Harmonic Analysis, Problem set 11

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Problem 1 (Small set maximal function)**.** For this problem, let $E \subset [-1,1]$ be an arbitrary measurable set, and denote by $(\chi_E)_a(x) = \frac{1}{a}\chi_E\left(\frac{x}{a}\right)$. We define the small set maximal function of $f \in L^1_{loc}(\mathbb{R})$ as

$$\mathcal{M}_\varepsilon(f)(x) := \sup_{a>0,\, m(E)\le\varepsilon} |f| * (\chi_E)_a(x).$$

(a) Denoting by M the standard maximal function, prove that

$$\{x \in \mathbb{R} \colon \mathcal{M}_\varepsilon(f)(x) > \alpha\} \subset \{x \in \mathbb{R} \colon M(f_1)(x) > \alpha/8\},$$

where $f_1 = f\chi_{\{|f|>\alpha/(2\varepsilon)\}}$. (Hint: prove that $\mathcal{M}_\varepsilon(1) \le \varepsilon$ and $\mathcal{M}_\varepsilon(f) \le 4 \cdot Mf$ pointwise.)

(b) Conclude that, for each $p \in (1,+\infty)$, there is $c_p > 0$ such that

$$\|\mathcal{M}_\varepsilon(f)\|_p \le c_p \cdot \varepsilon^{1-\frac{1}{p}}\|f\|_p.$$

Problem 2 (Oscillatory singular integrals)**.** For this problem, let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported in $[1/2, 2]$ and $f \in C_c^\infty(\mathbb{R})$.

(a) Prove that there is $C > 0$ and $\theta > 0$ such that, for all $\xi \in \mathbb{R}$, $j > 0$,

$$\left| \int_{\mathbb{R}} e^{i\cdot 2^{3j}(\eta^3-(\eta-\xi)^3)} \cdot \frac{\psi(\eta)}{\eta}\frac{\psi(\eta-\xi)}{\eta-\xi}\,\mathrm{d}\eta \right|$$

$$\le C \cdot \left(\chi_{[-2^{-\theta j},2^{-\theta j}]}(\xi) + 2^{-\theta j}\chi_{[-4,4]}(\xi)\right).$$

(b) Let $T_jf(x) = \int_{\mathbb{R}} f(x-t)e^{it^3}\psi(2^{-j}t)\,\frac{\mathrm{d}t}{t}$ for $j > 0$. Prove that $\|T_jf\|_2 \le 2^{-\frac{\theta}{10}j}\|f\|_2$. (Hint: Use the TT^* method from the lecture and Problem 1(c)).

(c) Conclude that the oscillatory singular integral

$$Tf(x) = \int_{\mathbb{R}} f(x-t)e^{it^3}\,\frac{\mathrm{d}t}{t}$$

is bounded in $L^p(\mathbb{R})$, $1 < p < \infty$.