WS 2025/26

Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Graduate Seminar on Advanced Topology (S4D4)

Higher Algebra

Tuesdays, 14:15, N 0.007

Preliminary meeting: July 15, 11:00 (s.t.!), Lipschitzsaal

Two of the most salient features of singular cohomology are its multiplicative structure and the Steenrod operations. From the point of view of modern homotopy theory, these are shadows of a so-called E_∞ *ring structure* on the spectrum $H\mathbb{Z}$ representing singular cohomology. There are many other examples of E_∞ rings, arising from geometry (e.g. topological K-theory or bordism) or algebra and category theory (algebraic K-theory, Hochschild homology, etc.).

In this seminar, we will build up the language for E_∞ ring spectra and their ∞-categories of modules, and then discuss analogues of some subjects of classical algebra, such as Morita theory and deformation theory, in this new higher algebraic context.

Prerequisites. We assume that all participants are familiar with the foundations of ∞-categories, and in particular with their model in terms of quasi-categories.

Note. In many cases, the references contain more material than you will be able to present in your talk, and you should decide on which parts you want to focus (apart from the results explicitly mentioned in the talk description below, of course). You should moreover meet with the assistant (Tobias Lenz, `lenz@math.uni-bonn.de`) at least two weeks before your talk to go through the material you want to present and to discuss any questions you might have.

Talks

1. **Presentable and stable ∞-categories** .... October 14

   Define accessible and presentable ∞-categories, show that the ∞-category Spc of spaces is presentable, and introduce the ∞-categories Pr^{L} and Pr^{R}. Afterwards, state the Adjoint Functor Theorem for presentable ∞-categories [Lur09, §5.4–5.5].

   Discuss the various equivalent definitions of stable ∞-categories [Lur17, §1.1] and introduce the stabilization of ∞-categories with finite limits [Lur17, §1.4.1]. Show that the stabilization of a presentable category is presentable again [Lur09, Prop. 1.4.4.4], and define the ∞-category Sp of spectra as the stabilization of Spc. Deduce its universal property [Lur17, Cor. 1.4.4.6].

2. **Stable ∞-categories and t-structures** .... October 21

   Introduce the notion of a t-structure on a stable ∞-category [Lur17, Def. 1.2.1.1], derive its basic properties, and define what it means for a t-structure to be accessible [Lur17, Def. 1.4.4.12] or left/right complete [Lur17, Rmk. 1.2.1.12]. Introduce the standard t-structures on Sp [Lur17, Prop. 1.4.3.4] and on the derived category $D(R)$ of a (discrete) ring R [Lur17, Prop. 1.3.2.19].

3. **Symmetric monoidal ∞-categories** .... October 28

   Describe the formulation of symmetric monoidal 1-categories in terms of cocartesian fibrations ('Grothendieck op-fibrations') over Fin_* [Lur17, beginning of Chapter 2] and introduce symmetric monoidal ∞-categories [Lur17, Def. 2.0.0.7]. Discuss cartesian and cocartesian symmetric monoidal structures [Lur17, §2.4.1, §2.4.3] and nerves of symmetric monoidal model categories [NS18, Thm. A.7].

4. **Classical operads** .... November 4

   Motivate operads with the Stasheff associahedra and the Recognition Theorem for (1-fold) loop spaces [Sta63]. Define operads in a symmetric monoidal 1-category as well as their algebras and modules over operadic algebras. Give many examples, and in particular introduce the little cubes operads and talk about the recognition theorem for iterated loop spaces [May72]. Mention colored operads (aka multicategories) as a generalization of operads.

As the starting point of your talk you should use [Lur17, §2.2.1], but you may want to consult other textbooks on operads as well.

5. **Algebras and modules over ∞-operads** .... November 11

   Define ∞-operads as well as their algebras and the corresponding notion of modules [Lur17, §2.1 and §3.3]. Discuss the relation to operadic algebras on the pointset level [Lur17, §4.1.8 and §4.5.4]. Finally, give an overview of the proof of [Lur17, Thm. 3.4.4.2].

6. **The Lurie tensor product** .... November 18

   Define the tensor products of ∞-categories with certain colimits [Lur17, §4.8.1] and how this gives a tensor product on Pr^{L}. Use this to show that Sp admits a unique presentably symmetric monoidal structure with unit given by the sphere spectrum $\mathbb{S}$ [Lur17, §4.8.2].

7. **Day convolution** .... November 25

   Introduce Day convolution and its universal property following [Lur17, §2.2.6] (paying special attention to the symmetric monoidal case). Afterwards, discuss the additional universal property for Day convolution on presheaf categories [Lur17, 4.8.1.10–13] and use this to show that presentably symmetric monoidal ∞-categories have presentations [NS17, Prop. 2.2].

   If you have time left, discuss the (localized) Day convolution structure on CMon(Spc) in analogy with [CHLL25, Cor. 3.3.5] (the 'Lydakis smash product') and use this to give a multiplicative refinement of the Recognition Theorem for connective spectra [GGN15].

8. **Structured ring spectra** .... December 2

   Define E_1 rings and E_∞ rings [Lur17, Def.s 4.1.1.6 and 7.1.0.1]. Give many examples, in particular the sphere, spherical monoid rings, and Eilenberg–Mac Lane spectra [Lur17, Prop. 7.1.3.18], and discuss general constructions like completions [Lur18, Cor. 7.3.5.2] and localizations [Lur17, Prop. 7.2.3.27].

9. **Modules over ring spectra** .... December 9

   Define the ∞-category LMod_R of modules over an E_1 ring R, show that it is stable, and introduce its standard t-structure [Lur17, §7.1.1]. Show that LMod_R for an E_∞ ring R has a preferred symmetric monoidal structure, and discuss how for connective R the standard t-structure yields truncations and connective covers on Alg_R and CAlg_R [Lur17, §7.1.3].

10. **Morita theory** .... December 16

    Prove the ∞-categorical Schwede–Shipley Theorem [Lur17, Thm. 7.1.2.1] as well as [Lur17, Prop. 7.1.2.7]. Discuss many examples, in particular the equivalence $D(R) \simeq \mathrm{LMod}_{HR}(\mathrm{Sp})$ for a discrete commutative ring R and the algebraic model for rational spectra [Lur17, Thm. 7.1.2.13].

11. **The cotangent complex I: General Theory** .... January 13

    Define the tangent bundle of a presentable ∞-category [Lur17, Def. 7.3.1.9], relative adjunctions [Lur17, Def. 7.3.2.2], and the relative contangent complex functor [Lur17, Def. 7.3.3.1]. Discuss the proof of [Lur17, Thm.s 7.3.4.13 and 7.3.4.18], paying special attention to the commutative case.

12. **The cotangent complex II: Kähler differentials** .... January 20

    Specialize the discussion from last week to the case of E_∞ rings. Discuss square-zero extensions, leading to the fact that the Postnikow tower of an E_∞ ring is comprised of square-zero extensions [Lur17, §7.4.1]. Prove the basic finiteness and connectivity properties as well as the relation to classical Kähler differentials [Lur17, §7.4.3].

References

[CHLL25] Bastiaan Cnossen, Rune Haugseng, Tobias Lenz, and Sil Linskens, *Homotopical commutative rings and bispans*, J. Lond. Math. Soc. **111** (2025), no. 6, 33, Article ID e70200.

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[Lur09] Jacob Lurie, *Higher topos theory*, Ann. Math. Stud., vol. 170, Princeton, NJ: Princeton University Press, 2009.

[Lur17] ______, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf` (2017).

[Lur18] ______, *Spectral Algebraic Geometry*, under construction (version dated February 2018), `www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf` (2018).

[May72] J. P. May, *The geometry of iterated loop spaces*, Lect. Notes Math., vol. 271, Springer, 1972.

[NS17] Thomas Nikolaus and Steffen Sagave, *Presentably symmetric monoidal ∞-categories are represented by symmetric monoidal model categories*, Algebr. Geom. Topol. **17** (2017), no. 5, 3189–3212.

[NS18] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409.

[Sta63] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.