □

24 Embedding Theorem for the Walsh model

2017-01-24

Before concluding the proof of the Embedding Theorem we recall the outer measure setting we are working in:

- $X = \mathcal{P} = \{I \times \omega \colon I, \omega \text{ dyadic intervals in } \mathbb{R}_{\geq 0}, |I||\omega| = 4\}$. The elements of X are called *quartiles*;
- $\mathcal{E} = \mathcal{T} = \{T \subset \mathcal{P} \colon \exists I_T, \xi_T \colon T = \{I \times \omega \colon I \subset I_T, \xi_T \in \omega_1 \cup \omega_2 \cup \omega_3\}\}$. The elements of $\mathcal{T}$ are called *trees*;



- $\sigma(T) = |I_T|$;
- for $i \in \{1, 2, 3\}$ we define the sizes

$$S_{e,i}(F)(T) = \sup_{\substack{I\subset I_T \\ \xi_T\in\omega_i}} F(I\times\omega) + \sum_{\substack{j\neq i \\ j\in\{1,2,3\}}} \left(\frac{1}{|I_T|} \sum_{I\subset I_T \xi_T\in\omega_j} |F(I\times\omega)|^2 |I| \right)^{\frac{1}{2}}.$$

Theorem 24.1. *Let $i \in \{1,2,3\}$, $2 < p \leq \infty$. Define*

$$F(I \times \omega) = \langle f, w_{I\times\omega_i}\rangle .$$

Then

$$\|F\|_{\mathcal{L}^p(S_{e,i})} \leq C_p \|f\|_p.$$

Proof. <u>*Case $p = \infty$:*</u> last time.
<u>*Case $p = 2$:*</u> In this case, we need to prove a weak type bound, namely

$$\|F\|_{\mathcal{L}^{2,\infty}(S_{e,i})} \leq C\|f\|_2,$$

We are going to prove that for all $\mathcal{P}' \subset \mathcal{P}, |\mathcal{P}'| < \infty$ then

$$\|F\mathbb{1}_{\mathcal{P}'}\|_{\mathcal{L}^{2,\infty}(S_{e,i})} \leq C\|f\|_2,$$

with C independent on $\mathcal{P}'$. Therefore we can recover the claim of the theorem by an approximation argument.
Once fixed $\mathcal{P}'$, we denote $\widetilde{F} = F\mathbb{1}_{\mathcal{P}'}$. We have to show that for all $\lambda > 0$ there exists a collection $\mathcal{T}' \subset \mathcal{T}$ such that

$$\sum_{T\in\mathcal{T}'} |I_T| \leq C\frac{\|f\|_2^2}{\lambda^2},$$

and for every $T \in \mathcal{T}$

$$S_{e,i}(\widetilde{F}\mathbb{1}_{(\cup_{\mathcal{T}'})^c})(T) \leq C\lambda.$$

We construct $\mathcal{T}'$. Pick $P_1 = I_1 \times \omega_1$ such that $|\widetilde{F}(P_1)| \geq \frac{\lambda}{10}$ and $|I_1|$ is maximal possible (it is possible since $\mathcal{P}'$ is finite).
Now suppose to have defined $P_1, \dots, P_n$. If it exists, pick $P_{n+1} = I_{n+1}\times\omega_{n+1}$ such that $|\widetilde{F}(P_{n+1})| \geq \frac{\lambda}{10}$, $|I_{n+1}|$ is maximal possible, and $I_{n+1} \times (\omega_{n+1})_i$ is disjoint from $I_1 \times (\omega_1)_i, \dots, I_n \times (\omega_n)_i$.
Eventually, because of the condition on finiteness of $\mathcal{P}'$, the process ends.
The rectangles P_i are disjoint. For each n we have

$$\sum_{k=1}^{n}|I_k| \leq 100\lambda^{-2}\sum_{k=1}^{n}|I_k||\langle f, w_{I_k\times(\omega_k)_i}\rangle|^2 \leq 100\lambda^{-2}\|f\|_2^2.$$

Define $T_{i,n}$ by $I_{T_{i,n}} = I_n$, $\xi_{T_{i,n}} \in (\omega_n)_i$, hence $P_n \in T_{i,n}$. We let $\widetilde{P} \in \mathcal{P} \setminus \bigcup_n T_{i,n}$ be dyadic rectangle and assume that, by contradiction,

$$\widetilde{F}(\widetilde{P}) \geq \frac{\lambda}{10}.$$

Then $\widetilde{P}$ intersects some P_n. Let n be the smallest index such that $P = I \times \omega_i \cap I_n \times (\omega_n)_i \neq \emptyset$. Thus $|I| \leq |I_n|$, otherwise we would have picked $\widetilde{P}$ instead of P_n. Hence $I \subset I_n$, $(\omega_n)_i \subset \omega_i$, therefore $\widetilde{P} \in T_{i,n}$. This gives a

contradiction with the definition of $\widetilde{P}$.
This argument takes care of the L^∞ part in i in the definition $S_{e,i}$. Now we want to take care of the L^2 part in $j \neq i$.
First assume $j \in \{1,2,3\}$, $j < i$. Pick $T_{j,1}$ such that

$$\frac{1}{|I_{T_{j,1}}|} \sum_{\substack{I \subset I_{T_{j,1}} \\ \xi_{T_{j,1}} \in \omega_j \\ I \times \omega \notin \cup_n T_{i,n}}} |I||\widetilde{F}(I \times \omega)|^2 \geq \lambda^2,$$

$\xi_{T_{j,1}} \in 2^{-M}\mathbb{Z}$ for a fixed M such that $2^{-M} \leq \frac{\lambda^2}{10\|f\|_2^2}$, and $\xi_{T_{j,1}}$ is maximal possible (once again the issue of maximality is solved by finiteness of $\mathcal{P}'$).
Why is the condition $\xi_{T_{j,1}} \in 2^{-M}\mathbb{Z}$ not an issue? Let

$$\mathcal{R}_1 = \bigcup_n T_{i,n} = \mathcal{R}_1^{\text{even}} \cup \mathcal{R}_1^{\text{odd}},$$

where

$$\mathcal{R}_1^{\text{even}} = \{I \times \omega \in \mathcal{R}_1 \colon \ \lg_2|I| \text{ is even}\}.$$

We claim that $I \times \omega_i$ are pairwise disjoint for $I \times \omega \in \mathcal{R}_1^{\text{even}}$. In fact assume that $(I \times \omega_i) \cap (I' \times \omega_i') \neq \emptyset$ for two elements of $\mathcal{R}_1^{\text{even}}$. If $|I| = |I'|$, then $I \times \omega_i = I' \times \omega_i'$. Therefore without loss of generality we have $I \subsetneq I'$, hence $4|I| \leq |I'|$, yielding $\omega_i' \subset \omega_i$, thus $\omega_j' \subset \omega_i$, and finally $\omega_j' \cap \omega_j = \emptyset$. This gives a contradiction, since $\xi_{T_{j,1}} \in \omega_j \cap \omega_j'$.
Therefore we observe that

$$\sum_{I \times \omega \in \mathcal{R}_1^{\text{even}}} |I||\langle f, w_{I \times \omega_i}\rangle|^2 \leq \|f\|_2^2.$$

The same argument for $\mathcal{R}_1^{\text{odd}}$ yields

$$\sum_{I \times \omega \in \mathcal{R}_1} |I||\langle f, w_{I \times \omega_i}\rangle|^2 \leq 2\|f\|_2^2,$$

hence

$$|I_{T_{j,1}}| \leq C\frac{\|f\|_2^2}{\lambda^2}, |\omega_{T_{j,1}}| \geq \frac{\lambda^2}{C\|f\|_2^2}.$$

Therefore there is no loss of trees by assuming $\xi_{T_{j,1}} \in 2^{-M}\mathbb{Z}$.
Suppose to have defined $T_{j,1}, \ldots, T_{j,n}$. If possible, pick $T_{j,n+1}$ such that

$$\frac{1}{|I_{T_{j,n+1}}|} \sum_{\substack{I \subset I_{T_{j,n+1}} \\ \xi_{T_{j,n+1}} \in \omega_j \\ I \times \omega \notin \cup_k T_{i,k} \cup \cup_{k \leq n} T_{j,k}}} |I||\widetilde{F}(I \times \omega)|^2 \geq \lambda^2,$$

$\xi_{T_{j,n+1}} \in 2^{-M}\mathbb{Z}$, and $\xi_{T_{j,n+1}}$ is maximal possible. Define

$$\mathcal{R}_n \coloneqq \bigcup_k T_{i,k} \cup \bigcup_{k\le n} T_{j,k},$$
$$\widetilde{\mathcal{R}}_n \coloneqq \{I\times\omega \in \mathcal{R}_n \colon \nexists I'\times\omega' \in \mathcal{R}_n \colon I' \subsetneq I\}.$$

Then

$$\frac{1}{|I_{T_{j,n}}|}\sum_{I\times\omega\in\widetilde{\mathcal{R}}_n} |I||\widetilde{F}(I\times\omega)|^2 \le \frac{1}{|I_{T_{j,n}}|}\frac{\lambda^2}{100}|I_{T_{j,n}}| \le \frac{\lambda^2}{100},$$

where we used the fact that $I\times\omega \in \widetilde{\mathcal{R}}_n$ are pairwise disjoint and $|\widetilde{F}(I\times\omega)| \le \frac{\lambda}{10}$ because of the choice of $T_{i,k}$. As a consequence

$$\frac{1}{|I_{T_{j,n}}|}\sum_{I\in\mathcal{R}_n\setminus\widetilde{\mathcal{R}}_n} |I||\widetilde{F}(I\times\omega)|^2 \ge \frac{99}{100}\lambda^2.$$

We claim that if

$$I\times\omega \in (\mathcal{R}_n\setminus\widetilde{\mathcal{R}}_n)^{\text{even}},$$
$$I'\times\omega' \in (\mathcal{R}_{n'}\setminus\widetilde{\mathcal{R}}_{n'})^{\text{even}},$$

with $n \ne n'$, then

$$I\times\omega_i \cap I'\times\omega_i' = \emptyset.$$

In fact, suppose not. Since $I\times\omega_i \ne I'\times\omega_i'$, without loss of generality we can assume $|I|<|I'|$. Then $I \subsetneq I'$, hence $\omega_i' \subsetneq \omega_i$, yielding $4|\omega_i'| \le |\omega_i|$, thus $\omega_j' \subset \omega_i$, and finally $\omega_j'\cap\omega_j = \emptyset$. We observe that ω_j is below ω_j' since $j<i$, hence $n'<n$. Therefore there is $I''\times\omega'' \in \widetilde{\mathcal{R}}_n$ such that $I'' \subsetneq I$, hence $\xi_{T_{j,n}} \in \omega_j \subsetneq \omega_j''$, yielding $4|\omega_j| \le |\omega_j''|$, thus $\omega_i \subsetneq \omega_j''$, implying $\omega_i' \subsetneq \omega_j''$, and finally $\xi_{T_{j,n'}} \in \omega_j' \subset \omega_j''$. But we have $I'' \subset I \subset I'$. Hence $I''\times w'' \in T_{j,n'}$, which yields a contradiction.
To conclude we observe that

$$\frac{99}{100}\lambda^2\sum_n |I_{T_{j,n}}| \le \sum_n\sum_{\mathcal{R}_n\setminus\widetilde{\mathcal{R}}_n} |I||\langle f, w_{I\times\omega_i}\rangle|^2 \le \|f\|_2^2,$$

therefore

$$\sum_n |I_{T_{j,n}}| \le C\frac{\|f\|_2^2}{\lambda^2}.$$

It remains to show the case $j>i$. In this case we use an analogous argument, but we choose $\xi_{T_{j,n}}$ to be minimal in defining $T_{j,n}$.
To conclude we observe that we used the condition on finiteness of $\mathcal{P}'$ only to assume that the processes of choice of $T_{i,n}$, $T_{j,n}$ end, but not in defining the constants. Therefore, by an approximation argument, we recover the weak type (2,2) embedding for F. □

This concludes the proof of the Embedding Theorem and, as a consequence, of the boundedness of the quartile operator.

25

2017-01-26

Let, as before, I, ω be dyadic intervals in $\mathbb{R}_{\geq 0}$, and $N : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a measurable function. We define then the *Walsh-Carleson (bilinear) form* as

$$\sum_{|I||\omega|=2} |I| \langle f, w_{I\times\omega_l} \rangle \langle g, w_{I\times\omega_l} \mathbb{1}_{\omega_r} \circ N \rangle = \Lambda(f,g). \tag{25.1}$$

We are going to use quite often a modified version of this operator, namely a *truncation* of it, as follows: let $\mathbb{P}$ be a finite set of bitiles (i.e., of dyadic rectangels $I \times \omega$ such that $|I||\omega| = 2$.), and let

$$\Lambda_{\mathbb{P}}(f,g) = \sum_{I\times\omega\in\mathbb{P}} |I| \langle f, w_{I\times\omega_l} \rangle \langle g, e_{I\times\omega_l} \mathbb{1}_{\omega_r} \circ N \rangle.$$

Our goal then will be to establish estimates on $\Lambda_{\mathbb{P}}$ that do *not* depend on the finite set $\mathbb{P}$, and therefore take a limit in the end to obtain the original bounds for our operator.
Therefore, let $X = \mathcal{P}$ be the set of all bitiles. Define also a *tree* on this set to be a collection of bitiles

$$\{I \times \omega, I \subset I_T, \xi_T \in \omega\}.$$

Let then $\mathcal{T}$ be the set of all trees, and $|I_T| = \sigma(T)$ is the pre-measure on this set. Moreover, we define a *size* on this set as

$$S_e(F)(T) = \sup_{I\subset I_T, \xi_T\in\omega_l} F(I\times\omega) + \left(\frac{1}{|I_T|} \sum_{I\subset I_T,\xi_T\in\omega_r} |F(I\times\omega)|^2 |I| \right)^{1/2}.$$

For this definition, we get that, from our last classe's theorem, if $F(I \times \omega) = \langle f, w_{I\times\omega_l} \rangle$,

$$\|F\|_{\mathcal{L}^p(S_e)} \leq C_p \|f\|_p,$$

for $2 \leq p \leq \infty$. A weak inequality must, in addition, hold at the endpoint. To prove today's bounds, however, we have to define some other a little more sophisticated objects. Therefore, let

$$S_m(G)(T) = \frac{1}{|I_T|} \sum_{I\subset I_T,\xi_T\in\omega_l} |G(I\times\omega)||I| + \left(\frac{1}{|I_T|} \sum_{I\subset I_T,\xi_T\in\omega_r} |I||G(I\times\omega)|^2 \right)^{1/2}.$$

By bounding directly, we just need to prove an embedding theorem for S_m :

Theorem 25.2. *Let $G(I \times \omega) = \langle g, w_{I\times\omega_r} \mathbb{1}_{w_r} \circ N \rangle$. Then, for $1 < p \leq \infty$,*

$$\|G\|_{\mathcal{L}^p(S_m)} \leq C_p \|g\|_p.$$

As a corollary, we get the following:

Theorem 25.3 (Boundedness of the Walsh-Carleson form)**.** *For* $2 < p \leq \infty$, *we have that*

$$\Lambda(f, g) \leq C_p \|f\|_p \|g\|_{p'}.$$

Moreover, there holds an analogous weak bound at the endpoint $p = 2$.

In what follows, our sums are all going to be considered (although not always explicitly stated) over a finite fixed set $\mathbb{P}$ of bitiles.

Proof of theorem 25.2. We do, as usual, an interpolation argument:
$\underline{p = \infty}$. In this case, we analyse differently the two summands defining S_m :

1.
$$\begin{aligned}
&\frac{1}{|I_T|} \sum_{I \subset I_T} |\langle g, w_{I \times \omega_l} \mathbb{1}_{w_r} \circ N \rangle| |I| \\
&\leq \frac{1}{|I_T|} \sum_{I \subset I_T} \int |g(x)| \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N(x) dx \\
&= \frac{1}{|I_T|} \int |g(x)| \left(\sum_{I \subset I_T, \xi_T \in \omega_l} \mathbb{1}_I \mathbb{1}_{w_r} \circ N \right) dx \\
&\overset{w_l \text{overlaps} \Rightarrow w_r \text{disjoint}}{\leq} \frac{1}{|I_T|} \int g(x) \mathbb{1}_{I_T}(x) dx \leq \|g\|_\infty,
\end{aligned}$$

which finalizes the proof for this part.

2. Let
$$\left(\frac{1}{|I_T|} \sum_{I \subset I_T, \xi_T \in \omega_r} |\langle g, w_{I \times \omega_l} \mathbb{1}_{\omega_r} \circ N \rangle|^2 |I| \right)^{1/2} = A.$$

Define a function h implicitly by requiring that h has the following expansion:
$$\begin{aligned}
h(x) &= \sum_{I \subset I_T, \xi_T \in \omega_r} |I| H(I \times \omega) w_{I \times \omega_l}(x) \\
&= \sum_{I \subset I_T, \xi_T \in \omega_r} |I| \langle h, w_{I \times \omega_r} \rangle w_{I \times \omega_l}(x),
\end{aligned}$$

where $H(I \times \omega) = \langle g, w_{I \times \omega_l} \mathbb{1}_{\omega_r} \circ N \rangle$. With this definition, we can estimate:

$$A^2 = \frac{1}{|I_T|} \sum_{I \subset I_T, \xi_T \in \omega_r} \langle g, w_{I\times\omega_r} \mathbb{1}_{\omega_r} \circ N\rangle \langle h, w_{I\times\omega_l}\rangle |I|$$

$$\leq \frac{1}{|I_T|} \|g\|_{L^2(I_T)} \left\| \sum_{I \subset I_T, \xi_T \in \omega_r} |I| \langle h, w_{I\times\omega_r}\rangle w_{I\times\omega_l} \mathbb{1}_{\omega_r} \circ N \right\|_2$$

$$\leq \frac{1}{|I_T|} \|g\|_2 \left\| \sum_{I \subset I_T, \xi_T \in \omega_r} |I| \langle h, w_{I\times\omega_r}\rangle w_{I\times\omega_l} \right\|_2$$

As all our sums until the present moment have been over a finite set of bitiles, all our intervals I must satisfy that $|I| \leq 2^k$, for some $k \in \mathbb{Z}$. This automatically shows the existence of a bitile $I_0 \times \omega_0$ such that

$$\left\| \sum_{I \subset I_T, \xi_T \in \omega_r} |I| \langle h, w_{I\times\omega_r}\rangle w_{I\times\omega_l} \right\|_2 =$$

$$\left\| \langle \sum_{I \subset I_T, \xi_T \in \omega_r} |I| \langle h, w_{I\times\omega_r}\rangle w_{I\times\omega_l}, w_{I_0\times\omega_0}\rangle w_{I_0\times\omega_0} \cdot \right\|_2 .$$

This last expression is, on the other hand, controlled by (a multiple of) the Hardy–Littlewood maximal function, which, in turn, satisfies that $\|Mf\|_2 \leq C'\|f\|_2$. This finishes the proof of this case.

$\underline{p = 1}$. In this case, we want a weak bound of the type $\|G\|_{\mathcal{L}^{1,\infty}(S_m)} \leq C\|g\|_1$. We need, therefore, to contruct for any $\lambda > 0$ a collection of trees $\mathcal{T}'$ such that

$$\sum_{T \in \mathcal{T}'} |I_T| \leq \frac{C\|f\|_1}{\lambda},$$

and, for all T,

$$S_m(G\mathbb{1}_{(\cup\mathcal{T}')^c})(T) \leq \lambda.$$

Define then an auxiliary function

$$\tilde{G}(I \times \omega) = \frac{1}{|I|} \int_I |g(x)| \mathbb{1}_\omega \circ N(x) dx.$$

(Compare with the definition of $G(I \times \omega) = \langle g, w_{I\times\omega_l} \mathbb{1}_{\omega_r} \circ N\rangle$) The idea goes on roughly like on the last lecture: Choose $I_1 \times \omega_1$ bitile such that $\tilde{G}(I_1 \times \omega_1) > \frac{\lambda}{10}$ and $|I_1|$ maximal possible. We iterate the process, by choosing $I_{n+1} \times \omega_{n+1}$ such that $\tilde{G}(I_{n+1} \times \omega_{n+1}) > \frac{\lambda}{10}$, $|I_{n+1}|$ maximal and $I_{n+1} \times \omega_{n+1}$ disjoint from $I_j \times \omega_j$, $j = 1, ..., n$. For each $n \geq 0$, we define also the tree T_n as spanned by the top interval $I_{T_n} = I_n$, and fixing any $\xi_{T_n} \in \omega_n$.

We notice also that there is a $n_0 \geq 0$ such that, for $n \geq n_0$, then actually $I_n = \omega_n = \emptyset$, as our finite set of tiles $\mathbb{P}$ has to be exhausted at some point. By the same classical reasons, we get that

$$\sum_{k=1}^{n} |I_k| \leq \frac{10}{\lambda} \int_{\mathbb{R}} |g(x)| \left(\sum_{k=1}^{n} \mathbb{1}_{I_k} \mathbb{1}_{\omega_k} \circ N(x) \right) dx$$
$$\leq \frac{10}{\lambda} \int_{\mathbb{R}} |g(x)| dx = \frac{10}{\lambda} \|g\|_1,$$

as for all $x \in \mathbb{R}$, there is at most one k such that $x \in I_k$ and $N(x) \in \omega_k$, by the selection above.
To prove the second part, notice first that if $I \times \omega \notin \cup_n T_n$, then $\tilde{G}(I \times \omega) \leq \frac{\lambda}{10}$. We pick then $\mathcal{T}' = \{T_n, n \geq 1\}$. We need then to estimate the remaining part, and as for the $\underline{p = \infty}$ case, we divide into two tasks:

1. Let, first of all, $\mathcal{J} = \{J \subset I_T \colon J \text{ maximal in } \cup_n T_n\}$. Therefore, we may write $I_T = \cup_{J \in \mathcal{J}} J \cup E$. We therefore estimate

   $$\frac{1}{|I_T|} \sum_{I \subset I_T, \xi_T \in \omega_l, I \times \omega \notin \cup T_n} |I| |\langle g, w_{I \times \omega_l} \mathbb{1}_{\omega_r} \circ N \rangle|$$
   $$\leq \frac{1}{|I_T|} \int_{I_T} |g(x)| \left(\sum_{I \subset I_T, \xi_T \in \omega_l, I \times \omega \notin \cup T_n} \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N \right) dx$$
   $$= \frac{1}{|I_T|} \times (\sum_{J \in \mathcal{J}} \int_J |g(x)| | \left(\sum_{I \subset I_T, \xi_T \in \omega_l, I \times \omega \notin \cup T_n} \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N \right) dx +$$
   $$\int_E |g(x)| | \left(\sum_{I \subset I_T, \xi_T \in \omega_l, I \times \omega \notin \cup T_n} \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N \right) dx)$$

   But then the summands accounting for $J \in \mathcal{J}$ can be bounded each by

   $$\int_{\tilde{J}} |g(x)| dx \leq \frac{\lambda |\tilde{J}|}{10} \leq c\lambda |J|,$$

   where $\tilde{J}$ stands for the parent dyadic interval of J. For the other summand, we estimate it by

   $$\sum_{I \subset I_T \setminus \cup \mathcal{J}, \xi_T \in \omega_l, I \times \omega \notin \cup T_n} |I| \tilde{G}(I \times \omega) \leq c\lambda |E| \leq c\lambda |I_T|.$$

   This is enough to complete the first part.

2. As in the $\underline{p = \infty}$ case, we let

   $$A^2 = \frac{1}{|I_T|} \sum_{T \notin \cup T_n} |\langle g, w_{I \times \omega_l} \mathbb{1}_{\omega_r} \circ N \rangle|^2 |I|.$$

We let h be also defined as in the $p = \infty$ case, and $\mathcal{J}$ as in the first part, and then we have

$$A^2 = \frac{1}{|I_T|}\langle g, \sum_{T \notin \cup T_n} |I| w_{I\times\omega_l} \mathbb{1}_{\omega_r} \circ N\langle h, w_{I\times\omega_l}\rangle\rangle$$
$$\leq \frac{1}{|I_T|}\sum_J \int_J |g(x)| \left(\sum_{I\subset I_T, \xi_T\in\omega_r, I\times\omega\notin\cup T_n} \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N|\langle h, w_{I\times\omega_r}\rangle| \right) dx +$$
$$\frac{1}{|I_T|}\int_E |g(x)| \left(\sum_{I\subset I_T, \xi_T\in\omega_r, I\times\omega\notin\cup T_n} \mathbb{1}_I \mathbb{1}_{\omega_r} \circ N|\langle h, w_{I\times\omega_r}\rangle| \right) dx.$$

The analysis then goes more or less as in the previous case, mixing together the ideas already presented. The details are left to the reader.

□

26 The Bilinear Hilbert Transform

2017-01-31

We introduce the *bilinear Hilbert transform.* For $f, g \in \mathcal{S}(\mathbb{R})$, we define

$$\begin{aligned} B(f,g)(x) :=& \text{p.v.} \int_{\mathbb{R}} f(x-t)g(x-2t)\frac{\mathrm{d}t}{t} \\ =& \lim_{\varepsilon\to 0}\int_{[-\varepsilon,\varepsilon]^c} f(x-t)g(x-2t)\frac{\mathrm{d}t}{t} = \\ =& \frac{1}{2}\int_{\mathbb{R}} \left(f(x-tg(x-2t))\right), \end{aligned}$$

where in the last passage we gained integrability of the argument by boundedness property of Schwartz functions.

To the transform we associate a trilinear form

$$\begin{aligned} \Lambda(f,g,h) :=& \int_{\mathbb{R}} B(f,g)(x)h(x)\mathrm{d}x = \\ =& \text{p.v.} \iint h(x)f(x-t)g(x-2t)\frac{\mathrm{d}t}{t}\mathrm{d}x. \end{aligned}$$

A related integral is given by

$$\int_{\mathbb{R}}\int_0^1 f(x)f(x-t)f(x-2t)\mathrm{d}t\mathrm{d}x.$$

If $f = \mathbb{1}_E$ for a set $E \subset \mathbb{R}$ it counts arithmetic progression of length 3 inside E of width at most 1.

More generally, for $\beta(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$, we define

$$\Lambda_\beta(f_1, f_2, f_3) = \text{p.v.} \iint f_1(x-\beta_1 t)f_2(x-\beta_2 t)f_3(x-\beta_3 t)\frac{\mathrm{d}t}{t}\mathrm{d}x.$$

The change of variables $x \mapsto x - \gamma t$ yields

$$\begin{aligned}\Lambda_\beta(f_1, f_2, f_3) &= \text{p.v.} \iint \prod_{j=1}^{3} f_j(x - \beta_j t) \frac{\mathrm{d}t}{t} \mathrm{d}x = \\ &= \text{p.v.} \iint \prod_{j=1}^{3} f_j(x - \gamma t - \beta_j t) \frac{\mathrm{d}t}{t} \mathrm{d}x.\end{aligned}$$

Thus we may add γ to β_j and, without loss of generality, assume $\beta_1 + \beta_2 + \beta_3 = 0$.

The change of variables $t \mapsto \lambda t$ yields

$$\begin{aligned}\Lambda_\beta(f_1, f_2, f_3) &= \text{p.v.} \iint \prod_{j=1}^{3} f_j(x - \beta_j t) \frac{\mathrm{d}t}{t} \mathrm{d}x = \\ &= \text{p.v.} \iint \prod_{j=1}^{3} f_j(x - \beta_j \lambda t) \frac{\mathrm{d}t}{t} \mathrm{d}x.\end{aligned}$$

Thus we may replace β_j by $\lambda\beta_j$ and, without loss of generality, assume $\beta_1^2 + \beta_2^2 + \beta_3^2 = 1$ (unless $\beta_1 = \beta_2 = \beta_3 = 0$, but then $\Lambda = \text{p.v.} \int_{\mathbb{R}} \frac{\mathrm{d}t}{t} = 0$). Therefore $\beta = (\beta_1, \beta_2, \beta_3)$ is a unit vector perpendicular to the vector $(1, 1, 1)$. We are down to a 1-parameter family. Moreover,

$$\Lambda_\beta(f_1, f_2, f_3) = \Lambda_{-\beta}(f_1, f_2, f_3),$$

so the parameter belongs to a projective line.

We can't get rid of this parameter dependence. In fact, consider the degenerate cases, i.e. when $\beta_i = \beta_j$ for some $i \neq j$, e.g. $\beta_1 = \beta_2$. The changes of variables described above allow us to assume $\beta_1 = \beta_2 = 0$, $\beta_3 = 1$. Therefore we get

$$\Lambda_{(0,0,1)}(f_1, f_2, f_3) = \text{p.v.} \iint f_1(x) f_2(x) f_3(x - t) \frac{\mathrm{d}t}{t} \mathrm{d}x = \int f_1 f_2 H f_3,$$

where Hf_3 is the Hilbert transform of f_3. In particular we have the bound

$$\Lambda_{(0,0,1)}(f_1, f_2, f_3) = \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3},$$

where $1 < p_1, p_2, p_3 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. The same bound in the non degenerate case can't be proven through a similar simple argument. This should tell us that we can't recover the non degenerate case from the degenerate one.

We look at the symmetries of the trilinear form:

- **Translations.** For $y \in \mathbb{R}$, $T_y f(x) = f(x - y)$. Then

$$\Lambda_\beta(T_y f_1, T_y f_2, T_y f_3) = \Lambda_\beta(f_1, f_2, f_3);$$

- **Dilations.** For $\lambda > 0$, $D_\lambda f(x) = f\left(\frac{x}{\lambda}\right)$. Then

$$\Lambda_\beta(D_\lambda f_1, D_\lambda f_2, D_\lambda f_3) = \lambda \Lambda_\beta(f_1, f_2, f_3);$$

- **Modulations.** For $\eta \in \mathbb{R}$, $M_\eta f(x) = e^{2\pi i \eta x} f(x)$. Then, for $\alpha \in \mathbb{R}^3$,

$$\begin{aligned}
&\Lambda_\beta(M_{\alpha_1\eta} f_1, M_{\alpha_2\eta} f_2, M_{\alpha_3\eta} f_3) = \\
&\quad = \text{p.v.} \iint f_1(x-\beta_1 t) f_2(x-\beta_2 t) f_3(x-\beta_3 t) \\
&\qquad\qquad \underbrace{e^{2\pi i \alpha_1 \eta(x-\beta_1 t) + 2\pi i \alpha_2 \eta(x-\beta_2 t) + 2\pi i \alpha_3 \eta(x-\beta_3 t)}}_{=1,\ \text{if } \alpha \perp (1,1,1),\ \alpha \perp \beta} \frac{\mathrm{d}t}{t} \mathrm{d}x = \\
&\quad = \Lambda_\beta(f_1, f_2, f_3).
\end{aligned}$$

We can define the Hilbert transform of the function f in terms of an integral of $\widehat{f}$ in the following way

$$\text{p.v.} \int f(x-t) \frac{\mathrm{d}t}{t} = c \int \widehat{f}(\eta) \operatorname{sgn}(\eta) \mathrm{d}\eta.$$

What is the analogous for the Bilinear Hilbert Transform? If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an odd Schwartz function such that $\int_0^\infty \varphi(s)\mathrm{d}s = 1$, then

$$\int_0^\infty \varphi(ts)\mathrm{d}s = \frac{1}{t} \int_0^\infty \varphi(u)\mathrm{d}u = \frac{1}{t}.$$

By substituting this equality inside the trilinear form we obtain

$$\Lambda_\beta(f,g,h) = \int_0^\infty \left[\iint f(x-\beta_1 t) g(x-\beta_2 t) h(x-\beta_3 t) \varphi(st) \mathrm{d}t \mathrm{d}x \right] \mathrm{d}s.$$

We want to express the integral in terms of an integral of the Fourier transform of

$$F(y_1, y_2, y_3, y_4) = f(y_1) g(y_2) h(y_3) \varphi(y_4),$$

Let

$$\Gamma = \operatorname{span}\{(1,1,1,0), (-\beta_1, -\beta_2, -\beta_3, s)\},$$

where the vectors are orthogonal and have length $\sqrt{3}$ and $\sqrt{1+s^2}$. We can continue the chain of equality above

$$= c \int_0^\infty \frac{1}{\sqrt{1+s^2}} \left(\iint_\Gamma f\ g\ h\ \varphi \right) \mathrm{d}s = c \int_0^\infty \frac{1}{\sqrt{1+s^2}} \left(\iint_{\Gamma^\perp} \widehat{f}\ \widehat{g}\ \widehat{h}\ \widehat{\varphi} \right) \mathrm{d}s.$$

In the last equality we used the following result

Claim 26.1. *Integrating in $F\colon \mathbb{R}^n \to \mathbb{R}$ over a subspace Γ is equivalent to integrating $\widehat{F}\colon \mathbb{R}^n \to \mathbb{R}$ over $\Gamma^\perp$.*

Proof.

$$\widehat{F}(\xi) = \int_{\mathbb{R}^n} F(x) e^{-2\pi i x\cdot\xi} \mathrm{d}x \qquad\qquad \widehat{F}(0) = \int_{\mathbb{R}^n} F(x)\mathrm{d}x$$
$$F(x) = \int_{\mathbb{R}^n} \widehat{F}(\xi) e^{2\pi i x\cdot\xi} \mathrm{d}\xi \qquad\qquad F(0) = \int_{\mathbb{R}^n} \widehat{F}(\xi)\mathrm{d}\xi.$$

These are already a first instance of the claim with $\Gamma = \mathbb{R}^n$ and $\Gamma = \{0\}$. For a general subspace Γ, we can assume without loss of generality that Γ is spanned by $x_1, \ldots, x_k$, and therefore $\Gamma^\perp$ is spanned by $x_{k+1}, \ldots, x_n$. Then

$$\int F(x_1, \ldots, x_k, 0, \ldots, 0)\mathrm{d}x_1 \ldots \mathrm{d}x_n = \widehat{F}^{1,\ldots,k}(0, \ldots, 0) =$$
$$= \int \widehat{F}(0, \ldots, 0, \xi_{k+1}, \ldots, \xi_n)\mathrm{d}\xi_{k+1} \ldots \mathrm{d}\xi_n,$$

where $\widehat{F}^{1,\ldots,k}$ is the Fourier transform only with respect to the first k coordinates. □

In our case, we have

$$\Gamma^\perp = \operatorname{span}\left\{(\alpha_1, \alpha_2, \alpha_3, 0), \left(\beta_1, \beta_2, \beta_3, \frac{1}{s}\right)\right\},$$

where $\alpha \perp \beta$, $\alpha \perp (1,1,1)$, $\|\alpha\| = 1$. The two vectors are orthogonal to each other and of length 1 and $\sqrt{1 + \frac{1}{s^2}}$. We can continue the chain of equality above

$$= c\int_0^\infty \frac{\sqrt{1+\frac{1}{s^2}}}{\sqrt{1+s^2}} \iint \widehat{f}(\alpha_1\xi + \beta_1\eta)\widehat{g}(\alpha_2\xi + \beta_2\eta)\widehat{h}(\alpha_3\xi + \beta_3\eta)\widehat{\varphi}\left(\frac{1}{s}\eta\right) \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}s.$$

But

$$\int_0^\infty \frac{1}{s}\widehat{\varphi}\left(\frac{1}{s}\eta\right) \mathrm{d}s = \operatorname{sgn}(\eta)\int_0^\infty \widehat{\varphi}(s|\eta|)\frac{\mathrm{d}s}{s} =$$
$$= \operatorname{sgn}(\eta)\int_0^\infty \widehat{\varphi}(s)\frac{\mathrm{d}s}{s} = \operatorname{sgn}(\eta)\text{const}.$$

Therefore we continue the chain of equalities above

$$= \text{const.} \iint \widehat{f}(\alpha_1\xi + \beta_1\eta)\widehat{g}(\alpha_2\xi + \beta_2\eta)\widehat{h}(\alpha_3\xi + \beta_3\eta)\operatorname{sgn}(\eta)\mathrm{d}\xi\mathrm{d}\eta =$$
$$= \text{const.} \iint_{\eta_1+\eta_2+\eta_3=0} \widehat{f}(\eta_1)\widehat{g}(\eta_2)\widehat{h}(\eta_3)\operatorname{sgn}(\eta\cdot\beta)\mathrm{d}\sigma.$$

In order to prove the wanted bound for the trilinear form we would like to use the Carleson embedding theorem we proved last time. We consider the embedding map into the upper 3-space defined, for $f\colon \mathbb{R} \to \mathbb{R}$, φ Schwartz function, by

$$F(y, \eta, \lambda) = \int f(x)\lambda^{-1}\varphi(\lambda^{-1}(y-x))e^{2\pi i\eta(y-x)}\mathrm{d}x.$$

In particular, we pick φ such that $\widehat{\varphi}$ has compact support contained in $[-10^{-1}, 10^{-1}]$ and it is nonnegative, and we consider

$$\widehat{\varphi}(\eta_1)\widehat{\varphi}(\eta_2)\widehat{\varphi}(\eta_3).$$

To recover the $\mathrm{sgn}(\eta \cdot \beta)$ we shift the support of the functions $\widehat{\varphi}(\eta_i)$ so that the centre is in β. In particular, for $\eta \in \mathbb{R}^3$ such that

$$\widehat{\varphi}(\eta_1 - \beta_1)\widehat{\varphi}(\eta_2 - \beta_2)\widehat{\varphi}(\eta_3 - \beta_3) > 0,$$

we have $\mathrm{sgn}(\eta \cdot \beta)$. To make value independent on the vector α we integrate the product with variables η_i translated by $s\alpha_i$, obtaining

$$\int \prod_{j=1}^{3} \widehat{\varphi}(\eta_j - \beta_j - s\alpha_j)\mathrm{d}s.$$

To make value independent on the dilations by factor λ we integrate the product with variables η_i dilated by a factor λ, obtaining

$$\int \prod_{j=1}^{3} \widehat{\varphi}(\lambda\eta_j - \beta_j - s\alpha_j)\mathrm{d}s\frac{\mathrm{d}\lambda}{\lambda} = c\,\mathrm{sgn}(\eta \cdot \beta),$$

where c is a constant.

As a consequence we can rewrite

$$\begin{aligned}
\widetilde{\Lambda}_\beta(f_1, f_2, f_3) &= \iint\limits_{\substack{\eta_1+\eta_2+\eta_3=0 \\ \mathrm{sgn}(\eta\cdot\beta)}} \prod_{j=1}^{3} \widehat{f}_j(\eta_j)\mathrm{sgn}(\eta \cdot \beta)\mathrm{d}\sigma = \\
&= \int_0^\infty \int_{\mathbb{R}} \iint\limits_{\eta_1+\eta_2+\eta_3=0} \prod_{j=1}^{3} \widehat{f}_j(\eta_j)\widehat{\varphi}(\lambda\eta_j - \beta_j - s\alpha_j)\mathrm{d}\sigma\mathrm{d}s\frac{\mathrm{d}\lambda}{\lambda} = \\
&\overset{\text{FT trick}}{=} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \Bigg[\int_{\mathbb{R}^3} \prod_{j=1}^{3} f_j(\eta_j)e^{-2\pi i y\eta_j} \\
&\qquad \widehat{\varphi}(\lambda\eta_j - \beta_j - s\alpha_j)\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3 \Bigg] \mathrm{d}y\mathrm{d}s\frac{\mathrm{d}\lambda}{\lambda} = \\
&= \cdots = \int_0^\infty \int_{\mathbb{R}^2} \prod_{j=1}^{3} F_j(y, \alpha_j s + \beta_j\lambda^{-1}, \lambda)\mathrm{d}s\mathrm{d}y\mathrm{d}\lambda.
\end{aligned}$$

Therefore we can prove the theorem

Theorem 26.2. *For* $2 < p_i < \infty$, $\sum \frac{1}{p_i} = 1$, *there exists* $C_{\beta,p}$ *such that*

$$\Lambda_\beta(f_1, f_2, f_3) \leq C_{\beta,p} \prod_{j=1}^{3} \|f_j\|_{p_j}.$$

by means of the Carleson embedding theorem. Note that a priori the constant depends on β, and the dependence make it blow up in a nonintegrable way near the degenerate cases. However, the bound is known to holds also in the degenerate case with a finite constant. This suggests that the estimate given by the theorem above is not optimal, in particular that an uniform bound with a constant independent on β can be proven. This has been done with the same conditions of the statement of the theorem.

The problem in the degenerate case is that, as in the proof of the boundedness of the Carleson operator, we need the translations of a tile to be disjoint. In the degenerate case this fails, with two translated copies overlapping. By taking a tile smaller inverse proportionally to the distance of these translations we can recover the necessary disjointness property. However, in this way the constant blows up morally like the inverse of the distance between these pieces, thus in a nonintegrable way near the degenerate cases.

We conclude the lecture describing an example of an application for the BHT bound, which is historically one of the starting point of the study of the bilinear Hilbert transform.

Consider the Cauchy integral over a Lipschitz curve $y \mapsto y + iA(y)$ given by

$$\int f(x) \frac{1}{y - x + i(A(y) - A(x))} \mathrm{d}x = \int f(x) \frac{1}{y-x} \left(\frac{1}{1 + i\frac{A(y)-A(x)}{y-x}} \right) \mathrm{d}x =$$

$$\overset{\text{Taylor}}{=} c \int f(x) \frac{1}{y-x} \frac{A(y) - A(x)}{y-x} \mathrm{d}x.$$

This is the so called *Calderon commutator* $\left[* \frac{1}{t^2}, A \right] f$. By expanding the last fraction to an integral we obtain

$$\frac{A(y) - A(x)}{y - x} = \int_0^1 A'(x + (y-x)\alpha) \mathrm{d}\alpha.$$

By substituting it in the integral above we get

$$\int_0^1 \left[\int f(x) \frac{1}{y-x} A'(x + (y-x)\alpha) \mathrm{d}x \right] \mathrm{d}\alpha,$$

where we recognize the bilinear Hilbert transform (in this case we artificially introduced the parameter α). Therefore the inner integral can be bounded by $C_{\alpha,p} \|f\|_p$. In order to conclude that the Cauchy integral over the Lipschitz curve is bounded by a norm of f, we need $C_{\alpha,p}$ to be integrable near $\alpha = 0$, which corresponds to the degenerate case for the trilinear form. A uniform bound for the bilinear Hilbert transform, i.e. if $C_{\alpha,p} = C_p$ was independent of α, would do the work, but even a weaker result is enough.

27 Uniform bounds for the BHT

2017-02-02

As anticipated in the previous lecture, there exist uniform bounds for the bilinear Hilbert transform

$$\int f(x-t)g(x-\beta t)\frac{\mathrm{d}t}{t},$$

where β is the one real parameter degree of freedom. To the BHT we associate the trilinear form

$$\iint f(x-t)g(x-\beta t)\frac{\mathrm{d}t}{t}h(x)\mathrm{d}x = \iiint_{\text{symmetry space}} \langle f, \_\rangle \langle g, \_\rangle \langle h, \_\rangle ,$$

where we used the Fourier transform trick and the wave packets $D_\lambda M_\eta T_y \varphi$ to rewrite the form as an integral over the symmetry space.

We define

$$\Pi_{I\times\omega} f = \sum_{\substack{I\times\omega' \\ |I||\omega'|=1 \\ \omega'\subset\omega}} |I| \langle f, w_{I\times\omega'}\rangle w_{I\times\omega'},$$

and

$$\Lambda_k(f,g,h) = \sum_{|I||\omega|=2} |I| \langle f, w_{I\times\omega_r}\rangle \left\langle \Pi_{I\times 2^k\omega_l} g \cdot \Pi_{I\times 2^k\omega_l} h, h_I \right\rangle ,$$

where h_I is the Haar function in the interval I, and $2^k[a,b) = [2^k a, 2^k b)$ for $k \in \mathbb{N}$.

Remark 27.1. In the definition of $\Pi_I f$ we can choose every decomposition of $I \times \omega$ into dyadic rectangles of area 1.

Consider $\Lambda_k(f,g,h)$ in some particular cases:

- $k = 0$, then $\Lambda_k = 0$;
- $k = 1$, then for $w_1 = w_{I\times(2\omega)_{ll}}$, $w_2 = w_{I\times(2\omega)_{lr}}$,

  $$\Pi_{I\times 2^k\omega_l} g = \langle g, w_1\rangle w_1 + \langle g, w_2\rangle w_2, \Pi_{I\times 2^k\omega_l} h \quad = \langle h, w_1\rangle w_1 + \langle h, w_2\rangle w_2.$$

  The only things that "survive" are the crossed product. What we obtain is similar to the quartile operator;

- $k = \infty$, then we obtain

$$\sum_I |I| \langle f, h_I \rangle \langle ghh_I, ghh_I \rangle ,$$

since for

We want to prove the following

Theorem 27.2. *For* $2 < p_i < \infty$, $\sum \frac{1}{p_i} = 1$, *there exists* C_p *independent of* k *such that*

$$\Lambda_k(f, g, h) \leq C_p \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}.$$

Proof. We use the following outer measure structure:

- $X = \mathcal{P} = \{\text{bitiles}\}$;
- $T = \{I \times \omega \in \mathcal{P} \colon I \subset I_T, \xi_T \in \omega_r\}$, the tree defined by I_T, ξ_T;
- $\mathcal{E} = \mathcal{T} = \{\text{trees}\}$;
- $\sigma(T) = |I_T|$;
- for a vector valued function $F \colon \mathcal{P} \to \mathbb{R}^n$

$$S(F)(T) = \sup_{P \in T} \|F(P)\|_2.$$

We need the following embedding theorem:

Theorem 27.3. *Let* $k \in \mathbb{N}$, $f \mapsto F$. *Define*

$$F(I \times \omega) = \Pi_{I \times 2^k \omega} f.$$

Then, for $2 < p < \infty$,

$$\|F\|_{\mathcal{L}^p(S)} \leq C_p \|f\|_p.$$

Proof. We interpolate between the cases:

$\underline{p = \infty.}$ $\|\Pi_{I \times 2^k \omega} f\|_2 \leq \|f \mathbb{1}_I\|_2 \leq |I|^{\frac{1}{2}} \|f\|_\infty$;

$\underline{p = 2.}$ We need a weak type 2 bound. Let $\lambda > 0$ and pick $P_1 = I_1 \times \omega_1$ such that

$$\|\Pi_{I_n \times 2^k \omega_n} f\|_2 > \lambda |I_1|^{\frac{1}{2}},$$

and I_1 is maximal. Now suppose to have defined $P_1, \ldots, P_n$, then pick P_{n+1} disjoint from them as above. In particular observe that

$$\sum_n \|\Pi_{I_n \times 2^k \omega_n} f\|_2^2 \leq \|f\|_2^2.$$

This ends the proof of the embedding theorem

□

To conclude the proof of the boundedness of Λ_k we need a modified version of the Hölder's inequality. □