Harmonic Analysis, Problem set 10

Mathematisches Institut
Prof. Dr. Christoph Thiele
João Pedro Ramos
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Problem 1 (Maximal operators and pointwise convergence). (a) Let $\mathcal{C}f(x) = \sup_N |H(e^{2\pi i N(\cdot)} f)(x)|$ denote the Carleson operator as discussed in the lecture. Prove that, if there is $A > 0$ such that

$$\sup_{\lambda>0} \lambda(m(\{\mathcal{C}f > \lambda\}))^{1/2} =: \|\mathcal{C}f\|_{L^{2,\infty}(\mathbb{R})} \le A\|f\|_2,$$

for all $f \in C_c^\infty(\mathbb{R})$, then the partial fourier integrals $S_N f(x) := \int_{-N}^N \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, \mathrm{d}\xi \to f(x)$ almost everywhere, for each $f \in L^2(\mathbb{R})$. (Hint: Analyze the measure of $E_\varepsilon := \{x \in \mathbb{R} \colon \limsup_{N\to\infty} S_N f(x) - \liminf_{N\to\infty} S_N f(x) > \varepsilon\}$; use that the assertion holds already for each $g \in C_c^\infty(\mathbb{R})$.)

(b) Let $Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, \mathrm{d}t$ denote the Hardy–Littlewood maximal function. Prove that $\|Mf\|_{L^p(\mathbb{R})} \le C_p \|f\|_p$, for all $1 < p \le +\infty$.

(c) Prove that, for each $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$,

$$\frac{1}{2t} \int_{x-t}^{x+t} |f(x) - f(y)|\, \mathrm{d}y \to 0$$

as $t \to 0$ for almost every $x \in \mathbb{R}$.

Problem 2 (Dirichlet integral and Fourier series). (a) Prove that the improper integral defined as

$$\int_0^\infty \frac{\sin t}{t}\, \mathrm{d}t := \lim_{s\to\infty} \int_0^s \frac{\sin t}{t}\, \mathrm{d}t$$

converges. (Hint: prove that there is $C > 0$ such that $\left|\int_{n\pi}^{(n+2)\pi} \frac{\sin t}{t}\, \mathrm{d}t\right| \le \frac{C}{n^2}$ for all $n \ge 1$.)

(b) Define the *Dirichlet kernel* as $D_N(x) := \sum_{k=-N}^N e^{2\pi i k x}$. Verify that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$ and that

$$\int_{-1/2}^0 D_N(x)\, \mathrm{d}x = \int_0^{1/2} D_N(x)\, \mathrm{d}x = \frac{1}{2},$$

as well as the fact that there is $C > 0$ so that for each $\alpha, \beta \in [-1/2, 1/2]$, $\left|\int_\alpha^\beta D_N(x)\, \mathrm{d}x\right| \le C$.

(c) Let $f \in L^2(\mathbb{T})$ be differentiable when restricted to the interval $(-1/2, 1/2)$ with a bounded derivative. Prove that $f = g - h$, where both g, h are non-decreasing differentiable functions in $(-1/2, 1/2)$.

(d) Prove that, under the hypotheses above, the (periodic) convolution

$$f * D_N(x) = \int_{-1/2}^{1/2} f(x-t) D_N(t)\, \mathrm{d}t \to f(x),\ \forall\, x \in (-1/2, 1/2).$$

(Hint: reduce to non-decreasing functions; split the integral into two parts and use the Riemann-Lebesgue lemma for Fourier series.)

(e) Considering the periodic continuation of the function $f(z) = \chi_{(-1/2,1/2)}(z) \cdot \frac{\sin(\pi z)}{2\pi z}$, prove that the Dirichlet integral

$$\int_0^\infty \frac{\sin t}{t}\, \mathrm{d}t = \frac{\pi}{2}.$$