The Boolean surface area of polynomial threshold functions

Abstract

Polynomial threshold functions (PTFs) are an important low-complexity class of Boolean functions, with strong connections to learning theory and approximation theory. Recent work on learning and testing PTFs has exploited structural and isoperimetric properties of the class, especially bounds on average sensitivity, one of the central themes in the study of PTFs since the Gotsman–Linial conjecture.

In this work we exhibit a new geometric sense in which PTFs are tightly constrained, by studying them through the lens of the *Boolean surface area* (or Talagrand boundary):

$$\mathbf{BSA}[f] = \mathbb{E}\,|\nabla f| = \mathbb{E}\,\sqrt{\mathrm{Sens}_f(x)},$$

which is a natural measure of vertex-boundary complexity on the discrete cube. Our main result is that every degree-d PTF f has subpolynomial Boolean surface area:

$$\mathbf{BSA}[f] \le \exp(C(d)\sqrt{\log n}).$$

This is a superpolynomial improvement over the previous bound of $n^{1/4}(\log n)^{C(d)}$ that follows from Kane's landmark bounds on average sensitivity of PTFs [**?**].

Degree-d PTFs thus satisfy a stronger form of geometric regularity than was previously visible from influence bounds alone. As an application, we obtain improved noise sensitivity estimates in the case of small noise parameter.