Harmonic Analysis, Problem set 13

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Extra problem set.

WARNING: This problem set is intended as practice for the exam. It should not be handed in unless in need of extra points for the exam admission.

Problem 1 (Dini criterion for pointwise convergence of Fourier series)**.** Let $f : [0,1] \to \mathbb{C}$ be a function such that the function $g(x) := (f(x) + f(1-x))/(1 - e^{2\pi i x})$ is integrable on $[0,1]$. Show that

$$\sum_{n=-N}^{N} \hat{f}(n) \to 0 \text{ as } N \to \infty.$$

Problem 2 (An oscillatory Carleson theorem)**.** Let $f \in C_c^\infty(\mathbb{R})$ and define the *maximally modulated oscillatory singular integral*

$$C_3 f(x) := \sup_{N \in \mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} f(x-t) e^{iNt} e^{it^3} \frac{\mathrm{d}t}{t} \right|.$$

Prove that $C_3 : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < +\infty$. (Hint: Mimic the strategy of proof of Problem 2 in Problem Set 11. You will need to use bounds on a truncated Carleson's operator as a black box. Grafakos's book is a good reference for that).

Definition. The *BMO (for "bounded mean oscillation") norm* of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_{\mathrm{BMO}} := \sup_I \inf_{c \in \mathbb{R}} |I|^{-1} \int_I |f - c|,$$

where the supremum is taken over all subintervals of $\mathbb{R}$. The *dyadic BMO norm* is defined similarly with a supremum over dyadic intervals I.
The space of functions with finite BMO (resp. dyadic BMO) norm is denoted by BMO (resp. BMO_d)

Problem 3. (a) Show that $\|f\|_{\mathrm{BMO}} \le \|f\|_\infty$

(b) Show that the function $\log|x|$ is in BMO.

(c) Show that the function $1_{x>0} \log|x|$ is in BMO_d, but not in BMO.

(d) Show that

$$\|f\|_{\mathrm{BMO}} \le \sup_I |I|^{-1} \int_I |f - f_I| \le 2\|f\|_{\mathrm{BMO}}, \quad f_I = |I|^{-1} \int_I f.$$

(e) Prove the (dyadic) *John–Nirenberg inequality*: there exist constants $C, c > 0$ such that for every dyadic interval I

$$|I \cap \{f - f_I > \lambda\}| \le C \exp(-c\lambda/\|f\|_{\mathrm{BMO}_d})|I|, \quad 0 \le \lambda < \infty$$

(Hint: one can assume $\|f\|_{\mathrm{BMO}_d} = 1$ and $f_I = 0$, and it suffices to consider $\lambda = 10N$, $N \in \mathbb{N}$. Construct inductively a sequence of subsets $I_N \subset I$ starting with $I_0 = I$ as follows: each $I_N = \cup_i I_{N,i}$ will be a disjoint union of dyadic intervals. Given I_N define

$$I_{N+1} := \cup_i (I_{N,i} \cap \{M_d(f - f_{I_{N,i}}) > 5\}),$$

where M_d is the dyadic Hardy–Littlewood maximal function. By induction on N show:

(e.1) $f \le 10N$ on $I \setminus I_N$,
(e.2) $|I_N| \le \exp(-cN)|I|$,
(e.3) $|f_{I_{N,i}}| \le 10N$)

Problem 4. Recall that the *Fejér kernel* is given by

$$F_t(x) = \int_{-t}^{t} (1 - |\xi|/t) e^{2\pi i x \xi} \, \mathrm{d}\xi = \frac{\sin(\pi t x)^2}{\pi^2 t x^2}, \quad t > 0.$$

Show that for $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have $F_t * f \to f$ as $t \to \infty$ pointwise almost everywhere. (Hint: consider first Schwartz functions f and use the Hardy–Littlewood maximal inequality.)

Problem 5 (Maximal functions and counterexamples)**.** Let $f \in L^1_{loc}(\mathbb{R}^n)$ be a locally integrable function. Let Mf denote its Hardy–Littlewood maximal function given by

$$Mf(x) := \sup_{r>0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

(a) Show that for all $f \in L^1(\mathbb{R}^n)$, it holds that $Mf \notin L^1(\mathbb{R}^n)$.

(b) Show that there is $f \in L^1(\mathbb{R}^n)$ such that $Mf \notin L^1_{loc}(\mathbb{R}^n)$.

(c) Show that, if $|f| \cdot \log(e + |f|) \in L^1$, then it holds that $Mf \in L^1_{loc}(\mathbb{R}^n)$ with

$$\int_B Mf(y) \, \mathrm{d}y \leq 2m(B) + C \cdot \int_{\mathbb{R}^n} |f(y)| \log(e + |f(y)|) \, \mathrm{d}y,$$

where $B \subset \mathbb{R}^n$ is a ball and $C > 0$ a constant independent of f. (Hint: prove first that $m(\{x \in B \colon Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\{x \in B \colon |f(x)| > \lambda\}} |f(x)| \mathrm{d}x$ and use the layer cake representation.)

Problem 6 (Products and Paraproducts)**.** The purpose of this exercise is to establish a connection between products, paraproducts, Fourier analysis and outter measure theory. Let, for $f, g \in \mathcal{S}(\mathbb{R})$, the *paraproduct* $\mathcal{P}(f,g)$ be defined by

$$\mathcal{P}(f,g)(x) = \int_0^\infty f * \varphi_t(x) g * \psi_t(x) \frac{\mathrm{d}t}{t},$$

where $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\psi}(0) = 0$ and $t^{-1}\varphi(x/t) = \varphi_t(x)$.

(a) Given $f, g \in \mathcal{S}(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\int \varphi = 1$, prove that

$$f(x)g(x) = \int_0^\infty f * \psi_t(x) g * \varphi_t(x) \, \frac{\mathrm{d}t}{t} + \int_0^\infty f * \varphi_t(x) \, g * \psi_t(x) \, \frac{\mathrm{d}t}{t},$$

where $\psi(x) = -\partial_x(x\varphi(x))$ and thus $\widehat{\psi}(0) = 0$. (Hint: use the fact that $f(x)g(x) = \lim_{t \to 0} f * \varphi_t(x) g * \varphi_t(x)$ and the fundamental theorem of calculus in the last expression.)

(b) Let $h \in \mathcal{S}(\mathbb{R})$. Prove that

$$\langle \mathcal{P}(f,g), h \rangle = \int_0^{+\infty} \int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{h}(-\xi - \eta)\widehat{\varphi}(t\xi)\widehat{\psi}(t\eta) \, \mathrm{d}\xi \mathrm{d}\eta \, \frac{\mathrm{d}t}{t},$$

where $\widehat{\ }$ denotes the one-dimensional Fourier transform. (Hint: use Fourier inversion in f, g and Plancherel).

(c) Split $\varphi = \varphi_1 + \varphi_2$ a sum of Schwartz functions, where the support of φ_1 is contained in the unit interval and that of φ_2 in the annulus $\{y \colon 1/4 < |y| < 2\}$. Define Φ, Ψ to be two Schwartz functions on $\mathbb{R}$ such that $\widehat{\Psi}(0) = 0$ and $\widehat{\Psi} \equiv 1$ on $\{y \colon 1/2 < |y| < 4\}$, and $\widehat{\Phi} \equiv 1$ on $[-4, 4]$. Prove that the expression from item (b) above equals

$$\int_0^{+\infty} \int_{\mathbb{R}} f * (\varphi_1)_t(x) g * \psi_t(x) h * \Psi_t(x) \, \mathrm{d}x \, \frac{\mathrm{d}t}{t} + \int_0^{+\infty} \int_{\mathbb{R}} f * (\varphi_2)_t(x) g * \psi_t(x) g * \Phi_t(x) \, \mathrm{d}x \frac{\mathrm{d}t}{t}.$$

(d) Prove that each of the terms in (c) is bounded by $\lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}$, where $p_i \in (1, +\infty)$ and $1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$. (Hint: Use atomicity, outer-Hölder and the embeddings from the lecture).

Problem 7 (Oscillatory integrals and the Spherical measure)**.** Let σ_{n-1} denote the $(n-1)-$dimensional spherical measure throughout this exercise.

(a) Prove that its Fourier transform $\widehat{\sigma_{n-1}}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi x} \, \mathrm{d}\sigma_{n-1}(x)$ is a well-defined bounded function.

(b) Prove the explicit formula

$$\widehat{\sigma_{n-1}}(\xi) = 2\pi \frac{J_{\frac{n}{2}-1}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}-1}},$$

where we define the Bessel function of order ν to be $J_\nu(t) = \frac{(s/2)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 e^{ist}(1 - s^2)^{\nu - \frac{1}{2}} \, \mathrm{d}s$. (Hint: use spherical coordinates).

(c) Prove that $J_\nu(t) = C_\nu \cdot s^\nu \int_0^\pi e^{is\cos\theta}(\sin\theta)^{2\nu} \, \mathrm{d}\theta$.

(d) Prove that $|J_\nu(t)| = O(t^{-1/2})$ for $t \to +\infty$. (Hint: split smoothly the interval $[-\pi, \pi]$ depending on where $(\cos\theta)' = -\sin\theta = 0$ and use the van der Corput Lemma in each of them)

(e) Conclude that $|\widehat{\sigma_{n-1}}(\xi)| = O((1 + |\xi|)^{-\frac{n}{2} + \frac{1}{2}})$