Harmonic Analysis, Problem set 5

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Problem 1 (Quasinormed spaces). Let X be a quasinormed vector space, that is, $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$, $x \in X$, and $\|x+y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$ and some $C \geq 1$.

(a) Let $x_1, \ldots, x_{2^n} \in X$. Show that

$$\|\sum_{i=1}^{2^n} x_i\| \leq C^n \sum_{i=1}^{2^n} \|x_i\|.$$

(b) Let $x_1, \ldots, x_n \in X$. Show that

$$\|\sum_{i=1}^{n} x_i\| \leq \sum_{i=1}^{n} C^i \|x_i\|.$$

(c) Let $(x_i)_{i=1}^{\infty} \subset X$ be a sequence with $\|x_i\| \leq a^{-i}$ for some $a > 1$. Show that the sequence of partial sums

$$\sum_{i=1}^{n} x_i$$

is Cauchy.

Problem 2 (Indicator functions of sparse collections are Carleson measures). Consider the outer measure space $X = \mathcal{D}$ with $\mathcal{E}$ consisting of the trees $T_I = \{J \subseteq I\}$ and the outer measure generated by $\sigma(T_I) = |I|$. Let the size S_1 be defined by

$$S_1(F)(T_I) := |I|^{-1} \sum_{J \subseteq I} |J| F(J).$$

Recall that a collection of dyadic intervals $\mathcal{S} \subset \mathcal{D}$ is called *η-sparse* if for every $I \in \mathcal{S}$ we have

$$|\cup \{J \in \mathcal{S}, J \subsetneq I\}| \leq (1-\eta)|I|. \tag{1}$$

Let $\mathcal{S} \subset \mathcal{D}$ be an η-sparse collection. Show that $\|1_{\mathcal{S}}\|_{L^\infty(S_1)} \leq 1/\eta$. Hint: the relative complements of the sets (1) inside I are pairwise disjoint.