Harmonic Analysis, Problem set 6

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The first problem establishes equivalence between classical and outer L^p spaces. In particular, this shows that the Marcinkiewicz interpolation theorem can be used with any combinations of outer and classical L^p spaces. We denote

$$\mu\{f > \lambda\} = \mu(\{x : f(x) > \lambda\}), \quad \mu(Sf > \lambda) = \inf_{\mathbf{E}\subset\mathcal{E}:\sup_{E'\in\mathcal{E}} S(f1_{(\cup\mathbf{E})^c})(E')\le\lambda} \sum_{E\in\mathbf{E}} \sigma(E).$$

(in the lecture notes the latter quantity is sometimes denoted by $\mu(\{Sf > \lambda\})$, but this is not intended and will hopefully be corrected soon).

Problem 1. Let $(X, \mathcal{E}, \sigma)$ be an outer measure space. It is known that the class $\mathcal{X}$ of Carathéodory measurable sets is a σ-algebra and the outer measure μ is σ-additive on $\mathcal{X}$. Assume that μ is also σ-finite (so that Fubini's theorem applies) and that $\mathcal{E} \subset \mathcal{X}$. Let $\mathcal{B}$ denote the set of $\mathcal{X}$-measurable functions from X to $\mathbb{R}_+$.

(a) For $f \in \mathcal{B}$ show that

$$\int_X f(x)^p \mathrm{d}\mu(x) = p\int_0^\infty \lambda^{p-1}\mu\{f > \lambda\}\mathrm{d}\lambda.$$

Hint: write $f(x)^p = \int_0^{f(x)} p\lambda^{p-1}\mathrm{d}\lambda$.

(b) Consider the size

$$S_\infty f(E) := \inf_{A:\mu(A)=0} \sup_{E\setminus A} f.$$

Show that $\mu\{f > \lambda\} = \mu(S_\infty f > \lambda)$ for all $f \in \mathcal{B}$ and $0 < \lambda < \infty$.

(c) Suppose additionally that $\sigma(E) < \infty$ for all $E \in \mathcal{E}$ and consider the size

$$S_1 f(E) := \sigma(E)^{-1}\int_E f\mathrm{d}\mu.$$

Show that $\mu\{f > \lambda\} = \mu(S_1 f > \lambda)$ for all $f \in \mathcal{B}$ and $0 < \lambda < \infty$.

Problem 2 (Hardy–Littlewood maximal inequality)**.** For this problem, let $Mf(x) := \sup_{I\ni x}\frac{1}{|I|}\int_I |f(t)|\,\mathrm{d}t$ denote the one-dimensional Hardy–Littlewood maximal function, and $M_r f(x) := \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(t)|\,\mathrm{d}t$ its (right) one-sided version.

(a) (Rising sun Lemma) Let $[a, b]$ be a compact interval on the real line, and $F : [a, b] \to \mathbb{R}$ a continuous function. Then there is a countable family of disjoint, non-empty intervals $I_k = (a_k, b_k)$ contained in $[a, b]$ so that (i) For each k, either $F(a_k) = F(b_k)$ or $a_k = a$ and $F(b_k) \ge F(a_k)$; (ii) If $x \in (a, b]$ does not lie in any of the I_k, then $F(y) \le F(x)$ for all $x \le y \le b$.

(b) (One-sided maximal inequality) Let m^* denote the Lebesgue outer measure on the real line and $f \in L^1(\mathbb{R})$. Prove that

$$m^*\left(\{x \in \mathbb{R}\colon M_r f(x) > \lambda\}\right) \le \frac{1}{\lambda}\int_{\mathbb{R}}|f(t)|\,\mathrm{d}t.$$

(Hint: use the rising sun Lemma for $F_a(x) := \int_{[a,x]}|f(t)|\,\mathrm{d}t - (x-a)\lambda$.)

(c) Conclude that

$$m^*\left(\{x \in \mathbb{R}\colon Mf(x) > \lambda\}\right) \le \frac{2}{\lambda}\int_{\mathbb{R}}|f(t)|\,\mathrm{d}t.$$

Show that 2 cannot be replaced by any smaller constant above by exhibiting $f \in L^1(\mathbb{R})$ so that

$$\sup_{\lambda>0}\lambda m^*(\{Mf > \lambda\}) = 2\int_{\mathbb{R}}|f|.$$