Real and Harmonic Analysis, Problem set 9

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Due on Tuesday, 2016-07-05
Problems marked as oral will not be graded.
Please submit your solutions in groups of two

Problem 1. Here f^* denotes the non-increasing rearrangement (say, on $\mathbb{R}^n$) and $f_+ = \max(f, 0)$ the positive part of a function.

(a) Show that $(f-t)_+^* = (f^* - t)_+$ for positive functions f.

(b) Show that the non-increasing rearrangement is L^p-contractive in the sense that $\|f^* - g^*\|_p \leq \|f - g\|_p$, $1 \leq p \leq \infty$. Hint: use the following version of the layer cake representation:

$$|f-g|^p = p \int_0^\infty (f-t)_+^{p-1} 1_{g \leq t} + (g-t)_+^{p-1} 1_{f \leq t} dt, \quad 1 \leq p < \infty.$$

Problem 2 (Helly's selection principle)**.** (a) Let $f : [0,\infty) \to [0, C]$ be a non-increasing function. Show that f has at most countably many discontinuities.

(b) Let $f_n : [0,\infty) \to [0, C]$ be a sequence of (uniformly bounded) non-increasing functions. Show that there exists a subsequence f_{n_k} that converges pointwise everywhere.

Problem 3 (oral)**.** Let K be a CZ distribution on $\mathbb{R}^d$, that is, it coincides with a function k away from the origin, and the function k satisfies

$$|\partial^\alpha k(x)| \lesssim_\alpha |x|^{-d-|\alpha|} \quad \text{for all } \alpha, \tag{1}$$

$$\exists n \geq 1 : \sup_{\phi \in B_n, r>0} |K(\phi_r)| < \infty, \tag{2}$$

where B_n is the set of all smooth functions ϕ supported in the unit ball with $\sup_{x,|\alpha|\leq n} |\partial^\alpha \phi(x)| \leq 1$ and $\phi_r(x) = \phi(rx)$.

(a) Show that the distribution $f \mapsto K(f(x)x_j)$ coincides with the function $x_j k$ for every $j = 1, \dots, d$.

(b) Show that the cancellation condition (2) in fact holds with $n = 1$ (assuming that it holds for some $n \geq 1$).

Problem 4 (oral)**.** Let $X_0 \subset \mathbb{R}^d$ be a hyperplane that does not contain the origin, X_+ the open half-space with boundary X_0 that contains the origin, and X_- the complementary open half-space. Denote by σ the reflection on X_0. For a non-negative function f on $\mathbb{R}^d$, the *polarization at* X_0 is the function

$$f^\sigma(x) = \begin{cases} \max(f(x), f(\sigma x)), & x \in X_+ \cup X_0, \\ \min(f(x), f(\sigma x)), & x \in X_- \cup X_0 \end{cases}$$

(note that both lines are equal for $x \in X_0$).

(a) Suppose that the function f has modulus of continuity ω, that is, $\omega : [0,\infty) \to [0,\infty]$ is a non-decreasing subadditive function and $|f(x) - f(y)| \leq \omega(|x-y|)$ for all $x, y \in \mathbb{R}^d$. Show that f^σ also has modulus of continuity ω.

(b) Show that

$$\int f^\sigma g^\sigma \geq \int fg$$

with equality if and only if $(f(x) - f(\sigma x))(g(x) - g(\sigma x)) \geq 0$ for a.e. $x \in \mathbb{R}^d$.

See "A short course on rearrangement inequalities" by Burchard (available online) for a proof of the special case of the Riesz rearrangement inequality that is used in the HLS inequality using polariaztion.

Problem 5 (*)**.** Let f be a non-negative function on $\mathbb{R}^d$ with modulus of continuity ω. It is true that the Steiner symmetrization Sf (that is, the radial non-increasing rearrangement in the last coordinate) also has modulus of continuity ω?