# Real and Harmonic Analysis, Problem set 1

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**Due on Thursday, 2016-04-21**
Problems marked as oral will not be graded, but will be discussed during the exercise class.
Please submit your solutions in groups of two

**Problem 1** (Minkowski integral inequality)**.** Let $X_1, X_2$ be $\sigma$-finite measure spaces, $f$ a non-negative measurable function on $X_1 \times X_2$, and $1 \leq p \leq \infty$. Show that

$$\| \int_{X_2} f(x_1, x_2) dx_2 \|_{L^p(x_1)} \leq \int_{X_2} \|f(x_1, x_2)\|_{L^p(x_1)} dx_2.$$

Hint: for $1 < p < \infty$ use Hölder's inequality and its converse.

**Problem 2.** Show that the following sets of functions are dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

(a) The set $L^\infty_c$ of bounded measurable functions with compact support,

(b) the set of finite linear combinations of characteristic functions of bounded measurable sets,

(c) the set of finite linear combinations of characteristic functions of rectangular boxes (with edges parallel to coordinate axes),

(d) the set $C^\infty_c$ of smooth functions with compact support.

Hint: in the penultimate step use the outer measure to approximate sets of finite measure by finite unions of cubes.

**Problem 3** (weak topology)**.** A sequence of functions $f_n$ in $L^p(\mathbb{R}^d)$, $1 < p \leq \infty$, is said to converge *weakly* to $f \in L^p$, in symbols $f_n \rightharpoonup f$ if

$$\int f_n g \to \int fg \tag{1}$$

for every $g \in L^{p'}$.
Let $(f_n)$ be a sequence in $L^p$ with $\|f_n\| \leq C < \infty$ for all $n$.

(a) Show that it suffices to verify (1) for $g$ in any given dense subset $D \subset L^{p'}$.

(b) Show that $L^{p'}$ is *separable*, that is, it contains a countable dense subset (hint: use part (c) of Problem 2).

(c) Conclude that the sequence $(f_n)$ has a weakly convergent subsequence.

**Problem 4** (Multiple term Hölder inequality, oral)**.** Let $1 \leq p_1, \ldots, p_n \leq \infty$ be such that $\sum_{j=1}^n p_j^{-1} = 1$. Show that

$$|\int \prod_{j=1}^n f_j| \leq \prod_{j=1}^n \|f_j\|_{p_j}.$$

What if $n = \infty$?