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# Partial Differential Equations and Modelling

Sheet Nr. 10 Due: 07.07.2017

## Exercise 1

Consider the ordinary differential equation

$$P(u) = \frac{d}{dx}u + \tanh(x)u = f, \quad x \in \mathbb{R}.$$

Determine the null space of $P$ and prove that for every $f$ there is a solution $u \in H^1$.
Prove that

$$Q(u) = \frac{d}{dx}u - \tanh(x)u = f$$

has at most one solution in $H^1$ and determine for which $f \in L^2$ the equation is solvable.
Can you generalize the results to more general equations

$$\frac{d}{dx}u + h(x)u = f?$$

Hint: Write down a formula for the general solution.

## Exercise 2

Consider the ordinary differential equation

$$u_{xx} + u = f$$

and consider a uniformly convex $C^3$ function $h$ with

$$h'' \geq 1, \qquad |h^{(3)}| \leq \frac{1}{10}h''.$$

Prove

$$\|e^h u\|_{L^2} \leq 2\|e^h f\|_{L^2}$$

for $u \in H^2$ with compact support.

Hint: Derive an equation for $v = e^h u$, multiply by $h'v'$ and integrate.

## Exercise 3

The strong unique continuation property states that if a solution to a linear pde in a connected open set $U$ vanishes in an open subset, then it vanishes in $U$. In which cases does the strong unique continuation property hold?

a) $\Delta u = Vu$, $V$ bounded.

b) The heat equation $u_t - \Delta u = 0$.

c) The wave equation $u_{tt} - \Delta u = 0$.

**Exercise 4**

Find all solutions to

$$-u_{xx} - \delta u = z^2 u$$

with $u = c_\pm e^{iz|x|}$ for $x \neq 0$: Pay attention that the product $\delta u$ can not be well-defined if $u$ is irregular enough, e.g. $u$ is discontinuous at $0$. Prove that the operator

$$H^1 \ni u \to -u_{xx} - \delta u \in H^{-1}$$

is well defined. Here $\delta$ is the Dirac measure and $z \in \mathbb{C}$.