Harmonic Analysis, Problem set 1

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Winter term 2016/17

Due on Thursday, 2016-10-27

Problem 1 (Approximations to the identity)**.** Let μ be a positive Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \mathrm{d}\mu = 1$. Define the measures μ_ϵ, $\epsilon > 0$, by

$$\int f \mathrm{d}\mu_\epsilon := \int f(\epsilon x) \mathrm{d}\mu(x).$$

Recall that the convolution of a function with a measure is defined by $\mu * f(x) = \int_{\mathbb{R}} f(x-y)\mathrm{d}\mu(y)$.

(a) Show that $\mu_\epsilon * f \to f$ pointwise as $\epsilon \to 0$ for every bounded Lipschitz function f.

(b) Assume in addition that $\int_{\mathbb{R}} |x| \mathrm{d}\mu(x) < \infty$. Show that $\mu_\epsilon * f \to f$ pointwise as $\epsilon \to 0$ for every (not necessarily bounded) Lipschitz function f.

(c) For a function $\phi : \mathbb{R} \to \mathbb{R}$ show that $(\phi \mathrm{d}\lambda)_\epsilon = \phi_\epsilon \mathrm{d}\lambda$ for $\epsilon > 0$, where $\phi_\epsilon(x) := \epsilon^{-1}\phi(x/\epsilon)$ and λ is the Lebesgue measure on $\mathbb{R}$.

Problem 2 (Heat semigroup)**.** Let $\phi(x) = (4\pi)^{-1/2} \exp(-x^2/4)$.

(a) Show that the function $\Phi(x,t) = \phi_{\sqrt{t}}(x)$ solves the heat equation $\partial_t F = \partial_x^2 F$ on $\mathbb{R} \times (0, \infty)$.

(b) Let f be a bounded Lipschitz function. Recall that the convolution of two functions is defined by $\phi * f(x) = \int_{\mathbb{R}} \phi(y) f(x-y)\mathrm{d}y$. Show that the function

$$F(x,t) := \begin{cases} \phi_{\sqrt{t}} * f(x), & t > 0, \\ f(x), & t = 0 \end{cases}$$

is continuous on $\mathbb{R} \times [0, \infty)$ and solves the heat equation on $\mathbb{R} \times (0, \infty)$.