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Partial Differential Equations and Modelling

Sheet Nr.1 Due: 28.04.2017

Exercise 1

Let $f \in L^1(\mathbb{R}^d; \mathbb{C})$ have compact support. Prove that the Fourier transform is the restriction of a holomorphic function on $\mathbb{C}^d$.

Hint: Prove that the formula for the Fourier transform make sense if we replace $k \in \mathbb{R}^d$ by $z \in \mathbb{C}^d$. Prove that the extension $\hat{f}(z)$ with $z = k + iw$ satisfies the Cauchy-Riemann equations

$$\partial_{k_j} \operatorname{Re} \hat{f} = \partial_{w_j} \operatorname{Im} \hat{f}$$
$$\partial_{w_j} \operatorname{Re} \hat{f} = -\partial_{k_j} \operatorname{Im} \hat{f}$$

Exercise 2

We want to compute the Fourier transform of $f(x) = e^{-\alpha x^2}$ where $x \in \mathbb{R}$ and $\alpha = a + ib$ with $a > 0$ and $b \in \mathbb{R}$.

- Prove that $f \in \mathcal{S}(\mathbb{R})$.
- Let
$$I(a + ib) = \int_{\mathbb{R}} e^{-(a+ib)x^2} dx, \quad a > 0, \quad b \in \mathbb{R}.$$
Compute $\partial_b[\sqrt{a + ib} I(a + ib)]$ and $\partial_a[\sqrt{a + ib} I(a + ib)]$.
Deduce a formula for $I(a + ib)$.
- Find a differential equation for $\hat{f}$ and use it to compute $\hat{f}$.

Exercise 3

Let A be an invertible $d \times d$ matrix and let $f \in L^1(\mathbb{R}^d)$. Express $\mathcal{F}(f \circ A)$ in terms of $\hat{f}$. Suppose that B is a positive definite $d \times d$ matrix. Compute the Fourier transform of $f(x) = e^{-x^t B x}$.

Exercise 4

Let $h \in \mathbb{R}^d$. Express the Fourier transform of $x \to f(x + h)$ in terms of the Fourier transform of f.