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UNIVERSITÄT BONN

Partial Differential Equations and Modelling

Sheet Nr.2 Due: 05.05.2017

Exercise 1

Prove the following statement: Every function f in $L^2(\mathbb{R}^d;\mathbb{C})$ can be written as

$$f = f_1 + f_{-1} + f_i + f_{-i}$$

so that

$$\|f\|_{L^2}^2 = \|f_1\|_{L^2}^2 + \|f_{-1}\|_{L^2}^2 + \|f_i\|_{L^2}^2 + \|f_{-i}\|_{L^2}^2$$

and

$$\hat{f}_1 = f_1, \quad \hat{f}_{-1} = -f_{-1}, \quad \hat{f}_i = if_i, \qquad \hat{f}_{-i} = -if_{-i}.$$

The decomposition is unique.
Hint: Consider the even part and odd part of the function f separately.

Exercise 2

The harmonic oscillator is defined by

$$L\psi = -\Delta\psi + |x|^2\psi, \quad x \in \mathbb{R}^d$$

for Schwartz functions.

- Prove that
$$L\psi = -\sum_{j=1}^{d}(\partial_j - x_j)(\partial_j + x_j)\psi + d\psi.$$
Here, we use the convention that $\partial_j x_j \psi = \partial_j(x_j\psi) = \psi + x_j\partial_j\psi$.
- Find a formula for the commutator
$$\Big[\frac{1}{2}(\partial_j + x_j), \frac{1}{2}(\partial_j - x_j)\Big]\psi := \Big(\frac{1}{2}(\partial_j + x_j)\Big)\Big(\frac{1}{2}(\partial_j - x_j)\Big)\psi - \Big(\frac{1}{2}(\partial_j - x_j)\Big)\Big(\frac{1}{2}(\partial_j + x_j)\Big)\psi.$$
- Compute Lh_α, where $h_\alpha = [\frac{1}{2}(x-\partial)]^\alpha e^{-\frac{|x|^2}{2}}$ for the multiindex α. Please start with the case $d = 1$.

Exercise 3

This exercise requires some knowledge about the Laplace operator and its fundamental solution.
Let $d = 3$ and

$$g = \frac{1}{3|B_1(0)|}|x|^{-1}e^{-|x|}$$

where $|B_1(0)|$ is the volume of the unit ball.

- Prove that

$$g - \Delta g = \delta_0.$$

  Hint: g is a radial function and there is an easy way to calculate $-\Delta g$ away from the origin.

- Derive from $g - \Delta g = \delta_0$ an integral formula for a solution to

$$u - \Delta u = f.$$

- Derive from $g - \Delta g = \delta_0$ the Fourier transform of g.

Exercise 4

Let $p = p(x) = p(x_1, \cdots, x_d)$ be a polynomial in d variables. Let

$$p(D)u$$

be the operator with constant coefficients obtained by replacing x_j in the argument by $\frac{1}{i}\partial_j$, e.g. if $p(x) = Cx_1$ then $p(D)u = C\frac{1}{i}\partial_1 u$.

Prove that

$$(\widehat{p(D)u})(k) = p(k)\hat{u}(k).$$

and

$$p(D)u = (p(D)\delta_0) * u.$$

We then identify $p(D)$ with the distribution $p(D)\delta_0$.