Harmonic Analysis, Problem set 9

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Definition. The cone in the upper half-space with vertex $x \in \mathbb{R}^d$ is the set $\Gamma(x) := \{(y,t) \in \mathbb{R}^d \times (0,\infty), |x-y| < t\}$. For a function $G : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ we define

$$A_q G(x) := \Big(\int_{\Gamma(x)} |G(y,t)|^q \frac{\mathrm{d}t \mathrm{d}y}{t^{d+1}} \Big)^{1/q}, \quad A_\infty G(x) := \sup_{(y,t) \in \Gamma(x)} |G(y,t)|,$$

where the supremum is taken in the almost everywhere sense.

Problem 1. (a) Show that $\|A_q G\|_{L^q} \lesssim \|G\|_{L^q(S_q)}$, $0 < q < \infty$. Hint: reduce to the case $q = 1$.

(b) Show that $\|A_2 G\|_{L^{1,\infty}} \lesssim \|G\|_{L^{1,\infty}(S_2)}$. Hint: remove an exceptional set from $\mathbb{R}^d \times (0,\infty)$ and use part a on the remaining set.

Remark. Using the embeddings from the lectures and the Marcinkiewicz interpolation theorem this implies that the *square function*

$$Sf(x) := A_2(\Delta F)(x)$$

is a bounded operator on $L^p(\mathbb{R}^d)$ for $1 < p \leq 2$. In fact the square function is also bounded for $2 < p < \infty$, but the endpoint at $p = \infty$ is an estimate in the space BMO that probably will not appear in this course.

Problem 2. Let $F, G : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$. Show that

$$\int |F(x,t) G(x,t)| \mathrm{d}x \frac{\mathrm{d}t}{t} \leq C \|F\|_{L^\infty(S^1)} \|A_\infty G\|_{L^1}.$$

Hint: reduce to the case of G being a characteristic function and use Vitali's covering lemma.