Harmonic Analysis, Problem set 2

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The set of harmonic functions on the upper half-plane from the lectures 1 and 2 is denoted by $\tilde{\mathcal{M}}$.

Problem 1. Let $\lambda : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be an non-decreasing, bounded function, so that

$$\lim_{x \to -\infty} \lambda(x) = 0.$$

Let $F \in \tilde{\mathcal{M}}$ be its harmonic extension to the upper half-plane. Show that $\lim_{t \to 0} F(x,t) = (\lambda_l(x) + \lambda_r(x))/2$ for every $x \in \mathbb{R}$, where $\lambda_r(y)$ and $\lambda_l(y)$ denote the lateral limits $\lim_{x \searrow y} \lambda(x)$ and $\lim_{x \nearrow y} \lambda(x)$, respectively.

Problem 2. Define *adjacent systems of dyadic intervals* by

$$\mathcal{D}^\alpha = \{2^{-k}([0,1) + m + (-1)^k \alpha/3), m, k \in \mathbb{Z}\},$$

where $\alpha = 0, 1, 2$. Not that $\mathcal{D}^0$ is the usual system of dyadic intervals.

(a) Show that each $\mathcal{D}^\alpha$ is nested in the sense that for $I, J \in \mathcal{D}^\alpha$ we have $I \cap J \in \{I, J, \emptyset\}$.

(b) Show that for every interval $I = [a,b] \subset \mathbb{R}$ there exists $\alpha \in \{0,1,2\}$ and $J \in \mathcal{D}^\alpha$ such that $I \subset J$ and $|J| \leq 4|I|$.

(c) Let $f \in C_0(\mathbb{R})$. The (continuous) Hardy–Littlewood maximal function is defined by

$$Mf(x) := \sup_{x \in I} |I|^{-1} \int_I f,$$

where the supremum is taken over all intervals containing x. Show that $Mf < \infty$ almost everywhere.

(d) Let $F \in \tilde{\mathcal{M}}$ and $MF(x) := \sup_{t>0} F(x,t)$. Show that $MF < \infty$ almost everywhere.