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Summer Term 2017

Partial Differential Equations and Modelling

Sheet Nr.9 Due: 30.06.2017

Exercise 1

Let $f \in C^2([0,\infty))$, $V(x) = f(|x|)$. Suppose that

$$-\Delta\Psi + V(x)\Psi = 0$$

on $B_2(0)\backslash\overline{B_{1/2}(0)}$ with $\Psi \in C^2$. Let h_m be a real homogeneous harmonic polynomial of degree m with $\|h_m\|_{L^2(\mathbb{S}^{d-1})} = 1$. We define

$$\Psi_m(r) = \int_{\mathbb{S}^{d-1}} \Psi(ry)h_m(y)dy.$$

Prove that for $r \in (1/2, 2)$,

$$-\Psi_m''(r) - \frac{d-1}{r}\Psi_m' + \frac{m(m+d-2)}{r^2}\Psi_m + f(r)\Psi_m = 0.$$

Exercise 2

Let Ψ be as in the last exercise. Prove that it can be written as

$$\Psi(x) = \sum_{m=0}^{\infty} \Psi_m(|x|)h_m(x/|x|)$$

for suitable homogeneous harmonic polynomials h_m. It suffices to prove that the sum converges in L^2_{loc}.

Exercise 3

Consider Kummer's confluent hypergeometric differential equation

$$z\frac{d^2u}{dz^2} + (b-z)\frac{du}{dz} - az = 0$$

where $a, b \in \mathbb{C}$, b not a negative integer. Prove the first part of Lemma 4.17: There exists a unique holomorphic solution $M(a, b; z)$ with $M(a, b; 0) = 1$.

Exercise 4

Assume in the setting of the last exercise that $\operatorname{Re} b > 1$. Prove that there is a second unbounded solution as stated, i.e. it is defined in $B_1 \backslash (-1, 0]$ and unbounded near 0.

Hint: Consider the equation for the Wronskian $W(z) = M(a, b; z)v' - M'(a, b; z)v$ in $(0, \varepsilon)$ where v is a second solution. Find the general solution and consider the equation

$$v' = M(a, b; z)^{-1}(M'(a, b; z)v + W(z)$$

in $(0, \varepsilon)$.