Harmonic Analysis, Problem set 3

Mathematisches Institut
Prof. Dr. Christoph Thiele
João Pedro Ramos
Summer term 2019

UNIVERSITÄT BONN

Due on Thursday, 25-04-2019

Problem 1. (a) (Special case of monotone convergence theorem) Let $\mathcal{I}_i$, $i \in \mathbb{N}$, be collections of disjoint dyadic intervals and write $U_i = \cup \mathcal{I}_i$. Assume that $U_i \supseteq U_{i+1}$ for all i and $S := \inf_i \sum_{I \in \mathcal{I}_i} |I| < \infty$. Show that the set $\cap_i U_i$ has outer measure at least S, that is, it cannot be covered by a collection of disjoint dyadic intervals $\mathcal{I}$ such that $\sum_{I \in \mathcal{I}} |I| < S$.

(b) (A version of Egorov's theorem) Let $F : \mathcal{D} \to \mathbb{R}$ be a function that (for simplicity) vanishes on dyadic intervals not contained in $[0,1]$. Assume that $\lim_{k \to -\infty} F(I_{k,x})$ exists almost everywhere. Show that for every $\epsilon > 0$ there exists a collection of dyadic intervals $\mathcal{I}_\epsilon$ and an integer k_ϵ such that $\cup_{I \in \mathcal{I}} |I| < \epsilon$ and for all $k, k' < k_\epsilon$ and all $x \notin \cup \mathcal{I}_\epsilon$ we have $|F(I_{k,x}) - F(I_{k',x})| < \epsilon$.

Hint: consider

$$\mathcal{I}_i = \{\text{maximal } J : \exists J \subset J', |J'| \leq 2^{-i}, |F(J) - F(J')| \geq \epsilon\}$$

Problem 2. A collection of dyadic intervals $\mathcal{S} \subset \mathcal{D}$ is called *η-sparse* if for every $I \in \mathcal{S}$ we have

$$|\cup \{J \in \mathcal{S}, J \subsetneq I\}| \leq (1 - \eta)|I|.$$

Let F be a dyadic martingale. Show that the collection $\mathcal{S} := \cup_{k \in \mathbb{Z}} \mathcal{I}_{4^k, F}$ is $\frac{1}{2}$-sparse. Here, we define

$$\mathcal{I}_{4^k, F} := \{I,\, F(I) > 4^k\}.$$