Real and Harmonic Analysis, Problem set 8

Mathematisches Institut
Dr. Diogo Oliveira e Silva
Dr. Pavel Zorin-Kranich
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Due on Tuesday, 2016-06-28
Problems marked as oral will not be graded.
Please submit your solutions in groups of two

Problem 1 (oral)**.** Let ν be a (positive) measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} (1+|x|)^N d\nu(x) < \infty$$

for some $N \in \mathbb{R}$.

(a) Show that the measure ν defines a tempered distribution by the formula $\nu(f) = \int f d\nu$.

(b) In the case $N \geq 0$ show that the Fourier transform $\hat{\nu}$ of this tempered distribution coincides with a function $g \in C^{\lfloor N \rfloor}(\mathbb{R}^d)$ such that $\partial^\alpha g = (-2\pi i)^\alpha \int x^\alpha d\nu$ for every multiindex α with $0 \leq |\alpha| \leq N$.

Problem 2 (Central Limit Theorem)**.** Let ν be a probability measure on the real line such that $\int x d\nu(x) = 0$ and the variance $\sigma^2 := \int x^2 d\nu(x)$ is finite.

(a) Show that

$$\lim_{N\to\infty} \hat{\nu}(\xi/\sqrt{N})^N = \exp(-2\pi^2\sigma^2\xi^2)$$

for every $\xi \in \mathbb{R}$.

(b) Let ν_N be the measure on $\mathbb{R}$ defined by

$$\int f(x) d\nu_N(x) := \int f(x/\sqrt{N}) d\nu^{*N}(x),$$

where ν^{*N} denotes the N-th convolution power. Show that

$$\nu_N \to (2\sigma^2\pi)^{1/2} \exp(-\frac{x^2}{2\sigma^2})$$

in the sense of tempered distributions.

Problem 3 (Uncertainty principle)**.** The following sharp version (due to Beckner) of the Hausdorff–Young inequality for the Fourier transform has been proved in class:

$$\|\widehat{f}\|_{p'} \leq (p^{1/2p}/p'^{1/2p'})^n \|f\|_p, \qquad f \in L^p(\mathbb{R}^n), 1 < p \leq 2.$$

(a) Let $\alpha, \beta > 1/2$ be such that $\alpha, \beta \neq 1$ and $1/\alpha + 1/\beta = 2$. Show that

$$H_\alpha(|f|^2) + H_\beta(|\hat{f}|^2) \geq \frac{n}{2}\Big(\frac{\log 2\alpha}{\alpha - 1} + \frac{\log 2\beta}{\beta - 1}\Big), \tag{1}$$

where $f \in L^{2\alpha}(\mathbb{R}^n)$ and $H_\alpha(f) = \frac{1}{1-\alpha} \log \int f^\alpha$ is the *Rényi entropy*. Hint: assume without loss of generality $\alpha < \beta$ and take the logarithm of the sharp Hausdorff–Young inequality.

(b) Let $f \in L^2(\mathbb{R}^n) \cap L^{2-\epsilon}(\mathbb{R}^n)$ for some $\epsilon > 0$. Taking the limit in (1) as $\alpha, \beta \to 1$ show that

$$H(|f|^2) + H(|\hat{f}|^2) \geq \|f\|_2^2(-2\log\|f\|_2^2 + n(\log e - \log 2)), \tag{2}$$

where $H(f) = -\int f \log f$ is the *Shannon entropy.*

(c) Using the Shannon entropy inequality $H(p) \leq \log\sqrt{2\pi e \mathrm{Var}(p)}$ for probability densities p on $\mathbb{R}$ (introduced in Shannon's 1948 article "A mathematical theory of communication") conclude that, for $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$, one has

$$\sqrt{\mathrm{Var}(|f|^2)\mathrm{Var}(|\hat{f}|^2)} \geq \frac{1}{4\pi}. \tag{3}$$

(A probability density p is a non-negative function with $\int p = 1$, its mean is $\mu = \int xp(x)d(x)$, and variance $\mathrm{Var}(p) = \int (x-\mu)^2 p(x)dx$.)

Problem 4 (oral)**.** Verify that in (2) and (3) equality is attained for Gaussians.