Harmonic Analysis, Problem set 8

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Due on Thursday, 30-05-2019.

Problem 1 (Fourier transform). (a) Let $f \in L^1(\mathbb{R})$ be such that $\widehat{f}$ is an odd function. Prove that

$$\left| \int_1^a \frac{\widehat{f}(x)}{x} \, \mathrm{d}x \right| \leq C,$$

for some $C > 0$, independently of $a > 1$.

(b) Find an odd $g \in C_0(\mathbb{R})$ such that

$$\left| \int_1^a \frac{g(x)}{x} \, \mathrm{d}x \right|$$

is unbounded as $a \to \infty$.

(c) Conclude that $\mathcal{F}(L^1) \subsetneq C_0(\mathbb{R})$.

Problem 2 (Hilbert transform). It has been sketched in the lecture that

$$\int_{-\infty}^{+\infty} (f + iHf)^k = 0$$

for all smooth compactly supported real-valued functions f and integers $k \geq 2$, where H denotes the Hilbert transform (in particular Hf is also real-valued).

(a) Let $k \geq 2$ be an even integer. Show that $\|Hf\|_{L^k} \leq C_k \|f\|_{L^k}$ for some C_k that does not depend on f. Hint: expand the real part of the above identity and use Hölder's inequality to separate L^k norms of f and Hf.

(b) Conclude that $\|Hf\|_{L^p} \leq C_p \|f\|_{L^p}$, whenever $2 \leq p < \infty$.

(c) Let $\tilde{f}(x) = f(-x)$. Show that for all smooth compactly supported functions $f, g \in C_0^\infty(\mathbb{R})$ we have $\int (Hf)g = \int (H\tilde{g})\tilde{f}$. Use duality between L^p and $L^{p'}$, $1/p + 1/p' + 1$, to show that $\|Hf\|_{L^p} \leq C_p \|f\|_{L^p}$, $1 < p < 2$.