Harmonic Analysis, Problem set 7

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Problem 1 (On the Hausdorff-Young inequality). (a) Suppose that T is a *linear* operator from $L^{p_1}(X,\mu) + L^{p_1}(X,\mu)$ to $\nu-$measurable functions, so that

$$\|Tf\|_{L^{q_1}(Y,\nu)} \le \|f\|_{L^{p_1}(X,\mu)}, \; \|Tf\|_{L^{q_2}(Y,\nu)} \le \|f\|_{L^{p_2}(X,\mu)},$$

with $p_i, q_i \in [1,+\infty]$, $p_i \le q_i, i = 1,2$. Then, for $\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}$, $\frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_2}$, $t \in [0,1]$, we have

$$\|Tf\|_{L^q(Y,\nu)} \le \|f\|_{L^p(X,\mu)}.$$

(Hint: you are allowed to use here the generalized version of the Marcinkiewicz interpolation theorem without proving it to get the result with some constant, then consider tensor powers $T^{\otimes^m}$ of T to reduce it to 1)

(b) Prove the Hausdorff-Young inequality: for $1 \le p \le 2$ and $f \in L^p(\mathbb{R})$,

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}.$$

Here, $\frac{1}{p} + \frac{1}{p'} = 1$.

(c) Prove that a *necessary* condition for the inequality

$$\|\widehat{f}\|_{L^q(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}$$

to hold is that $\frac{1}{p} + \frac{1}{q} = 1$. By considering the functions $g_a(x) := e^{-\pi a|x|^2}$, with $a = 1 + it$, $t \in \mathbb{R}$, prove that $p \le 2$.

Problem 2 (Summation formulas and extremal problems). (a) Suppose that the function $f \in L^1(\mathbb{R})$ is such that there are $C, \varepsilon > 0$so that $|f(x)| + |\widehat{f}(x)| \le C(1+|x|)^{-(1+\varepsilon)}$, for all $x \in \mathbb{R}$. Prove that the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi i m x}$$

holds pointwise.

(b) (One delta problem)** For the following problem, you are allowed to use the following result as a black box:

Theorem 1 (Vaaler Interpolation). Let $f \in L^2(\mathbb{R})$ be a function such that $\text{supp}(f) \subseteq [-1,1]$, and let also $F := \widehat{f}$ denote its Fourier transform. We have that

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left(\sum_{k\in\mathbb{Z}} \frac{F(k)}{(z-k)^2} + \sum_{j\in\mathbb{Z}} \frac{F'(j)}{z-j} \right),$$

where the right hand side converges uniformly on compact sets of $\mathbb{C}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative, integrable function on $\mathbb{R}$ such that $f(0) \ge 1$. Suppose that $\text{supp}(\widehat{f}) \subset [-1,1]$. Find, with proof, the minimal value of

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Exhibit a function that attains the minimum and prove that this function is, in fact, *unique*. (Hint: such an extremal f has to vanish at k, $\forall k \in \mathbb{Z}\backslash\{0\}$.)