

# Harmonic Analysis, Problem set 12

Mathematisches Institut  
Prof. Dr. Christoph Thiele  
João Pedro Ramos  
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**Definition.** We denote the embeddings introduced in the lecture by

$$Af(x, t) := \int_{\mathbb{R}^d} |f(z)| \frac{t}{(t + \|x - z\|)^{d+1}} dz,$$

$$Df(x, t) := \sup \left\{ \left| \int_{\mathbb{R}^d} f(z) \phi(z) dz \right| : \int \phi = 0, |\phi(z)| \leq \frac{t}{(t + \|x - z\|)^{d+1}}, \|\nabla \phi(z)\| \leq \frac{t}{(t + \|x - z\|)^{d+2}} \right\}.$$

**Problem 1.** (a) Show that if  $\|x - y\| \leq C \cdot t$ , then  $|Df(x, t)| \leq C' \cdot |Df(y, t)|$ , for some universal  $C' > 0$ .

(b) For  $f \in L^p(\mathbb{R}^d)$ ,  $2 < p < \infty$ , consider the square function

$$Sf(x) := \left( \int_0^\infty Df(x, t)^2 \frac{dt}{t} \right)^{1/2}.$$

Show that  $\|Sf\|_p \lesssim \|f\|_p$  by estimating  $\int (Sf)^2 g$ ,  $g \in L^{(p/2)'}(\mathbb{R}^d)$ , using the outer Hölder inequality and the embeddings from the lectures.

**Problem 2** (Calderón reproducing formula). Let  $\phi, \psi$  be functions on  $\mathbb{R}^d$  with  $\int \phi = \int \psi = 0$ ,  $|\phi(x)|, |\psi(x)| \lesssim (1 + \|x\|)^{-d-1}$ ,  $|\nabla \phi(x)|, |\nabla \psi(x)| \lesssim (1 + \|x\|)^{-d-2}$ .

(a) Show that the operators

$$T_R f := \int_{1/R}^R \int_{\mathbb{R}^d} (f * \psi_t)(y) \phi_t(\cdot - y) dy \frac{dt}{t},$$

where  $\phi_t(x) = t^{-d} \phi(x/t)$  and analogously for  $\psi$ , are bounded on  $L^2(\mathbb{R}^d)$  uniformly in  $R > 1$ . (Hint: use duality and the  $L^2$  almost orthogonality result from the lecture)

(b) Show that  $\lim_{R, R' \rightarrow \infty} \|T_R f - T_{R'} f\|_2 = 0$ .

(c) Show that the limit  $Tf := \lim_{R \rightarrow \infty} T_R f$  exists and defines a bounded linear operator on  $L^2(\mathbb{R}^d)$ .

(d) Assuming that  $\int_0^\infty \hat{\phi}(t\xi) \hat{\psi}(t\xi) \frac{dt}{t} = 1$  for all  $\xi \in \mathbb{R}^d \setminus \{0\}$  show that  $Tf = f$ .