SEMINAR S4D3
GRADUATE SEMINAR ON ADVANCED GEOMETRY
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An introduction to automorphisms of Kähler manifolds

It is known (but difficult to prove) that the group $\mathrm{Aut}(M)$ of biholomorphic automorphisms of a compact complex manifold M of general type (this notion will be introduced in the seminar) is finite. The easiest and classical examples of such manifolds are Riemann surfaces of genus $g \geq 2$. Indeed, the maximal nummber of automorphisms of such a surface S is $84(g-1)$. This can be shown using the fact that such surfaces admit a metric of constant negative Gauss curvature.

On the other hand, the automorphism group of $\mathbb{C}P^n$ (which has Kodaira dimension $-\infty$) is the group $\mathrm{GL}(n+1, \mathbb{C})$ acting as a group of projective transformations. While this is a big group, the dynamics of the action of an infinite cyclic subgroup is not very interesting.

Surprisingly, there are examples of $K3$-surfaces which admit automorphisms with complicated dynamical behavior. The goal of this seminar is to give a gentle introduction to the study of automorphisms of Kähler manifolds, in particular $K3$ surfaces.

We start with introducing automorphisms and computing the automorphism group of $\mathbb{C}P^n$. We also estimate the size of the automorphism group of a closed Riemann surface of genus $g \geq 2$.

Kähler geometry can be used to control automorphisms. The pertinent tool for surfaces with positive or trivial canonical bundle is the solution to the Calabi conjecture by Aubin and Yau. In the case of positive canonical bundle, it shows that there exists a unique Kähler Einstein metric up to a constant which then turns automorphisms of manifolds with positive canonical bundle into isometries. The case of a positive canonical bundle (the easier case of Aubin and Yau's theorem) will be discussed (not in all detail) including the introduction of the pertinent curvature notions.

We then turn to $K3$-surfaces (which have trivial canonical bundle) and study some of their properties, like the Neron Severi group and Torelli theorems (without proof). Kummer surfaces will be discussed in more detail. Using the Torelli theorem, we then present examples of Kummer surfaces with dynamically interesting automorphisms.

Prerequisites are a one semester course on Kähler manifolds. No additional knowledge in algebraic geometry or related areas is assumed.

Talks: The literature for the talks is not complete at this point and will filled in shortly.

(1) Talk 1. Automorphism groups of compact complex manifolds are complex Lie groups; examples, in particular the automorphism groups of elliptic curves.
[Mo07], Def. 8.6 and Lemma 8.7
[Ko95] Theorem 3.1 in Chapter I, Theorem 1.1 in Chapter III (in slight contrast of the presentation in the book, we'll assume that we know that the dimension of the space of holomorphic sections of a holomorphic vector bundle is finite)
Examples: Riemann surfaces of genus $g \geq 2$ with ample canonical bundle, elliptic curves with trivial canonical bundle and a transitive group of automorphisms acting by translations
(2) Talk 2. The automorphism group of $\mathbb{C}P^n$
[RS25] Summary of intersection: Section 4.1.1 and Example (iii), p.111-112, Def. 4.2.7 and Thm. 4.2.8 on p.125
[GH78] 60-64.
(3) Talk 3. Automorphism groups of compact Riemann surfaces of genus at least 2 are finite via finite branched covers.
[Fo91] p. 20,29,140
[Ko95] p.82-86, 88-90.
(4) Talk 4: Curvature of Kähler manifolds, the Ricci form, examples
[L24], p. 239-247.
(5) Talk 5: Manifolds with trivial or positive canonical bundle: The theorem of Aubin and Yau I
[Bl06], 88-93, see also the excellent more comprehensive treatment of the material in Chapter 5 of [J00].
(6) Talk 6: The theorem of Aubin and Yau II.
[Bl06], p. 94-101, see also the excellent more comprehensive treatment of the material in Chapter 5 of [J00].
(7) Talk 7: $K3$ surfaces I: Neron Severi group and Hodge decomposition
[Hy05] Section 3.3, p. 132–134, [Ko15], p. 35–36, p. 39–41
[Vo04], p. 330–313,
(8) Talk 8: $K3$ surfaces II: The Kummer construction and analytic families
[?] p. 35 (irregularity), p. 55-56, p. 62-62, Section 5.2 (p. 80-83)
(9) Talk 9: Torelli theorems I
[Ko15] p. 87-94.
(10) Talk 10: Torelli theorems II
[Ko15], p.94-103.
(11) Talk 11: Automorphisms of complex 2-tori and automorphisms of $K3$ surfaces; the action on the Kähler cone
(12) Talk 12: The theorem of Yomdin and Gromov.

References

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[Fi19] S. Filip, *An introduction to $K3$ surfaces and their dynamics*, preprint 2019.

[Fo91] O. Forster, *Lectures on Riemann surfaces*, Grad. Texts in Math. 81, Springer, New York 1991.

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