There are several treatments of weighted theory in book form: [GCF85] [Duo01] [CUMP11] [LN15]. Our focus is on the basics. Some parts of these notes are stubs, in these cases references or sufficiently common names of results are provided to make the full statements locatable.

1 Fefferman–Stein inequality

Definition 1.1 (Hardy–Littlewood maximal operator)**.**

Theorem 1.2 (Marcinkiewicz interpolation)**.**

Remark (Layer cake formula).

Definition 1.3 (Adjacent dyadic grids)**.**

Theorem 1.4 ([FS71])**.**

$$\sup_{\lambda>0} \lambda w\{Mf > \lambda\} \lesssim \int |f| Mw.$$

2 The A_p condition

Definition 2.1. A *dyadic grid* $\mathscr{D}$ is a collections of measurable sets such that for all $Q, Q' \in \mathscr{D}$

$$Q \cap Q' \in \{\emptyset, Q, Q'\}.$$

We fix a measure space (X, μ) and a dyadic grid $\mathscr{D}$.

Definition 2.2. The *dyadic maximal operator* is given by

$$Mf(x) := \sup_{x \in Q \in \mathscr{D}} (|f|)_Q, \quad \text{where} \quad (f)_Q = |Q|^{-1} \int_Q f. \tag{2.3}$$

In this lecture we consider the dependence of the constant in weighted weak and strong type (p,p) estimates for the dyadic maximal operator on the weights for $1 < p < \infty$. We begin with the weak type estimates since in this case there is an explicit formula for arbitrary pairs of weights.

Definition 2.4. A *weight* is a non-negative measurable function.

In order to avoid measurability and summability issues we will assume throughout that the dyadic grid $\mathscr{D}$ is finite. Since none of the estimates will depend on the cardinality of $\mathscr{D}$ one can then pass to an infinite $\mathscr{D}$ using the monotone convergence theorem. For notational simplicity we also assume that all functions are positive.

Theorem 2.5 ([Muc72])**.** *Let $1 < p < \infty$ and let v, w be weights. Then*

$$\|M(fv)\|_{L^{p,\infty}(w)} \leq [v,w]^{(1/p',1/p)} \|f\|_{L^p(v)},$$

where

$$[v,w]^{(1/p',1/p)} = \sup_{Q \in \mathscr{D}} (v)_Q^{1/p'} (w)_Q^{1/p}$$

is a two-weight characteristic. *The above inequality is sharp in the sense that for a given pair v, w the constant cannot be improved.*

The above version of the square bracket notation for weight characteristics has been recently introduced in [LN15] and is not universally adopted yet. I personally find it very convenient.

Proof. In order to see that the constant is sharp consider $f = 1_Q$. Then

$$M(fv) \geq (1_Q v)_Q = (v)_Q \text{ on } Q,$$

so

$$\|M(fv)\|_{L^{p,\infty}(w)} \geq (v)_Q w(\{M(fv) \geq (v)_Q\})^{1/p} \geq (v)_Q w(Q)^{1/p} = (v)_Q^{1/p'} (w)_Q^{1/p} \|f\|_{L^p(v)}.$$

Taking a supremum over Q we see that the constant cannot be improved.

Now we prove the estimate. The set $\{M(fv) > \lambda\}$ is the union of the family $\mathscr{Q}$ of the maximal cubes $Q \in \mathscr{D}$ with $(fv)_Q > \lambda$. Notice that the members of $\mathscr{Q}$ are pairwise disjoint. Therefore

$$\begin{aligned}
w\{M(fv) > \lambda\} &= \sum_{Q \in \mathscr{Q}} w(Q) \\
&< \sum_{Q \in \mathscr{Q}} w(Q)\Big(\frac{(fv)_Q}{\lambda}\Big)^p \\
&\leq \lambda^{-p} \sum_{Q \in \mathscr{Q}} w(Q)(f^p v)_Q (v)_Q^{p/p'} && \text{by Hölder} \\
&= \lambda^{-p} \sum_{Q \in \mathscr{Q}} (w)_Q (v)_Q^{p/p'} \int_Q f^p v \\
&\leq \lambda^{-p} [v,w]^{p/p',1} \sum_{Q \in \mathscr{Q}} \int_Q f^p v \\
&\leq \lambda^{-p} [v,w]^{p/p',1} \int_X f^p v. && \square
\end{aligned}$$

It looks peculiar to estimate $M(fv)$ and not just $M(f)$. However, in the range $1 < p < \infty$ one can pass between these two objects using the fact that $\|f\|_{L^p(v)} = \|fv\|_{L^p(v^{1-p})}$. Hence the above theorem can be restated in the equivalent form

$$\|M(f)\|_{L^{p,\infty}(w)} \leq [v,w]^{(1/p',1/p)} \|f\|_{L^p(v^{1-p})}.$$

When the weights on both sides coincide: $w = v^{1-p}$ (or equivalently $v^{1/p'} w^{1/p} \equiv 1$), this inequality becomes

$$\|M(f)\|_{L^{p,\infty}(w)} \leq [w]_{A_p}^{1/p} \|f\|_{L^p(w)}, \quad [w]_{A_p} = \sup_{Q \in \mathscr{D}} (w)_Q (w^{-p'/p})_Q^{p/p'}.$$

The latter quantity is called the A_p *characteristic* of the weight w. Inequalities like this are called *one-weight estimates* in order to emphasize that there is only one independent weight in contrast to *two-weight estimates* in which the weights on the left-hand side and the right-hand side are independent.

Remark (Nestedness of A_p classes). It is an immediate consequence of Jensen's inequality that

$$1 < p < r < \infty \implies [w]_{A_r} \leq [w]_{A_p},$$

which is consistent with the fact that we can interpolate a weighted L^p estimate with the trivial weighted L^∞ estimate and obtain weighted L^r estimates.

We turn to strong type estimates. An important tool in this context is the uniform (in the underlying measure μ) boundedness of the maximal function M on $L^p(\mu)$ (which we have proved in the first lecture). In particular, we may use this with the measure μ replaced by $v\mathrm{d}\mu$, where v is a weight. Notice however that the maaximal function itself also depends on the measure μ, and the modified maximal function with respect to the measure $v\mathrm{d}\mu$ has the form

$$M_v f(x) = \sup_{x \in Q \in \mathscr{D}} (f)_{Q,v}, \quad \text{where} \quad (f)_{Q,v} = v(Q)^{-1} \int_Q f v = \frac{(fv)_Q}{(v)_Q}.$$

For these maximal functions we have the estimate

$$\|M_v f\|_{L^p(v)} \lesssim_p \|f\|_{L^p(v)}, \quad 1 < p < \infty,$$

where the implicit constant does not depend on v.

There is no full characteriztion of two-weight strong type estimates for the maximal function. Thee one-weight situation is substantially simpler, so we restrict ourselves to this setting.

Theorem 2.6 ([Buc93, Theorem 2.5]). *Let $1 < p < \infty$ and let w, v be weights with $v^{1/p'} w^{1/p} \equiv 1$. Then*

$$\|M(fv)\|_{L^p(w)} \lesssim [w]_{A_p}^{p'/p} \|f\|_{L^p(v)}.$$

Proof. We construct a stopping time $\mathscr{S}$ in the following way. Initialize

$$\begin{aligned} STOCK &:= \mathscr{D}, \\ \mathscr{S} &:= \emptyset. \end{aligned}$$

While $STOCK$ is non-empty let

$$\begin{aligned} \mathscr{S} &:= \mathscr{S} \cup \{Q \in STOCK \text{ maximal}\}, \\ STOCK &:= STOCK \setminus \{Q' \in STOCK : \exists Q \in \mathscr{S} \text{ with } Q' \subseteq Q \text{ and } (fv)_{Q'} \leq 2(fv)_Q\}. \end{aligned}$$

This process terminates after finitely many steps because at each step we remove at least the maximal elements from $STOCK$. For $Q \in Q$ let $\mathrm{ch}_{\mathscr{S}}(Q)$ be the set of maximal cubes $Q' \in \mathscr{S}$ with $Q' \subsetneq Q$, called *children of* Q. Since the children Q' have been chosen after Q, we have $(fv)_{Q'} > 2(fv)_Q$. It follows that

$$\sum_{Q' \in \mathrm{ch}_{\mathscr{S}}(Q)} |Q'| = \sum_{Q' \in \mathrm{ch}_{\mathscr{S}}(Q)} (fv)_{Q'}^{-1} \int_{Q'} fv < \frac{1}{2} \sum_{Q' \in \mathrm{ch}_{\mathscr{S}}(Q)} (fv)_Q^{-1} \int_{Q'} fv \leq \frac{1}{2} (fv)_Q^{-1} \int_Q fv \leq \frac{1}{2} |Q|.$$

Therefore the sets $E(Q) := Q \setminus \cup \mathrm{ch}_{\mathscr{S}}(Q)$ satisfy $|E(Q)| \geq \frac{1}{2}|Q|$, and they are pairwise disjoint. Moreover, by construction

$$M(fv) \leq 2 \sum_{Q \in \mathscr{S}} (fv)_Q 1_{E(Q)}.$$

Write

$$\begin{aligned} \|M(fv)\|_{L^p(w)} &\leq 2 \Big(\int \big(\sum_{Q \in \mathscr{S}} (fv)_Q 1_{E(Q)} \big)^p \mathrm{d}w \Big)^{1/p} \\ &= 2 \Big(\sum_{Q \in \mathscr{S}} (fv)_Q \int 1_{E(Q)} \mathrm{d}w \Big)^{1/p} \\ &= 2 \Big(\sum_{Q \in \mathscr{S}} (f)_{Q,v} (v)_Q w(E(Q)) \Big)^{1/p} \\ &\leq 2 \Big(\sum_{Q \in \mathscr{S}} (f)_{Q,v} v(E(Q)) \Big)^{1/p} \sup_Q \big(v(E(Q))^{-1} (v)_Q w(E(Q)) \big)^{1/p}. \end{aligned}$$

By disjointness of Q's and the maximal inequality for the martingale maximal function the first term is bounded by

$$\Big(\int M_v(f) \mathrm{d}v \Big)^{1/p} \lesssim \|f\|_{L^p(v)}.$$

In the second term we use the hypothesis on the weights v, w and Hölder's inequality in the form

$$|Q| \leq 2|E(Q)| = \int_{E(Q)} v^{1/p'} w^{1/p} \leq v(E(Q))^{1/p'} w(E(Q))^{1/p}.$$

Raising this inequality to power p' we estimate the above supremum by

$$\begin{aligned}\sup_Q \big(v(E(Q))^{-1}(v)_Q w(E(Q))\big)^{1/p} &= \sup_Q \big(|Q|^{-p'}|Q|^{p'} v(E(Q))^{-1}(v)_Q w(E(Q))\big)^{1/p}\\ &\lesssim \sup_Q \big(|Q|^{-p'}(v)_Q w(E(Q))^{1+p'/p}\big)^{1/p}\\ &= \sup_Q \big(|Q|^{-p'}(v)_Q w(E(Q))^{p'}\big)^{1/p}\\ &\le \sup_Q \big((v)_Q (w)_Q^{p'}\big)^{1/p}\\ &= [w]_{A_p}^{p'/p}. \qquad \square\end{aligned}$$

Theorems 2.5 and 2.6 can be extended to the Hardy–Littlewood maximal operator using adjacent dydic grids. In this case the weight characteristics have to be modified to allow suprema over all cubes $Q \subset \mathbb{R}^n$.

2.1 Power weights

The basic examples of A_p weights on $\mathbb{R}^n$ are power weights.

Lemma 2.7. *Let $w : \mathbb{R}^n \to (0,\infty)$, $w(x) = |x|^\alpha$. Then*

$$[w]_{A_p} \sim_{n,p} \begin{cases} (\alpha+n)^{-1}(-\alpha p'/p+n)^{-p/p'} & \text{if } -n < \alpha < n(p-1),\\ \infty & \text{otherwise.}\end{cases}$$

In the above statement notice that $p-1 = p/p'$ for Hölder conjugate exponents.

Proof. The A_p characteristic is a supremum over all cubes $Q \subset \mathbb{R}^n$. On the cubes such that $\operatorname{dist}(Q,0) > \operatorname{diam}(Q)$ the function w is constant (up to a multiplicative factor), so their contribution is benign.

If $\operatorname{dist}(Q,0) \le \operatorname{diam}(Q)$, then $Q \subset B = B(0,R)$, where $R = 2\operatorname{diam}(Q)$, and $|B| \lesssim |Q|$. It follows that

$$(w)_Q (w^{-p'/p})_Q^{p/p'} \lesssim (w)_B (w^{-p'/p})_B^{p/p'}.$$

A computation shows that the right-hand side does not depend on R, so the supremum over Q is finite if and only if both w and $w^{-p'/p}$ are locally integrable, which is equivalent to the claimed condition on α. $\square$

Example ([Buc93]). The following example shows that the dependence on the A_p characteristic in Theorem 2.6 cannot be improved.

Fix $1 < p < \infty$ and consider the power weights $w(x) = |x|^{(p-1)(1-\delta)}$ on $\mathbb{R}^1$ for small δ. Then $[w]_{A_p} \sim \delta^{-p/p'}$. Let $f(x) = |x|^{-(1-\delta)}\chi_{[0,1]}(x)$. Then $f^p w = |x|^{-(1-\delta)}\chi_{[0,1]}$ is integrable, so $f \in L^p(w)$. Moreover, it is easy to see that $Mf \ge \delta^{-1} f$, so

$$\|Mf\|_{L^p(w)} \ge \delta^{-1}\|f\|_{L^p(w)} \gtrsim [w]_{A_p}^{p'/p}\|f\|_{L^p(w)}.$$

3 Extrapolation

This section follows [Duo11].

Definition 3.1. The A_1 characterisitc of a weight w is

$$[w]_{A_1} := \sup \frac{Mw}{w} = \sup_{x\in Q\in\mathscr{D}} \frac{(w)_Q}{w(x)}.$$

Recall the identity $p/p' = p-1$ and the definition

$$[w]_{A_p} = \sup_{Q\in\mathscr{D}} (w)_Q (w^{-p'/p})_Q^{p/p'}, \quad 1<p<\infty.$$

Lemma 3.2. *If $1<p<p_0<\infty$ then for any weights w,u we have*

$$[wu^{p-p_0}]_{A_{p_0}} \leq [w]_{A_p} [u]_{A_1}^{p_0-p} \tag{3.3}$$

Proof. Let $Q\in\mathscr{D}$. Using $1/u(x) \leq [u]_{A_1}/(u)_Q$ for $x\in Q$ we obtain

$$(wu^{p-p_0})_Q \leq [u]_{A_1}^{p_0-p} (u)_Q^{p-p_0} (w)_Q,$$

and by Hölder's inequality with the exponent $q = (p'/p)(p_0/p_0')$ we have

$$\begin{aligned}((wu^{p-p_0})^{-p_0'/p_0})_Q^{p_0/p_0'} &\leq (w^{-p'/p})_Q^{p/p'} (u^{-(p-p_0)(p_0'/p_0)\cdot q'})_Q^{(p_0/p_0')/q'}\\ &= (w^{-p'/p})_Q^{p/p'} (u)_Q^{p_0-p}.\end{aligned}$$

Multiplying these inequalities and taking a supremum over Q we obtain the claim. □

Lemma 3.4. *If $1<p_0<p<\infty$ then for any weights w,u we have*

$$[(w^{p_0-1}u^{p-p_0})^{1/(p-1)}]_{A_{p_0}} \leq [w]_{A_p}^{(p_0-1)/(p-1)} [u]_{A_1}^{(p-p_0)/(p-1)}. \tag{3.5}$$

Proof. Let $Q\in\mathscr{D}$. Then similarly as above

$$(((w^{p_0-1}u^{p-p_0})^{1/(p-1)})^{-p_0'/p_0})_Q^{p_0/p_0'} \leq [u]_{A_1}^{(p-p_0)/(p-1)} (u)_Q^{-(p-p_0)/(p-1)} (w^{-p'/p})_Q^{p_0/p_0'}$$

and by Hölder's inequality with the exponent $q = (p-1)/(p_0-1)$ we have

$$((w^{p_0-1}u^{p-p_0})^{1/(p-1)})_Q \leq (w)_Q^{(p_0-1)/(p-1)} (u^{(p-p_0)q'/(p-1)})_Q^{1/q'} = (w)_Q^{(p_0-1)/(p-1)} (u)_Q^{(p-p_0)/(p-1)}.$$

Substituting these inequalities into the definition of the A_{p_0} characteristic and taking the supremum over Q we obtain the claim. □

Definition 3.6 (Rubio de Francia construction, [RdF84])**.** Let $1<p\leq\infty$ and assume that M is bounded on $L^p(w)$. Then the operator

$$Rf := \sum_{k=0}^{\infty} \Big(\frac{M}{2\|M\|_{L^p(w)}}\Big)^k f$$

has the following properties.

$$f \leq Rf \tag{3.7}$$

$$\|Rf\|_{L^p(w)} \leq 2\|f\|_{L^p(w)} \tag{3.8}$$

$$[Rf]_{A_1} \leq 2\|M\|_{L^p(w)}. \tag{3.9}$$

Theorem 3.10. *Let f,g be nonnegative functions, $1<p_0<\infty$, and assume*

$$\Big(\int g^{p_0} v\Big)^{1/p_0} \leq N([v]_{A_{p_0}}) \Big(\int f^{p_0} v\Big)^{1/p_0}$$

for some non-decreasing function N and all weights $v\in A_{p_0}$. Then for every $1<p<\infty$ and $w\in A_p$ we have

$$\Big(\int g^p w\Big)^{1/p} \leq K(w) \Big(\int f^p w\Big)^{1/p},$$

where

$$K(w) = \begin{cases} 2N([w]_{A_p}(2\|M\|_{L^p(w)})^{p_0-p}), & p < p_0, \\ 2^{p'(1/p_0-1/p)}N([w]_{A_p}^{(p_0-1)/(p-1)}(2\|M\|_{L^{p'}(w^{1-p'})})^{(p-p_0)/(p-1)}), & p_0 < p. \end{cases}$$

In particular, $K(w) \leq CN(C[w]_{A_p}^{\max(1,(p_0-1)/(p-1))})$ *if* M *is the dyadic or the Hardy–Littlewood maximal function.*

Corollary 3.11. *Let* $1 < p_0 < \infty$. *If* T_n *are operators (not necessary linear) that are bounded on all* $L^{p_0}(w)$ *with* $w \in A_{p_0}$ *with constant depending only on the weight characteristic (but not on n), then*

$$\|(\sum_n |T_n f_n|^{p_0})^{1/p_0}\|_{L^p(w)} \leq K([w]_{A_p})\|(\sum_n |f_n|^{p_0})^{1/p_0}\|_{L^p(w)}, \quad 1 < p < \infty.$$

Proof. Apply Theorem 3.10 with functions $(\sum_n |f_n|^{p_0})^{1/p_0}$ and $(\sum_n |T_n f_n|^{p_0})^{1/p_0}$. Note that the hypothesis of that Theorem is given by Fubini's theorem. □

Applying the corollary twice we obtain

Corollary 3.12. *If* T_n *are operators (not necessary linear) that are bounded on all* $L^{p_0}(w)$ *with* $w \in A_{p_0}$ *with constant depending only on the weight characteristic (but not on n), then*

$$\|(\sum_n |T_n f_n|^{q})^{1/q}\|_{L^p(w)} \leq K([w]_{A_p})\|(\sum_n |f_n|^{q})^{1/q}\|_{L^p(w)}, \quad 1 < p, q < \infty.$$

Remark. Theorem 3.10 does not necessarily recover the best dependence on the weight characteristic. Consider for instance the Hardy–Littlewood maximal function that is bounded on $L^2(w)$ with norm $\lesssim [w]_{A_2}$. Inserting this information into Theorem 3.10 yields that for $p > 2$ the maximal function is bounded on $L^2(w)$ with norm $\lesssim [w]_{A_p}$, which is worse than the conclusion of Theorem 2.6 in that range. On the other hand, Theorem 4.4 can be recovered from the special case $p = 2$ using Theorem 3.10.

Remark. Hölder's inequality for strictly positive functions $f_i > 0$ can be formulated as follows:

$$\int \prod_i f_i^{a_i} \leq \prod_j (\int \prod_j f_i^{a_{i,j}})^{b_j}$$

if $0 \leq b_j$, $\sum_j b_j = 1$, $a_i, a_{i,j} \in \mathbb{R}$, and $\sum_j a_{i,j} b_j = 1$. This is convenient if we know some of the terms one wants to obatain on the right-hand side and want to calculate the missing exponents.

Proof. Consider the case $p < p_0$.

$$\begin{aligned} \int g^p w &\leq \Big(\int g^{p_0} w (Rf)^{p-p_0}\Big)^{p/p_0} \Big(\int w(Rf)^p\Big)^{1-p/p_0} && \text{by Hölder} \\ &\leq N([w(Rf)^{p-p_0}]_{A_{p_0}})^p \Big(\int f^{p_0} w (Rf)^{p-p_0}\Big)^{p/p_0} \Big(\int w(Rf)^p\Big)^{1-p/p_0} && \text{by hypothesis} \\ &\leq N([w]_{A_p}[Rf]_{A_1}^{p_0-p})^p \int w(Rf)^p && \text{by (3.3) and (3.7)} \\ &\leq 2^p N([w]_{A_p}(2\|M\|_{L^p(w)})^{p_0-p})^p \int f^p w && \text{by (3.9) and (3.8).} \end{aligned}$$

In the case $p > p_0$ let $H = wg^{p-1}$ so that $\|H\|_{L^{p'}(w^{1/(1-p)})}^{p'} = \|g\|_{L^p(w)}^p$ and write

$$\int g^p w = \int g^{p_0} w^{(p_0-1)/(p-1)} H^{(p-p_0)/(p-1)}$$

$$
\begin{aligned}
&\le \int g^{p_0} w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)} && \text{by (3.}\\
&\le N([w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)}]_{A_{p_0}})^{p_0} \int f^{p_0} w^{(p_0-1)/(p-1)} (RH)^{(p-p_0)/(p-1)} && \text{by hypothe}\\
&\le N([w]_{A_p}^{\frac{(p_0-1)}{(p-1)}} [RH]_{A_1}^{\frac{(p-p_0)}{(p-1)}})^{p_0} \Big(\int f^p w \Big)^{p_0/p} \Big(\int w^{1/(1-p)} (RH)^{p/(p-1)} \Big)^{1-p_0/p} && \text{by (3.5) and Höld}\\
&\le 2^{p'(1-p_0/p)} N([w]_{A_p}^{\frac{(p_0-1)}{(p-1)}} (2\|M\|_{L^{p'}(w^{1/(1-p)})})^{\frac{(p-p_0)}{(p-1)}})^{p_0} \Big(\int f^p w \Big)^{p_0/p} \Big(\int w^{1/(1-p)} H^{p'} \Big)^{1-p_0/p} && \text{by (3.9) and (3.8}
\end{aligned}
$$

By construction the last bracket equals the left-hand side, and the claim follows. □

4 Sparse operators

So far we only had one example of an operator that is bounded on A_p weighted spaces, namely the maximal operator. Now we introduce the second basic example.

Definition 4.1. Let $\mathscr{D}$ be a dyadic grid and $0 < \eta \le 1$. A collection of cubes $\mathscr{S} \subset \mathscr{D}$ is called *η-sparse* if there exist pairwise disjoint subsets $E(Q) \subset Q$ such that $|E(Q)| \ge \eta |Q|$.

Usually we will not specify the parameter η and just talk about "sparse collections". In this case we assume a universal lower bound on η.

In the previous lecture we have seen one example of a sparse collection that was constructed using a stopping time.

Definition 4.2. The *sparse operator* associated to a sparse collection $\mathscr{S}$ is given by

$$
A_{\mathscr{S}} f := \sum_{Q \in \mathscr{S}} (f)_Q 1_Q.
$$

This operator is strictly larger than the operator that we have encountered in the estimate for the maximal operator. The difference is that we have replaced $1_{E(Q)}$ by $E(Q)$, so that the individual terms are no longer disjointly supported.

Over the last few years people have found out that sparse operators control many different interesting oeprators, beginning with Calderón–Zygmund operators.

Exercise 4.3. Identify the sparse collection in the proof of [JN61, Lemma 1].

For the time being we concentrate on weighted estimates for sparse operators.

Theorem 4.4. *For $1 < p < \infty$ and an η-sparse collection $\mathscr{S}$ we have*

$$
\|A_{\mathscr{S}} f\|_{L^p(w)} \lesssim_{p,\eta} [w]_{A_p}^{\max(1,1/(p-1))} \|f\|_{L^p(w)}.
$$

The implicit constant does not depend on the collection $\mathscr{S}$ or the weight w.

The proof below is from [Moe12].

Proof. With the dual weight $v = w^{1/(1-p)} = w^{-p'/p}$ and by duality it suffices to show

$$
\int A_{\mathscr{S}}(f v) g w \lesssim_p \eta^{-1} [w]_{A_p}^{\max(1,1/(p-1))} \|f\|_{L^p(v)} \|g\|_{L^{p'}(w)}
$$

(all functions positive). To this end write the left-hand side as

$$
\sum_{Q \in \mathscr{S}} (f v)_Q (g w)_Q |Q| = \sum_{Q \in \mathscr{S}} (f)_{Q,v} v(E(Q))^{1/p} (g)_{Q,w} w(E(Q))^{1/p'} |Q| (v)_Q (w)_Q v(E(Q))^{-1/p} w(E(Q))^{-1/p'}.
$$

By Hölder's inequality for sums this is bounded by

$$\Big(\sum_{Q\in\mathscr{S}}(f)_{Q,v}^{p}v(E(Q))\Big)^{1/p}\Big(\sum_{Q\in\mathscr{S}}(g)_{Q,w}^{p'}w(E(Q))\Big)^{1/p'}\sup_{Q}|Q|(v)_Q(w)_Q v(E(Q))^{-1/p}w(E(Q))^{-1/p'}.$$

The first thow terms are bounded by $\|M_v f\|_{L^p(v)}$ and $\|M_w g\|_{L^{p'}(w)}$, respectively, and we can use the martingale maximal inequality in both. It remains to estimate the supremum over Q. Fix Q. Recall

$$|Q| \leq \eta^{-1}|E(Q)| \leq \eta^{-1}v(E(Q))^{1/p'}w(E(Q))^{1/p}.$$

There are now two cases, $p \leq p'$ and $p \geq p'$ (equivalently, $p \leq 2$ and $p \geq 2$). In the former case take the last inequality to the power p'/p and obtain the estimate

$$\eta^{-p'/p}|Q|^{1-p'/p}(v)_Q(w)_Q w(E(Q))^{p'/p^2-1/p'} \lesssim |Q|^{1-p'/p}(v)_Q(w)_Q w(Q)^{p'/p^2-1/p'} = (v)_Q(w)_Q^{p'/p} \leq [w]_{A_p}^{p'/p}.$$

The case $p \geq 2$ is analogous with roles of v and w interchanged. □

Time permitting: show that $\mathscr{S}$ is η-sparse iff $1_{\mathscr{S}}$ is $1/\eta$-Carleson (reference [LN15, Lemma 6.3], easier [ZK16]).

5 Calderón–Zygmund (CZ) theory

In this lecture we cover some standard material which can be found e.g. in [Gra14] or [Ste93].

Definition 5.1. A *modulus of continuity* is a function $\omega : [0,\infty) \to [0,\infty)$ that is subadditive in the sense $u \leq s+t \implies \omega(u) \leq \omega(s)+\omega(t)$. The *Dini norm* of a modulus of continuity is

$$\|\omega\|_{\mathrm{Dini}} = \int_0^1 \omega(t)\frac{\mathrm{d}t}{t}.$$

Notice that a Dini modulus of continuity is monotonically increasing, and it follows that $\|\omega\|_{\mathrm{Dini}} \sim \sum_{k\in\mathbb{N}}\omega(2^{-k})$.

Definition 5.2. Let ω be a modulus of continuity. An *ω-CZ kernel* is a function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ such that

1. $|K(x,y)| \leq C_K|x-y|^{-d}$ for some $C_K < \infty$ and all $x,y \in \mathbb{R}^n$ with $x \neq y$,
2. $|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)| \leq \omega(|y-y'|/|y-x|)|x-y|^{-d}$ for all $x,y,y' \in \mathbb{R}^n$ with $|y-y'| < |y-x|/2$.

An *ω-CZ operator* is a linear operator, initially defined on bounded compactly supported measurable functions on $\mathbb{R}^n$ with values in $L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$ that has an associated ω-CZ kernel K such that for all functions f and points $x \notin \mathrm{supp}(f)$ we have

$$Tf(x) = \int K(x,y)f(y)\mathrm{d}y.$$

Remark. The use of the constant C_K is traditional; it can be replaced by a qualitative off-diagonal decay. Our quantitative estimates will depend on $\|\omega\|_{\mathrm{Dini}}$, and since one can show $C_K \lesssim \|\omega\|_{\mathrm{Dini}}$ the constant C_K will not appear in the estimates.

5.1 CZ decomposition

Theorem 5.3 (CZ decomposition). *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then there exists a decomposition $f = g + \sum_{Q\in\mathcal{Q}} b_Q$, where*

1. $\|g\|_1 \le \|f\|_1$,
2. $\|g\|_\infty \le 2^n\lambda$,
3. *$\mathcal{Q}$ is a collection of pairwise disjoint dyadic cubes,*
4. $\sum_{Q\in\mathcal{Q}} |Q| \le \lambda^{-1}\|f\|_1$,
5. $\|b_Q\|_1 \le 2^{n+1}\lambda|Q|$ *for every* $Q \in \mathcal{Q}$,
6. $\int b_Q = 0$ *for every* $Q \in \mathcal{Q}$.

Sketch of proof. Let $\mathcal{Q}$ be the collection of maximal dyadic cubes with $(|f|)_Q > \lambda$. Let $b_Q = \mathbf{1}_Q(f - \fint_Q f)$. Then

$$g(x) = \begin{cases} \fint_Q f, & x \in Q \in \mathcal{Q}, \\ f(x) & \text{otherwise.} \end{cases}$$

It follows that $\|g\|_\infty \lesssim \lambda$ (proved using the fact that $|f| \le Mf$ pointwise a.e. that is obtained by a density argument/Lebesgue differentiation theorem), and $\|g\|_1 \le \|f\|_1$. □

Lemma 5.4. *Let T be an ω-CZ operator. Then*

$$\|T\|_{L^1\to L^{1,\infty}} \lesssim_d \|T\|_{L^2\to L^2} + \|\omega\|_{\mathrm{Dini}}.$$

Proof. The claim is non-void only if T is bounded on L^2. Multiplying T and its kernel by a scalar we may normalize $\|T\|_{L^2\to L^2} = 1$. By homogeneity if suffices to show

$$|\{Tf > 1\}| \lesssim (1 + \|\omega\|_{\mathrm{Dini}})\|f\|_1$$

for bounded compactly supported functions f. Consider the CZ decomposition

$$f = g + \sum_{Q\in\mathcal{Q}} b_Q$$

at level 1. This expansion in fact converges (unconditionally) in L^2 by the qualitative assumptions on f, so $Tf = Tg + \sum_{Q\in\mathcal{Q}} Tb_Q$ with unconditional convergence in L^2.

Using the mean zero property of the bad functions b_Q we obtain

$$\begin{aligned}
\|Tb_Q\|_{L^1((10Q)^c)} &\le \sum_{k\ge 0} \int_{2^k r_Q \le |x-x_Q| < 2^{k+1} r_Q} \Big| \int_Q K(x,y) b_Q(y) \mathrm{d}y \Big| \mathrm{d}x \\
&\le \sum_{k\ge 0} \int_{2^k r_Q \le |x-x_Q| < 2^{k+1} r_Q} \int_Q |K(x,y) - K(x,x_Q)| |b_Q(y)| \mathrm{d}y \mathrm{d}x \\
&\le \sum_{k\ge 0} \int_{2^k r_Q \le |x-x_Q| < 2^{k+1} r_Q} \int_Q |K(x,y) - K(x,x_Q)| |b_Q(y)| \mathrm{d}y \mathrm{d}x \\
&\lesssim \sum_{k\ge 0} \omega(2^{-k}) \int_Q |b_Q| \\
&\lesssim \|\omega\|_{\mathrm{Dini}} |Q|.
\end{aligned}$$

Hence with $\Omega = \cup_{Q\in\mathscr{Q}} 10Q$ we have

$$\|\sum_{Q} Tb_Q\|_{L^1(\mathbb{R}^d\setminus\Omega)} \lesssim \|\omega\|_{\mathrm{Dini}} \sum_{Q\in\mathscr{Q}} |Q| \lesssim \|\omega\|_{\mathrm{Dini}} \|f\|_1$$

Therefore

$$\begin{aligned}|\{Tf > 1\}| &\le |\Omega| + |\{\sum_{Q} Tb_Q > 1/2\} \cap \Omega^c| + |\{Tg > 1/2\}| \\ &\lesssim \|f\|_1 + \|\sum_{Q} Tb_Q\|_{L^1(\mathbb{R}^d\setminus\Omega)} + \|Tg\|_2^2 \\ &\lesssim (1 + \|\omega\|_{\mathrm{Dini}})\|f\|_1,\end{aligned}$$

where we have used $\|g\|_2^2 \le \|g\|_1 \|g\|_\infty \lesssim \|f\|_1$. □

5.2 Cotlar's inequality

Define the maximally truncated operator

$$T_\sharp f(x) := \sup_{\varepsilon>0, |x-x'|\le\varepsilon/2} \int_{B(x',\varepsilon)^c} K(x',y) f(y) \mathrm{d}y.$$

This maximal truncation is usually considered without the supremum in x' (i.e. with $x' = x$), but the above version is more convenient for us.

Lemma 5.5 (Cotlar's inequality)**.**

$$T_\sharp f \lesssim_{d,\delta} (\|T\|_{L^2\to L^2} + \|\omega\|_{\mathrm{Dini}}) Mf + M_\delta Tf. \tag{5.6}$$

Here $M_\delta f = (M(f^\delta))^{1/\delta}$, $0 < \delta < 1$, where M is the usual Hardy–Littlewood maximal function.

In particular $T_\sharp$ has weak type $(1,1)$.

Proof. For $x', x'' \in B(x, \varepsilon/2)$ write

$$\begin{aligned}\int_{B(x',\varepsilon)^c} K(x',y) f(y) \mathrm{d}y = \int_{B(x',\varepsilon)^c \setminus B(x,2\varepsilon)^c} K(x',y) f(y) \mathrm{d}y + \int K(x'',y)(f\mathbf{1}_{B(x,2\varepsilon)^c})(y) \mathrm{d}y \\ - \int (K(x'',y) - K(x',y)) \mathbf{1}_{B(x,2\varepsilon)^c} f(y) \mathrm{d}y.\end{aligned}$$

The first term is estimated using the kernel bound by $C_K Mf(x)$. The last term is estimated by

$$\sum_{k>0} \int_{2^k\varepsilon \le |x-y| < 2^{k+1}\varepsilon} |K(x'',y) - K(x',y)| |f(y)| \mathrm{d}y \lesssim_d \sum_{k>0} \int_{2^k\varepsilon \le |x-y| < 2^{k+1}\varepsilon} \frac{\omega(2^{-k})}{(2^k\varepsilon)^d} |f(y)| \mathrm{d}y \lesssim \|\omega\|_{\mathrm{Dini}} Mf(x).$$

The middle term equals

$$T(f\mathbf{1}_{B(x,2\varepsilon)^c})(x'') = T(f)(x'') - T(f\mathbf{1}_{B(x,2\varepsilon)})(x''),$$

where we have used that T is associated to K and linearity of T. In both terms we take the L^δ average over $x'' \in B := B(x, \varepsilon/2)$. The contribution of the former term is then clearly bounded by $M_\delta Tf(x)$. The contribution of the latter term is bounded by

$$(\fint_B |T(\mathbf{1}_{4B} f)|^\delta)^{1/\delta} \lesssim |B|^{-1} \|T(\mathbf{1}_{4B} f)\|_{L^{1,\infty}(B)} \le \|T\|_{L^1\to L^{1,\infty}} |B|^{-1} \|\mathbf{1}_{4B} f\|_{L^1} \lesssim \|T\|_{L^1\to L^{1,\infty}} Mf(x),$$

and we conclude using Lemma 5.4. □

Exercise 5.7. Replace $M_\delta Tf$ in (5.6) by $\mathscr{M}_{1/2} Tf$, where

$$\mathscr{M}_\lambda f(x) = \sup_{x\in Q} (f\mathbf{1}_Q)^*(\lambda|Q|)$$

and f^* denotes the non-increasing rearrangement of f.

5.3 Marcinkiewicz interpolation theorem, $L^{p,\infty}$ version

We need the following version of the Marcinkiewicz interpolation theorem in which the conclusion is a bound on a weak L^p space.

Theorem 5.8. *Let T be a quasisubadditive operator and assume $T : L^{p_j} \to L^{p_j,\infty}$ for $j = 0, 1$ with $1 \le p_0 < p_1 \le \infty$. Let $0 < \theta < 1$ and $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. Then $T : L^{p_\theta,\infty} \to L^{p_\theta,\infty}$.*

Proof. Similarly as in the proof of the strong type estimate split $f = f_{0,\lambda} + f_{1,\lambda}$ with $f_{1,\lambda} = f 1_{|f|\le\lambda}$. Then

$$\begin{aligned}
\{|Tf| > \eta\} &\le \{|Tf_{0,\lambda}| > \eta/(2C)\} + \{|Tf_{1,\lambda}| > \eta/(2C)\} \\
&\lesssim \eta^{-p_0} \|Tf_{0,\lambda}\|_{p_0,\infty}^{p_0} + \eta^{-p_1} \|Tf_{1,\lambda}\|_{p_1,\infty}^{p_1} \\
&\lesssim \eta^{-p_0} \|f_{0,\lambda}\|_{p_0}^{p_0} + \eta^{-p_1} \|f_{1,\lambda}\|_{p_1}^{p_1} \\
&\le \eta^{-p_0} \int_{|f|>\lambda} |f|^{p_0} + \eta^{-p_1} \int_{|f|\le\lambda} |f|^{p_1} \\
&\le \eta^{-p_0} \sum_{k\ge 0} \int_{|f|\sim 2^k\lambda} |f|^{p_0} + \eta^{-p_1} \sum_{k\le 0} \int_{|f|\sim 2^k\lambda} |f|^{p_1} \\
&\le \eta^{-p_0} \sum_{k\ge 0} (2^k\lambda)^{p_0-p_\theta} \|f\|_{p_\theta,\infty} + \eta^{-p_1} \sum_{k\le 0} (2^k\lambda)^{p_1-p_\theta} \|f\|_{p_\theta,\infty}.
\end{aligned}$$

Since $p_0 < p_\theta < p_1$, both series are geometric and dominated by the $k = 0$ terms. Hence

$$\{|Tf| > \eta\} \lesssim \eta^{-p_0} (\lambda)^{p_0-p_\theta} \|f\|_{p_\theta,\infty} + \eta^{-p_1} (\lambda)^{p_1-p_\theta} \|f\|_{p_\theta,\infty}.$$

Choosing $\lambda = \eta$ we obtain the claim

$$\{|Tf| > \eta\} \lesssim \eta^{-p_\theta} \|f\|_{p_\theta,\infty}.$$

□

Corollary 5.9. *The maximal operator M_δ is bounded on $L^{1,\infty}$ for $0 < \delta < 1$.*

Proof. By Theorem 5.8 the Hardy-Littlewood maximal operator M is bounded on $L^{1/\delta,\infty}$. Hence

$$\|M_\delta f\|_{1,\infty} = \|M(f^\delta)\|_{1/\delta,\infty} \lesssim \|f^\delta\|_{1/\delta,\infty} = \|f\|_{1,\infty}.$$ □

6 Sparse domination of CZ operators

The fierst proof of sharp weighted estimates for CZ operators was quite complicated [Hyt12]. Many simplifications have been made since then The two key simplifications were the introduction of sparse domination by Lerner [Ler13] and a simple algorithm for constructing sparse collections by Lacey [Lac15], a streamlined version of which appears in [HRT15]. We have followed [Ler16].

The main example that I am aware of where sharp weighted estimates are useful is the regularity theory for solutions of the Beltrami equation in [AIS01].

7 A_∞ weights

Let $\mathscr{D}$ be a dyadic grid (in $\mathbb{R}^d$) and M the associated dyadic maximal operator. The associated A_∞ characteristic is defined by

$$[w]_{A_\infty} := \sup_{Q\in\mathscr{D}} w(Q)^{-1} \int_Q M(w\mathbf{1}_Q).$$

Lemma 7.1. *For every weight w and $1 < p < \infty$ we have*

$$[w]_{A_\infty} \lesssim_p [w]_{A_p}.$$

Proof. Let v be the dual weight given by $w^{1/p} v^{1/p'} \equiv 1$. Fix $Q_0 \in \mathscr{D}$ and construct the (minimal) stopping collection $\mathscr{S}$ by the rules

1. $Q_0 \in \mathscr{S}$,
2. If $Q \in \mathscr{S}$, then the maximal cubes $Q' \subset Q$ with $(w)_{Q'} \geq 2(w)_Q$ are in $\mathscr{S}$.

Then the collection $\mathscr{S}$ is sparse. The pairwise disjoint major subsets $E(Q) \subset Q \in \mathscr{S}$ satisfy

$$|Q| \sim |E(Q)| = \int_{E(Q)} 1 = \int_{E(Q)} w^{1/p} v^{1/p'} \leq w(E(Q))^{1/p} v(E(Q))^{1/p'} \tag{7.2}$$

by Hölder's inequality.

Now

$$\begin{aligned}
\int_{Q_0} M(w\mathbf{1}_{Q_0}) &\lesssim \int_{Q_0} \sum_{Q \in \mathscr{S}} \mathbf{1}_{E(Q)} (w)_Q \\
&= \sum_{Q \in \mathscr{S}} |E(Q)| (w)_Q \\
&= \sum_{Q \in \mathscr{S}} w(E(Q)) \frac{|E(Q)|(w)_Q}{w(E(Q))} \\
&\leq \sup_{Q \in \mathscr{S}} \frac{|E(Q)|(w)_Q}{w(E(Q))} \sum_{Q \in \mathscr{S}} w(E(Q)) \\
&\leq \sup_{Q \in \mathscr{S}} \frac{|E(Q)|(w)_Q}{w(E(Q))} w(Q).
\end{aligned}$$

Hence it suffices to show

$$\frac{|E(Q)|(w)_Q}{w(E(Q))} \lesssim [w]_{A_p}$$

uniformly in Q. To this end multiply the left-hand side by (7.2) taken to the power p:

$$\begin{aligned}
\frac{|E(Q)|(w)_Q}{w(E(Q))} &\lesssim \frac{|E(Q)|(w)_Q}{w(E(Q))} \Big(\frac{w(E(Q))^{1/p} v(E(Q))^{1/p'}}{|E(Q)|} \Big)^p \\
&\leq |E(Q)|^{1-p} (w)_Q v(E(Q))^{p/p'} \\
&\lesssim |Q|^{1-p} (w)_Q v(Q)^{p/p'} \\
&= (w)_Q (v)_Q^{p/p'} \\
&\leq [w]_{A_p}.
\end{aligned}$$

□

Lemma 7.3. *Let w be a weight, $\lambda > 0$, and $Q \in \mathscr{D}$ maximal with $(w)_Q > \lambda$. Then $Mw = M(w\mathbf{1}_Q)$ on Q. Moreover,*

$$\int_Q Mw \leq 2^d [w]_{A_\infty} |Q| \lambda$$

and

$$w(Q) \leq 2^d |Q| \lambda.$$

Proof. The first conclusion is clear because cubes that strictly contain Q have a smaller contribution to the maximal function than Q. For the second conclusion let $\hat{Q}$ be the dyadic parent of Q, then

$$\int_Q Mw = \int_Q M(w\mathbf{1}_Q) \leq [w]_{A_\infty} \int_Q w \leq [w]_{A_\infty} \int_{\hat{Q}} w \leq [w]_{A_\infty} |\hat{Q}|(w)_{\hat{Q}} \leq 2^d [w]_{A_\infty} |Q|\lambda.$$

The third conclusion is even easier:

$$w(Q) \leq w(\hat{Q}) = (w)_{\hat{Q}} |\hat{Q}| \leq 2^d \lambda |Q|. \qquad \square$$

Lemma 7.4 ([HPR12, Lemmas 2.2 and 2.3]). *Let $w \in A_\infty$ and $\varepsilon = \frac{1}{2^{d+1}[w]_{A_\infty} - 1}$. Then for every $Q_0 \in \mathscr{D}$ we have*

$$\fint_{Q_0} M(w\mathbf{1}_{Q_0})^{1+\varepsilon} \leq 2(w)_Q^{\varepsilon} \fint_{Q_0} M(w\mathbf{1}_{Q_0}) \leq 2[w]_{A_\infty} (w)_Q^{1+\varepsilon}, \tag{7.5}$$

$$\fint_{Q_0} M(w\mathbf{1}_{Q_0})^{\varepsilon} w \leq 2(w)_Q^{1+\varepsilon}. \tag{7.6}$$

Proof. For notational convenience suppose $w = w\mathbf{1}_{Q_0}$ and Q_0 is the unique maximal element of $\mathscr{D}$. The layer cake formula for the reference measure that will be useful for both conclusions:

$$\int_{Q_0} (Mw)^{1+\varepsilon} = (1+\varepsilon) \int_{Q_0} \int_0^{Mw} \lambda^{\varepsilon} \mathrm{d}\lambda = (1+\varepsilon) \int_0^{\infty} \lambda^{\varepsilon} |\{Mw > \lambda\}| \mathrm{d}\lambda. \tag{7.7}$$

By the layer cake formula with a measure v (we will later use $v = w$ or $v = Mw$) we have

$$\int_{Q_0} M(w\mathbf{1}_{Q_0})^{\varepsilon} v = \varepsilon \int_{Q_0} (\int_0^{Mw} \lambda^{\varepsilon-1} \mathrm{d}\lambda) v = \varepsilon \int_0^{\infty} \lambda^{\varepsilon-1} v\{Mw > \lambda\} \mathrm{d}\lambda.$$

We split this integral at $\lambda = (w)_{Q_0}$. The part with $\lambda \leq (w)_{Q_0}$ we estimate by

$$\varepsilon \int_0^{(w)_{Q_0}} \lambda^{\varepsilon-1} v\{Mw > \lambda\} \mathrm{d}\lambda = \varepsilon \int_0^{(w)_{Q_0}} \lambda^{\varepsilon-1} v(Q_0) \mathrm{d}\lambda = (w)_{Q_0}^{\varepsilon} v(Q_0).$$

For $\lambda > (w)_{Q_0}$ the superlevel set $\{Mw > \lambda\}$ is the union of the collection $\mathscr{Q}_\lambda$ of maximal dyadic cubes $Q \subset Q_0$ with

$$(w)_Q > \lambda.$$

Consider first the case $v = Mw$ that corresponds to (7.5). It follows from Lemma 7.3 that

$$Mw\{Mw > \lambda\} \leq 2^d [w]_{A_\infty} |\{Mw > \lambda\}| \lambda,$$

so using (7.7) we obtain

$$\begin{aligned} \varepsilon \int_{(w)_Q}^{\infty} \lambda^{\varepsilon-1} Mw\{Mw > \lambda\} \mathrm{d}\lambda &\leq 2^d [w]_{A_\infty} \varepsilon \int_{(w)_{Q_0}}^{\infty} \lambda^{\varepsilon} |\{Mw > \lambda\}| \mathrm{d}\lambda \\ &\leq \frac{2^d [w]_{A_\infty} \varepsilon}{1+\varepsilon} \int_{Q_0} (Mw)^{1+\varepsilon}. \end{aligned}$$

Notice that the fraction on the right-hand side equals $1/2$ by the hypothesis. Substituting this above we obtain

$$\int_{Q_0} M(w\mathbf{1}_{Q_0})^{1+\varepsilon} \leq (w)_{Q_0}^{\varepsilon} Mw(Q_0) + \frac{1}{2} \int_{Q_0} (Mw)^{1+\varepsilon},$$

and the conclusion (7.5) follows.

Consider now the case $v = w$ that corresponds to (7.6). By Lemma 7.3 we obtain

$$w\{Mw > \lambda\} \leq 2^d \lambda |\{Mw > \lambda\}|.$$

Hence using (7.7) and (7.5) we obtain

$$\begin{aligned}\varepsilon \int_{(w)_{Q_0}}^{\infty} \lambda^{\varepsilon-1} w\{Mw > \lambda\} \mathrm{d}\lambda \leq 2^d \varepsilon \int_{(w)_{Q_0}}^{\infty} \lambda^{\varepsilon} |\{Mw > \lambda\}| \mathrm{d}\lambda &\leq \frac{2^d \varepsilon}{1+\varepsilon} \int_{Q_0} (Mw)^{1+\varepsilon} \\ &\leq \frac{2^{d+1}[w]_{A_\infty} \varepsilon}{1+\varepsilon} |Q_0| (w)_{Q_0}^{1+\varepsilon} = |Q_0| (w)_{Q_0}^{1+\varepsilon}\end{aligned}$$

by the choice of ε. The conclusion follows. □

Corollary 7.8 (Open property). *Let $1 < p < \infty$ and $w \in A_p$. Then $[w]_{A_{\tilde{p}}} \lesssim [w]_{A_p}$, where $\tilde{p} = p - \frac{p-1}{2^{d+1}[v]_{A_\infty}} < p$ and v is the dual weight: $w^{1/p} v^{1/p'} \equiv 1$.*

Proof. The exponent $\tilde{p}$ is chosen in such a way that $1 + \varepsilon = (p/p')(\tilde{p}'/\tilde{p})$, where ε is as in Lemma 7.4 for the weight v. Consider the dual weight $\tilde{v} = w^{-\tilde{p}'/\tilde{p}}$. Then by (7.6) applied to the weight v we have

$$(\tilde{v})_Q = (v^{1+\varepsilon})_Q \leq 2(v)_Q^{1+\varepsilon}.$$

Hence for every $Q \in \mathscr{D}$ we have

$$(w)_Q (\tilde{v})_Q^{\tilde{p}/\tilde{p}'} \lesssim (w)_Q (v)_Q^{(1+\varepsilon)\tilde{p}/\tilde{p}'} = (w)_Q (v)_Q^{p/p'} \leq [w]_{A_p}.$$ □

7.1 Embedding of A_∞ into A_p

We call a weight *(C_{db}-)doubling* if

$$w(2Q) \leq C_{db} w(Q)$$

for some *doubling constant* $C_{db} < \infty$ and all cubes $Q \subset \mathbb{R}^d$.

It is not hard to show that A_p weights are doubling if $p < \infty$. The case $p = \infty$ is more subtle.

Exercise 7.9. Find a weight that is A_∞ with respect to the standard dyadic filtration but not $A_\infty(\mathbb{R}^d)$.

Exercise 7.10. Find a weight on $\mathbb{R}$ that is A_∞ with respect to the three 1/3-shifted dyadic grids but not $A_\infty(\mathbb{R})$.

To combat these difficulties we define the $A_\infty(\mathbb{R}^d)$ by

$$[w]_{A_\infty(\mathbb{R}^d)} = \sup_Q w(Q)^{-1} \int_Q M(w \mathbf{1}_Q),$$

where the supremum is taken over *all* non-empty axis-parallel cubes in $\mathbb{R}^d$.

Lemma 7.11. *For every $d \geq 1$ there exists $C = C(d)$ such that for every $w \in A_\infty(\mathbb{R}^d)$ we have*

$$C_{db}(w) \leq C^{C^{[w]_{A_\infty(\mathbb{R}^d)}}}.$$

The converse is not true: there exist doubling weights that are not A_∞, see [FM74] (a different version of the A_∞ condition was used there).

Proof. Let $k > C[w]_{A_\infty(\mathbb{R}^d)}$ be an integer, where C is a large constant to be chosen later. We first show that

$$w(\tilde{Q}) \lesssim w(Q),$$

where Q is a cube and $\tilde{Q} = (1+2^{-k})Q$. The claim then follows by iterating this estimate $\log_2 k$ times.

By scaling invariance we may assume that Q has side length 1. Also, it suffices to estimate $w(P)$, where P is a parallelepiped of dimensions $1 \times \cdots \times 1 \times 2^{-k}$ at the boundary of $\tilde{Q}$ since $\tilde{Q}$ is the union of finitely many such parallelepipeds and the cube Q.

Consider now the strip $S_l \subset Q$ of width 2^{-l} at distance 2^{-l} from P. Estimating the maximal function on this strip by the averages of scale 2^{-l} we obtain

$$\int_{S_l} M(w1_{\tilde{Q}}) \geq \int_{S_l} M(w1_P) \gtrsim \int w1_P.$$

Since there are $\sim k$ pairwise disjoint strips S_l, it follows that

$$\int_{\tilde{Q}} M(w1_{\tilde{Q}}) \gtrsim k \int_P w.$$

Summing up these estimates for finitely many P's we obtain

$$\int_{\tilde{Q}} M(w1_{\tilde{Q}}) \gtrsim k \int_{\tilde{Q}\setminus Q} w.$$

Using the definition of the A_∞ characteristic it follows that

$$k \int_{\tilde{Q}\setminus Q} w \lesssim [w]_{A_\infty} w(\tilde{Q}).$$

If k was chosen sufficiently large in terms of $[w]_{A_\infty}$ this implies $w(\tilde{Q}\setminus Q) \leq w(\tilde{Q})/2$ and consequently $w(\tilde{Q}) \leq 2w(Q)$. □

At this point we return to dyadic weights and assume henceforth the doubling condition

$$w(\hat{Q}) \leq C_{db} w(Q),$$

where $\hat{Q}$ is the dyadic parent of a cube Q.

Recall that A_∞ weights satisfy the reverse Hölder inequality

$$(w^{1+\varepsilon})_Q \leq 2(w)_Q^{1+\varepsilon}, \quad 1/\varepsilon \sim [w]_{A_\infty}.$$

Theorem 7.12. *Let $w > 0$ be a dyadic weight that satisfies the doubling condition with some $C_{db} < \infty$ and the reverse Hölder inequality with some $\varepsilon > 0$. Then $w \in A_r$ provided $r \geq C4^{1/\varepsilon}\log(2C_{db})$.*

In the case of A_∞ weights this holds in particular if $r \geq C^{[w]_{A_\infty(\mathbb{R}^d)}}$. This quantitative dependence has been noted in [HP14, Theorem 1.3] with the remark that it has been implicitly known before (i.e. it follows from previous results and/or their proofs). We follow the proof in [Ste93, p. V.5.1].

Proof. Let $\mathcal{M} = M_{\mathcal{D},w}$ be the dyadic maximal function with respect to the weight w. Fix $Q_0 \in \mathcal{D}$ and let $f = w^{-1}\mathbf{1}_{Q_0}$. We want to estimate $(w)_{Q_0}(w^{1-r'})_{Q_0}^{1-r}$. We may normalize $|Q_0| = 1$ by scaling and $w(Q_0) = 0$ by multiplying w by an absolute constant.

Let $N > 1$ be chosen later and write $\mathcal{Q}_k$, $k \in \mathbb{N}$, for the collection of maximal cubes Q such that

$$N^k \leq w(Q)^{-1} \int_Q fw.$$

Notice that for $Q \in \mathcal{Q}_k$ we have

$$w(Q)^{-1} \int_Q f w \le w(Q)^{-1} \int_{\hat{Q}} f w = w(Q)^{-1} w(\hat{Q}) w(\hat{Q})^{-1} \int_{\hat{Q}} f w \le C_{db} N^k.$$

Let $k > 0$ and $Q \in \mathcal{Q}_{k-1}$. Then

$$\sum_{Q' \in \mathcal{Q}_k : Q' \subset Q} w(Q') \le N^{-k} \sum_{Q' \in \mathcal{Q}_k : Q' \subset Q} \int_{Q'} f w \le N^{-k} \int_Q f w \le N^{-k} C_{db} N^{k-1} w(Q) = N^{-1} C_{db} w(Q).$$

Choose $N = 2C_{db}$ and let $E_k = \cup \mathcal{Q}_k$. Then $w(E_k \cap Q) \le w(Q)/2$.

By the Hölder and the reverse Hölder inequality for every set $E \subset Q$ with $w(E) \le w(Q)/2$ we have

$$(w)_Q/2 \le (w 1_{Q \setminus E})_Q \le (w^{1+\varepsilon})_Q^{1/(1+\varepsilon)} (1_{Q \setminus E})^{\varepsilon/(1+\varepsilon)} \le 2^{1/(1+\varepsilon)} (w)_Q (1 - |E|/|Q|)^{\varepsilon/(1+\varepsilon)},$$

so that

$$2^{-2-\varepsilon} \le (1 - |E|/|Q|)^{\varepsilon},$$

so

$$|E|/|Q| \le 1 - 2^{-2/\varepsilon - 1}.$$

Summing over $Q \in \mathcal{Q}_{k-1}$ we obtain

$$|E_k|/|E_{k-1}| \le 1 - 2^{-2/\varepsilon - 1}.$$

Hence

$$\int_{Q_0} w^{1-r'} \le \int_{Q_0} (Mf)^{r'-1} \le \sum_{k=0}^{\infty} N^{(k+1)(r'-1)} |E_k \setminus E_{k+1}| \le N^{r'-1} \sum_{k=0}^{\infty} N^{(r'-1)k} (1 - 2^{-2/\varepsilon - 1})^k < \infty$$

provided $N^{r'-1}(1 - 2^{-2/\varepsilon - 1}) < 1$, which follows from

$$(r' - 1) \log N \lesssim 4^{-1/\varepsilon},$$

or in other words

$$r - 1 = \frac{1}{r' - 1} \gtrsim 4^{1/\varepsilon} \log N.$$

□

Corollary 7.13. $A_\infty(\mathbb{R}^d) = \cup_{p<\infty} A_p(\mathbb{R}^d)$.

8 Mixed A_p–A_∞ estimates

In Lecture 2 we have proved weighted estimates for the dyadic maximal operator. Now we refine these estimates following [HP13].

Proposition 8.1. *Let $1 < p < \infty$ and let v, w be weights. Then*

$$\|M(fv)\|_{L^p(w)} \lesssim_p [v, w]^{1/p', 1/p} [v]_{A_\infty}^{1/p} \|f\|_{L^p(v)}.$$

This is indeed a refinement of our previous result because if the weights are related by $v^{1/p'} w^{1/p} \equiv 1$, then the product of the above characteristics is bounded by $[w]_{A_p}^{p'/p}$. Moreover, the A_∞ characteristic above can be substantially smaller than $[v]_{A_{p'}}$ (example: power weights).

Proof. We start with the stopping collection $\mathscr{S}$ as in Lecture 2 which is sparse with respect to the reference measure μ and such that

$$M(fv) \lesssim \sum_{Q\in\mathscr{S}} (fv)_Q \mathbf{1}_{E(Q)}.$$

Next we construct a second stopping collection $\mathscr{F}$ with respect to the measure $v\mathrm{d}\mu$ as follows. Denote $(f)_{Q,v} = v(Q)^{-1}\int_Q fv = (fv)_Q/(v)_Q$. Assuming without loss of generality that $\mathscr{D}$ is finite we put the maximal elements of $\mathscr{D}$ into $\mathscr{F}$. Then for each $Q\in\mathscr{F}$ we add all maximal $Q'\subset Q$ with $Q'\in\mathscr{D}$ and $(f)_{Q',v} > 2(f)_{Q,v}$ to $\mathscr{F}$. Then

$$\begin{aligned}
\|M(fv)\|_{L^p(w)}^p &\lesssim \int \sum_{Q\in\mathscr{S}} (fv)_Q^p \mathbf{1}_{E(Q)} w \\
&= \sum_{Q\in\mathscr{S}} (f)_{Q,v}^p (v)_Q^p \int_{E(Q)} w \\
&= \sum_{F\in\mathscr{F}} \sum_{Q\in\mathscr{S}:\pi_{\mathscr{F}}(Q)=F} (f)_{Q,v}^p (v)_Q^p \int_{E(Q)} w \\
&\lesssim \sum_{F\in\mathscr{F}} (f)_{F,v}^p \sum_{Q\in\mathscr{S},Q\subseteq F} (v)_Q^p w(Q),
\end{aligned}$$

where $\pi_{\mathscr{F}}(Q)$ is the smallest member of $\mathscr{F}$ containing Q. Notice that

$$\sum_{F\in\mathscr{F}} (f)_{F,v}^p v(F) \lesssim \sum_{F\in\mathscr{F}} (f)_{F,v}^p v(\tilde{E}(F)) \lesssim \int (M_v f)^p v \lesssim \int (f)^p v$$

by the L^p estimate for the weighted maximal function M_v. Hence it remains to estimate

$$\begin{aligned}
v(F)^{-1} \sum_{Q\in\mathscr{S},Q\subseteq F} (v)_Q^p w(Q) &\le [v,w]^{p-1,1} v(F)^{-1} \sum_{Q\in\mathscr{S},Q\subseteq F} (v)_Q |Q| \\
&\lesssim [v,w]^{p-1,1} v(F)^{-1} \sum_{Q\in\mathscr{S},Q\subseteq F} (v)_Q |E(Q)| \\
&\lesssim [v,w]^{p-1,1} v(F)^{-1} \int_F M(v\mathbf{1}_F) \\
&\le [v,w]^{p-1,1} [v]_{A_\infty}. \qquad \square
\end{aligned}$$

The next objective is a similar estimate for sparse operators. Since the maximal function is dominated by sparse operators, we cannot expect the estimate for sparse operators to be better than for maximal. Moreover, the estimate for sparse operators should be symmetric in the weights v and w (by duality). Hence the following result may be expected.

Proposition 8.2 ([HL15]). *Let $1<p<\infty$ and v,w be weights. Let $\mathscr{S}$ be a sparse collection. Then*

$$\|A_{\mathscr{S}}(fv)\|_{L^p(w)} \lesssim [v,w]^{1/p',1/p}([v]_{A_\infty}^{1/p} + [w]_{A_\infty}^{1/p'})\|f\|_{L^p(v)}.$$

Proof. By duality $\|A_{\mathscr{S}}(fv)\|_{L^p(w)} = \sup_{g:\|g\|_{L^{p'}(w)}=1} \int A_{\mathscr{S}}(fv) gw$, so it suffices to show

$$\sum_{Q\in\mathscr{S}} (fv)_Q (gw)_Q |Q| \lesssim [v,w]^{1/p',1/p}([v]_{A_\infty}^{1/p} + [w]_{A_\infty}^{1/p'})\|f\|_{L^p(v)}\|g\|_{L^{p'}(w)}.$$

Construct the stopping family $\mathscr{F}$ as before and $\mathscr{G}$ similarly for the function g and the measure $w\mathrm{d}\mu$. Then the left-hand side above is bounded by

$$\sum_{F\in\mathscr{F}} (f)_{F,v} \sum_{G\in\mathscr{G}} (g)_{G,w} \sum_{Q\in\mathscr{S}:\pi_{\mathscr{F}}(Q)=F,\pi_{\mathscr{G}}(Q)=G} (v)_Q (w)_Q |Q|.$$

We can split the sum into two parts: $F \subseteq G$ and $F \supsetneq G$. Since both parts are symmetric (under interchanging f with g, v with w, and p with p') we consider only the second. We rewrite that part as

$$\int \sum_F (f)_{F,v} \sum_{G:\pi_{\mathscr{F}}(G)=F} (g)_{G,w} \mathbf{1}_G \sum_{Q\in\mathscr{S}:\pi_{\mathscr{F}}(Q)=F,\pi_{\mathscr{G}}(Q)=G} (v)_Q \mathbf{1}_Q w$$

By Hölder's inequality this is bounded by

$$\Big(\int \sum_F \big(\sum_{G:\pi_{\mathscr{F}}(G)=F} (g)_{Q,w} \mathbf{1}_G \big)^{p'} w \Big)^{1/p'} \Big(\int \sum_F (f)_{F,v}^p \big(\sum_{G:\pi_{\mathscr{F}}(G)=F} \sum_{Q\in\mathscr{S}:\pi_{\mathscr{F}}(Q)=F,\pi_{\mathscr{G}}(Q)=G} (v)_Q \mathbf{1}_Q \big)^p w \Big)^{1/p}.$$

The first bracket is bounded by $\|M_w g\|_{L^{p'}(w)} \lesssim \|g\|_{L^{p'}(w)}$. We write the second bracket as

$$\Big(\sum_F (f)_{F,v}^p \int \big(\sum_{Q\in\mathscr{S}:\pi_{\mathscr{F}}(Q)=F} (v)_Q \mathbf{1}_Q \big)^p w \Big)^{1/p}.$$

Multiplying and dividing each summand by $v(F)$ and observing that

$$\sum_F (f)_{F,v}^p v(F) \lesssim \int (M_v f)^p v \lesssim \|f\|_{L^p(v)}$$

it remains to show

$$v(F)^{-1} \int \big(\sum_{Q\in\mathscr{S}:Q\subset F} (v)_Q \mathbf{1}_Q \big)^p w \lesssim [v,w]^{p-1,1} [v]_{A_\infty}.$$

This is more difficult than the corresponding step in the estimate for the maximal operator since now the sum is inside the power p. To get some feeling for what is going on let us first consider the case $p=2$. Then the left-hand side is

$$\begin{aligned}
\leq 2v(F)^{-1} \sum_{Q'\subseteq Q\subseteq F} (v)_{Q'} (v)_Q w(Q') &\leq [v,w]^{1,1} v(F)^{-1} \sum_{Q'\subseteq Q\subseteq F} (v)_Q |Q'| \\
&\lesssim [v,w]^{1,1} v(F)^{-1} \sum_{Q\subseteq F} (v)_Q |Q| \\
&\lesssim [v,w]^{1,1} [v]_{A_\infty},
\end{aligned}$$

where we have used sparseness of $\mathscr{S}$ in the penultimate step and argued as in the estimate for maximal operator in the last step.

For general p we use the numerical inequality

$$\big(\sum_i a_i\big)^p \lesssim \sum_{i_1\geq i_2 \leq \dots \geq i_{\lfloor p\rfloor}} a_{i_1} \cdots a_{i_{\lfloor p\rfloor}} \big(\sum_{i_{\lfloor p\rfloor}\geq i} a_i \big)^{\{p\}} 7 \tag{8.3}$$

to estimate

$$\int \big(\sum_{Q\in\mathscr{S}:Q\subset F} (v)_Q \mathbf{1}_Q \big)^p w \lesssim \int \sum_{F\supseteq Q_1 \supseteq \dots \supseteq Q_{\lfloor p\rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p\rfloor}} \mathbf{1}_{Q_{\lfloor p\rfloor}} \big(\sum_{Q\subseteq Q_{\lfloor p\rfloor}} (v)_Q \mathbf{1}_Q \big)^{\{p\}} w.$$

By Jensen's inequality this is bounded by

$$\begin{aligned}
&\sum_{F\supseteq Q_1 \supseteq \dots \supseteq Q_{\lfloor p\rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p\rfloor}} w(Q_{\lfloor p\rfloor}) \big(w(Q_{\lfloor p\rfloor})^{-1} \int \sum_{Q\subseteq Q_{\lfloor p\rfloor}} (v)_Q \mathbf{1}_Q w \big)^{\{p\}} \\
&= \sum_{F\supseteq Q_1 \supseteq \dots \supseteq Q_{\lfloor p\rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p\rfloor}} w(Q_{\lfloor p\rfloor}) \big(w(Q_{\lfloor p\rfloor})^{-1} \sum_{Q\subseteq Q_{\lfloor p\rfloor}} (v)_Q w(Q) \big)^{\{p\}}.
\end{aligned}$$

Consider first the case $p < 2$, so that $\lfloor p \rfloor = 1$ and $1/p > 1/p'$. Then we estimate this by

$$= \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \big(w(Q_1)^{-1} \sum_{Q \subseteq Q_1} (v)_Q (w)_Q |Q| \big)^{\{p\}}$$

$$\le [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \big(w(Q_1)^{-1} \sum_{Q \subseteq Q_1} (v)_Q^{1-p/p'} |Q| \big)^{\{p\}}$$

Using Lemma 8.4

$$\lesssim [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \big(w(Q_1)^{-1} (v)_{Q_1}^{1-p/p'} |Q_1| \big)^{\{p\}}$$

$$= [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}^{1+(1-p/p')\{p\}} (w)_{Q_1}^{1-\{p\}}$$

$$\le [v,w]^{(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}^{1+(1-p/p')\{p\}-(p/p')(1-\{p\})}$$

$$= [v,w]^{(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}$$

and using sparseness of $\mathscr{Q}$

$$\lesssim [v,w]^{(p/p',1)} [v]_{A_\infty} v(F).$$

Consider now the case $p \ge 2$, so that $1/p \le 1/p'$. Then we estimate

$$= \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \big(w(Q_{\lfloor p \rfloor})^{-1} \sum_{Q \subseteq Q_{\lfloor p \rfloor}} |Q| (v)_Q (w)_Q \big)^{\{p\}}$$

$$\le [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \big(w(Q_{\lfloor p \rfloor})^{-1} \sum_{Q \subseteq Q_{\lfloor p \rfloor}} |Q| (w)_Q^{1-p'/p} \big)^{\{p\}}$$

Using Lemma 8.4

$$\lesssim [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \big(w(Q_{\lfloor p \rfloor})^{-1} |Q_{\lfloor p \rfloor}| (w)_{Q_{\lfloor p \rfloor}}^{1-p'/p} \big)^{\{p\}}$$

$$= [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) (w)_{Q_{\lfloor p \rfloor}}^{-\{p\}p'/p}$$

$$\le [v,w]^{\{p\}(1,p'/p)+(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - 1}} |Q_{\lfloor p \rfloor}| (w)_{Q_{\lfloor p \rfloor}}^{1-\{p\}p'/p-p'/p}$$

Using Lemma 8.4 again

$$\lesssim [v,w]^{\{p\}(1,p'/p)+(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor - 1}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - 1}} |Q_{\lfloor p \rfloor - 1}| (w)_{Q_{\lfloor p \rfloor - 1}}^{1-\{p\}p'/p-p'/p}.$$

Continuing in this manner we obtain inductively

$$\lesssim [v,w]^{\{p\}(1,p'/p)+m(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor - m}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - m}} |Q_{\lfloor p \rfloor - m}| (w)_{Q_{\lfloor p \rfloor - m}}^{1-\{p\}p'/p-mp'/p}.$$

For $m = \lfloor p \rfloor - 1$ this gives the estimate

$$\lesssim [v,w]^{\{p\}(1,p'/p)+(\lfloor p \rfloor - 1)(1,p'/p)} \sum_{F \supseteq Q_1} (v)_{Q_1} |Q_1| (w)_{Q_{\lfloor p \rfloor - m}}^{1-\{p\}p'/p-(\lfloor p \rfloor - 1)p'/p}$$

$$= [v,w]^{(p-1)(1,p'/p)} \sum_{F \supseteq Q_1} (v)_{Q_1} |Q_1| (w)_{Q_{\lfloor p \rfloor - m}}^{1-(p-1)p'/p}$$

$$= [v,w]^{(p/p',1)} \sum_{F \supseteq Q_1} (v)_{Q_1} |Q_1|$$

and using sparseness of $\mathscr{S}$ again

$$\lesssim [v,w]^{(p/p',1)} [v]_{A_\infty} v(F).$$

□

In the above proof we have used repeatedly the following fact.

Lemma 8.4. *Let* $0 \le \beta < 1$ *and let* $\mathscr{S}$ *be sparse. Then for every non-negative function* v *we have*

$$\sum_{Q \in \mathscr{S}, Q \subset F} |Q| (v)_Q^\beta \lesssim |F| (v)_F^\beta.$$

In the case $\beta = 1$ the implicit constant in this lemma has to be replaced by A_∞.

Proof. The left-hand side is bounded by

$$\begin{aligned}
\sum_Q |E(Q)| (v)_Q^\beta \le \int_F (M(v\mathbf{1}_F))^\beta \lesssim \sum_{k \in \mathbb{Z}} 2^{k\beta} |F \cap \{M(v\mathbf{1}_F) > 2^k\}| \\
\le \sum_{k \in \mathbb{Z}} 2^{k\beta} \min(|F|, |\{M(v\mathbf{1}_F) > 2^k\}|) \le \sum_{k \in \mathbb{Z}} 2^{k\beta} \min(|F|, 2^{-k} \|v\|_{L^1(F)}) \\
= |F| \sum_{k \in \mathbb{Z}} 2^{k\beta} \min(1, 2^{-k} (v)_F) \lesssim |F| \sum_{k \in \mathbb{Z}} (v)_F^\beta,
\end{aligned}$$

where in the last inequality we have used that a geometric series is dominated by its larges term. □

Proof of (8.3). The claim (8.3) follows by $\lfloor p \rfloor$ applications of the following inequality (valid for $p \ge 1$):

$$(\sum_i a_i)^p \le p \sum_{i_1} a_{i_1} (\sum_{i_1 \ge i} a_i)^{p-1}. \tag{8.5}$$

To show this inequality notice that it suffices to consider finite sequences (a_i). For real $a, b \ge 0$ we have

$$(a+b)^p = a^p + \int_a^{a+b} p t^{p-1} \mathrm{d}t \le a^p + p(a+b)^{p-1} \int_a^{a+b} \mathrm{d}t = a^p + p(a+b)^{p-1} b.$$

Using this inequality with $a = \sum_{i=1}^m a_i$, $b = a_{m+1}$ we obtain

$$(\sum_{i=1}^{m+1} a_i)^p \le (\sum_{i=1}^{m} a_i)^p + p a_{m+1} (\sum_{i=1}^{m+1} a_i)^{p-1},$$

and the claim (8.5) follows by induction on m. □

9 Weighted weak type $(1,1)$ for sparse operators

9.1 Orlicz spaces

Definition 9.1. A *Young function* is a convex increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Lemma 9.2. *Let φ be a Young function with $\lim_{t\to\infty} \varphi(t)/t = \infty$. Then*

$$\psi(s) = \sup_{t>0}(st - \varphi(t))$$

is also a Young function, called the complementary Young function of φ.

Proof. All properties are easy to verify with the possible exception of convexity. Let $0 \le s_0 < s_1 < \infty$ and $0 < \lambda < 1$. Then

$$\begin{aligned}
\psi((1-\lambda)s_0 + \lambda s_1) &= \sup_{t>0}(((1-\lambda)s_0 + \lambda s_1)t - \varphi(t)) \\
&= \sup_{t>0}((1-\lambda)(s_0 t - \varphi(t)) + \lambda(s_1 t - \varphi(t))) \\
&\ge \sup_{t>0}(1-\lambda)(s_0 t - \varphi(t)) + \sup_{t>0} \lambda(s_1 t - \varphi(t)) \\
&= (1-\lambda)\psi(s_0) + \lambda\psi(s_1).
\end{aligned}$$

□

Example. If $\varphi(t) = t^p$, then $\psi(s) = s^{p'}$ (exercise).

Definition 9.3. Let (X,μ) be a measure space and φ a Young functional. The Orlicz space $\varphi(L)(X,\mu)$ is defined by

$$\|f\|_\varphi = \inf\{\Lambda > 0 : \int_X \varphi(|f|/\Lambda) \le 1\}.$$

It is clear that this defines a homogeneous functional, and quasisubadditivity is also not hard to verify.

Lemma 9.4. *Let φ be a continuous Young function and ψ its complementary Young functions. Then*

$$\|f\|_\varphi \sim \sup_{g:\|g\|_\psi \le 1} \int_X |fg| \mathrm{d}\mu.$$

Proof. First we show $\gtrsim$. Notice that $\psi(s)+\varphi(t) \ge ts$ for all $t,s > 0$. Suppose $\|f\|_\varphi < 1$ and $\|g\|_\psi < 1$. Then

$$\int |fg| \le \int_{\{fg\ne 0\}} \varphi(|f|) + \psi(|g|) \le 2.$$

For the converse we notice

$$\psi(\varphi(u)/u) = \sup_{t>0} t\varphi(u)/u - \varphi(t) = \sup_{0<t\le u} t\varphi(u)/u - \varphi(t) \le \sup_{0<t\le u} t\varphi(u)/u = \varphi(u),$$

where we have restricted the parameter in the supremum using concavity of the argument and the fact that the argument vanishes at 0 and at u. Suppose now that $\|f\|_\varphi > 1$ and without loss of generality $f \ge 0$. Then for every $\Lambda > 1$ we have $\int \varphi(f/\Lambda) < 1$, so with $g_\Lambda = \varphi(f/\Lambda)/(f/\Lambda)$ we obtain

$$\int \psi(g_\Lambda) \le \int \varphi(f/\Lambda) < 1.$$

On the other hand,

$$\int f g_\Lambda = \int \Lambda\varphi(f/\Lambda) \to \int \varphi(f) \ge 1$$

as $\Lambda \to 1$. □

Theorem 9.5 ([DSLR16, Theorem 1.6]). *Let φ be a Young function and ψ its complementary function. Let $\mathscr{S} \subset \mathscr{D}$ be a $1/16$-sparse collection. Then*

$$\sup_{\lambda>0} \lambda w\{A_{\mathscr{S}} f > \lambda\} \lesssim \sum_{k=1}^{\infty} \frac{1}{\psi^{-1}(2^{2^k})} \int |f| M_{\varphi} w,$$

where ψ^{-1} denotes the inverse function of ψ and

$$M_{\varphi} w(x) = \sup_{x \in Q \in \mathscr{D}} (w)_{Q,\varphi}, \quad (w)_{Q,\varphi} = \inf\{v > 0 : \fint_Q \varphi(w/v) \leq 1\}.$$

Corollary 9.6. *Theorem 9.5 applies if $\varphi(t) = tL(t)$ with $0 \leq sL'(s) \leq C$ and $\sum_{k\geq 1} 1/L(2^{2^k}) < \infty$, in particular e.g. if $L(t) = \ln\ln t(\ln\ln\ln t)^{1+\varepsilon}$.*

Proof.

$$\begin{aligned}\psi(L(t)) = \sup_{\tau>0}(L(t)\tau - \varphi(\tau)) = \sup_{0<\tau\leq t} \tau(L(t) - L(\tau)) &\leq \sup_{0<\tau\leq t} \tau(L(t) - L(\tau)) \\ &\lesssim \sup_{0<\tau\leq t} \tau \int_{\tau}^{t} s^{-1} \mathrm{d}s = \sup_{0<\tau\leq t} \tau \ln(t/\tau) \lesssim t.\end{aligned}$$

It follows that $L(t) \leq \psi^{-1}(Ct)$. □

It is known that Theorem 9.5 fails if $\varphi(t) = o(t \ln\ln t)$ [CLO17]. We will prove an earlier result that it fails if $\varphi(t) = t$ [RT12]. It also seems to be known that the norm of the Hilbert transform from $L^1(w)$ to $L^{1,\infty}(w)$ grows faster than linearly in $[w]_{A_1}$ [NRVV15; NRVV16], but this is a more difficult result.

Lemma 9.7 (cf. [CUP00]). *Let T be a linear operator on $L^2(\mathbb{R}^d)$ and T' its adjoint. Assume that*

$$\|T'f\|_{L^{1,\infty}(w)} \lesssim \|f\|_{L^1(Mw)}.$$

Then

$$\int |Tw|^2 (Mw)^{-2} w \lesssim \int w.$$

Proof. Let w be a weight. Note

$$M(w\mathbf{1}_{\Omega})(x) \lesssim \sup_{x \in I \in \cup_\alpha \mathscr{D}^\alpha : w(I) \neq 0} \frac{w(I)}{I} \Big(w(I)^{-1} \int_I \mathbf{1}_{\Omega} w \big) \leq \sup_{\alpha} M_{\mathscr{D}^\alpha}(w)(x) M_{\mathscr{D}^\alpha, w}(\mathbf{1}_{\Omega})(x).$$

Let w be a weight, f a function supported on $W = \operatorname{supp} w$, and $\Omega = \{|T'f| > 1\}$. Then

$$\begin{aligned} w(\Omega) \lesssim \int_W |f| M(w\mathbf{1}_{\Omega}) \leq \sum_{\alpha} (\int_W |f|^2 M_{\mathscr{D}^\alpha}(w)^2 w^{-1})^{1/2} (\int_W M_{\mathscr{D}^\alpha, w}(\mathbf{1}_{\Omega}) w)^{1/2} \\ \lesssim (\int_W |f|^2 M(w)^2 w^{-1})^{1/2} w(\Omega)^{1/2},\end{aligned}$$

where we have used the L^2 estimate for the weighted dyadic maximal function. Dividing both sides by $w(\Omega)^{1/2}$ and using homogeneity we obtain

$$\|T'f\|_{L^{2,\infty}(w)} \lesssim \|f\|_{L^2(W,(Mw)^2 w^{-1})}.$$

By duality for functions g supported in W we have

$$\|Tg\|_{L^2(W,(Mw)^{-2}w)} = \|(Mw)^{-1}w^{1/2}Tg\|_{L^2(W)} = \sup_{\|f\|_{L^2(W)}=1} |\int f(Mw)^{-1}w^{1/2}Tg|$$
$$= \sup_{\|h\|_{L^2(W,(Mw)^2w^{-1})}=1} |\int hTg| = \sup_{\|h\|_{L^2(W,(Mw)^2w^{-1})}=1} |\int (T'h)g|$$
$$\leq \sup_{\|f\|_{L^{2,\infty}(w)}\lesssim 1} |\int fg|.$$

Substituting $g = w$ we obtain on the right-hand side

$$\sup_{\|f\|_{L^{2,\infty}(w)}\lesssim 1} |\int fw| \lesssim (\int w)^{-1/2}(\int w) + \sup_{\|f\|_{L^{2,\infty}(w)}\lesssim 1} \sum_{k\in\mathbb{Z}:2^k\geq(\int w)^{-1/2}} 2^{k+1} \int_{\{2^k<|f|\leq 2^{k+1}\}} w$$
$$\lesssim (\int w)^{1/2} + \sum_{k\in\mathbb{Z}:2^k\geq(\int w)^{-1/2}} 2^{-k} \lesssim (\int w)^{1/2}. \quad \square$$

Construct collections of intervals in $\mathbb{R}$ as follows. Fix large $k \in \mathbb{N}$. Let $\mathscr{J}_1 = \{[0,1]\}$. For each l and each $J \in \mathscr{J}_l$ subdivide $\frac{1}{3}J$ into 3^{k-1} intervals of length $3^{-k}|J|$ (call the set of these intervals $\mathrm{ch}(J)$) and add all these intervals to $\mathscr{J}_{l+1}$.

For each l and $J \in \mathscr{J}_l$ let $P(J)$ be an interval of length $3^{-k}|J|$ situated either to the left or to the right from $\frac{1}{3}J$ (we will decide later on which side each $P(J)$ is situated). Then the intervals $P(J), J \in \cup_l \mathscr{J}_l$ are pairwise disjoint. Let $\Omega_l = \cup_{J\in\mathscr{J}_l} P(J)$, $\Omega_l' = \cup_{J\in\mathscr{J}_l} \frac{1}{3}P(J)$, and consider the weight

$$w = \sum_{l=1}^{\infty} \Big(\frac{3^k}{3^{k-1}+1}\Big)^l \mathbf{1}_{\Omega_l}.$$

Lemma 9.8. *$Mw \lesssim w$ on $\cup_l \Omega_l'$ no matter how $P(J)$ are chosen.*

Proof. Let $x \in \frac{1}{3}I$, $I = P(J)$, $J \in \mathscr{J}_l$. On $\Omega_{l'}$ with $l' \leq l$ we have $w \leq w(x)$, so it suffices to consider contributions of $\Omega_{l'}$ with $l' > l$. The point x is separated from $\Omega_{l'}$ at least by $\frac{1}{3}|I|$, so we may restrict the supremum in the definition of maximal function to intervals whose endpoints are multiples of $|I|$. By construction $w(J') = w(P(J))$ for each J and $J' \in \mathrm{ch}(J)$, and it follows that on each interval of length $|I|$ the mass of w does not exceed $w(I)$. $\square$

Lemma 9.9. *One can choose $P(J)$ in such a way that $|Hw| \geq (k/3)w$ on $\cup_l \Omega_l'$, where H is the Hilbert transform.*

Proof. Let $l \in \mathbb{N}$, $J \in \mathscr{J}_l$, $I = P(J)$, $x \in \frac{1}{3}I$. Split

$$\int \frac{w(y)}{y-x}\mathrm{d}y = \int_I \frac{w(y)}{y-x}\mathrm{d}y + \int_{\frac{1}{3}J} \frac{w(y)}{y-x}\mathrm{d}y$$
$$+ \int_{J^c} \frac{w(y)}{y-c(J)}\mathrm{d}y + \int_{J^c} \Big(\frac{w(y)}{y-x} - \frac{w(y)}{y-c(J)}\Big)\mathrm{d}y.$$

The first and the last summands are bounded by $Cw(x)$. The third summand only depends on the choices of $P(J)$ for $J \in \mathscr{J}_{l'}$ with $l' < l$. The absolute value of the second term is bounded below by $kw(x)$, and this term can be positive or negative depending on the choice of $P(J)$. Choose $P(J)$ so that the sign of this term matches the sign of the third term. $\square$

Using these two results and assuming that the Hilbert transform satisfies $\|Hf\|_{L^{1,\infty}(w)} \lesssim \|f\|_{L^1(Mw)}$, using also Lemma 9.7, we obtain

$$k^2 \int w \lesssim k^2 \int_{\cup_l \Omega_l'} w \lesssim \int |Hw|^2 (Mw)^{-2} w \lesssim \int w$$

and this is a contradiction for large k.

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