Real and Harmonic Analysis, Problem set 4

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Due on Tuesday, 2016-05-24
May 26 is a holiday, so there are no oral problems this time.
Please submit your solutions in groups of two

Problem 1 (Common Fourier transforms)**.** Express the Fourier transform of a function of the form $f(x/t)$ in terms of $\hat{f}$. Compute the Fourier transforms of the following functions on $\mathbb{R}$.

(a) $f(x) = 1_{[-t,t]}(x)$, $t > 0$ ($D_R = \hat{f}$ is the *Dirichlet kernel*)

(b) $f(x) = \max(1 - |x|/t, 0)$, $t > 0$ ($F_R = \hat{f}$ is the *Fejér kernel*)

(c) $f(x) = \frac{1}{t}\frac{1}{1+(x/t)^2}$, $t > 0$ (f is the *Poisson kernel*). Hint: use the Cauchy integral formula for the contour consisting of the line segment $[-R, R]$ and a half-circle.

(d) $f(x) = \operatorname{sech}(x) = \frac{2}{e^x+e^{-x}}$. Hint: use the Cauchy integral formula for a rectangle of height π and large width.

Problem 2 (Convergence of Fourier integrals)**.** Notice that

$$\widehat{D_R * f}(\xi) = 1_{[-R,R]}(\xi)\widehat{f}(\xi)$$

and

$$\widehat{F_R * f}(\xi) = \max(1 - |\xi|/R, 0)\widehat{f}(\xi)$$

holds for all Schwartz functions f.

(a) Show that the operators $f \mapsto D_R * f$, $R > 0$, are uniformly bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

(b) Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Show that $D_R * f \to f$ in L^p as $R \to \infty$.

(c) Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Show that $F_R * f \to f$ pointwise almost everywhere as $R \to \infty$.

It is also true that $D_R * f \to f$ pointwise almost everywhere for $f \in L^p(\mathbb{R})$, $1 < p < \infty$. This is a difficult result due to Carleson.