Proper Equivariant Stable Homotopy Theory

Dieter Degrijse

Markus Hausmann

Wolfgang Lück

Irakli Patchkoria

Stefan Schwede

Author address:

Keylane, Copenhagen, Denmark
Email address: dieterdegrijse@gmail.com

Mathematisches Institut, Universität Bonn, Germany
Email address: hausmann@math.uni-bonn.de

Mathematisches Institut, Universität Bonn, Germany
Email address: wolfgang.lueck@him.uni-bonn.de

Department of Mathematics, University of Aberdeen, UK
Email address: irakli.patchkoria@abdn.ac.uk

Mathematisches Institut, Universität Bonn, Germany
Email address: schwede@math.uni-bonn.de

Contents

Abstract

This monograph introduces a framework for genuine proper equivariant stable homotopy theory for Lie groups. The adjective 'proper' alludes to the feature that equivalences are tested on compact subgroups, and that the objects are built from equivariant cells with compact isotropy groups; the adjective 'genuine' indicates that the theory comes with appropriate transfers and Wirthmüller isomorphisms, and the resulting equivariant cohomology theories support the analog of an $RO(G)$-grading.

Our model for genuine proper G-equivariant stable homotopy theory is the category of orthogonal G-spectra; the equivalences are those morphisms that induce isomorphisms of equivariant stable homotopy groups for all compact subgroups of G. This class of π_*-isomorphisms is part of a symmetric monoidal stable model structure, and the associated tensor triangulated homotopy category is compactly generated. Consequently, every orthogonal G-spectrum represents an equivariant cohomology theory on the category of G-spaces. These represented cohomology theories are designed to only depend on the 'proper G-homotopy type', tested by fixed points under all compact subgroups.

An important special case of our theory are infinite discrete groups. For these, our genuine equivariant theory is related to finiteness properties in the sense of geometric group theory; for example, the G-sphere spectrum is a compact object in our triangulated equivariant homotopy category if the universal space for proper G-actions has a finite G-CW-model. For discrete groups, the represented equivariant cohomology theories on finite proper G-CW-complexes admit a more explicit description in terms of parameterized equivariant homotopy theory, suitably stabilized by G-vector bundles. Via this description, we can identify the previously defined G-cohomology theories of equivariant stable cohomotopy and equivariant K-theory as cohomology theories represented by specific orthogonal G-spectra.

2010 *Mathematics Subject Classification.* Primary 55P91.

Key words and phrases. Lie group, equivariant homotopy theory; proper action.

All five authors were supported by the Hausdorff Center for Mathematics at the University of Bonn (DFG GZ 2047/1, project ID 390685813) and by the Centre for Symmetry and Deformation at the University of Copenhagen (CPH-SYM-DNRF92); we would like to thank these two institutions for their hospitality, support and the stimulating atmosphere. Hausmann, Patchkoria and Schwede were partially supported by the DFG Priority Programme 1786 'Homotopy Theory and Algebraic Geometry'. Work on this monograph was funded by the ERC Advanced Grant 'KL2MG-interactions' of Lück (Grant ID 662400), granted by the European Research Council. Patchkoria was supported by the Shota Rustaveli National Science Foundation Grant 217-614. Patchkoria and Schwede would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme 'Homotopy harnessing higher structures', when work on this paper was undertaken (EPSRC grant number EP/R014604/1). We would also like to thank Bob Oliver and Søren Galatius for helpful conversations on topics related to this project, and the anonymous referee for his or her careful reading and the many useful comments.

Introduction

This monograph explores proper equivariant stable homotopy theory for Lie groups. The theory generalizes the well established 'genuine' equivariant stable homotopy theory for *compact* Lie groups, and the adjective 'proper' indicates that the theory is only sensible to fixed point information for compact subgroups. In other words, hardwired into our theory are bootstrap arguments that reduce questions to equivariant homotopy theory for compact Lie groups. Nevertheless, there are many new features that have no direct analog in the compact case.

Equivariant stable homotopy theory has a long tradition, originally motivated by geometric questions about symmetries of manifolds. Certain kinds of features can be captured by 'naive' stable equivariant theories, obtained one way or other by formally inverting suspension on some category of equivariant spaces. The naive stable theory works in broad generality for general classes of topological groups, and it can be modeled by sequential spectra of G-spaces, or functors from a suitable orbit category to spectra (interpreted either in a strict sense, or in an ∞-categorical context).

A refined version of equivariant stable homotopy theory, usually referred to as 'genuine', was traditionally only available for compact Lie groups of equivariance. The genuine theory has several features not available in the naive theory, such as transfer maps, dualizability, stability under twisted suspension (i.e., smash product with linear representation spheres), an extension of the $\mathbb{Z}$-graded cohomology groups to an $RO(G)$-graded theory, and an equivariant refinement of additivity (the so called Wirthmüller isomorphism). The homotopy theoretic foundations of this theory were laid by tom Dieck [**72, 73, 74, 75**], May [**35, 22, 23**] and Segal [**65, 66, 67**] and their students and collaborators since the 70s. A spectacular recent application was the solution, by Hill, Hopkins and Ravenel [**24**], to the Kervaire invariant 1 problem. This monograph extends genuine equivariant stable homotopy theory to Lie groups that need not be compact; this includes infinite discrete groups as an important special case.

A major piece of our motivation for studying equivariant homotopy theory for infinite discrete groups and not necessarily compact Lie groups comes from the Baum-Connes Conjecture and the Farrell-Jones Conjectures. The Baum-Connes Conjecture was originally formulated in [**2**], and subsequently considered in the formulation stated in [**3**, Conjecture 3.15 on page 254]. The Farrell-Jones Conjecture was formulated in [**20**, 1.6 on page 257]; two survey articles about these isomorphism conjectures are [**43**] and [**77**]. Roughly speaking, these conjectures identify the theory of interest – topological K-groups of reduced group C^*-algebras or algebraic K-and L-groups of group rings – with certain equivariant homology theories, applied to classifying spaces of certain families of subgroups. Many applications of these conjectures to group homology, geometry, or classification results of C^*-algebras are based on computations of the relevant equivariant homology theories, and of their cohomological analogues. Even if one is interested only in non-equivariant (co)homology of classifying spaces, it is useful to invoke equivariant homotopy theory on the level of spectra; examples are [**18, 33, 40**]. Our book provides a systematic framework for such calculations, capable of capturing extra pieces of structure like induction, transfers, restriction, multiplication, norms, global equivariant features, and gradings beyond naive $\mathbb{Z}$-grading.

For example, one might wish to use the Atiyah-Hirzebruch spectral sequence to compute K-theory or stable cohomotopy of BG for an infinite discrete group G; this will often be a spectral sequence with differentials of arbitrary length. However, the completion theorems for equivariant K-theory [**42**] and for equivariant stable cohomotopy [**41**] provide an alternative line of attack: one may instead compute equivariant cohomology of $\underline{E}G$ by the equivariant Atiyah-Hirzebruch spectral sequence (see Construction 3.2.14), and then complete the result to obtain the non-equivariant cohomology of BG. Even if BG is infinite dimensional, $\underline{E}G$ might well be finite-dimensional and relatively small, in which case the equivariant Atiyah-Hirzebruch spectral sequence is easier to analyze. For example, for a virtually torsion free group, this spectral sequence has a vanishing line at the virtual cohomological dimension.

Among other things, our formalism provides a definition of the equivariant homotopy groups π_*^G for infinite groups G. These equivariant homotopy groups have features which reflect geometric group theoretic properties of G. As the group $G = \mathbb{Z}$ already illustrates, the sphere spectrum need not be connective with respect to π_*^G, compare Example 2.3.8. If the group G is virtually torsion free, then the equivariant Atiyah-Hirzebruch spectral sequence shows that the equivariant homotopy groups $\pi_*^G(\mathbb{S})$ vanish below the negative of the virtual cohomological dimension of G.

We conclude this introduction with a summary of the highlights of this monograph.

• Our model for proper equivariant stable homotopy theory of a Lie group G is the category of orthogonal G-spectra, i.e., orthogonal spectra equipped with a continuous action of G. This pointset level model is well-established, explicit, and has nice formal properties; for example, orthogonal G-spectra are symmetric monoidal under the smash product of orthogonal spectra, endowed with the diagonal G-action.

• All the interesting homotopy theory is encoded in the notion of stable equivalences for orthogonal G-spectra. We use the π_**-isomorphisms*, defined as those morphisms of orthogonal G-spectra that induce isomorphisms on $\mathbb{Z}$-graded H-equivariant homotopy groups, for all compact subgroups H of G. These H-equivariant homotopy groups are based on a complete universe of orthogonal H-representations. In [**21**, Prop. 6.5], Fausk has extended these π_*-isomorphisms to a stable model structure on the category of orthogonal G-spectra, via an abstract Bousfield localization procedure. We develop a different (but Quillen equivalent) model structure in Theorem 1.2.22, also with the π_*-isomorphisms as weak equivalences, that gives better control over the stable fibrations; in particular, the stably fibrant objects in our model structure are the G-Ω-spectra as defined in Definition 1.2.15 below. Our model structure is compatible with the smash product of orthogonal G-spectra, and compatible with restriction to closed subgroups, see Theorem 1.2.22.

• A direct payoff of the stable model structure is that the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ comes with a triangulated structure. This structure is made so that mapping cone sequences of proper G-CW-complexes become distinguished triangles in $\mathrm{Ho}(\mathrm{Sp}_G)$. As a consequence, every orthogonal G-spectrum represents a G-equivariant cohomology theory on proper G-spaces, see Construction 3.2.3. The triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ comes with a distinguished set of small generators, the suspension spectra of the homogeneous spaces G/H for all compact subgroups H of G, see Corollary 1.3.11. This again has certain direct payoffs, such as Brown representability of homology and cohomology theories on $\mathrm{Ho}(\mathrm{Sp}_G)$, and a non-degenerate t-structure.

• The triangulated categories $\mathrm{Ho}(\mathrm{Sp}_G)$ enjoy a large amount of functoriality in the group G: every continuous homomorphism $\alpha : K \longrightarrow G$ gives rise to a restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ that in turn admits an exact total left derived functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$, see Theorem 1.4.17. These derived functors assemble to a contravariant pseudo-functor from the category of Lie groups and continuous homomorphisms to the category of triangulated categories and exact functors. Moreover, conjugate homomorphisms and homotopic homomorphisms induce isomorphic derived functors.

• The proper equivariant stable homotopy theory should be thought of as a 'weak homotopy invariant' of the Lie group G. More precisely, we show in Theorem 1.4.31 that for every continuous homomorphism $\alpha : K \longrightarrow G$ between Lie groups that is a weak equivalence of underlying spaces, the derived restriction functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ is an equivalence of tensor triangulated categories. A special case is the inclusion of a maximal compact subgroup M of an almost connected Lie group G. In this case, the restriction functor $\mathrm{res}^G_M : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_M)$ is an equivalence. So for almost connected Lie groups, our theory reduces to the classical case of compact Lie groups. In this sense, the new mathematics in this memoir is mostly about infinite versus finite component groups.

• When G is discrete, the heart of the preferred t-structure on $\mathrm{Ho}(\mathrm{Sp}_G)$ has a direct and explicit algebraic description: it is equivalent to the abelian category of G-Mackey functors in the sense of Martinez-Pérez and Nucinkis [**49**], see Theorem 2.2.9. In particular, every G-Mackey functor is realized, essentially uniquely, by an Eilenberg-Mac Lane spectrum in $\mathrm{Ho}(\mathrm{Sp}_G)$. The cohomology theory represented by the Eilenberg-Mac Lane spectrum is Bredon cohomology, see Example 3.2.16.

• For discrete groups G, the *rational* G-equivariant stable homotopy theory is completely algebraic: the rationalization of $\mathrm{Ho}(\mathrm{Sp}_G)$ is equivalent to the derived category of rational G-Mackey functors, see Theorem 2.3.4. When G is infinite, the abelian category of rational G-Mackey functors is usually not semisimple, so in contrast to the well-known case of finite groups, a rational G-spectrum is not generally classified in $\mathrm{Ho}(\mathrm{Sp}_G)$ by its homotopy group Mackey functors alone.

• Our theory is the analog, for general Lie groups, of 'genuine' equivariant stable homotopy theory; for example, the equivariant cohomology theories arising from orthogonal G-spectra have a feature analogous to an '$RO(G)$-grading' in the compact case. In the present generality, however, representations should be replaced by equivariant real vector bundles over the universal space $\underline{E}G$ for proper

G-actions, and so the analog of an $RO(G)$-grading is a grading by the Grothendieck group $KO_G(\underline{E}G)$ of such equivariant vector bundles, see Remark 3.2.10.

• For discrete groups we identify the cohomology theories represented by G-spectra in more concrete terms via fiberwise equivariant homotopy theory, see Theorem 3.2.7. This allows us to compare our approach to equivariant cohomology theories that were previously defined by different means. For example, we show in Example 3.2.9 that for discrete groups the theory represented by the G-sphere spectrum coincides, for finite proper G-CW-complexes, with equivariant cohomotopy as defined by the third author in [**38**]. In Theorem 3.4.22 we show that if G is a discrete group, then the equivariant K-theory based on G-vector bundles defined by the third author and Oliver in [**42**] is also representable in $\mathrm{Ho}(\mathrm{Sp}_G)$.

Conventions. Throughout this memoir, a *space* is a *compactly generated space* in the sense of [**51**], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition. Two extensive resources with background material about compactly generated spaces are Section 7.9 of tom Dieck's textbook [**76**] and Appendix A of the fifth author's book [**60**]. Two other influential – but unpublished – sources about compactly generated spaces are the Appendix A of Gaunce Lewis's thesis [**34**] and Neil Strickland's preprint [**71**]. We denote the category of compactly generated spaces and continuous maps by $\mathbf{T}$.

CHAPTER 1

Equivariant spectra

1.1. Orthogonal G-spectra

In this section we recall the basic objects of our theory, orthogonal spectra and orthogonal G-spectra, where G is a Lie group. We start in Proposition 1.1.3 with a quick review of the $\mathcal{C}om$-model structure for G-spaces, i.e., the model structure where equivalences and fibrations are tested on fixed points for compact subgroups of G. The homotopy category of this model structure is equivalent to the category of proper G-CW-complexes and equivariant homotopy classes of G-maps. Proposition 1.1.6 records how the $\mathcal{C}om$-model structures interact with restriction along a continuous homomorphism between Lie groups. We recall orthogonal G-spectra in Definition 1.1.9, and we end this section with several examples.

We let G be a topological group, which we take to mean a group object in the category $\mathbf{T}$ of compactly generated spaces. So a topological group is a compactly generated space equipped with an associative and unital multiplication

$$\mu \ : \ G \times G \ \longrightarrow \ G$$

that is continuous with respect to the compactly generated product topology, and such that the shearing map

$$G \times G \ \longrightarrow \ G \times G \ , \quad (g,h) \ \longmapsto \ (g, gh)$$

is a homeomorphism (again for the compactly generated product topology). This implies in particular that inverses exist in G, and that the inverse map $g \mapsto g^{-1}$ is continuous. A *G-space* is then a compactly generated space X equipped with an associative and unital action

$$\alpha \ : \ G \times X \ \longrightarrow \ X$$

that is continuous with respect to the compactly generated product topology. We write $G\mathbf{T}$ for the category of G-spaces and continuous G-maps. The forgetful functor from G-spaces to compactly generated spaces has both a left and a right adjoint, and hence limits and colimits of G-spaces are created in the underlying category $\mathbf{T}$.

Remark 1.1.1. We mostly care about the case when G is a Lie group. Then the underlying space of G is locally compact Hausdorff, and for every compactly generated space X, the space $G \times X$ is a k-space (and hence compactly generated) in the usual product topology. So for Lie groups, the potential ambiguity about continuity of the action disappears.

For G-spaces X and Y we write $\mathrm{map}(X, Y)$ for the space of continuous maps with the function space topology internal to $\mathbf{T}$ (i.e., the Kelleyfied compact-open topology). The group G acts continuously on $\mathrm{map}(X,Y)$ by conjugation, and these constructions are related by adjunctions, i.e., natural homeomorphisms of G-spaces

$$\mathrm{map}(X,\mathrm{map}(Y,Z)) \cong \mathrm{map}(X\times Y,Z) ;$$

in particular, the category of G-spaces is cartesian closed.

The category $G\mathbf{T}$ is enriched, tensored and cotensored over the category $\mathbf{T}$ of spaces, as follows. The tensor and cotensor of a G-space X and a space K are the product $X\times K$ and the function space $\mathrm{map}(K,X)$, respectively, both with trivial G-action on K. The enrichment in $\mathbf{T}$ is given by the G-fixed point space $\mathrm{map}^G(X,Y)$, i.e., the space of G-equivariant continuous maps.

A G-space X is *proper* if the map

$$G\times X \longrightarrow X\times X , \quad (g,x) \longmapsto (gx,x)$$

is proper, i.e., inverse images of compact subsets are compact. Because the inverse image of a diagonal point (x,x) is homeomorphic to the isotropy group of x, all isotropy groups of proper G-spaces are compact. Any irrational rotation action of the group $\mathbb{Z}$ on a circle shows that the converse is not generally true, i.e., a continuous G-action with compact stabilizers need not be proper in general.

A *G-CW-complex* is a G-space X equipped with an exhaustive increasing filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

by closed G-invariant subspaces such that

- each X_n is obtained from X_{n-1} by attaching a disjoint union of G-spaces $G/H_i\times D^n$ along $G/H_i\times S^{n-1}$, for some set $\{H_i\}_{i\in I}$ of closed subgroups of G, and
- the space X has the colimit topology (weak topology) of the sequence.

A G-CW-complex is *finite-dimensional* if the filtration stabilizes, i.e., $X_n=X$ for some $n\geq 0$. A G-CW-complex is *of finite type* if in each dimension n, the number of equivariant cells is finite. A G-CW-complex is *finite* if it is finite-dimensional and of finite type, i.e., if the total number of equivariant cells is finite. Equivalently, a G-CW-complex X is finite if and only if the action is cocompact, i.e., the orbit space X/G is compact. For a G-CW-complex, properness of the action is equivalent to the requirement that all of the isotropy groups are compact, see for example [**36**, Theorem 1.23].

DEFINITION 1.1.2. Let G be a Lie group.

(i) A map $f\colon X\longrightarrow Y$ of G-spaces is a *$\mathcal{C}om$-equivalence* if for every compact subgroup H of G the map $f^H\colon X^H\longrightarrow Y^H$ is a weak equivalence of topological spaces.
(ii) A map $f\colon X\longrightarrow Y$ of G-spaces is a *$\mathcal{C}om$-fibration* if for every compact subgroup H of G the map $f^H\colon X^H\longrightarrow Y^H$ is a Serre fibration of topological spaces.
(iii) A map $i\colon A\longrightarrow B$ of G-spaces is a *$\mathcal{C}om$-cofibration* if it has the left lifting property with respect to every map which is a $\mathcal{C}om$-equivalence and $\mathcal{C}om$-fibration.
(iv) A map $i\colon A\longrightarrow B$ of G-spaces is a *G-cofibration* if it has the left lifting property with respect to every map $f : X\longrightarrow Y$ such that $f^\Gamma : X^\Gamma\longrightarrow Y^\Gamma$ is a weak equivalence and Serre fibration for every closed subgroup Γ of G.

We alert the reader that our use of the expression 'G-cofibration' is different from the usage in some older papers on the subject, where this term refers to the larger class of G-maps with the equivariant homotopy extension property. In this book, morphisms with the homotopy extension property will be referred to as *h-cofibrations*, see Definition 1.2.8 below.

Clearly, every $\mathcal{C}om$-cofibration of G-spaces is a G-cofibration. The following proposition is a special case of [**21**, Prop. 2.11] or [**60**, Prop. B.7]. For the definition of a cofibrantly generated model category see for example [**25**, Sec. 2.1].

Proposition 1.1.3. *Let G be a Lie group.*

(i) *The $\mathcal{C}om$-equivalences, $\mathcal{C}om$-fibrations and $\mathcal{C}om$-cofibrations form a proper, topological, cofibrantly generated model structure on the category of G-spaces, the $\mathcal{C}om$-model structure. A morphism $i : A \longrightarrow B$ is a $\mathcal{C}om$-cofibration if and only if it is a G-cofibration and the stabilizer group of every point in $B - i(A)$ is compact.*

(ii) *The set of maps*

$$G/H \times i^k : G/H \times \partial D^k \longrightarrow G/H \times D^k$$

serves as a set of generating cofibrations for the $\mathcal{C}om$-model structure, as H ranges over all compact subgroups of G and $k \geq 0$. The set of maps

$$G/H \times j^k : G/H \times D^k \times \{0\} \longrightarrow G/H \times D^k \times [0,1]$$

serves as a set of generating acyclic cofibrations, as H ranges over all compact subgroups of G and $k \geq 0$.

(iii) *For every G-cofibration $i : A \longrightarrow B$ and every $\mathcal{C}om$-cofibration $j : K \longrightarrow L$ of based G-spaces, the pushout product map*

$$i \,\square\, j = (i \times L) \cup (B \times j) \ : \ (A \times L) \cup_{A \times K} (B \times K) \ \longrightarrow \ B \times L$$

is a $\mathcal{C}om$-cofibration. If moreover one of i or j is a $\mathcal{C}om$-equivalence, then $i \,\square\, j$ is also a $\mathcal{C}om$-equivalence.

Proof. As we already mentioned, parts (i) and (ii) are proved in detail in [**60**, Prop. B.7]. Since smash product has an adjoint in each variable, it preserves colimits in each variable. So it suffices to check the pushout product properties in (iii) when the maps f and g are from the sets of generating (acyclic) cofibrations, compare [**25**, Cor. 4.2.5]. The set of inclusions of spheres into discs is closed under pushout product, in the sense that $i^k \square i^l$ is homeomorphic to i^{k+l}. Similarly, the pushout product of i^k with j^l is isomorphic to j^{k+l}. So all claims reduce to the fact that for every pair of closed subgroups Γ and H of G such that H is compact, the G-space $G/\Gamma \times G/H$ with the diagonal G-action is $\mathcal{C}om$-cofibrant. Indeed, this product is G-homeomorphic to $G \times_H (G/\Gamma)$, with G-action only on the left factor of G. Illman's theorem [**27**, Thm. 7.1] implies that G/Γ admits the structure of an H-CW complex; hence $G \times_H (G/\Gamma)$ admits the structure of a G-CW-complex, and its isotropy groups are compact. $\square$

Every proper G-CW-complex is in particular $\mathcal{C}om$-cofibrant. On the other hand, every $\mathcal{C}om$-cofibrant G-space is G-equivariantly homotopy equivalent to a proper G-space that admits the structure of a G-CW-complex. So for all practical purposes, $\mathcal{C}om$-cofibrant G-spaces are as good as proper G-CW-complexes. Since all G-spaces are fibrant in the $\mathcal{C}om$-model structure, the abstract homotopy category $\mathrm{Ho}^{\mathcal{C}om}(G\mathbf{T})$, defined as the localization at the class of $\mathcal{C}om$-equivalences, is equivalent to the concrete homotopy category of proper G-CW-complexes and equivariant homotopy classes of G-maps.

We denote by $\underline{E}G$ a universal proper G-space, i.e., a universal G-space for the family of compact subgroups of G. It is characterized up to G-homotopy equivalence by the following properties:

(i) $\underline{E}G$ admits the structure of a G-CW complex.
(ii) The H-fixed point space $(\underline{E}G)^H$ is contractible if H is compact, and empty otherwise.

The existence of $\underline{E}G$ follows for example from [**39**, Thm. 1.9]. We note that for every G-space X, the projection $\underline{E}G \times X \longrightarrow X$ is a $\mathcal{C}om$-equivalence. Indeed, taking H-fixed points for a compact subgroup H of G we have

$$(\underline{E}G \times X)^H \ \cong\ (\underline{E}G)^H \times X^H$$

which maps by a homotopy equivalence to X^H since $(\underline{E}G)^H$ is contractible. The unit of the cartesian product (the one-point G-space) is not $\mathcal{C}om$-cofibrant unless the group G is compact; since the unique map $\underline{E}G \longrightarrow *$ is a $\mathcal{C}om$-equivalence, $\underline{E}G$ is a cofibrant replacement of the one-point G-space.

Here are some examples of universal spaces for proper actions.

- If the Lie group G is compact, then the one-point G-space is a model for $\underline{E}G$.
- Suppose that G has no non-trivial compact subgroups; for example, G could be discrete and torsion free. Then any universal G-space for free G-actions, usually denoted by EG, is also a universal G-space for proper G-actions.
- If G is discrete and can be exhausted by an ascending sequence of finite subgroups, then it has a 1-dimensional model for $\underline{E}G$, see Example 1.4.5.
- Suppose that G is *almost connected*, i.e., its group of path components is finite. Then K has a maximal compact subgroup, i.e., a compact subgroup K such that every compact subgroup is subconjugate to K. In this situation, the orbit space G/K is a model for $\underline{E}G$. We refer to the discussion immediately preceding Theorem 1.4.4 for more details and references.

Example 1.1.4 (A universal proper D_∞-space). We offer the case $G = D_\infty = \mathbb{Z} \ltimes \mathbb{Z}/2$ of the infinite dihedral group as an explicit example to keep in mind; the group D_∞ is also isomorphic to the free product $\mathbb{Z}/2 * \mathbb{Z}/2$. Besides the trivial subgroup, D_∞ has two other conjugacy classes of finite subgroups, both of order 2; the subgroup H_1 generated by $(0, 1 + 2\mathbb{Z})$ and the subgroup H_2 generated by $(1, 1 + 2\mathbb{Z})$ are representatives.

The groups $\mathbb{Z}$ and $\mathbb{Z}/2$ act on the real line by translation and reflection at the origin, respectively. These two actions conspire into an action of the semidirect product $D_\infty = \mathbb{Z} \ltimes \mathbb{Z}/2$ on $\mathbb{R}$. The real line has a 1-dimensional D_∞-CW-structure with 3 equivariant cells, as follows. The 0-skeleton is the union of the transitive subsets $\mathbb{Z}$ and $1/2 + \mathbb{Z}$; so each of these two invariant subspaces is an equivariant 0-cell, and the respective stabilizers are the conjugacy classes of H_1 and H_2. There is a single equivariant 1-cell, with trivial stabilizer. The fixed point spaces of this D_∞-action on $\mathbb{R}$ are either empty (for infinite subgroups), they are single points (for two-element subgroups), or all of $\mathbb{R}$ (for the trivial subgroup). So the real line is a universal proper D_∞-space.

We briefly discuss how change of group functors interact with the $\mathcal{C}om$-model structures. We let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups. Restriction of scalars along α is a functor

$$\alpha^* \ : \ G\mathbf{T} \ \longrightarrow \ K\mathbf{T} \ ;$$

here $\alpha^*(X)$ has the same underlying space as X, with K acting through the homomorphism α. The restriction functor α^* has a left adjoint

$$G \times_\alpha - \ : \ K\mathbf{T} \ \longrightarrow \ G\mathbf{T}$$

and a right adjoint

$$\mathrm{map}^{K,\alpha}(G, -) \ : \ K\mathbf{T} \ \longrightarrow \ G\mathbf{T} \ .$$

For any based K-space X, the G-space $G \times_\alpha X$ is the quotient of $G \times X$ by the equivalence relation $(g\alpha(k), x) \sim (g, kx)$. The G-space $\mathrm{map}^{K,\alpha}(G, X)$ is the space of those continuous maps $f : G \longrightarrow X$ that satisfy $k \cdot f(g) = f(\alpha(k) \cdot g)$ for all $(k, g) \in K \times G$, with G acting by

$$(\gamma \cdot f)(g) \ = \ f(g\gamma) \ .$$

An important special case is when α is the inclusion of a closed subgroup Γ of G. In that case we write res^G_Γ for the restriction functor, and we simplify the notation for the left and right adjoint to $G \times_\Gamma -$ and $\mathrm{map}^\Gamma(G, -)$, respectively.

Proposition 1.1.5. *Let H and Γ be closed subgroups of a Lie group G, such that moreover H is compact. Then the $(H \times \Gamma)$-action on G given by $(h, \gamma) \cdot g = hg\gamma^{-1}$ underlies an $(H \times \Gamma)$-CW-complex.*

Proof. We claim that the $(H \times \Gamma)$-action is *proper*, i.e., the map

$$(H \times \Gamma) \times G \ \longrightarrow \ G \times G \ , \quad ((h, \gamma), g) \ \longmapsto \ (hg\gamma^{-1}, g)$$

is a proper map in the sense that preimages of compact sets are compact. Indeed, we can factor this map as the composite of three proper maps, namely the inclusion of the closed subspace $H \times \Gamma \times G$ into $H \times G \times G$, followed by the homeomorphism

$$H \times G \times G \ \xrightarrow{\cong} \ H \times G \times G, \quad (h, \gamma, g) \ \longmapsto \ (h, hg\gamma^{-1}, g)$$

and the projection of $H \times G \times G$ to the last two factors. Since the $(H \times \Gamma)$-action on G is also smooth, Theorem I of [**28**] provides an $(H\times\Gamma)$-equivariant triangulation of G, and hence the desired equivariant CW-structure, by [**28**, Prop. 11.5]. $\square$

We recall that a *Quillen adjunction* is an adjoint functor pair (F, G) between model categories such that the left adjoint F preserves cofibrations and the right adjoint G preserves fibrations. Equivalent conditions are to require that the left adjoint F preserves cofibrations and acyclic cofibrations; or that the right adjoint G preserves fibrations and acyclic fibrations.

Proposition 1.1.6. *Let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups.*

(i) *The restriction functor $\alpha^* : G\mathbf{T} \longrightarrow K\mathbf{T}$ takes $\mathcal{C}om$-equivalences of G-spaces to $\mathcal{C}om$-equivalences of K-spaces.*
(ii) *The adjoint functor pair*

$$G \times_\alpha - \ : \ K\mathbf{T} \ \rightleftarrows \ G\mathbf{T} \ : \ \alpha^*$$

is a Quillen adjunction with respect to the $\mathcal{C}om$-model structures.

(iii) *If the image of α is closed in G and the kernel of α is compact, then the adjoint functor pair*

$$\mathrm{map}^{K,\alpha}(G,-) \ : \ K\mathbf{T} \rightleftarrows G\mathbf{T} \ : \ \alpha^*$$

is a Quillen adjunction with respect to the $\mathcal{C}om$-model structures.

PROOF. For every compact subgroup $L \leq K$ the image $\alpha(L)$ is a compact subgroup of G, and for every based G-space X, we have the equality $(\alpha^*(X))^L = X^{\alpha(L)}$. So the restriction functor takes $\mathcal{C}om$-equivalences of G-spaces to $\mathcal{C}om$-equivalences of K-spaces, and it takes $\mathcal{C}om$-fibrations of G-spaces to $\mathcal{C}om$-fibrations of K-spaces. This establishes parts (i) and (ii).

As a left adjoint, the restriction functor α^* preserves colimits. So for part (iii) we only have to check that the restriction of each of the generating G-$\mathcal{C}om$-cofibrations specified in Proposition 1.1.3 (ii) is a $\mathcal{C}om$-cofibration of K-spaces. This amounts to the claim that for every compact subgroup H of G the K-space $\alpha^*(G/H)$ is $\mathcal{C}om$-cofibrant. We let $\Gamma = \alpha(K)$ denote the image of α, which is a closed subgroup of G by hypothesis. Since G admits a $(\Gamma \times H)$-CW-structure by Proposition 1.1.5, the orbit space G/H inherits a Γ-CW-structure. We let $\beta : K \longrightarrow \Gamma$ denote the same homomorphism as α, but now considered as a continuous epimorphism onto its image. For every closed subgroup Δ of Γ we have

$$\beta^*(\Gamma/\Delta) \ \cong \ K/\beta^{-1}(\Delta) \ .$$

Since the restriction functor $\beta^* : \Gamma\mathbf{T} \longrightarrow K\mathbf{T}$ commutes with colimits and products with spaces, this shows that it takes Γ-CW-complexes to K-CW-complexes. In particular, $\alpha^*(G/H) = \beta^*(\mathrm{res}^G_\Gamma(G/H))$ admits the structure of a K-CW-complex. The K-stabilizer group of a coset gH is $\alpha^{-1}(H^g)$. Since H is compact and the kernel of α is compact by hypothesis, all stabilizer groups of $\alpha^*(G/H)$ are compact. So $\alpha^*(G/H)$ is $\mathcal{C}om$-cofibrant as a K-space. $\square$

REMARK 1.1.7. One should beware that restriction to a closed subgroup does *not* preserve general equivariant cofibrations without an isotropy condition. This should be contrasted with the fact that h-cofibrations (i.e., maps with the equivariant homotopy extension property) are stable under restriction to closed subgroups. For example, the left translation action makes $\mathbb{R}/\mathbb{Z}$ an $\mathbb{R}$-CW-complex and a cofibrant $\mathbb{R}$-space; the $\mathbb{R}$-space $\mathbb{R}/\mathbb{Z}$ is *not* $\mathcal{C}om$-cofibrant, however, because the stabilizer group $\mathbb{Z}$ is not compact. On the other hand, if Γ is the additive subgroup of $\mathbb{R}$ generated by an irrational number, then the underlying Γ-action on the circle $\mathbb{R}/\mathbb{Z}$ is not proper, and $\mathbb{R}/\mathbb{Z}$ is neither a Γ-CW-complex nor cofibrant as a Γ-space.

In the application to orthogonal G-spectra, we will also need the *based* version of the $\mathcal{C}om$-model structure, and the modification of some of the previous results to the based context. We write $\mathbf{T}_*$ for the category of based compactly generated spaces. A *based G-space* is a G-space equipped with a G-fixed basepoint; we write $G\mathbf{T}_*$ for the category of based G-spaces and based continuous G-maps.

A standard result in model category theory lets us lift the $\mathcal{C}om$-model structure from unbased to based G-spaces. A morphism in $G\mathbf{T}_*$ is a $\mathcal{C}om$-equivalence, $\mathcal{C}om$-fibration or $\mathcal{C}om$-cofibration if and only if it is so as an unbased G-map, see [**25**, Prop. 1.1.8]. We will freely use the based version of Propositions 1.1.3 and 1.1.6 in what follows.

After discussing equivariant spaces, we now move on to equivariant spectra. An *inner product space* is a finite-dimensional real vector space equipped with a scalar product. We denote by $\mathbf{L}(V,W)$ the space of linear isometric embeddings between two inner product spaces, topologized as the Stiefel manifold of $\dim(V)$-frames in W.

CONSTRUCTION 1.1.8. We let V and W be inner product spaces. The *orthogonal complement vector bundle* over the space $\mathbf{L}(V,W)$ of linear isometric embeddings has total space

$$\xi(V,W) \ = \ \{ (w,\varphi) \in W \times \mathbf{L}(V,W) \mid w \perp \varphi(V) \} \ .$$

The structure map $\xi(V,W) \longrightarrow \mathbf{L}(V,W)$ is the projection to the second factor. The vector bundle structure of $\xi(V,W)$ is as a vector subbundle of the trivial vector bundle $W \times \mathbf{L}(V,W)$, and the fiber over $\varphi : V \longrightarrow W$ is the orthogonal complement $W - \varphi(V)$ of the image of φ.

We let $\mathbf{O}(V,W)$ be the Thom space of the orthogonal complement bundle, i.e., the one-point compactification of the total space of $\xi(V,W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V,W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V,W)$ is empty and $\mathbf{O}(V,W)$ consists of a single point at infinity. If $\dim V = m$ and $\dim W = m+n$, then $\mathbf{L}(V,W)$ is homeomorphic to the homogeneous space $O(m+n)/O(n)$ and $\mathbf{O}(V,W)$ is homeomorphic to $O(m+n) \ltimes_{O(n)} S^n$.

The Thom spaces $\mathbf{O}(V,W)$ are the morphism spaces of a based topological category $\mathbf{O}$. Given a third inner product space U, the bundle map

$$\xi(V,W) \times \xi(U,V) \ \longrightarrow \ \xi(U,W) \ , \quad ((w,\varphi),\, (v,\psi)) \ \longmapsto \ (w+\varphi(v),\, \varphi\psi)$$

covers the composition map $\mathbf{L}(V,W) \times \mathbf{L}(U,V) \longrightarrow \mathbf{L}(U,W)$. Passage to Thom spaces gives a based map

$$\circ \ : \ \mathbf{O}(V,W) \wedge \mathbf{O}(U,V) \ \longrightarrow \ \mathbf{O}(U,W)$$

which is clearly associative, and is the composition in the category $\mathbf{O}$. The identity of V is $(0,\mathrm{Id}_V)$ in $\mathbf{O}(V,V)$.

DEFINITION 1.1.9. Let G be a Lie group. An *orthogonal G-spectrum* is a based continuous functor from $\mathbf{O}$ to the category $G\mathbf{T}_*$ of based G-spaces. A *morphism* of orthogonal spectra is a natural transformation of functors. We denote the category of orthogonal G-spectra by Sp_G.

A continuous functor to based G-spaces is the same as a G-object of continuous functors. So orthogonal G-spectra could equivalently be defined as orthogonal spectra equipped with a continuous G-action. Since we will not consider any other kind of spectra in this memoir, we will often drop the adjective 'orthogonal'; in other words, we use 'G-spectrum' as a synonym for 'orthogonal G-spectrum'.

If V and W are inner product spaces, we define a distinguished based continuous map

$$i_{V,W} \ : \ S^V \ \longrightarrow \ \mathbf{O}(W, V\oplus W) \quad \text{by} \quad v \ \longmapsto \ ((v,0),(0,-)) \ , \tag{1.1.10}$$

the one-point compactification of the fiber over the embedding $(0,-) : W \longrightarrow V \oplus W$ as the second summand. If X is an orthogonal spectrum, we refer to the composite

$$\sigma_{V,W} \ : \ S^V \wedge X(W) \ \xrightarrow{i_{V,W} \wedge X(W)} \ \mathbf{O}(W, V\oplus W) \wedge X(W) \ \longrightarrow \ X(V\oplus W)$$

as the *structure map* of X.

Limits and colimits in enriched functor categories are created objectwise. In particular, all small limits and colimits in Sp_G exist and are created 'levelwise'. Moreover, limits and colimits of based G-spaces are created on underlying non-equivariant based spaces; hence all limits and colimits in Sp_G are created in the category of underlying non-equivariant orthogonal spectra. By [**31**, Sec. 3.8] we conclude that the category Sp_G is enriched complete and cocomplete.

REMARK 1.1.11. If G is compact, then the above definition is equivalent to the original definition of orthogonal G-spectra given by Mandell and May in [**48**], in the following sense. In [**48**, II.2], Mandell and May define G-equivariant orthogonal spectra indexed on a G-universe $\mathcal{U}$. Such a G-spectrum is a collection of G-spaces indexed on those representations that embed into $\mathcal{U}$, together with certain equivariant structure maps. It follows from [**48**, Thm. II.4.3, Thm. V.1.5] that for any G-universe $\mathcal{U}$, the category of orthogonal G-spectra indexed on $\mathcal{U}$ and the category of orthogonal G-spectra as in Definition 1.1.9 are equivalent. This shows that universes are not really relevant for the pointset level definition of an orthogonal G-spectrum. However, they become important when one considers the homotopy theory of orthogonal G-spectra.

Here are some basic examples of orthogonal G-spectra; further examples will be discussed along the way.

EXAMPLE 1.1.12 (Suspension spectra). The *sphere spectrum* $\mathbb{S}$ is the orthogonal spectrum given by $\mathbb{S}(V) = S^V$. The orthogonal group $O(V)$ acts on V and hence on the one-point compactification S^V. The structure maps are the canonical homeomorphisms $S^V \wedge S^W \cong S^{V\oplus W}$. The *$G$-sphere spectrum* $\mathbb{S}_G$ is the orthogonal sphere spectrum equipped with trivial G-action. We will show in Example 3.2.9 that for discrete groups, the sphere spectrum represents G-equivariant stable cohomotopy as defined by the third author in [**38**].

More generally we consider a based G-space A. The *suspension spectrum* $\Sigma^\infty A$ is defined by $(\Sigma^\infty A)(V) = S^V \wedge A$. The group G acts through the second factor and the orthogonal groups act through the first factor. The structure maps are given by the canonical homeomorphisms $S^V \wedge (S^W \wedge A) \cong S^{V\oplus W} \wedge A$. The sphere spectrum $\mathbb{S}$ is isomorphic to $\Sigma^\infty S^0$.

EXAMPLE 1.1.13 (Trivial G-spectra). Every orthogonal spectrum becomes an orthogonal G-spectrum by letting G act trivially. For example, the G-sphere spectrum $\mathbb{S}_G$ arises in this way. This construction derives to an exact functor from global stable homotopy theory to G-equivariant stable homotopy theory, compare Theorem 3.3.3 below. The G-equivariant cohomology theories that arise in this way from global stable homotopy types have additional structure and special properties, i.e., they form 'equivariant cohomology theories' for all Lie groups and not just for a particular group and its subgroups. We return to this class of examples in more detail in Section 3.3.

CONSTRUCTION 1.1.14. We let G and K be Lie groups. Every continuous based functor $F : G\mathbf{T}_* \longrightarrow K\mathbf{T}_*$ between the categories of based equivariant spaces gives rise to a continuous functor

$$F \circ - \ : \ \mathrm{Sp}_G \ \longrightarrow \ \mathrm{Sp}_K$$

from orthogonal G-spectra to orthogonal K-spectra by postcomposition: it simply takes an orthogonal G-spectrum X to the composite

$$\mathbf{O} \xrightarrow{X} G\mathbf{T}_* \xrightarrow{F} K\mathbf{T}_* .$$

If A is a based G-space, then smashing with A and taking based maps out of A are two such functors (for $K = G$). So for every orthogonal G-spectrum X, we can define two new orthogonal G-spectra $X \wedge A$ and X^A by smashing with A or taking maps from A levelwise. More explicitly, we have

$$(X \wedge A)(V) = X(V) \wedge A \quad \text{respectively} \quad (X^A)(V) = X(V)^A = \mathrm{map}_*(A, X(V))$$

for an inner product space V. The structure maps and actions of the orthogonal groups do not interact with A. Just as the functors $- \wedge A$ and $\mathrm{map}_*(A, -)$ are adjoint on the level of based G-spaces, the two functors just introduced are an adjoint pair on the level of orthogonal G-spectra.

The previous Construction 1.1.14 provides tensors and cotensors for the category of orthogonal G-spectra over the closed symmetric monoidal category (under smash product) of based G-spaces. There is also enrichment of orthogonal G-spectra in based G-spaces as follows. The mapping space $\mathrm{map}(X, Y)$ between two orthogonal G-spectra is the space of morphisms between the underlying non-equivariant orthogonal spectra of X and Y; on this mapping space, the group G acts by conjugation.

Example 1.1.15 (Free spectra and evaluation on representations). Every real inner product space V gives rise to a representable functor $\mathbf{O}(V, -) : \mathbf{O} \longrightarrow \mathbf{T}_*$, which we denote by F_V. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the representable functor $\mathbf{O}(0, -)$. We turn F_V into a G-orthogonal spectrum by giving it the trivial action. As a consequence of the enriched Yoneda lemma, the functor

$$G\mathbf{T}_* \longrightarrow \mathrm{Sp}_G , \quad A \longmapsto F_V \wedge A$$

is left adjoint to the evaluation functor at V.

More generally, we let H be a closed subgroup of G and V an H-representation. Then the evaluation $X(V)$ is an $(H \times H)$-space by the 'external' H-action on X and the 'internal' H-action from the action on V and the $O(V)$-functoriality of X. We consider $X(V)$ as an H-space via the diagonal H-action. Via this action, if W is another H-representation, the structure map

$$S^V \wedge X(W) \longrightarrow X(V \oplus W)$$

becomes H-equivariant, with H acting diagonally on the domain. Moreover, the resulting evaluation functor

$$-(V) \ : \ \mathrm{Sp}_G \longrightarrow H\mathbf{T}_*$$

also has a left adjoint which sends a based H-space A to the orthogonal G-spectrum $G \ltimes_H (F_V \wedge A)$. Here H acts on F_V by precomposition with the H-action on V.

1.2. The stable model structure

In this relatively long section we establish the stable model structure on the category Sp_G of orthogonal G-spectra and investigate how it interacts with the smash product. We begin by recalling the equivariant homotopy groups π_*^H for *compact* Lie groups H in Construction 1.2.2, which are used to define the π_*-isomorphisms of orthogonal G-spectra, see Definition 1.2.4. As an application of our theory we will later also define equivariant homotopy groups for non-compact Lie groups, see Definition 2.1.1 below, but these do not play a role in the construction of the model structure. Theorem 1.2.9 and Corollary 1.2.10 show that certain induction constructions preserve π_*-isomorphisms. The stable model structure on the category of orthogonal G-spectra in Theorem 1.2.22 is the main result of this section. This model structure is Quillen equivalent to the one previously obtained by Fausk in [**21**, Prop. 6.5]; our model structure has more cofibrations and we provide an explicit characterization of the stable fibrations by certain homotopy pullback requirements. The proof of the stable model structure proceeds by localizing a certain level model structure.

We also review the smash product of orthogonal spectra in Definition 1.2.25 and establish various homotopical properties of the smash product of orthogonal G-spectra, with diagonal G-action. Theorem 1.2.27 shows that smashing with a fairly broad class of orthogonal G-spectra that we call 'quasi-flat' preserves π_*-isomorphisms; Proposition 1.2.28 ('pushout product property') and Proposition 1.2.30 ('monoid axiom') show that the smash product of orthogonal G-spectra interacts well with the stable model structure. The final result of this section is Proposition 1.2.33, which shows that the Thom space of a G-vector bundle over the universal G-space $\underline{E}G$ for proper actions is smash invertible in $\mathrm{Ho}(\mathrm{Sp}_G)$. This result is the first indication that the role of the representation ring $RO(G)$ in the realm of compact Lie groups is now taken by $KO_G(\underline{E}G)$, the Grothendieck group of G-vector bundles over $\underline{E}G$.

Definition 1.2.1. Let H be a compact Lie group. A *complete H-universe* is an orthogonal H-representation of countably infinite dimension such that every finite-dimensional H-representation embeds into it.

For every compact Lie group H, we choose a complete H-universe $\mathcal{U}_H$. Up to equivariant isometry, such a complete H-universe is given by

$$\mathcal{U}_H \;\cong\; \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mathbb{N}} \lambda \ ,$$

where Λ is a set of representatives of all irreducible H-representations. We let $s(\mathcal{U}_H)$ denote the poset, under inclusion, of finite-dimensional H-subrepresentations of $\mathcal{U}_H$.

Construction 1.2.2 (Equivariant homotopy groups). Let k be any integer, X an orthogonal G-spectrum and H a compact subgroup of G; we define the H-equivariant homotopy group $\pi_k^H(X)$. We start with the case $k \geq 0$. We recall that for an orthogonal H-representation V, we let H act diagonally on $X(V)$, through the two H-actions on X and on V. For every $V \in s(\mathcal{U}_H)$ we consider the set

$$[S^{V\oplus\mathbb{R}^k}, X(V)]^H$$

of H-equivariant homotopy classes of based H-maps from $S^{V\oplus\mathbb{R}^k}$ to $X(V)$. We can stabilize by increasing $V \subset W$ along the maps

$$[S^{V\oplus\mathbb{R}^k}, X(V)]^H \;\longrightarrow\; [S^{W\oplus\mathbb{R}^k}, X(W)]^H$$

defined as follows. We let $V^\perp = W - V$ denote the orthogonal complement of V in W. The stabilization sends the homotopy class of $f : S^{V\oplus\mathbb{R}^k} \longrightarrow X(V)$ to the homotopy class of the composite

$$\begin{aligned} S^{W\oplus\mathbb{R}^k} \ \cong \ S^{V^\perp} \wedge S^{V\oplus\mathbb{R}^k} &\xrightarrow{S^{V^\perp}\wedge f} S^{V^\perp} \wedge X(V) \\ &\xrightarrow{\sigma_{V^\perp,V}} X(V^\perp \oplus V) \ \cong \ X(W) \ , \end{aligned}$$

where the two unnamed homeomorphisms use the preferred linear isometry

$$V^\perp \oplus V \ \cong \ W \ , \quad (w,v) \ \longmapsto \ (w+v) \ .$$

These stabilization maps define a functor on the poset $s(\mathcal{U}_H)$. The *k-th equivariant homotopy group* $\pi_k^H(X)$ is then defined as

$$\pi_k^H(X) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_H)} \, [S^{V\oplus\mathbb{R}^k}, X(V)]^H \ . \tag{1.2.3}$$

The abelian group structure arises from the pinch addition in the source variable, based on a G-fixed unit vector in V, for large enough V. For $k<0$, the definition of $\pi_k^H(X)$ is the same, but with $[S^{V\oplus\mathbb{R}^k}, X(V)]^H$ replaced by $[S^V, X(V\oplus\mathbb{R}^{-k})]^H$.

Definition 1.2.4. Let G be a Lie group. A morphism $f\colon X\longrightarrow Y$ of orthogonal G-spectra is a *π_*-isomorphism* if for every compact subgroup H of G and every integer k, the induced map

$$\pi_k^H(f) \ : \ \pi_k^H(X) \ \longrightarrow \ \pi_k^H(Y)$$

is an isomorphism.

In the case of compact groups, this definition recovers the notion of π_*-isomorphism from [**48**, Sec. III.3] or [**60**, Def. 3.1.12].

Construction 1.2.5 (Loop and suspension isomorphism). An important special case of Construction 1.1.14 is when $A=S^1$ is a 1-sphere with trivial action. The *suspension* $X\wedge S^1$ is defined by

$$(X\wedge S^1)(V) \ = \ X(V)\wedge S^1 \ ,$$

the smash product of the V-th level of X with the sphere S^1. The *loop spectrum* $\Omega X = X^{S^1}$ is defined by

$$(\Omega X)(V) \ = \ \Omega X(V) \ = \ \operatorname{map}_*(S^1, X(V)) \ ,$$

the based mapping space from S^1 to the V-th level of X.

We define the *loop isomorphism*

$$\alpha \ : \ \pi_k^H(\Omega X) \ \longrightarrow \ \pi_{k+1}^H(X) \ . \tag{1.2.6}$$

For $k\geq 0$, we represent a given class in $\pi_k^H(\Omega X)$ by a based H-map $f: S^{V\oplus\mathbb{R}^k} \longrightarrow \Omega X(V)$ and let $\tilde f : S^{V\oplus\mathbb{R}^{k+1}} \longrightarrow X(V)$ denote the adjoint of f, which represents an element of $\pi_{k+1}^H(X)$. For $k<0$, we represent a class in $\pi_k^H(\Omega X)$ by a based H-map $f: S^V \longrightarrow \Omega X(V\oplus\mathbb{R}^{-k})$ and let $\tilde f : S^{V\oplus\mathbb{R}} \longrightarrow X(V\oplus\mathbb{R}^{-k}) \cong X((V\oplus\mathbb{R})\oplus\mathbb{R}^{-(k+1)})$ denote the adjoint of f, which represents an element of $\pi_{k+1}^H(X)$. Then we can set

$$\alpha[f] \ = \ [\tilde f] \ .$$

The loop isomorphism is indeed bijective, by straightforward adjointness.

Next we define the *suspension isomorphism*

$$-\wedge S^1 \ : \ \pi_k^H(X) \ \longrightarrow \ \pi_{k+1}^H(X\wedge S^1) \ . \tag{1.2.7}$$

For $k \geq 0$ we represent a given class in $\pi_k^H(X)$ by a based H-map $f : S^{V\oplus\mathbb{R}^k} \longrightarrow X(V)$; then $f \wedge S^1 : S^{V\oplus\mathbb{R}^{k+1}} \longrightarrow X(V) \wedge S^1$ represents a class in $\pi_{k+1}^H(X \wedge S^1)$. For $k < 0$ we represent a given class in $\pi_k^H(X)$ by a based H-map $f : S^V \longrightarrow X(V\oplus\mathbb{R}^{-k})$; then $f\wedge S^1 : S^{V\oplus\mathbb{R}} \longrightarrow X(V\oplus\mathbb{R}^{-k})\wedge S^1 \cong X((V\oplus\mathbb{R})\oplus\mathbb{R}^{-(k+1)})\wedge S^1$ represents a class in $\pi_{k+1}^H(X \wedge S^1)$. Then we set

$$[f] \wedge S^1 \;=\; [f \wedge S^1] \ .$$

The suspension isomorphism is indeed bijective, see for example [**60**, Prop. 3.1.30].

Next we recall the concept of an h-cofibration of orthogonal G-spectra; this notion occurs in the proof of the Theorem 1.2.9, and we will use it at several later points. In the case when G is a compact Lie group, the basic properties of h-cofibrations are discussed in [**48**, III Thm. 3.5].

Definition 1.2.8. A morphism $i\colon A \longrightarrow B$ of orthogonal G-spectra is an *h-cofibration* if it has the homotopy extension property: for every orthogonal G-spectrum X, every morphism $\varphi : B \longrightarrow X$ and every homotopy $H\colon A \wedge [0,1]_+ \longrightarrow X$ starting with $\varphi \circ i$, there exists a homotopy $\bar{H} : B \wedge [0,1]_+ \longrightarrow X$ starting with φ that satisfies $\bar{H} \circ (i \wedge [0,1]_+) = H$.

There is a universal test case for the homotopy extension property, and a morphism $i\colon A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism $(A\wedge[0,1]_+)\cup_i B \longrightarrow B\wedge[0,1]_+$ admits a retraction. For every continuous homomorphism $\alpha : K \longrightarrow G$ between Lie groups, the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ preserves colimits and smash products with based spaces; so if i is an h-cofibration of G-spectra, then $\alpha^*(i)$ is an h-cofibration of K-spectra. In particular, restriction to a closed subgroup preserves h-cofibrations.

Similarly, if $i : A \longrightarrow B$ is an h-cofibration of orthogonal G-spectra, then for every compact subgroup H of G and every orthogonal H-representation V, the H-equivariant map $i(V)\colon A(V) \longrightarrow B(V)$ is an h-cofibration of based H-spaces. This uses that the evaluation functors also commute with colimits and smash products with based spaces.

In the next theorem we consider two Lie groups K and Γ. We call a $(K \times \Gamma)$-space *bifree* if the underlying K-action is free and the underlying Γ-action is free. Equivalently, the stabilizer group of every point intersects both of the two subgroups $K \times \{1\}$ and $\{1\} \times \Gamma$ only in the neutral element. The compact subgroups of $K \times \Gamma$ with this property are precisely the graphs of all continuous monomorphisms $\alpha : L \longrightarrow \Gamma$, defined on compact subgroups of K. In the following theorem we turn the left Γ-action on A into a right action by setting $a \cdot \gamma = \gamma^{-1} \cdot a$, for $(a, \gamma) \in A \times \Gamma$.

Theorem 1.2.9. *Let Γ and K be Lie groups and A a bifree $\mathcal{C}om$-cofibrant $(K \times \Gamma)$-space. Then the functor*

$$A_+ \wedge_\Gamma - \;:\; \mathrm{Sp}_\Gamma \;\longrightarrow\; \mathrm{Sp}_K$$

takes π_*-isomorphisms of orthogonal Γ-spectra to π_*-isomorphisms of orthogonal K-spectra.

Proof. We start with the special case when the group K is compact. Since the functor $A_+ \wedge_\Gamma -$ commutes with mapping cones, and since mapping cone sequences give rise to long exact sequences of equivariant homotopy groups [**60**, Prop. 3.1.36], it suffices to show the following special case: we let X be any orthogonal Γ-spectrum that is Γ-π_*-trivial, i.e., all of whose equivariant homotopy groups, for all compact subgroups of Γ, vanish. Then $A_+ \wedge_\Gamma X$ is K-π_*-trivial. For this we assume first that A is a finite-dimensional proper $(K \times \Gamma)$-CW-complex, with skeleta A^n. We argue by induction over the dimension of A. The induction starts with A^{-1}, which is empty, and there is nothing to show. Then we let $n \geq 0$ and assume the claim for A^{n-1}. By hypothesis there is a pushout square of $(K \times \Gamma)$-spaces:

$$\begin{tikzcd} \coprod_{j\in J} (K \times \Gamma)/\Delta_j \times \partial D^n \arrow[r] \arrow[d] & \coprod_{j\in J} (K \times \Gamma)/\Delta_j \times D^n \arrow[d] \\ A^{n-1} \arrow[r] & A^n \end{tikzcd}$$

Here J is an indexing set of the n-cells of the equivariant CW-structure and Δ_j is a compact subgroup of $K \times \Gamma$. Since the $(K \times \Gamma)$-action on A is bifree, each of the subgroups Δ_j must be the graph of a continuous monomorphism $\alpha_j : L_j \longrightarrow \Gamma$ defined on a compact subgroup L_j of K.

The inclusion $A^{n-1} \longrightarrow A^n$ is an h-cofibration of $(K \times \Gamma)$-spaces, so the morphism $A_+^{n-1} \wedge_\Gamma X \longrightarrow A_+^n \wedge_\Gamma X$ is an h-cofibration of orthogonal K-spectra. The long exact homotopy group sequence [**60**, Cor. 3.1.38] thus reduces the inductive step to showing that the K-equivariant homotopy groups of the cofiber $(A_+^n \wedge_\Gamma X)/(A_+^{n-1} \wedge_\Gamma X)$ vanish. This cofiber is isomorphic to

$$\bigvee_{j\in J} (K \times \Gamma/\Delta_j)_+ \wedge_\Gamma X \wedge S^n \ .$$

Since equivariant homotopy groups take wedges to sums [**60**, Cor. 3.1.37 (i)] and reindex upon smashing with S^n (by the suspension isomorphism (1.2.7), compare [**60**, Prop. 3.1.30]), it suffices to consider an individual wedge summand without any suspension. In other words, we may show that the orthogonal K-spectrum

$$(K \times \Gamma)/\Delta_+ \wedge_\Gamma X \ \cong \ K \ltimes_L \alpha^*(X)$$

is K-π_*-trivial, where L is a closed subgroup of K and Δ is the graph of a continuous monomorphism $\alpha : L \longrightarrow \Gamma$. Now X is Γ-π_*-trivial by hypothesis, so $\alpha^*(X)$ is L-π_*-trivial. Since K and L are compact, [**60**, Cor. 3.2.21] lets us conclude that $K \ltimes_L \alpha^*(X)$ is K-π_*-trivial. This completes the inductive step.

Now we suppose that A is a proper $(K \times \Gamma)$-CW-complex, possibly infinite dimensional. As already noted above, the morphisms

$$A_+^{n-1} \wedge_\Gamma X \ \longrightarrow \ A_+^n \wedge_\Gamma X$$

induced by the skeleton inclusions are h-cofibrations of orthogonal K-spectra. Since $A_+ \wedge_\Gamma X$ is a colimit of the sequence of spectra $A_+^n \wedge_\Gamma X$, each $A_+^n \wedge_\Gamma X$ is K-π_*-trivial, and a colimit of π_*-isomorphisms over a sequence of h-cofibrations is another π_*-isomorphism (compare [**48**, III Thm. 3.5 (v)] or [**60**, Prop. 3.1.41]), we conclude that $A_+ \wedge_\Gamma X$ is K-π_*-trivial. A general $\mathcal{C}om$-cofibrant $(K \times \Gamma)$-space is $(K \times \Gamma)$-homotopy equivalent to a proper $(K \times \Gamma)$-CW-complex, so this concludes the proof in the special case where K is compact.

Now we treat the general case. We let L be any compact subgroup of K. The underlying $(L \times \Gamma)$-space of A is again bifree, and it is $\mathcal{C}om$-cofibrant as an $(L \times \Gamma)$-space by Proposition 1.1.6 (iii). So the composite functor $\mathrm{res}_L^K \circ (A_+ \wedge_\Gamma -) : \mathrm{Sp}_\Gamma \longrightarrow \mathrm{Sp}_L$ preserves π_*-isomorphisms by the special case above. Since π_*-isomorphisms of K-spectra can be tested on all compact subgroups of K, this proves the claim. $\square$

Corollary 1.2.10. *Let Γ be a closed subgroup of a Lie group G. The induction functor*
$$G\ltimes_\Gamma \ : \ \mathrm{Sp}_\Gamma \ \longrightarrow \ \mathrm{Sp}_G$$
takes π_*-isomorphisms of orthogonal Γ-spectra to π_*-isomorphisms of orthogonal G-spectra.

Proof. We let H be a compact subgroup of G. Then the $(H\times\Gamma)$-action on G given by $(h,\gamma)\cdot g = hg\gamma^{-1}$ underlies a proper $(H\times\Gamma)$-CW-complex, by Proposition 1.1.5. In particular, G is $\mathcal{C}om$-cofibrant as an $(H\times\Gamma)$-space. The action is bifree, so Theorem 1.2.9 applies and shows that the functor $\mathrm{res}^G_H\circ(G\ltimes_\Gamma -)$ takes π_*-isomorphisms of orthogonal Γ-spectra to π_*-isomorphisms of orthogonal H-spectra. Since H was an arbitrary compact subgroup of G, this proves the claim. $\square$

We let H be a compact subgroup of a Lie group G, and X is an orthogonal G-spectrum. We recall from Example 1.1.15 that if V is an H-representation, then we equip the evaluation $X(V)$ with the diagonal H-action of the two H-actions on X and on V.

Definition 1.2.11. Let G be a Lie group and $f\colon X\longrightarrow Y$ a morphism of orthogonal G-spectra.

(i) The morphism f is a *level equivalence* if $f(V)^H\colon X(V)^H\longrightarrow Y(V)^H$ is a weak equivalence for every compact subgroup H of G and every orthogonal H-representation V.
(ii) The morphism f is a *level fibration* if $f(V)^H\colon X(V)^H\longrightarrow Y(V)^H$ is a Serre fibration for every compact subgroup H of G and every orthogonal H-representation V.

It follows from the definition that if $f\colon X\longrightarrow Y$ is a level equivalence (level fibration), then $f(V)\colon X(V)\longrightarrow Y(V)$ is an H-weak equivalence (H-fibration) for every compact subgroup H, simply because every orthogonal H-representation is also a K-representation for every $K\leq H$.

In order to define the cofibrations of orthogonal G-spectra, we recall the *skeleton filtration*, a functorial way to write an orthogonal spectrum as a sequential colimit of spectra which are made from the information below a fixed level. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal spectrum need not be injective.

Construction 1.2.12 (Skeleton filtration of orthogonal spectra). We let $\mathbf{O}_{\leq m}$ denote the full topological subcategory of $\mathbf{O}$ whose objects are the inner product spaces of dimension at most m. We write $\mathrm{Sp}_{\leq m}$ for the category of continuous based functors from $\mathbf{O}_{\leq m}$ to $\mathbf{T}_*$. Restriction to the subcategory $\mathbf{O}_{\leq m}$ defines a functor
$$(-)_{\leq m} \ : \ \mathrm{Sp} \ \longrightarrow \ \mathrm{Sp}_{\leq m} \ .$$
This functor has a left adjoint
$$l_m \ : \ \mathrm{Sp}_{\leq m} \ \longrightarrow \ \mathrm{Sp} \ ,$$
an enriched left Kan extension. The *m-skeleton* of an orthogonal spectrum X is
$$\mathrm{sk}^m X \ = \ l_m(X_{\leq m}) \ ,$$
the extension of the restriction of X to $\mathbf{O}_{\leq m}$. The skeleton comes with a natural morphism $i_m\ :\ \mathrm{sk}^m X\longrightarrow X$, the counit of the adjunction $(l_m,(-)_{\leq m})$. Kan

extensions along a fully faithful functor do not change the values on the given subcategory [**31**, Prop. 4.23], so the value

$$i_m(V) \ : \ (\mathrm{sk}^m X)(V) \ \longrightarrow \ X(V)$$

is an isomorphism for all inner product spaces V of dimension at most m. The *m-th latching space* of X is the based $O(m)$-space

$$L_m X \ = \ (\mathrm{sk}^{m-1} X)(\mathbb{R}^m) \ ;$$

it comes with a natural based $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) \ : \ L_m X \ \longrightarrow \ X(\mathbb{R}^m) \ ,$$

the *m-th latching map*. We set $\mathrm{sk}^{-1} X = *$, the trivial orthogonal spectrum, and $L_0 X = *$, a one-point space.

The different skeleta are related by natural morphisms $j_m \ : \ \mathrm{sk}^{m-1} X \longrightarrow \mathrm{sk}^m X$, for all $m \geq 0$, such that $i_m \circ j_m = i_{m-1}$. The sequence of skeleta stabilizes to X in a strong sense: the maps $j_m(V)$ and $i_m(V)$ are isomorphisms as soon as $m > \dim(V)$. In particular, $X(V)$ is a colimit, with respect to the maps $i_m(V)$, of the sequence of maps $j_m(V)$. Since colimits in the category of orthogonal spectra are created objectwise, the orthogonal spectrum X is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Moreover, each skeleton is built from the previous one in a systematic way controlled by the latching map. We write G_m for the left adjoint to the evaluation functor

$$\mathrm{ev}_{\mathbb{R}^m} \ : \ \mathrm{Sp} \ \longrightarrow \ O(m)\mathbf{T}_* \ .$$

Then the commutative square

$$\begin{tikzcd} G_m L_m X \arrow[r, "G_m \nu_m"] \arrow[d] & G_m X(\mathbb{R}^m) \arrow[d] \\ \mathrm{sk}^{m-1} X \arrow[r, "j_m"'] & \mathrm{sk}^m X \end{tikzcd} \tag{1.2.13}$$

is a pushout of orthogonal spectra, see [**60**, Prop. C.17]. The two vertical morphisms are instances of the adjunction counit.

Since the skeleta and latching objects are continuous functors in the orthogonal spectrum, and since the latching morphisms are natural, actions of groups go along free for the ride. More precisely, the skeleta of (the underlying orthogonal spectrum of) an orthogonal G-spectrum inherit a continuous G-action by functoriality. In other words, the skeleta and the various morphisms between them lift to endofunctors and natural transformations on the category of orthogonal G-spectra. If X is an orthogonal G-spectrum, then the $O(m)$-space $L_m X$ comes with a commuting action by G, again by functoriality of the latching space. Moreover, the latching morphism $\nu_m \ : \ L_m X \longrightarrow X(\mathbb{R}^m)$ is $(G \times O(m))$-equivariant. Since colimits of orthogonal G-spectra are created in the underlying category of orthogonal spectra, the square (1.2.13) is a pushout square of orthogonal G-spectra.

DEFINITION 1.2.14. Let G be a Lie group. A morphism $i : A \longrightarrow B$ of orthogonal G-spectra is a *cofibration* if for every $m \geq 0$ the latching map

$$\nu_m i \ = \ \nu_m^B \cup i(\mathbb{R}^m) \ : \ L_m B \cup_{L_m A} A(\mathbb{R}^m) \ \longrightarrow \ B(\mathbb{R}^m)$$

is a $\mathcal{C}om$-cofibration of $(G \times O(m))$-spaces and, moreover, the action of $O(m)$ is free away from the image of $\nu_m i$.

If X is an orthogonal spectrum and V and W are inner product spaces, we write

$$\tilde{\sigma}_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}_*(S^V, X(V \oplus W))$$

for the adjoint of the structure map $\sigma_{V,W} : S^V \wedge X(W) \longrightarrow X(V \oplus W)$. Here $\mathrm{map}_*(-,-)$ denotes the space of *based* continuous maps.

DEFINITION 1.2.15. A morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is a *stable fibration* if it is a level fibration, and moreover, for every compact subgroup H of G and all H-representations V and W the square

$$\begin{tikzcd} X(W)^H \arrow[r, "(\tilde{\sigma}_{V,W})^H"] \arrow[d, "f(V)^H"'] & \mathrm{map}_*^H(S^V, X(V \oplus W)) \arrow[d, "\mathrm{map}_*^H(S^V, f(V\oplus W))"] \\ Y(W)^H \arrow[r, "(\tilde{\sigma}_{V,W})^H"'] & \mathrm{map}_*^H(S^V, Y(V \oplus W)) \end{tikzcd} \tag{1.2.16}$$

is homotopy cartesian. An orthogonal G-spectrum is a *G-Ω-spectrum* if for every compact subgroup H of G and all H-representations V and W the map

$$(\tilde{\sigma}_{V,W})^H \ : \ X(W)^H \ \longrightarrow \ \mathrm{map}_*^H(S^V, X(V \oplus W))$$

is a weak equivalence.

We note that an orthogonal G-spectrum is a G-Ω-spectrum precisely when the unique morphism to any trivial spectrum is a stable fibration. In other words, G-Ω-spectra come out as the fibrant objects in the stable model structure on Sp_G.

PROPOSITION 1.2.17. *Let G be a Lie group. Every π_*-isomorphism that is also a stable fibration is a level equivalence.*

PROOF. This is a combination of Proposition 4.8 and Corollary 4.11 of [**48**, Ch. III]. In more detail, we let $f : X \longrightarrow Y$ be a π_*-isomorphism and a stable fibration, and we consider a compact subgroup H of G. Since f is a stable fibration, [**48**, III Prop. 4.8] shows that the morphism $\mathrm{res}^G_H(f)$ of underlying orthogonal H-spectra has the right lifting property with respect to a certain set K of morphisms specified in [**48**, III Def. 4.6]. Since f is also a π_*-isomorphism, [**48**, III Cor. 4.11] then shows that for every H-representation V the map $f(V)^H : X(V)^H \longrightarrow Y(V)^H$ is a weak equivalence. $\square$

Now we name explicit sets of generating cofibrations and generating acyclic cofibrations for the stable model structure on Sp_G. We fix once and for all a complete set $\mathcal{V}_H$ of representatives of isomorphism classes of finite-dimensional orthogonal H-representations, for every compact Lie group H. We let I^G_{lv} denote the set of morphisms

$$(G \ltimes_H F_V) \wedge \partial D^k_+ \ \longrightarrow \ (G \ltimes_H F_V) \wedge D^k_+ \ ,$$

for all $k \geq 0$, where H runs through all compact subgroups of G and V runs through all representations in $\mathcal{V}_H$. Here F_V is the free spectrum in level V, see Example 1.1.15. Similarly, we let J^G_{lv} denote the set of morphisms

$$(G \ltimes_H F_V) \wedge (D^k \times \{0\})_+ \longrightarrow (G \ltimes_H F_V) \wedge (D^k \times [0,1])_+ \ , \tag{1.2.18}$$

with (H, V, k) running through the same set as for I^G_{lv}.

Every morphism of orthogonal G-spectra $j : A \longrightarrow B$ factors through the mapping cylinder as the composite

$$A \xrightarrow{c(j)} Z(j) = (A \wedge [0,1]_+) \cup_j B \xrightarrow{r(j)} B$$

where $c(j)$ is the 'front' mapping cylinder inclusion and $r(j)$ is the projection, which is a homotopy equivalence. In our applications we will assume that both A and B are cofibrant, and then the morphism $c(j)$ is a cofibration, compare [**26**, Lemma 3.4.10]. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$c(j)\Box i^k_+ \ : \ A \wedge D^k_+ \cup_{A \wedge \partial D^k_+} Z(j) \wedge \partial D^k_+ \ \longrightarrow \ Z(j) \wedge D^k_+$$

for $k \geq 0$, where $i^k : \partial D^k \longrightarrow D^k$ is the inclusion.

Let H be a compact subgroup of the Lie group G. For a pair of H-representations V and W, a morphism of orthogonal H-spectra

$$\lambda_{H,V,W} \colon F_{V \oplus W} S^V \ \longrightarrow \ F_W \tag{1.2.19}$$

is defined as the adjoint of the H-map (1.1.10)

$$i_{V,W} \ : \ S^V \ \longrightarrow \ \mathbf{O}(W, V \oplus W) \ = \ F_W(V \oplus W) \ ,$$

compare [**48**, Sec. III.4]. So $\lambda_{H,V,W}$ represents taking H-fixed points of the adjoint structure map:

$$(\tilde{\sigma}_{V,W})^H \ : \ X(W)^H \ \longrightarrow \ \mathrm{map}^H(S^V, X(V \oplus W)) \ .$$

The morphism $\lambda_{H,V,W}$ is a π_*-isomorphism of orthogonal H-spectra by [**48**, III Lemma 4.5].

We set

$$\mathcal{K}^G \ = \ \bigcup_{H,V,W} \mathcal{Z}(G \ltimes_H \lambda_{H,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $\partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $G \ltimes_H \lambda_{H,V,W}$. Here the union is over a set of triples (H, V, W) consisting of a compact subgroup H of G and two H-representations V and W from the set $\mathcal{V}_H$ of representatives of isomorphism classes of H-representations. We let

$$J^G_{\mathrm{st}} \ = \ J^G_{\mathrm{lv}} \cup \mathcal{K}^G$$

stand for the union of J^G_{lv} and $\mathcal{K}^G$. The sets I^G_{lv} and J^G_{st} will serve as sets of generating cofibrations and acyclic cofibrations for the stable model structure on Sp_G.

Proposition 1.2.20. *A morphism of orthogonal G-spectra is a stable fibration if and only if it has the right lifting property with respect to the set J^G_{st}.*

Proof. The right lifting property with respect to J^G_{lv} is equivalent to being a level fibration. By [**60**, Prop. 1.2.16], the right lifting property of a morphism $f : X \longrightarrow Y$ with respect to $J^G_{\mathrm{st}} = J^G_{\mathrm{lv}} \cup \mathcal{K}^G$ is then equivalent to the additional requirement that the square of mapping spaces

$$\begin{tikzcd}
\mathrm{map}^G(G \ltimes_H F_W, X) \arrow[r, "\mathrm{map}^G(G\ltimes_H \lambda_{H,V,W}, X)"] \arrow[d, "\mathrm{map}^G(G\ltimes_H F_W, f)"'] & \mathrm{map}^G(G \ltimes_H F_{V\oplus W} S^V, X) \arrow[d, "\mathrm{map}^G(G\ltimes_H F_{V\oplus W} S^V, f)"] \\
\mathrm{map}^G(G \ltimes_H F_W, Y) \arrow[r, "\mathrm{map}^G(G\ltimes_H \lambda_{H,V,W}, Y)"'] & \mathrm{map}^G(G \ltimes_H F_{V\oplus W} S^V, Y)
\end{tikzcd}$$

is homotopy cartesian, where $\mathrm{map}^G(-,-)$ is the space of morphisms of orthogonal G-spectra. This proves the claim because the orthogonal G-spectrum $G \ltimes_H F_W$ represents the functor $X \mapsto X(W)^H$, the orthogonal G-spectrum $G \ltimes_H (F_{V\oplus W} S^V)$ represents the functor $X \mapsto \mathrm{map}_*^H(S^V, X(V \oplus W))$, and the morphism $G \ltimes_H \lambda_{H,V,W}$ represents H-fixed points of the adjoint structure map $\tilde{\sigma}_{V,W} : X(W) \longrightarrow \mathrm{map}_*(S^V, X(V \oplus W))$. $\square$

PROPOSITION 1.2.21. *Every morphism in J^G_{st} is a π_*-isomorphism and a cofibration.*

PROOF. All morphisms in J^G_{lv} are cofibrations. They are also level equivalences, and hence also π_*-isomorphisms by [**48**, III Lemma 3.3], applied to the underlying orthogonal H-spectra, for all compact subgroups H.

Since for any compact subgroup H of G, the orthogonal G-spectra $G \ltimes_H F_{V\oplus W} S^V$ and $G \ltimes_H F_W$ are cofibrant, the morphisms in $\mathcal{K}^G$ are cofibrations. The morphism $\lambda_{H,V,W}$ is a π_*-isomorphism of orthogonal H-spectra by [**48**, III Lemma 4.5]. So the mapping cylinder inclusion $c(\lambda_{H,V,W})$ is then also a π_*-isomorphism, because it differs from $\lambda_{H,V,W}$ only by a homotopy equivalence of orthogonal H-spectra. The pushout product $c(G \ltimes_H \lambda_{H,V,W}) \,\square\, i^k_+$ is isomorphic to $G \ltimes_H (c(\lambda_{H,V,W}) \,\square\, i^k_+)$. By [**48**, III Sec. 4], the morphism $c(\lambda_{H,V,W}) \,\square\, i^k_+$ is a π_*-isomorphism of orthogonal H-spectra. Now Corollary 1.2.10 implies that $G \ltimes_H (c(\lambda_{H,V,W}) \,\square\, i^k_+)$ is a π_*-isomorphism. $\square$

Now we assemble the ingredients and construct the stable model structure. As we already mentioned, the following model structure is Quillen equivalent to the one established by Fausk in [**21**, Prop. 6.5]. We refrain from comparing the two model structures, since that is not relevant for our purposes.

THEOREM 1.2.22 (Stable model structure). *Let G be a Lie group.*

(i) *The π_*-isomorphisms, stable fibrations and cofibrations form a model structure on the category of orthogonal G-spectra, the* stable model structure.
(ii) *Every cofibration of orthogonal G-spectra is an h-cofibration.*
(iii) *Let $i : A \longrightarrow B$ be a cofibration of orthogonal G-spectra and $j : K \longrightarrow L$ a G-cofibration of based G-spaces. Then the pushout-product morphism*

$$i \,\square\, j \;=\; (i \wedge L) \cup (B \wedge j) \;:\; A \wedge L \cup_{A\wedge K} B \wedge K \;\longrightarrow\; B \wedge L$$

is a cofibration of orthogonal G-spectra. If moreover i is a π_*-isomorphism or j is a $\mathcal{C}om$-equivalence, then $i \,\square\, j$ is a π_*-isomorphism.
(iv) *The stable model structure is proper, stable, topological and cofibrantly generated.*
(v) *The fibrant objects in the stable model structure are the G-Ω-spectra.*

PROOF. (i) We start by establishing a level model structure for orthogonal G-spectra. Orthogonal G-spectra are continuous based functors from $\mathbf{O}$ to the category $G\mathbf{T}_*$, so we can employ the machinery from [**60**, Prop. C.23] that produces level model structures on enriched functor categories. Here the base category is $\mathcal{V} = G\mathbf{T}_*$, with smash product as monoidal structure. The index category is $\mathcal{D} = \mathbf{O}$, where the group G acts trivially on all morphism spaces. The category $\mathcal{D}^*$ of enriched functors from $\mathbf{O}$ to $G\mathbf{T}_*$ then becomes Sp_G. The dimension function on $\mathbf{O}$ is the vector space dimension, and then the abstract skeleta of [**60**, Con. C.13] specialize to the skeleta above.

To apply [**60**, Prop. C.23] we need to specify a model structure on the category of $O(m)$-objects in $G\mathbf{T}_*$, i.e., on the category of based $(G \times O(m))$-spaces. We use the $\mathcal{C}(m)$-projective model structure, in the sense of [**60**, Prop. B.7], where $\mathcal{C}(m)$ is the family of those closed subgroups of $G \times O(m)$ that are the graph of a continuous homomorphism to $O(m)$ defined on some compact subgroup of G. Equivalently, a closed subgroup Δ of $G \times O(m)$ belongs to $\mathcal{C}(m)$ if and only if it is compact and $\Delta \cap (1 \times O(m))$ consists only of the neutral element. The restriction functor from $(G \times O(m+n))$-spaces to $(G \times O(m))$-spaces takes $\mathcal{C}(m+n)$-fibrations to $\mathcal{C}(m)$-fibrations, so its left adjoint

$$O(m+n) \ltimes_{O(m)} - \ : \ (G \times O(m))\mathbf{T}_* \ \longrightarrow \ (G \times O(m+n))\mathbf{T}_*$$

preserves acyclic cofibrations. This establishes the consistency condition of [**60**, Def. C.22]. With respect to the $\mathcal{C}(m)$-projective model structures on $(G \times O(m))$-spaces, the level equivalences, level fibrations and cofibrations in the sense of [**60**, Prop. C.23] are precisely the level equivalences, level fibrations, and cofibrations of orthogonal G-spectra. So [**60**, Prop. C.23] shows that the level equivalences, level fibrations and cofibrations form a model structure on the category Sp_G, the *level model structure*.

The right lifting property against the set I^G_{lv} detects the level acyclic fibrations, simply by the adjunction

$$\mathrm{Sp}_G((G \ltimes_H F_V) \wedge K, X) \ \cong \ \mathbf{T}_*(K, X(V)^H) \ ,$$

where K is a non-equivariant based space. So the set I^G_{lv} serves as a set of generating cofibrations.

Before proceeding with the stable model structure, we prove part (ii). If X is any orthogonal G-spectrum, then evaluation at the point $0 \in [0,1]$ is a level equivalence and level fibration $X^{[0,1]} \longrightarrow X$, by direct inspection. So every cofibration has the left lifting property with respect to this evaluation morphism, which means that every cofibration is an h-cofibration.

Now we continue with the stable model structure. The π_*-isomorphisms satisfy the 2-out-of-3 property (MC2) and the classes of π_*-isomorphisms, stable fibrations and cofibrations are closed under retracts (MC3). The level model structure shows that every morphism of orthogonal G-spectra can be factored as a cofibration followed by a level equivalence that is also a level fibration. Level equivalences are in particular π_*-isomorphisms, and for them the square (1.2.16) is homotopy cartesian. So level acyclic fibrations are also stable fibrations. Hence the level model structure provides one of the factorizations as required by MC5.

For the other half of the factorization axiom MC5 we exploit that the set J^G_{st} detects the stable fibrations, compare Proposition 1.2.20. We apply the small object argument (see for example [**16**, 7.12] or [**25**, Thm. 2.1.14]) to the set J^G_{st}.

All morphisms in J^G_{st} are cofibrations and π_*-isomorphisms by Proposition 1.2.21. The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal G-spectra as a composite

$$X \xrightarrow{\ i\ } W \xrightarrow{\ q\ } Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in J^G_{st}, and q has the right lifting property with respect to J^G_{st}; in particular, the morphism q is a stable fibration. All morphisms in J^G_{st} are π_*-isomorphisms and cofibrations, hence also h-cofibrations. The class of h-cofibrations that are simultaneously π_*-isomorphisms is closed under coproducts, cobase changes and sequential compositions by [**48**, III Thm. 3.5]. So the morphism i is a cofibration and a π_*-isomorphism.

Now we show the lifting properties MC4. By Proposition 1.2.17 a morphism that is both a stable fibration and a π_*-isomorphism is a level equivalence, and hence an acyclic fibration in the level model structure. So every morphism that is simultaneously a stable fibration and a π_*-isomorphism has the right lifting property with respect to cofibrations. Now we let $j : A \longrightarrow B$ be a cofibration that is also a π_*-isomorphism and we show that it has the left lifting property with respect to stable fibrations. We factor $j = q \circ i$, via the small object argument for J^G_{st}, where $i : A \longrightarrow W$ is a J^G_{st}-cell complex and $q : W \longrightarrow B$ is a stable fibration, see Proposition 1.2.20. Then q is a π_*-isomorphism since j and i are, so q is an acyclic fibration in the level model structure, again by Proposition 1.2.17. Since j is a cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to stable fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms for the stable model structure.

(iii) Pushouts of orthogonal spectra and smash products with based spaces are formed levelwise. So

$$\begin{aligned} L_m(B \wedge L) \cup_{L_m(A\wedge L \cup_{A\wedge K} B\wedge K)} & (A \wedge L \cup_{A\wedge K} B \wedge K)(\mathbb{R}^m) \\ &= L_m(B) \wedge L \cup_{L_m(A)\wedge L \cup_{L_m(A)\wedge K} L_m(B)\wedge K} \\ &\qquad\qquad (A(\mathbb{R}^m) \wedge L \cup_{A(\mathbb{R}^m)\wedge K} B(\mathbb{R}^m) \wedge K) \\ &\cong (L_m(B) \cup_{L_m(A)} A(\mathbb{R}^m)) \wedge L \cup_{(L_m(B)\cup_{L_m(A)} A(\mathbb{R}^m))\wedge K} B(\mathbb{R}^m) \wedge K \ . \end{aligned}$$

Moreover, $(B\wedge L)(\mathbb{R}^m) = B(\mathbb{R}^m)\wedge L$. Under these identifications, the m-th latching map for the morphism $i \,\square\, j$ becomes the pushout product of $\nu_m i : L_m B \cup_{L_m A} A(\mathbb{R}^m) \longrightarrow B(\mathbb{R}^m)$ with the map $j : K \longrightarrow L$. By hypothesis, $\nu_m(i)$ is a $\mathcal{C}om$-cofibration of $(G\times O(m))$-spaces. Since j is a G-cofibration, it is also a $(G\times O(m))$-cofibration for the trivial $O(m)$-action. So

$$\nu_m(i \,\square\, j) \ = \ \nu_m(i) \,\square\, j$$

is a $\mathcal{C}om$-cofibration of $(G \times O(m))$-spaces by Proposition 1.1.3 (iii). Also by hypothesis, the group $O(m)$ acts freely away from the image of $\nu_m(i)$; hence it also acts freely off the image of $\nu_m(i) \,\square\, j$. This proves that $i \,\square\, j$ is a cofibration of orthogonal G-spectra.

Now we suppose in addition that i is a π_*-isomorphism or j is a $\mathcal{C}om$-equivalence. Since the morphism $i \,\square\, j$ is a cofibration by the above, it is in particular an h-cofibration. So to show that $i \,\square\, j$ is a π_*-isomorphism, the long exact homotopy group sequence allows us to show that its cofiber $(B/A) \wedge (K/L)$ is π_*-isomorphic to the trivial spectrum. Now we let H be a compact subgroup of G. In the case where i is a π_*-isomorphism, its long exact homotopy group sequence shows that B/A is H-π_*-trivial; since K/L is a cofibrant based H-space, the smash product $(B/A) \wedge (K/L)$ is H-π_*-trivial, by [**48**, III Thm. 3.11] or [**60**, Prop. 3.2.19]. If j is a $\mathcal{C}om$-equivalence, then the underlying based H-space of K/L is equivariantly contractible. So $(B/A) \wedge (K/L)$ is H-equivariantly contractible, and hence π_*-trivial.

(iv) Alongside with the proof of the model structure we have also specified sets of generating cofibrations I^G_{lv} and generating acyclic cofibrations J^G_{st}. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of cofibrations. So the model structure is cofibrantly generated.

Left properness of the stable model structure follows from the fact that every cofibration is in particular an h-cofibration of orthogonal G-spectra. So for every compact subgroup H, the underlying morphism of orthogonal H-spectra is an h-cofibration, and pushout along it preserves π_*-isomorphisms of orthogonal H-spectra, by [**48**, III Thm. 3.5 (iii)] or [**60**, Cor. 3.1.39]. Right properness follows from the fact that stable fibrations are in particular level fibrations, and hence the natural morphism from the strict fiber to the homotopy fiber is a level equivalence. Source, target and homotopy fiber of any morphism of orthogonal G-spectra are related by a long exact sequences of equivariant homotopy groups (see for example [**60**, Prop. 3.1.36]), so the five lemma concludes the argument.

The loop functor $\Omega : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_G$ and the suspension functor $-\wedge S^1 : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_G$ preserve π_*-isomorphisms by the loop isomorphism (1.2.6) and the suspension isomorphism [**60**, Prop. 3.1.30]. Moreover, the loop functor preserves stable fibrations by direct inspection. So the adjoint functor pair

$$- \wedge S^1 \;:\; \mathrm{Sp}_G \rightleftarrows \mathrm{Sp}_G \;:\; \Omega$$

is a Quillen adjunction with respect to the stable model structure, and these functors model the model categorical suspension and loop functors. Furthermore, the unit $\eta : X \longrightarrow \Omega(X \wedge S^1)$ and counit $(\Omega X) \wedge S^1 \longrightarrow X$ of the adjunction are π_*-isomorphisms [**60**, Prop. 3.1.25]. Hence the adjunction is a Quillen equivalence, which proves stability of the stable model structure.

Every cofibration of non-equivariant spaces is in particular a G-cofibration when given the trivial G-action. So the stable model structure is topological as a special case of part (iii). Part (v) is clear from the definitions. $\square$

Remark 1.2.23 (Relation to previous stable model structures). For compact Lie groups, the proper equivariant stable homotopy theory reduces to the 'genuine' equivariant stable homotopy theory. In this special case, several stable model structures have already been constructed that complement the π_*-isomorphisms by different classes of cofibrations and fibrations. We explain how our stable model structure relates to the previous ones for compact Lie groups.

For compact Lie groups H, model structures on orthogonal H-spectra with π_*-isomorphisms as weak equivalences were established by Mandell and May [**48**], Stolz [**70**], Brun, Dundas and Stolz [**9**], and Hill, Hopkins and Ravenel [**24**].

The cofibrations in these model structures each admit a characterization in terms of the latching maps, with different conditions on the allowed isotropy away from the image; however, these characterizations are not explicitly stated in the other papers. A morphism of orthogonal H-spectra $i : A \longrightarrow B$ is an '$\mathbb{S}$-cofibration' in the sense of Stolz [**70**, Def. 2.3.4] and Brun-Dundas-Stolz [**9**, Def. 2.9.11] precisely when the latching morphism

$$\nu_m i \ = \ \nu_m^B \cup i(\mathbb{R}^m) \ : \ L_m B \cup_{L_m A} A(\mathbb{R}^m) \ \longrightarrow \ B(\mathbb{R}^m)$$

is a cofibration of $(H \times O(m))$-spaces, with no additional constraint on the isotropy. The morphism i is a 'q-cofibration' in the sense of Mandell and May [**48**, III Def. 2.3] precisely when the latching morphism $\nu_m i$ is a cofibration of $(H \times O(m))$-spaces and additionally the isotropy group of every point that is not in the image of $\nu_m i$ is the graph of a continuous homomorphism $K \longrightarrow O(m)$, for some closed subgroup K of H, that admits an extension to a continuous homomorphism defined on H. In particular, Stolz' $\mathbb{S}$-model structure has more cofibrations than our model structure, and we have more cofibrations than Mandell and May. For finite groups, our cofibrations of orthogonal H-spectra specialize to the 'complete cofibrations' in the sense of Hill, Hopkins and Ravenel, i.e., to the variant of the *positive complete cofibrations* of [**24**, B.63] where the positivity condition is dropped. So for finite groups H, the stable model structure specified in Theorem 1.2.22 is 'essentially' the positive complete model structure of [**24**, B.4.1].

Example 1.2.24 (No compact subgroups). We already emphasized that when G is compact, our theory just returns the well-known G-equivariant stable homotopy theory, based on a complete G-universe. There is another extreme where we also recover a well-known homotopy theory. Indeed, suppose that the only compact subgroup of G is the trivial subgroup. For example, G could be discrete and torsion free, or the additive group of $\mathbb{R}^n$. The category of orthogonal G-spectra is isomorphic to the category of module spectra over the spherical group ring $\mathbb{S}[G]$ – this is a pointset level statement and holds for all Lie groups G. But if the trivial group is the only compact subgroup of G, then a morphism of orthogonal G-spectra is a π_*-isomorphism or stable fibration if and only if the underlying morphism of non-equivariant orthogonal spectra is a π_*-isomorphism or stable fibration, respectively. So not only is the category Sp_G isomorphic to module spectra over $\mathbb{S}[G]$, also the model structure is the one on modules over an orthogonal ring spectrum, lifted along the forgetful functor, compare [**47**, Thm. 12.1 (i)]. So in particular,

$$\mathrm{Ho}(\mathrm{Sp}_G) \ \cong \ \mathrm{Ho}(\mathbb{S}[G]\text{-mod})$$

i.e., the stable G-equivariant homotopy category 'is' the homotopy category of module spectra over $\mathbb{S}[G]$.

The category Sp of orthogonal spectra supports a symmetric monoidal *smash product*, which is an example of a convolution product considered by category theorist Day in [**14**]; like the tensor product of abelian groups, it can be introduced via a universal property or as a specific construction. The indexing category $\mathbf{O}$ for orthogonal spectra was introduced in Construction 1.1.8. A based continuous functor

$$\oplus \ : \ \mathbf{O} \wedge \mathbf{O} \ \longrightarrow \ \mathbf{O}$$

is defined on objects by orthogonal direct sum, and on morphism spaces by

$$\begin{aligned} \mathbf{O}(V,W) \wedge \mathbf{O}(V',W') &\longrightarrow \mathbf{O}(V\oplus V', W\oplus W') \\ (w,\varphi)\wedge(w',\varphi') &\longmapsto ((w,w'),\varphi\oplus\varphi')\ . \end{aligned}$$

A *bimorphism* $b : (X,Y) \longrightarrow Z$ from a pair of orthogonal spectra (X,Y) to an orthogonal spectrum Z is a natural transformation

$$b \ : \ X\bar{\wedge}Y \ \longrightarrow \ Z\circ\oplus$$

of continuous functors $\mathbf{O}\wedge\mathbf{O} \longrightarrow \mathbf{T}_*$; here $X\bar{\wedge}Y$ is the 'external smash product' defined by $(X\bar{\wedge}Y)(V,W) = X(V)\wedge Y(W)$. A bimorphism thus consists of based continuous maps

$$b_{V,W} \ : \ X(V)\wedge Y(W) \ \longrightarrow \ Z(V\oplus W)$$

for all inner product spaces V and W that form morphisms of orthogonal spectra in each variable separately. A smash product of two orthogonal spectra is now a universal example of a bimorphism from (X,Y).

Definition 1.2.25. A *smash product* of two orthogonal spectra X and Y is a pair $(X\wedge Y, i)$ consisting of an orthogonal spectrum $X\wedge Y$ and a universal bimorphism $i : (X,Y) \longrightarrow X\wedge Y$, i.e., a bimorphism such that for every orthogonal spectrum Z the map

$$\mathrm{Sp}(X\wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X,Y),Z)\ , \ f\mapsto fi = \{f(V\oplus W)\circ i_{V,W}\}_{V,W}$$

is bijective.

A smash product of two orthogonal spectra can be constructed as an enriched Kan extension of the external smash product $X\bar{\wedge}Y : \mathbf{O}\wedge\mathbf{O} \longrightarrow \mathbf{T}_*$ along the continuous functor $\oplus : \mathbf{O}\wedge\mathbf{O} \longrightarrow \mathbf{O}$. This boils down to presenting $(X\wedge Y)(\mathbb{R}^n)$ as a quotient space of the wedge, over $0\le k\le n$, of the $O(n)$-spaces

$$O(n)\ltimes_{O(k)\times O(n-k)} X(\mathbb{R}^k)\wedge Y(\mathbb{R}^{n-k})\ .$$

However, we feel that this explicit construction does not give much insight beyond showing the existence of an object with the desired universal property. Anyhow, Day's general theory [**14**] shows that the smash product $X\wedge Y$ supports preferred natural associativity isomorphisms $(X\wedge Y)\wedge Z \cong X\wedge(Y\wedge Z)$, symmetry isomorphisms $X\wedge Y\cong Y\wedge X$ and unit isomorphisms $X\wedge\mathbb{S}\cong X\cong\mathbb{S}\wedge X$, see also [**60**, Con. C.9]. Moreover, there exists an adjoint internal function orthogonal spectrum. All this data makes the smash product into a closed symmetric monoidal structure on the category of orthogonal spectra.

If X and Y are orthogonal G-spectra, then the smash product $X\wedge Y$ inherits the diagonal G-action, and G acts on the internal function spectrum by conjugation. So the category Sp_G forms a closed symmetric monoidal category under smash product. We will now show that the category Sp_G of orthogonal G-spectra, equipped with the stable model structure and the smash product, is a monoidal model category in the sense of [**25**, Def. 4.2.6]. We also show that the stable model structure satisfies the monoid axiom [**62**, Def. 3.3]. This allows us to automatically lift the stable model structure to the categories of module G-spectra and ring G-spectra.

Definition 1.2.26. Let G be a Lie group. An orthogonal G-spectrum X is *quasi-flat* if for every compact subgroup H of G and every $m\ge 0$, the latching map $\nu_m : L_mX \longrightarrow X(\mathbb{R}^m)$ is an $(H\times O(m))$-cofibration.

An orthogonal G-spectrum X is quasi-flat precisely if for every compact subgroup H of G, the underlying orthogonal H-spectrum is H-flat in the sense of [**60**, Def. 3.5.7]. When G is compact, 'quasi-flat' is the same as 'G-flat'; in this case, the G-flat orthogonal spectra are the cofibrant objects in the $\mathbb{S}$-model structure of Stolz [**70**, Thm. 2.3.27] and Brun-Dundas-Stolz [**9**, Def. 2.9.11]. Every quasi-cofibrant orthogonal G-spectrum in the sense of Definition 1.4.14 below is in particular quasi-flat.

THEOREM 1.2.27. *Let G be a Lie group. For every quasi-flat orthogonal G-spectrum X, the functor $-\wedge X$ preserves π_*-isomorphisms of orthogonal G-spectra.*

PROOF. We let H be any compact subgroup of G. Then the underlying orthogonal H-spectrum of X is H-flat in the sense of [**60**, Def. 3.5.7]. Now we let $f\colon A \longrightarrow B$ be a π_*-isomorphism of orthogonal G-spectra. Since the G-action on a smash product is defined diagonally, we have $\mathrm{res}^G_H(A \wedge X) = \mathrm{res}^G_H(A) \wedge \mathrm{res}^G_H(X)$, and similarly for $B \wedge X$. Since X is H-flat, the morphism $\mathrm{res}^G_H(f \wedge X)$ is a π_*-isomorphism of orthogonal H-spectra by [**60**, Thm. 3.5.10]. Since H was an arbitrary compact subgroup of G, this proves the claim. $\square$

PROPOSITION 1.2.28. *Let G be a Lie group.*

(i) *Let $i : A \longrightarrow B$ and $j : K \longrightarrow L$ be cofibrations of orthogonal G-spectra. Then the pushout-product morphism*

$$i \,\square\, j \;=\; (i \wedge L) \cup (B \wedge j) \;:\; A \wedge L \cup_{A \wedge K} B \wedge K \;\longrightarrow\; B \wedge L$$

is a cofibration of orthogonal G-spectra. If, in addition, i or j is a π_*-isomorphism, then so is $i \,\square\, j$.

(ii) *The category Sp_G equipped with the stable model structure is a monoidal model category under the smash product of orthogonal G-spectra.*

PROOF. (i) We start with the claim that only involves cofibrations. It suffices to check the statement for the generating cofibrations. The pushout product, in the category of spaces, of two sphere inclusions is homeomorphic to another sphere inclusion. So it suffices to show that for all compact subgroups H and K of G, all H-representations V and all K-representations W, the G-spectrum

$$(G \ltimes_H F_V) \wedge (G \ltimes_K F_W) \;\cong\; \Delta^*((G \times G) \ltimes_{H \times K} F_{V \oplus W}) \tag{1.2.29}$$

is cofibrant, where $\Delta : G \longrightarrow G \times G$ is the diagonal embedding. Since $(G \times G) \ltimes_{H \times K} F_{V \oplus W}$ is a cofibrant $(G \times G)$-spectrum, Theorem 1.4.1 (ii) below shows that the orthogonal G-spectrum (1.2.29) is cofibrant.

Now we suppose that in addition the morphism i is a π_*-isomorphism, the other case being analogous. Since i is a cofibration, the long exact homotopy group sequence (see [**60**, Cor. 3.1.38]) shows that its cofiber B/A is π_*-trivial. Since j is a cofibration, its cofiber L/K is cofibrant, hence G-flat, so the smash product $(B/A) \wedge (K/L)$ is π_*-trivial by Theorem 1.2.27. Since the morphism $i \,\square\, j$ is a G-cofibration with cofiber isomorphic to $(B/A) \wedge (K/L)$, its long exact homotopy group sequence shows that $i \,\square\, j$ is a π_*-isomorphism.

(ii) The pushout product property is established in part (i). The suspension spectrum $\Sigma^\infty_+ \underline{E}G$ is a cofibrant replacement of the G-sphere spectrum $\mathbb{S}_G$, the monoidal unit object. For every compact subgroup H of G the underlying H-space of $\underline{E}G$ is H-equivariantly contractible. So for every orthogonal G-spectrum X, the projection

$$X \wedge \underline{E}G_+ \;\cong\; X \wedge \Sigma^\infty_+ \underline{E}G \;\longrightarrow\; X$$

is a homotopy equivalence of underlying orthogonal H-spectra, and thus induces an isomorphism on π_*^H. Since H was any compact subgroup, the projection is a π_*-isomorphism. This establishes the unit axiom of [**25**, Def. 4.2.6]. $\square$

Proposition 1.2.30 (Monoid axiom). *Let G be a Lie group and $i\colon A \longrightarrow B$ a cofibration of orthogonal G-spectra which is also a π_*-isomorphism.*

(i) *For every orthogonal G-spectrum Y, the morphism $i \wedge Y\colon A \wedge Y \longrightarrow B \wedge Y$ is an h-cofibration and a π_*-isomorphism.*
(ii) *Let $\mathcal{D}$ denote the class of maps of the form $i \wedge Y$, where i is a stably acyclic cofibration and Y any orthogonal G-spectrum. Then any map in the class $\mathcal{D}$-cell (maps obtained as transfinite compositions of cobase changes of small coproducts of morphisms in $\mathcal{D}$) is a π_*-isomorphism.*

Proof. The class of h-cofibrations which are also π_*-isomorphisms is closed under transfinite compositions, coproducts and cobase changes by [**48**, III Thm. 3.5]. Hence part (ii) is a consequence of part (i).

The proof of part (i) is very similar to the proof of the corresponding statement in the non-equivariant case, compare [**47**, Prop. 12.5]. For the sake of completeness we provide the details here. Since $i\colon A \longrightarrow B$ is a cofibration, the cofiber B/A is cofibrant. Let $\alpha\colon Y^c \longrightarrow Y$ be a cofibrant approximation of Y. Then $(B/A)\wedge\alpha$ is a π_*-isomorphism by Theorem 1.2.27. Furthermore the cofiber B/A is π_*-isomorphic to the trivial G-spectrum, by the long exact sequence of homotopy groups, see [**48**, III Thm. 3.5 (vi)]. Using again Theorem 1.2.27, we see that $(B/A) \wedge Y^c$ and hence $(B/A) \wedge Y$ are π_*-isomorphic to the trivial G-spectrum. Now the morphism $i \wedge Y\colon A \wedge Y \longrightarrow B \wedge Y$ is an h-cofibration and its cofiber is isomorphic to $B/A \wedge Y$. Since $(B/A)\wedge Y$ is π_*-isomorphic to the trivial G-spectrum, the long exact homotopy group sequence [**60**, Cor. 3.1.28] shows that the map $i \wedge Y$ is a π_*-isomorphism. $\square$

The previous proposition almost immediately implies that the stable model structure on Sp_G lifts to the category of orthogonal ring G-spectra and the category of module spectra over an orthogonal ring G-spectrum R, by the results of the fifth author and Shipley [**62**, Thm. 4.1]. We will not go into further details here.

For compact Lie groups, the spheres of linear representations become invertible objects in the genuine equivariant stable homotopy category. In our more general context, the role of linear representations is taken up by equivariant vector bundles over $\underline{E}G$, the universal G-space for proper actions. We recall that a *G-vector bundle* is a map of G-spaces $\xi\colon E \longrightarrow X$ equipped with the structure of a real vector bundle, and such that the map $g \cdot - : \xi_x \longrightarrow \xi_{gx}$ is $\mathbb{R}$-linear for all $(g, x) \in G \times X$. By one-point compactifying the fibers we obtain a G-equivariant fiber bundle $S^\xi \longrightarrow X$ with fibers the spheres of dimension equal to the dimension of ξ. This bundle has two preferred G-equivariant sections

$$s_0, s_\infty \;:\; X \;\longrightarrow\; S^\xi$$

which send a point in X to the zero element and the point at infinity, respectively, in the corresponding fiber.

Proposition 1.2.31. *Let G be a Lie group and (X,A) a relative proper G-CW-pair. Then for every G-vector bundle ξ over X, the G-map*

$$s_0 \cup s_\infty \cup \mathrm{incl} \ : \ X \times \{0,\infty\} \cup_{A\times\{0,\infty\}} S^{\xi|_A} \ \longrightarrow \ S^\xi$$

is a $\mathcal{C}om$-cofibration of G-spaces.

Proof. The G-vector bundle ξ admits a G-invariant euclidean metric by the real analog of [**42**, Lemma 1.4]. We choose a relative G-CW-structure on (X,A) with skeleta X^n and such that $X^{-1} = A$. We let $\xi^n : E^n \longrightarrow X^n$ denote the restriction of the given euclidean vector bundle to X^n. In a first step we show that the inclusion $S(E^{n-1}) \longrightarrow S(E^n)$ of the total spaces of the sphere bundles is a $\mathcal{C}om$-cofibration of G-spaces. Lemma 1.1 (iii) of [**42**] provides a pushout square of G-spaces:

$$\begin{tikzcd}
\coprod_{j\in J} (G \times_{H_j} S(V_j)) \times \partial D^n \arrow[r] \arrow[d] & S(E^{n-1}) \arrow[d] \\
\coprod_{j\in J} (G \times_{H_j} S(V_j)) \times D^n) \arrow[r] & S(E^n)
\end{tikzcd}$$

Here J is an indexing set of the equivariant n-cells of X, H_j is the stabilizer group of the cell indexed by j, V_j is an orthogonal representation of H_j, and $S(V_j)$ is its unit sphere. In particular, each of the groups H_j is compact by our hypotheses. The pushout arises from choices of characteristic maps for the equivariant n-cells of X and choices of trivializations of ξ over each equivariant cell. Since H_j is compact, the unit sphere $S(V_j)$ admits an H_j-CW-structure by Illman's theorem [**27**, Thm. 7.1], so it is H-cofibrant. Hence $G \times_{H_j} S(V_j)$ is a $\mathcal{C}om$-cofibrant G-space, and so the left and right vertical maps in the pushout square are $\mathcal{C}om$-cofibrations of G-spaces. Since $\mathcal{C}om$-cofibrations are closed under sequential colimits, this proves that the inclusion $S(\xi|_A) = S(E^{-1}) \longrightarrow \mathrm{colim}_n\, S(E^n) = S(E)$ is a $\mathcal{C}om$-cofibration.

The fiberwise one-point compactification participates in a pushout square of G-spaces:

$$\begin{tikzcd}
S(E) \times \{0,\infty\} \cup_{S(\xi|_A)\times\{0,\infty\}} S(\xi|_A) \times [0,\infty] \arrow[r] \arrow[d] & X \times \{0,\infty\} \cup_{A\times\{0,\infty\}} S^{\xi|_A} \arrow[d, "s_0\cup s_\infty\cup\mathrm{incl}"] \\
S(E) \times [0,\infty] \arrow[r] & S^\xi
\end{tikzcd}$$

Here the lower horizontal map crushes $S(E) \times \{0\}$ and $S(E) \times \{\infty\}$ to the sections at 0 and ∞, respectively. Since the inclusion $S(\xi|_A) \longrightarrow S(E)$ is a $\mathcal{C}om$-cofibration, so is the left vertical map, and hence also the right vertical map. □

We let $\xi : E \longrightarrow X$ be a G-vector bundle over a G-space X. By dividing out the image of the section at infinity $s_\infty : X \longrightarrow S^\xi$, we get a based G-space

$$\mathrm{Th}(\xi) \ = \ S^\xi / s_\infty(X) \ , \tag{1.2.32}$$

the Thom space of ξ.

Proposition 1.2.33. *Let G be a Lie group and ξ a G-vector bundle over $\underline{E}G$.*

(i) *For every compact subgroup H of G and every H-fixed point $x \in (\underline{E}G)^H$, the composite map*

$$S^{\xi_x} \ \xrightarrow{\mathrm{incl}} \ S^\xi \ \xrightarrow{\mathrm{proj}} \ \mathrm{Th}(\xi)$$

is a based H-equivariant homotopy equivalence.

(ii) *The endofunctors* $-\wedge \mathrm{Th}(\xi)$ *and* $\mathrm{map}_*(\mathrm{Th}(\xi), -)$ *of the category of orthogonal G-spectra preserve and detect π_*-isomorphisms.*

(iii) *For every orthogonal G-spectrum X the adjunction unit*

$$\eta_X \ : \ X \ \longrightarrow \ \mathrm{map}_*(\mathrm{Th}(\xi), X \wedge \mathrm{Th}(\xi))$$

is a π_*-isomorphism.

(iv) *The adjoint functor pair*

$$- \wedge \mathrm{Th}(\xi) \ : \ \mathrm{Sp}_G \rightleftarrows \mathrm{Sp}_G \ : \ \mathrm{map}_*(\mathrm{Th}(\xi), -)$$

is a Quillen equivalence. Consequently, the suspension spectrum of the Thom space $\mathrm{Th}(\xi)$ *is an invertible object in* $\mathrm{Ho}(\mathrm{Sp}_G)$.

Proof. (i) The restriction of the G-space $\underline{E}G$ is H-equivariantly contractible. Therefore, by the homotopy invariance theorem [**42**, Thm. 1.2], the underlying H-vector bundle of ξ is H-equivariantly isomorphic to the trivial bundle $\xi_x \times \underline{E}G \longrightarrow \underline{E}G$. This implies that the underlying H-space of the Thom space $\mathrm{Th}(\xi)$ is H-equivariantly isomorphic to $S^{\xi_x} \wedge \underline{E}G_+$. Since $\underline{E}G$ is H-equivariantly contractible, the claim follows.

(ii) We let H be any compact subgroup of G. We choose an H-fixed point $x \in (\underline{E}G)^H$. Part (i) shows that the fiber inclusion induces an H-equivariant based homotopy equivalence

$$\varphi\colon S^{\xi_x} \ \longrightarrow \ \mathrm{res}^G_H(\mathrm{Th}(\xi)) \ .$$

This map induces homotopy equivalences of orthogonal H-spectra

$$\varphi_* \ : \ \mathrm{res}^G_H(X) \wedge S^{\xi_x} \ \longrightarrow \ \mathrm{res}^G_H(X \wedge \mathrm{Th}(\xi))$$

and

$$\varphi_* \ : \ \mathrm{res}^G_H(\mathrm{map}_*(\mathrm{Th}(\xi), X)) \longrightarrow \Omega^{\xi_x}(\mathrm{res}^G_H(X)) \ .$$

Since H is compact, the fiber ξ_x can be endowed with an H-invariant inner product, making it an orthogonal H-representation. Now the representation sphere S^{ξ_x} admits the structure of a finite based H-CW-complex. So the functor $- \wedge S^{\xi_x}$ preserves π_*-isomorphisms by [**48**, III Thm. 3.11] or [**60**, Prop. 3.2.19 (ii)], and the functor Ω^{ξ_x} preserves π_*-isomorphisms by [**48**, III Prop. 3.9] or [**60**, Prop. 3.1.40 (ii)].

Furthermore, if $f : X \longrightarrow Y$ is a morphism of orthogonal H-spectra such that $f \wedge S^{\xi_x} : X \wedge S^{\xi_x} \longrightarrow Y \wedge S^{\xi_x}$ is a π_*-isomorphism, then $\Omega^{\xi_x}(f \wedge S^{\xi_x})$ is a π_*-isomorphism by the previous paragraph. Since the adjunction unit $\eta^{\xi_x}_X : X \longrightarrow \Omega^{\xi_x}(X \wedge S^{\xi_x})$ is a π_*-isomorphism (by [**48**, III Lemma 3.8] or [**60**, Prop. 3.1.25 (ii)]), the original morphism f is a π_*-isomorphism. So smashing with S^{ξ_x} detects π_*-isomorphisms. By the same argument, using that the adjunction counit $\epsilon^{\xi_x}_X : (\Omega^{\xi_x} X) \wedge S^{\xi_x} \longrightarrow X$ is a π_*-isomorphism (see [**60**, Prop. 3.1.25 (ii)]), it follows that Ω^{ξ_x} detects π_*-isomorphisms of H-orthogonal spectra.

Since H was any compact subgroup of G, this shows that smashing with $\mathrm{Th}(\xi)$ and taking $\mathrm{map}_*(\mathrm{Th}(\xi), -)$ detect and preserve π_*-isomorphisms of orthogonal G-spectra.

(iii) Again we let H be any compact subgroup of G, and we choose an H-fixed point $x \in (\underline{E}G)^H$. The fiber inclusion $\varphi\colon S^{\xi_x} \longrightarrow \mathrm{res}^G_H(\mathrm{Th}(\xi))$ is a based H-equivariant homotopy equivalence by part (i). The following square of orthogonal

H-spectra commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{map}_*(\mathrm{Th}(\xi), X \wedge \mathrm{Th}(\xi)) \arrow[d, "\mathrm{map}_*(\varphi,\mathrm{Id})"] \\ \mathrm{map}_*(S^{\xi_x}, X \wedge S^{\xi_x}) \arrow[r, "\mathrm{map}_*(\mathrm{Id}, X\wedge\varphi)"'] & \mathrm{map}_*(S^{\xi_x}, X \wedge \mathrm{Th}(\xi)) \end{tikzcd}$$

The two morphisms starting at X are the adjunction units. The left vertical morphism is a π_*-isomorphism by [**48**, III Lemma 3.8] or [**60**, Prop. 3.1.25 (ii)]; the right vertical and lower horizontal morphisms are homotopy equivalences, hence π_*-isomorphisms. So the upper horizontal morphism is also a π_*-isomorphism.

(iv) The section at infinity is a $\mathcal{C}om$-cofibration of G-spaces by Proposition 1.2.31 for $A = \emptyset$; so the Thom space $\mathrm{Th}(\xi)$ is $\mathcal{C}om$-cofibrant as a based G-space. The adjoint functors thus form a Quillen pair by Theorem 1.2.22 (iii). Since the right adjoint $\mathrm{map}_*(\mathrm{Th}(\xi), -)$ preserves and detects all π_*-isomorphisms and the adjunction unit is a π_*-isomorphism, the Quillen pair is a Quillen equivalence. □

1.3. The G-equivariant stable homotopy category

In Section 1.2 we showed that the *G-equivariant stable homotopy category* $\mathrm{Ho}(\mathrm{Sp}_G)$ is the homotopy category of a stable model structure, so it is naturally a triangulated category, for example by [**25**, Sec. 7.1] or [**58**, Thm. A.12]. This section discusses those aspects of our theory that are most conveniently phrased in terms of the triangulated structure.

We record in Proposition 1.3.10 that for every compact subgroup H of a Lie group G, the suspension spectrum of G/H represents the functor π_0^H on $\mathrm{Ho}(\mathrm{Sp}_G)$; a direct consequence is the fact that for varying compact subgroups H, the suspension spectra of the orbits G/H form a set of small weak generators for the stable G-homotopy category, see Corollary 1.3.11. In other words, the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$ is *compactly generated*. As we explain thereafter, this has various formal, but rather useful, consequences, such as Brown representability (see Corollary 1.3.12), a t-structure (see Corollary 1.3.16), and Postnikov sections (see Remark 1.3.17). In Section 2.2 we will return to this preferred t-structure in the special case of *discrete* groups G; we will then show that its heart is equivalent to the abelian category of G-Mackey functors. In particular, for discrete groups G, every G-Mackey functor has an associated Eilenberg-Mac Lane spectrum.

In the following we write

$$\gamma_G \ : \ \mathrm{Sp}_G \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G) \tag{1.3.1}$$

for the localization functor at the class of π_*-isomorphisms; so γ_G initial among functors from Sp_G that send π_*-isomorphisms to isomorphisms.

Construction 1.3.2 (Triangulated structure on $\mathrm{Ho}(\mathrm{Sp}_G)$). The suspension isomorphism (1.2.7) between $\pi_k^G(X)$ and $\pi_{k+1}^G(X \wedge S^1)$ shows that the pointset level suspension of orthogonal G-spectra preserves π_*-isomorphism, so it passes to a functor

$$[1] \ = \ \mathrm{Ho}(- \wedge S^1) \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G)$$

by the universal property of the localization. In other words, the shift functor is characterized by the relation

$$[1] \circ \gamma_G \ = \ \gamma_G \circ (- \wedge S^1) \ : \ \mathrm{Sp}_G \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G) \ .$$

By [**48**, III Lemma 3.8] or [**60**, Prop. 3.1.25 (ii)], the adjunction unit $\eta : X \longrightarrow \Omega(X \wedge S^1)$ is a π_*-isomorphism of orthogonal G-spectra; so at the level of the stable homotopy category, suspension becomes inverse to looping; in particular, the shift functor [1] is an auto-equivalence of the category $\mathrm{Ho}(\mathrm{Sp}_G)$.

The distinguished triangles in $\mathrm{Ho}(\mathrm{Sp}_G)$ are defined from mapping cone sequences as follows. We let $f : X \longrightarrow Y$ be a morphism of orthogonal G-spectra. The *reduced mapping cone* Cf is defined by

$$Cf \;=\; (X \wedge [0,1]) \cup_f Y \ .$$

Here the unit interval $[0,1]$ is based by $0 \in [0,1]$, so that $X \wedge [0,1]$ is the reduced cone of X. The mapping cone comes with an embedding $i : Y \longrightarrow Cf$ and a projection $p : Cf \longrightarrow X \wedge S^1$. As S^1 is the one-point compactification of $\mathbb{R}$, the projection sends Y to the basepoint and is given on $X \wedge [0,1]$ by $p(x,z) = x \wedge t(z)$ where

$$t \;:\; [0,1] \;\longrightarrow\; S^1 \qquad \text{is} \qquad t(z) \;=\; \frac{2z-1}{z(1-z)} \ . \tag{1.3.3}$$

What is relevant about the map t is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and $S^1 = \mathbb{R} \cup \{\infty\}$. Then the image in $\mathrm{Ho}(\mathrm{Sp}_G)$ of the sequence

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ i\ } Cf \xrightarrow{\ p\ } X \wedge S^1 \tag{1.3.4}$$

is a distinguished triangle. More generally, a triangle in $\mathrm{Ho}(\mathrm{Sp}_G)$ is distinguished if and only if it is isomorphic in $\mathrm{Ho}(\mathrm{Sp}_G)$ to such a mapping cone triangle for some morphism of orthogonal G-spectra f.

REMARK 1.3.5 (Integer shifts in $\mathrm{Ho}(\mathrm{Sp}_G)$). The shift functor on $\mathrm{Ho}(\mathrm{Sp}_G)$ is an auto-equivalence, but *not* an automorphism of $\mathrm{Ho}(\mathrm{Sp}_G)$, so we fix a convention of what we mean by integer shifts. For $k \geq 0$ we set

$$[k] \;=\; \mathrm{Ho}(- \wedge S^k) \;:\; \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G) \ .$$

For $k < 0$ we observe that the functor $\Omega^{-k} : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_G$ also preserves π_*-isomorphisms, because looping shifts equivariant homotopy groups by the loop isomorphism (1.2.6). So for negative values of k we define

$$[k] \;=\; \mathrm{Ho}(\Omega^{-k}) \;:\; \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G) \ .$$

Since positive and negative shift are not inverse to each other on the nose, we specify natural isomorphisms

$$s_k \;:\; X[k][1] \;\cong\; X[k+1] \tag{1.3.6}$$

as endofunctors on $\mathrm{Ho}(\mathrm{Sp}_G)$, for all integers k. For $k \geq 0$, we let s_k be induced by the canonical homeomorphism

$$S^k \wedge S^1 \;\cong\; S^{k+1} \ , \quad x \wedge y \;\longmapsto\; (x,y) \ .$$

For $k < 0$, we let s_k be induced by the natural morphism of orthogonal G-spectra

$$\mathrm{eval} \;:\; (\Omega^{-k}X) \wedge S^1 \;\longrightarrow\; \Omega^{-(k+1)}X \ , \quad \mathrm{eval}(f \wedge t)(z) \;=\; f(t \wedge z)$$

that evaluates the last coordinate, where $f \in \Omega^{-k}X$, $t \in S^1$ and $z \in S^{-(1+k)}$. The evaluation morphism is a π_*-isomorphism, see for example [**60**, Prop. 3.1.25 (ii)].

It is now a formal procedure to extend the isomorphisms s_k to a preferred system of natural isomorphisms $X[k][l] \cong X[k+l]$. We omit the proof of the following proposition.

PROPOSITION 1.3.7. *Let G be a Lie group. There is a unique collection of natural isomorphisms*

$$t_{k,l} \ : \ X[k][l] \xrightarrow{\cong} X[k+l] \ ,$$

for all integers k and l, of endofunctors of $\mathrm{Ho}(\mathrm{Sp}_G)$*, subject to the following conditions:*

(a) $t_{1,-1} \star [1] = [1] \star t_{-1,1} : [1] \circ [-1] \circ [1] \longrightarrow [1]$;
(b) $t_{k,0} = t_{0,k} = \mathrm{Id}_{[k]}$ *for every integer k;*
(c) $t_{k,1} = s_k$ *for every integer k; and*
(d) *for every triple of integers k, l, m, the following square commutes:*

$$\begin{tikzcd} X[k][l][m] \arrow[r, "t^{X[k]}_{l,m}"] \arrow[d, "t^X_{k,l}[m]"'] & X[k][l+m] \arrow[d, "t^X_{k,l+m}"] \\ X[k+l][m] \arrow[r, "t^X_{k+l,m}"'] & X[k+l+m] \end{tikzcd}$$

Now we turn to the topic of 'compact generation' for triangulated categories. The small objects in the sense of the following definition are most commonly called 'compact' objects; since we already use the adjective 'compact' in a different sense, we prefer to use 'small'.

DEFINITION 1.3.8. Let $\mathcal{T}$ be a triangulated category which has all set indexed sums. An object C of $\mathcal{T}$ is *small* (sometimes called *finite* or *compact*) if for every family $\{X_i\}_{i\in I}$ of objects the canonical map

$$\bigoplus_{i\in I} \mathcal{T}(C,\, X_i) \ \longrightarrow \ \mathcal{T}(C,\, \bigoplus_{i\in I} X_i)$$

is an isomorphism. A set $\mathcal{S}$ of objects of $\mathcal{T}$ is a set of *weak generators* if the following condition holds: if X is an object such that the groups $\mathcal{T}(C[k], X)$ are trivial for all $k \in \mathbb{Z}$ and all $C \in \mathcal{S}$, then X is a zero object. The triangulated category $\mathcal{T}$ is *compactly generated* if it has all set indexed sums and a set of small weak generators.

For every compact subgroup H of G we define a tautological homotopy class

$$u_H \ \in \ \pi_0^H(\Sigma^\infty_+ G/H) \tag{1.3.9}$$

as the class represented by the distinguished coset eH in G/H; indeed, eH is an H-fixed point of G/H, so it gives rise to a based H-map

$$S^0 \ \longrightarrow \ G/H_+ \ = \ (\Sigma^\infty_+ G/H)_0$$

by sending the non-basepoint to eH. For orthogonal G-spectra X and Y, we will denote the Hom abelian group $\mathrm{Ho}(\mathrm{Sp}_G)(X,Y)$ by $[X,Y]^G$.

PROPOSITION 1.3.10. *Let G be a Lie group and H a compact subgroup of G. Then for every orthogonal G-spectrum X, the evaluation map*

$$[\Sigma^\infty_+ G/H, X]^G \ \cong \ \pi_0^H(X) \ , \quad [f] \ \longmapsto \ f_*(u_H)$$

is an isomorphism. The suspension spectrum $\Sigma^\infty_+ G/H$ is a small object in $\mathrm{Ho}(\mathrm{Sp}_G)$.

PROOF. Source and target of the evaluation map take π_*-isomorphisms of orthogonal G-spectra to isomorphisms of groups, so it suffices to show the claim in the special case when X is fibrant in the stable model structure, i.e., a G-Ω-spectrum. Since H is compact, the orthogonal G-spectrum $\Sigma^\infty_+ G/H$ is cofibrant; so for stably fibrant X the localization functor $\gamma_G : \mathrm{Sp}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ induces a bijection

$$\mathrm{Sp}_G(\Sigma^\infty_+ G/H, X)/\text{homotopy} \xrightarrow{\ \cong\ } [\Sigma^\infty_+ G/H, X]^G$$

from the set of homotopy classes of morphisms in Sp_G to the set of morphisms in the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$. The suspension spectrum $\Sigma^\infty_+ G/H$ represents the H-fixed points in level 0, so the left hand side bijects with the path components of the space $X(0)^H$. Since X is a G-Ω-spectrum, all the maps in the colimit system for $\pi_0^H(X)$ are bijections, and hence the canonical map

$$\pi_0(X(0)^H) \;=\; [S^0, X(0)]^H \;\longrightarrow\; \mathrm{colim}_{V\in s(\mathcal{U}_H)}\, [S^V, X(V)]^H \;=\; \pi_0^H(X)$$

is bijective. We omit the straightforward verification that the combined bijection between $[\Sigma^\infty_+ G/H, X]^G$ and $\pi_0^H(X)$ coincides with evaluation at the class u_H.

For every compact Lie group H, the functor π_*^H takes wedges of orthogonal G-spectra to directs sums [**48**, III Thm. 3.5 (ii)]. So formation of wedges preserves π_*-isomorphisms, and the wedge of any family $\{X_i\}_{i\in I}$ of orthogonal G-spectra is a coproduct in $\mathrm{Ho}(\mathrm{Sp}_G)$. We have a commutative square

$$\begin{tikzcd}
\bigoplus_{i\in I}[\Sigma^\infty_+ G/H,\, X_i]^G \arrow[r] \arrow[d] & {[\Sigma^\infty_+ G/H,\, \bigvee_{i\in I} X_i]^G} \arrow[d] \\
\bigoplus_{i\in I}\pi_0^H(X_i) \arrow[r] & \pi_0^H\left(\bigvee_{i\in I} X_i\right)
\end{tikzcd}$$

in which the vertical maps are evaluation at u_H. The lower horizontal map is an isomorphism, hence so is the upper horizontal map. This shows that $\Sigma^\infty_+ G/H$ is small as an object of the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$. $\square$

Essentially by definition, an orthogonal G-spectrum is a zero object in $\mathrm{Ho}(\mathrm{Sp}_G)$ if and only if its H-equivariant homotopy groups vanish for all compact subgroups H of G. So Proposition 1.3.10 directly implies:

COROLLARY 1.3.11. *Let G be a Lie group. The triangulated stable homotopy category* $\mathrm{Ho}(\mathrm{Sp}_G)$ *has infinite sums and the suspension spectra $\Sigma^\infty_+ G/H$ for all compact subgroups H of G form a set of small weak generators. In particular, the triangulated stable homotopy category* $\mathrm{Ho}(\mathrm{Sp}_G)$ *is compactly generated.*

A contravariant functor E from a triangulated category $\mathcal{T}$ to the category of abelian groups is called *cohomological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(\Sigma A) \xrightarrow{E(h)} E(C) \xrightarrow{E(g)} E(B) \xrightarrow{E(f)} E(A)$$

is exact. Dually, a covariant functor F from $\mathcal{T}$ to the category of abelian groups is called *homological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Sigma A)$$

is exact. The fact that the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$ is compactly generated has various useful consequences that we summarize in the next corollary.

COROLLARY 1.3.12. *Let G be a Lie group.*

(i) *Every cohomological functor E on $\mathrm{Ho}(\mathrm{Sp}_G)$ that takes sums to products is representable, i.e., there is an orthogonal G-spectrum Y and a natural isomorphism $E \cong [-,Y]^G$.*
(ii) *Every homological functor F on $\mathrm{Ho}(\mathrm{Sp}_G)$ that takes products to products is representable, i.e., there is an orthogonal G-spectrum X and a natural isomorphism $F \cong [X,-]^G$.*
(iii) *An exact functor $F : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathcal{S}$ to another triangulated category has a right adjoint if and only if it preserves sums.*
(iv) *An exact functor $F : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathcal{S}$ to another triangulated category has a left adjoint if and only if it preserves products.*

PROOF. Part (i) is a direct consequence of being compactly generated, see for example [**53**, Thm. 3.1] or [**32**, Thm. A]. A proof of part (ii) of this form of Brown representability can be found in [**54**, Thm. 8.6.1] or [**32**, Thm. B]. Part (iii) is a formal consequences of part (i): if F preserves sums, then for every object X of $\mathcal{S}$ the functor

$$\mathcal{S}(F(-),X) \ : \ \mathrm{Ho}(\mathrm{Sp}_G)^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

is cohomological and takes sums to products. Hence the functor is representable by an orthogonal G-spectrum RX and an isomorphism

$$[A, RX]^G \ \cong \ \mathcal{S}(FA, X) \ ,$$

natural in A. Once this representing data is chosen, the assignment $X \mapsto RX$ extends canonically to a functor $R : \mathcal{S} \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ that is right adjoint to F. In much the same way, part (iv) is a formal consequence of part (ii). □

The preferred set of generators $\{\Sigma^\infty_+ G/H\}$ of the stable G-homotopy category has another special property, it is 'positive' in the following sense: for all compact subgroups H and K of G, and all $n < 0$,

$$[\Sigma^\infty_+ G/K[n],\ \Sigma^\infty_+ G/H]^G \ \cong \ \pi_n^K(\Sigma^\infty_+ G/H) \ = \ 0 \ , \tag{1.3.13}$$

because the underlying orthogonal K-spectrum of $\Sigma^\infty_+ G/H$ is the suspension spectrum of a K-space. A set of positive compact generators in this sense automatically gives rise to a non-degenerate t-structure, as we shall now recall. When G is discrete, the heart of the t-structure is equivalent to the abelian category of G-Mackey functors, see Theorem 2.2.9 below.

A 't-structure' as introduced by Beilinson, Bernstein and Deligne in [**5**, Def. 1.3.1] axiomatizes the situation in the derived category of an abelian category given by cochain complexes whose cohomology vanishes in positive respectively negative dimensions.

DEFINITION 1.3.14. A *t-structure* on a triangulated category $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ satisfying the following three conditions, where $\mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n]$ and $\mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n]$:

(1) For all $X \in \mathcal{T}_{\geq 0}$ and all $Y \in \mathcal{T}_{\leq -1}$ we have $\mathcal{T}(X,Y) = 0$.
(2) $\mathcal{T}_{\geq 0} \subset \mathcal{T}_{\geq -1}$ and $\mathcal{T}_{\leq 0} \supset \mathcal{T}_{\leq -1}$.

(3) For every object X of $\mathcal{T}$ there is a distinguished triangle
$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$
such that $A \in \mathcal{T}_{\geq 0}$ and $B \in \mathcal{T}_{\leq -1}$.

A t-structure is *non-degenerate* if $\bigcap_{n\in\mathbb{Z}} \mathcal{T}_{\leq n} = \{0\}$ and $\bigcap_{n\in\mathbb{Z}} \mathcal{T}_{\geq n} = \{0\}$. The *heart* of the t-structure is the full subcategory
$$\mathcal{H} \ = \ \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0} \ ;$$
it is an abelian category by [**5**, Thm. 1.3.6].

The original definition of t-structures is formulated slightly differently in 'cohomological' notation, motivated by derived categories of cochain complexes as the main examples. We are mainly interested in spectra, where a homological (as opposed to *co*homological) grading is more common, and the definition above is adapted to the homological setting.

Definition 1.3.15. Let G be a Lie group. An orthogonal G-spectrum X is *connective* if the homotopy group $\pi_n^H(X)$ is trivial for every compact subgroup H of G and every $n < 0$. An orthogonal G-spectrum X is *coconnective* if the homotopy group $\pi_n^H(X)$ is trivial for every compact subgroup H of G and every $n > 0$.

Corollary 1.3.16. *Let G be a Lie group. The classes of connective G-spectra and coconnective G-spectra form a non-degenerate t-structure on* $\mathrm{Ho}(\mathrm{Sp}_G)$ *whose heart consists of those orthogonal G-spectra X such that $\pi_n^H(X) = 0$ for all compact subgroups H of G and all $n \neq 0$.*

Proof. We use the more general arguments of Beligiannis and Reiten [**6**, Ch. III] who systematically investigate torsion pairs and t-structures in triangulated categories that are generated by small objects. By Corollary 1.3.11 the set
$$\mathcal{P} \ = \ \{\Sigma^\infty_+ G/H\}_{H \in \mathcal{C}om}$$
is a set of small weak generators for the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$. We let $\mathcal{Y}$ be the class of G-spectra Y such that
$$[P[n], Y]^G \ = \ 0$$
for all $P \in \mathcal{P}$ and all $n \geq 0$. The representability result of Proposition 1.3.10 shows that these are precisely those G-spectra such that $\pi_n^H(Y) = 0$ for all compact subgroups H of G and all $n \geq 0$. Hence $\mathcal{Y}[1]$ is the class of coconnective G-spectra. We let $\mathcal{X}$ be the 'left orthogonal' to $\mathcal{Y}$, i.e., the class of G-spectra X such that $[X, Y]^G = 0$ for all $Y \in \mathcal{Y}$. Since the objects of $\mathcal{P}$ are small in $\mathrm{Ho}(\mathrm{Sp}_G)$ by Proposition 1.3.10, Theorem III.2.3 of [**6**] shows that the pair $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in the sense of [**6**, Def. I.2.1]. This simply means that the pair $(\mathcal{X}, \mathcal{Y}[1])$ is a t-structure in the sense of Definition 1.3.14, see [**6**, Prop. I.2.13].

It remains to show that $\mathcal{X}$ coincides with the class of connective G-spectra. This needs the positivity property (1.3.13) of the set $\mathcal{P}$ of small generators, which lets us apply [**6**, Prop. III.2.8], showing that $\mathcal{X}$ coincides with the class of those G-spectra X such that $[\Sigma^\infty_+ G/H, X[n]]^G = 0$ for all $H \in \mathcal{C}om$ and $n \geq 1$. Since the latter group is isomorphic to $\pi_{-n}^H(X)$, this shows that $\mathcal{X}$ is precisely the class of connective G-spectra. The t-structure is non-degenerate because spectra with trivial $\mathcal{C}om$-equivariant homotopy groups are zero objects in $\mathrm{Ho}(\mathrm{Sp}_G)$. □

Remark 1.3.17 (Postnikov sections). In every t-structure and for every integer n, the inclusion $\mathcal{T}_{\leq n} \longrightarrow \mathcal{T}$ has a left adjoint $\tau_{\leq n} : \mathcal{T} \longrightarrow \mathcal{T}_{\leq n}$, by [**5**, Prop. 1.3.3]. For the standard t-structure on the $\mathrm{Ho}(\mathrm{Sp}_G)$, given by the connective and coconnective G-spectra, the truncation functor

$$\tau_{\leq n} \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G)_{\leq n} \ ,$$

left adjoint to the inclusion, provides a 'Postnikov section': For every orthogonal G-spectrum X and every compact subgroup H of G, the G-spectrum $\tau_{\leq n} X$ satisfies $\pi_k^H(\tau_{\leq n} X) = 0$ for $k > n$ and the adjunction unit $X \longrightarrow X_{\leq n}$ induces an isomorphism on π_k^H for every $k \leq n$.

1.4. Change of groups

This longish section is devoted to studying how the proper equivariant stable homotopy theory varies with the ambient Lie group. More precisely, we investigate different aspects of functoriality of the stable model structure on Sp_G and of the triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ for continuous group homomorphisms between Lie groups.

Theorem 1.4.1 records how the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ along a continuous homomorphism $\alpha : K \longrightarrow G$ interacts with the stable model structures: on the one hand, α^* preserves π_*-isomorphisms whenever the kernel of α has no non-trivial compact subgroup, and then α^* is a right Quillen functor for the stable model structures. On the other hand, α^* is a left Quillen functor for the stable model structures whenever the image of α is closed and its kernel is compact. Corollary 1.4.3 specializes this to the inclusion of a closed subgroup, in which case the restriction functor and its left adjoint both preserve π_*-isomorphisms.

Theorem 1.4.4 explains why for almost connected Lie groups (i.e., those with finitely many path components), our theory reduces to the classical case of *compact* Lie groups: almost connected Lie groups have maximal compacts subgroups, and restriction to a maximal compact subgroup is a Quillen equivalence between the stable model categories of equivariant spectra.

In the remaining part of this section we switch our focus to functoriality of the triangulated equivariant stable homotopy category. Despite the fact that the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ along a continuous homomorphism $\alpha : K \longrightarrow G$ need not be a Quillen functor for our model structures in general, we show in Theorem 1.4.17 that it always admits a total left derived functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$; moreover, $L\alpha^*$ is an exact functor of triangulated categories and has a right adjoint. Morally speaking, the functor $L\alpha^*$ is determined by these properties and the fact that it 'commutes with suspension spectra', see Theorem 1.4.17 for the precise statement. When the kernel of α has non-trivial compact subgroups, $L\alpha^*$ will typically *not* preserve products, as Remark 1.4.19 illustrates. The derived functors $L\alpha^*$ enjoy a specific kind of lax functoriality: pairs of composable continuous homomorphisms give rise to a preferred natural transformation $\langle \alpha, \beta \rangle : L\beta^* \circ L\alpha^* \Longrightarrow L(\alpha\beta)^*$, see Construction 1.4.24; and these transformations in turn satisfy a coherence condition, see Proposition 1.4.28.

Finally, we show that proper G-equivariant stable homotopy theory is 'homotopy invariant' in the Lie group G, in two specific ways. On the one hand, if $\alpha : K \longrightarrow G$ is a continuous homomorphism and a weak equivalence of underlying topological spaces, then $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ is an equivalence of triangulated categories, see Theorem 1.4.31. On the other hand, a homotopy through continuous group homomorphisms gives rise to a specific natural isomorphism between the left derived functors, see Theorem 1.4.34. Both of these homotopy invariance statements are new phenomena, since they are trivially true for *compact* Lie groups. Indeed, a multiplicative weak equivalence between compact Lie groups is already an isomorphism; and two homotopic continuous homomorphisms between compact Lie groups are already conjugate.

As we discussed in Proposition 1.1.6, a continuous homomorphism $\alpha : K \longrightarrow G$ between Lie groups gives rise to functors between the associated categories of equivariant spaces:

$$\begin{tikzcd} K\mathbf{T} \arrow[rr, bend left, "G \ltimes_\alpha -"] \arrow[rr, bend right, "\mathrm{map}^{K,\alpha}(G,-)"'] & & G\mathbf{T} \arrow[ll, "\alpha^*"'] \end{tikzcd}$$

Here $G \times_\alpha -$ is left adjoint to α^*, and α^* is left adjoint to $\mathrm{map}^{K,\alpha}(G,-)$. Levelwise application of these functors gives rise to analogous adjoint functor pairs between categories of equivariant spectra

$$\begin{tikzcd} \mathrm{Sp}_K \arrow[rr, bend left, "G \ltimes_\alpha -"] \arrow[rr, bend right, "\mathrm{map}^{K,\alpha}(G,-)"'] & & \mathrm{Sp}_G \arrow[ll, "\alpha^*"'] \end{tikzcd} ,$$

see Construction 1.1.14. The following theorem records how these functors interact with the stable model structures on Sp_K and Sp_G.

We call a continuous homomorphism $\alpha : K \longrightarrow G$ between Lie groups *quasi-injective* if the restriction of α to every compact subgroup of K is injective. Equivalently, the kernel of α has no non-trivial compact subgroups.

Theorem 1.4.1. *Let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups.*

(i) *If α is quasi-injective, then the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ preserves π_*-isomorphisms and stable fibrations. In particular, the adjoint functor pair $(G \ltimes_\alpha -, \alpha^*)$ is a Quillen pair with respect to the stable model structures.*

(ii) *If α has a closed image and a compact kernel, then the adjoint functor pair $(\alpha^*, \mathrm{map}^{K,\alpha}(G,-))$ is a Quillen pair with respect to the stable model structures.*

Proof. (i) We let $f : X \longrightarrow Y$ be a π_*-isomorphism or a stable fibration of orthogonal G-spectra. The definitions of π_*-isomorphism and stable fibrations only refer to compact subgroups, so to show that $\alpha^*(f)$ is a π_*-isomorphism or a stable fibration of orthogonal K-spectra, we can restrict to all compact subgroups L of K. Since L is compact, the restriction of α to L is injective, hence a closed embedding, and hence an isomorphism of Lie groups onto its image $H = \alpha(L)$. So α induces a natural isomorphism between $\pi_*^L(\alpha^*(X))$ and $\pi_*^H(X)$, which shows that $\pi_*^L(\alpha^*(f))$ is an isomorphism. Since L was an arbitrary compact subgroup of K, this proves that $\alpha^*(f)$ is a π_*-isomorphism. The argument for stable fibrations is similar, by using the isomorphism $\alpha : L \cong H$ to translate the commutative square (1.2.16) for given L-representations V and W into an analogous square for the H-representations $(\alpha^{-1})^*(V)$ and $(\alpha^{-1})^*(W)$.

(ii) In a first step we show that the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ takes all cofibrations of orthogonal G-spectra to cofibrations of orthogonal K-spectra. Restriction along α only changes the group actions, but it does not change the underlying orthogonal spectra. Hence the skeleta and latching objects of $\alpha^*(X)$ are the same as for X, but with action restricted along the homomorphism $\alpha \times O(m) : K \times O(m) \longrightarrow G \times O(m)$. Since the image of α is closed in G, the image of $\alpha \times O(m)$ is closed in $G \times O(m)$. The kernel of $\alpha \times O(m)$ is $\ker(\alpha) \times 1$, which is compact by hypothesis. So restriction along $\alpha \times O(m)$ takes $\mathcal{C}om$-cofibrations of $(G \times O(m))$-spaces to $\mathcal{C}om$-cofibrations of $(K \times O(m))$-spaces, by Proposition 1.1.6. So if $i : A \longrightarrow B$ is a cofibration of orthogonal G-spectra, then $\nu_m i$ is a $\mathcal{C}om$-cofibration of $(G \times O(m))$-spaces, and $\nu_m(\alpha^*(i)) = (\alpha \times O(m))^*(\nu_m i)$ is a $\mathcal{C}om$-cofibration of $(K \times O(m))$-spaces. Moreover, the $O(m)$-action is unchanged, so it still acts freely off the image of $\nu_m(\alpha^*(i))$. This shows that $\alpha^*(i)$ is a cofibration of orthogonal K-spectra.

It remains to show that α^* takes cofibrations of orthogonal G-spectra that are also π_*-isomorphisms to π_*-isomorphisms of orthogonal K-spectra. Here we treat two special cases first. If α is the inclusion of a closed subgroup Γ of G, then $\alpha^* = \mathrm{res}^G_\Gamma$ preserves all π_*-isomorphisms by part (i). If α is surjective, then we verify that α^* takes the generating acyclic cofibrations of the stable model structure on orthogonal G-spectra to acyclic cofibrations of orthogonal K-spectra. The generating acyclic cofibrations J^G_{lv} of the level model structure (1.2.18) are G-equivariant homotopy equivalences, so α^* takes them to K-equivariant homotopy equivalences, which are in particular π_*-isomorphisms. For the other generating acyclic cofibrations in $\mathcal{K}^G$ we recall from (1.2.19) the π_*-isomorphism

$$G \ltimes_H \lambda_{H,V,W} \ : \ G \ltimes_H (F_{V\oplus W} S^V) \ \longrightarrow \ G \ltimes_H F_W \ ;$$

here H is a compact subgroup of G, and V and W are H-representations. Since the kernel of α is compact, the group $L = \alpha^{-1}(H)$ is then a compact subgroup of K. An isomorphism of orthogonal K-spectra

$$K \ltimes_L F_{(\alpha|_L)^*(V)} \ \cong \ \alpha^*(G \ltimes_H F_V)$$

is given levelwise by sending $k \wedge x$ to $\alpha(k) \wedge x$. We conclude that α^* takes the π_*-isomorphism $G \ltimes_H \lambda_{H,V,W}$ to the π_*-isomorphism of orthogonal K-spectra $K \ltimes_L \lambda_{L,\alpha^*(V),\alpha^*(W)}$. The inflation functor α^* commutes with formation of mapping cylinders and levelwise smash product with spaces, so we conclude that $\alpha^*(\mathcal{K}^G) \subset \mathcal{K}^K$. This completes the proof that α^* preserves stable acyclic cofibrations if α is surjective with compact kernel.

In the general case we factor α as the composite

$$K \ \xrightarrow{\ \beta\ } \ \Gamma \ \xrightarrow{\ \mathrm{incl}\ } \ G \ ,$$

where $\Gamma = \alpha(K)$ is the image of α, and β is the same map as α, but with image Γ. The restriction homomorphism factors as $\alpha^* = \beta^* \circ \mathrm{res}^G_\Gamma$, and each of the two functors is a left Quillen functor by the special cases treated above. □

Remark 1.4.2. Both hypotheses on the continuous homomorphism α imposed in Theorem 1.4.1 (ii) are really necessary. For example, if $\alpha : \mathbb{Z} \longrightarrow U(1)$ is the continuous homomorphism that takes the generator to $e^{2\pi i x}$ for an irrational real number x, then $\alpha^*(U(1))$ is not cofibrant as a $\mathbb{Z}$-space, and $\alpha^*(\Sigma^\infty_+ U(1))$ is not cofibrant as an orthogonal $\mathbb{Z}$-spectrum. So we cannot drop the hypothesis that the image of α is closed.

If $\alpha : G \longrightarrow e$ is the unique homomorphism to the trivial group, then $\alpha^*(\mathbb{S}) = \mathbb{S}_G$ is the G-sphere spectrum. This G-sphere spectrum is cofibrant as an orthogonal G-spectrum precisely when G is compact. So we cannot drop the hypothesis that the kernel of α is compact.

For easier reference we spell out the important special case of Theorem 1.4.1 for the inclusion of a closed subgroup. The induction functor $G \ltimes_\Gamma -$ preserves π_*-isomorphisms by Corollary 1.2.10.

COROLLARY 1.4.3. *Let Γ be a closed subgroup of a Lie group G. The restriction functor $\mathrm{res}^G_\Gamma \colon \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_\Gamma$ preserves π_*-isomorphisms, cofibrations and stable fibrations. Hence the two adjoint functor pairs*

$$\mathrm{Sp}_\Gamma \underset{\mathrm{res}^G_\Gamma}{\overset{G\ltimes_\Gamma}{\rightleftarrows}} \mathrm{Sp}_G \qquad \text{and} \qquad \mathrm{Sp}_\Gamma \underset{\mathrm{map}^\Gamma(G,-)}{\overset{\mathrm{res}^G_\Gamma}{\leftrightarrows}} \mathrm{Sp}_G$$

are Quillen adjunctions with respect to the two stable model structures. Moreover, the induction functor $G\ltimes_\Gamma \colon \mathrm{Sp}_\Gamma \longrightarrow \mathrm{Sp}_G$ preserves π_*-isomorphisms.

A celebrated theorem of Cartan, Iwasawa [**29**] and Malcev [**45**] says that every connected Lie group G has a maximal compact subgroup, i.e., a compact subgroup K such that every compact subgroup is subconjugate to K. Moreover, G is homeomorphic as a topological space to $K \times \mathbb{R}^n$ for some $n \geq 0$ (but there is typically no Lie group isomorphism between G and $K \times \mathbb{R}^n$). In particular, the inclusion $K \longrightarrow G$ is a homotopy equivalence of underlying spaces. A comprehensive exposition with further references can be found in Borel's *Séminaire Bourbaki* article [**7**]. A maximal compact subgroup with these properties exists more generally when the Lie group G is *almost connected*, i.e., when it has finitely many path components, and even for locally compact topological groups whose component group is compact, see [**1**, Thm. A.5].

THEOREM 1.4.4 (Reduction to maximal compact subgroups). *Let G be an almost connected Lie group, and K a maximal compact subgroup of G. Then the restriction functor*

$$\mathrm{res}^G_K \colon \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$$

is a left and right Quillen equivalence for the stable model structures.

PROOF. Since every compact subgroup of G is subconjugate to K, the restriction functor detects π_*-isomorphisms. Since restriction and induction preserve π_*-isomorphisms in full generality, we may show that for every orthogonal K-spectrum Y, the adjunction unit $Y \longrightarrow G \ltimes_K Y$ is a π_*-isomorphism of orthogonal K-spectra.

At this point we need additional input, namely a theorem of Abels [**1**, Thm. A.5] that provides a subspace E of G with the following properties:

(i) The space E is invariant under conjugation by K.
(ii) Under the conjugation action, the space E is K-equivariantly homeomorphic to a finite-dimensional linear K-representation.
(iii) The multiplication map $E \times K \longrightarrow G$, $(e, k) \mapsto ek$ is a homeomorphism.

The multiplication homeomorphism $E \times K \cong G$ is left K-equivariant for the diagonal K-action on the source, by conjugation on E and by left translation on K. The multiplication homeomorphism is right K-equivariant for the trivial K-action on E and the right translation action on K. So in particular, the morphism

$$E_+ \wedge Y \longrightarrow G \ltimes_K Y \ , \quad (e,y) \longmapsto [e,y]$$

is an isomorphism of orthogonal K-spectra, where K acts diagonally on the source, by conjugation on E, and by the given action on Y. Since E is K-homeomorphic to a finite-dimensional linear K-representation, it is K-equivariantly contractible. So the adjunction unit is a K-equivariant homotopy equivalence of underlying orthogonal K-spectra. □

EXAMPLE 1.4.5. By the very construction, the proper G-equivariant homotopy theory is assembled from the equivariant homotopy theories of the compact subgroups. We now discuss a situation where this relationship is especially tight, and where morphisms in $\mathrm{Ho}(\mathrm{Sp}_G)$ can be calculated directly from morphisms in $\mathrm{Ho}(\mathrm{Sp}_H)$ for finite subgroups H.

Let G be a countable locally finite group, i.e., a discrete group that has an exhaustive sequence of finite subgroups

$$H_0 \subseteq H_1 \subseteq \dots \subseteq H_n \subseteq \dots \ ,$$

i.e., so that $G = \bigcup H_n$. The various restriction functors

$$\mathrm{res}^G_{H_n} \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{H_n})$$

are then compatible. For all orthogonal G-spectra X and Y, the restriction maps thus assemble into a group homomorphism

$$\mathrm{res}^G \ : \ [X,Y]^G \longrightarrow \lim\, [X,Y]^{H_n} \ ,$$

where the inverse limit on the right hand side is formed along restriction maps. We have simplified the notation by suppressing the restriction functors $\mathrm{res}^G_{H_n}$ on the right hand side. The hypothesis that the sequence H_n exhausts G implies that for every finite subgroup K of G the colimit of the sequence of sets

$$(G/H_0)^K \longrightarrow (G/H_1)^K \longrightarrow \dots \longrightarrow (G/H_n)^K \longrightarrow \dots$$

is a single point. Thus the mapping telescope, in the category of G-spaces, of the sequence of G-spaces G/H_n is $\mathcal{C}om$-equivalent to the one-point G-space. Since the mapping telescope comes to us as a 1-dimensional G-CW-complex, it is a 1-dimensional G-CW-model for $\underline{E}G$.

For all cofibrant G-spectra X, the mapping telescope, in the category of orthogonal G-spectra, of the sequence

$$X \wedge (G/H_0)_+ \longrightarrow X \wedge (G/H_1)_+ \longrightarrow \dots \longrightarrow X \wedge (G/H_n)_+ \longrightarrow \dots$$

is hence π_*-isomorphic to X. The mapping telescope models an abstract homotopy colimit in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$. So applying the functor $[-,Y]^G$ yields a short exact sequence of abelian groups

$$\begin{aligned} 0 \longrightarrow \lim{}^1\, [X \wedge (G/H_n)_+ \wedge S^1, Y]^G \longrightarrow [X,Y]^G \\ \longrightarrow \lim\, [X \wedge (G/H_n)_+, Y]^G \longrightarrow 0 \ . \end{aligned}$$

We rewrite $X \wedge (G/H_n)_+$ as $G \ltimes_{H_n} (\mathrm{res}^G_{H_n} X)$ and use the adjunction isomorphism between restriction and induction to identify the group $[X \wedge (G/H_n)_+, Y]^G$ of morphisms in $\mathrm{Ho}(\mathrm{Sp}_G)$ with the group $[X, Y]^{H_n}$ of morphisms in $\mathrm{Ho}(\mathrm{Sp}_{H_n})$. The maps in the tower then become restriction maps, so we have shown:

COROLLARY 1.4.6. *Let G be a discrete group and $\{H_n\}_{n\geq 0}$ an ascending exhaustive sequence of finite subgroups of G. Then for all orthogonal G-spectra X and Y there is a short exact sequence*

$$0 \longrightarrow \lim{}^1 [X \wedge S^1, Y]^{H_n} \longrightarrow [X, Y]^G \longrightarrow \lim [X, Y]^{H_n} \longrightarrow 0 \ .$$

Here the inverse and derived limit are formed along restriction maps, and so is the map from $[X, Y]^G$ to the inverse limit.

REMARK 1.4.7. The *$\mathcal{F}in$-orbit category* of a discrete group G is the full subcategory of the category of G-sets with objects G/H for all finite subgroups H of G. As we hope to make precise in future work, the underlying ∞-category of the stable model category of orthogonal G-spectra is a limit, for G/H ranging through the $\mathcal{F}in$-orbit category, of the ∞-categories of genuine H-spectra.

If G is an ascending union of finite subgroups H_n as in Example 1.4.5, the limit diagram can be modified to show that the underlying ∞-category of orthogonal G-spectra is an inverse limit of the tower of ∞-categories associated to orthogonal H_n-spectra. The short exact sequence of Corollary 1.4.6 is a consequence of this more refined relationship.

An interesting special case of this is the Prüfer group C_{p^∞} for a prime number p, i.e., the group of p-power torsion elements in $U(1)$, with the discrete topology. Since C_{p^∞} is the union of its subgroups C_{p^n} for $n \geq 0$, the underlying ∞-category of orthogonal C_{p^∞}-spectra is an inverse limit of the ∞-categories of genuine C_{p^n}-spectra. Hence the ∞-category of genuine proper orthogonal C_{p^∞}-spectra is equivalent to the ∞-category of genuine C_{p^∞}-spectra in the sense of Nikolaus and Scholze [**55**, Def. II.2.15], on which their notion of *genuine p-cyclotomic spectra* is based, compare [**55**, Def. II.3.1].

Now we compare the equivariant stable homotopy categories for varying Lie groups. We let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups. By Theorem 1.4.1 (i), the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ preserves π_*-isomorphisms if α happens to be quasi-injective, but not when the kernel of α has a non-trivial compact subgroup. In that case there cannot be an induced functor $\mathrm{Ho}(\alpha^*) : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ such that $\mathrm{Ho}(\alpha^*) \circ \gamma_G$ is equal to $\gamma_K \circ \alpha^*$, where γ_G and γ_K are the localization functors of (1.3.1). However, the next best thing is true: restriction along α has a total left derived functor, see Theorem 1.4.17 below. For the convenience of the reader we briefly review this concept. The arguments we use in the rest of this section have a substantial overlap with Chapter VII on 'Deformable functors and their approximations' of the book [**17**] by Dwyer, Hirschhorn, Kan and Smith; for the convenience of the reader, we give a largely self-contained exposition of the relevant parts, adapted to our context.

We let $(\mathcal{C}, w)$ be a relative category, i.e., a category $\mathcal{C}$ equipped with a distinguished class w of morphisms that we call *weak equivalences*. An important special case of relative categories are the ones underlying model categories. A functor between relative categories is *homotopical* if it takes weak equivalences to weak equivalences.

DEFINITION 1.4.8. The *homotopy category* of a relative category $(\mathcal{C}, w)$ is a functor $\gamma_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ that sends all weak equivalences to isomorphisms and initial among such functors.

Explicitly, the universal property of the homotopy category $\gamma_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is as follows. For every functor $\Phi : \mathcal{C} \longrightarrow \mathcal{X}$ that sends all weak equivalences to isomorphisms, there is a unique functor $\bar{\Phi}$ such that $\bar{\Phi} \circ \gamma_{\mathcal{C}} = \Phi$. In the generality of relative categories, a homotopy category need not always exist (with small hom set, or in the same Grothendieck universe, that is). In the examples we care about, the relative category is underlying a model category, and then already Quillen [**57**, I Thm. 1'] constructed a homotopy category as the quotient of the category of cofibrant-fibrant objects by an explicit homotopy relation on morphisms.

There are many interesting functors between relative categories that are not homotopical, but still induce interesting functors between the homotopy categories. Often, extra structure on the relative categories is used to define and study such 'derived' functors, for example a model category structure. We are particularly interested in 'left derived functors'.

In the following we will compose (or 'paste') functors and natural transformations, and we introduce notation for this. Let $\nu : F \Longrightarrow F' : \mathcal{C} \longrightarrow \mathcal{D}$ be a natural transformation between two functors, and let $E : \mathcal{B} \longrightarrow \mathcal{C}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ be functors. We write $\nu \star E : F \circ E \Longrightarrow F' \circ E$ and $G \star \nu : G \circ F \Longrightarrow G \circ F'$ for the natural transformation with components

$$(\nu \star E)_X \ = \ \nu_{E(X)} \ : F(E(X)) \ \longrightarrow \ F'(E(X))$$

and

$$(G \star \nu)_Y \ = \ G(\nu_Y) \ : G(F(Y)) \ \longrightarrow \ G(F'(Y)) \ ,$$

respectively. If $\mu : G \Longrightarrow G'$ is another natural transformation between functors from $\mathcal{D}$ to $\mathcal{E}$, then the following *interchange relation* holds:

$$(G' \star \nu) \circ (\mu \star F) \ = \ (\mu \star F') \circ (G \star \nu) \ ; \tag{1.4.9}$$

this relation is just a restatement of naturality.

DEFINITION 1.4.10. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between relative categories. A *total left derived functor* of F is a pair (L, τ) consisting of a functor $L : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ and a natural transformation $\tau : L \circ \gamma_{\mathcal{C}} \Longrightarrow \gamma_{\mathcal{D}} \circ F$ with the following universal property: for every pair (Φ, κ) consisting of a functor $\Phi : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ and a natural transformation $\kappa : \Phi \circ \gamma_{\mathcal{C}} \Longrightarrow \gamma_{\mathcal{D}} \circ F$, there is a unique natural transformation $\bar{\kappa} : \Phi \Longrightarrow L$ such that $\kappa = \tau \circ (\bar{\kappa} \star \gamma_{\mathcal{C}})$. A functor between relative categories is *left derivable* if it admits a total left derived functor.

EXAMPLE 1.4.11. Suppose that $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopical functor between relative categories. Then the composite $\gamma_{\mathcal{D}} \circ F$ takes all weak equivalences to isomorphisms. Hence the universal property of the homotopy category provides a unique functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ such that $\mathrm{Ho}(F) \circ \gamma_{\mathcal{C}} = \gamma_{\mathcal{D}} \circ F$. Then the pair $(\mathrm{Ho}(F), \mathrm{Id})$ is a total left derived functor of F. If $G : \mathcal{D} \longrightarrow \mathcal{E}$ is another homotopical functor, then the composite $GF : \mathcal{C} \longrightarrow \mathcal{E}$ is also homotopical. Moreover,

$$\mathrm{Ho}(G) \circ \mathrm{Ho}(F) \circ \gamma_{\mathcal{C}} \ = \ \mathrm{Ho}(G) \circ \gamma_{\mathcal{D}} \circ F \ = \ \gamma_{\mathcal{E}} \circ G \circ F \ = \ \mathrm{Ho}(GF) \circ \gamma_{\mathcal{C}} \ .$$

The universal property of the homotopy category thus shows that $\mathrm{Ho}(G) \circ \mathrm{Ho}(F) = \mathrm{Ho}(GF)$.

Conversely, suppose that $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between model categories which admits a total left derived functor (L, τ), and such that τ is a natural isomorphism. We claim that then F must be homotopical. Indeed, for every weak equivalence f in $\mathcal{C}$, the morphism $\gamma_{\mathcal{C}}(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$, and hence $L(\gamma_{\mathcal{C}}(f))$ is an isomorphism in $\mathrm{Ho}(\mathcal{D})$. Since $\tau : L \circ \gamma_{\mathcal{C}} \Longrightarrow \gamma_{\mathcal{D}} \circ F$ is a natural isomorphism, the morphism $\gamma_{\mathcal{D}}(F(f))$ is an isomorphism in $\mathrm{Ho}(\mathcal{D})$. In model categories, the localization functor detects weak equivalences, so $F(f)$ is a weak equivalence in $\mathcal{D}$.

We recall an important result of Quillen that implies that a functor between model categories admits a total left derived functor if it takes weak equivalences between cofibrant object to weak equivalences. Moreover, Maltsiniotis observed that any such total left derived functor is automatically an *absolute* right Kan extension, i.e., it remains a right Kan extension after postcomposition with any functor:

Theorem 1.4.12 (Quillen [**57**], Maltsiniotis [**46**]). *Let $\mathcal{C}$ be a model category and $F : \mathcal{C} \longrightarrow \mathcal{X}$ a functor that takes weak equivalences between cofibrant objects to isomorphisms. Then F admits a right Kan extension (L, τ) along the localization functor $\gamma_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$. Moreover:*

(i) *The morphism $\tau_X : L(\gamma_{\mathcal{C}}(X)) \longrightarrow FX$ is an isomorphism for every cofibrant object X of $\mathcal{C}$.*
(ii) *Every right Kan extension of F along $\gamma_{\mathcal{C}}$ is an absolute right Kan extension.*

The following proposition is a direct consequence of the absolute Kan extension property.

Proposition 1.4.13. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ be composable functors between relative categories such that F is absolutely left derivable and G is homotopical. Let (LF, τ_F) be a total left derived functor of F. Then the pair $(\mathrm{Ho}(G) \circ LF, \mathrm{Ho}(G) \star \tau_F)$ is an absolute left derived functor of GF.*

Proof. We let $\psi : \mathrm{Ho}(\mathcal{E}) \longrightarrow \mathcal{Y}$ be any functor. Since (LF, τ_F) is an absolute right Kan extension of $\gamma_{\mathcal{D}} \circ F$ along $\gamma_{\mathcal{C}}$, the pair $(\psi \circ \mathrm{Ho}(G) \circ LF, (\psi \circ \mathrm{Ho}(G)) \star \tau_F)$ is a right Kan extension of $\psi \circ \mathrm{Ho}(G) \circ \gamma_{\mathcal{D}} \circ F = \psi \circ \gamma_{\mathcal{E}} \circ G \circ F$ along $\gamma_{\mathcal{C}}$. This precisely means that $(\mathrm{Ho}(G) \circ LF, \mathrm{Ho}(G) \star \tau_F)$ is an absolute total left derived functor of GF. □

Our next goal is to show that restriction of equivariant spectra along a continuous homomorphism between Lie groups has a total left derived functor. To this end, the following class of equivariant spectra will be useful.

Definition 1.4.14. Let G be a Lie group. An orthogonal G-spectrum X is *quasi-cofibrant* if for every compact subgroup H of G the underlying H-spectrum of X is cofibrant.

Example 1.4.15. Restriction to a closed subgroup preserves cofibrancy, by Corollary 1.4.3; so every cofibrant orthogonal G-spectrum is quasi-cofibrant. If the Lie group G is itself compact, then every quasi-cofibrant G-spectrum is already cofibrant, so in the compact case the two notions coincide.

If the Lie group G is not compact, then 'quasi-cofibrant' is a strictly more general concept. Indeed, the G-equivariant sphere spectrum $\mathbb{S}_G$ is cofibrant if and only if G is compact; hence for every Lie group G, $\mathbb{S}_G$ is quasi-cofibrant. More generally, we let Γ be a closed subgroup of G. For every compact subgroup H of G, the H-action on the coset space G/Γ by translation is smooth; so Illman's theorem [**27**, Thm. 7.1] provides an H-CW-structure on G/Γ. In particular, G/Γ is cofibrant as an H-space, and hence the suspension spectrum $\Sigma^\infty_+ G/\Gamma$ is quasi-cofibrant.

The next proposition provides a characterization of quasi-cofibrant spectra in terms of cofibrant spectra.

PROPOSITION 1.4.16. *Let G be a Lie group. An orthogonal G-spectrum X is quasi-cofibrant if and only if for every $\mathcal{C}om$-cofibrant G-space B the orthogonal G-spectrum $X \wedge B_+$ is cofibrant.*

PROOF. For one implication we let X be an orthogonal G-spectrum X such that $X \wedge B_+$ is cofibrant as an orthogonal G-spectrum for every $\mathcal{C}om$-cofibrant G-space B. For every compact subgroup H of G, the homogeneous G-space $B = G/H$ is $\mathcal{C}om$-cofibrant, so in particular the G-spectrum $X \wedge G/H_+$ is cofibrant. The underlying H-spectrum of $X \wedge G/H_+$ is then cofibrant by Corollary 1.4.3. The two H-equivariant morphisms

$$X \xrightarrow{x \mapsto x \wedge eH} X \wedge G/H_+ \xrightarrow{x \wedge gH \mapsto x} X$$

witness that X is an H-equivariant retract of $X \wedge G/H_+$, and so X is itself cofibrant as an H-spectrum. Hence X is quasi-cofibrant.

For the other implication we let X be a quasi-cofibrant orthogonal G-spectrum, and we let K denote the class of those morphisms $i : A \longrightarrow B$ of G-spaces such that $X \wedge f_+ : X \wedge A_+ \longrightarrow X \wedge B_+$ is a cofibration of orthogonal G-spectra. We claim that K contains all $\mathcal{C}om$-cofibrations of G-spaces; for $A = \emptyset$ this proves that $X \wedge B_+$ is cofibrant.

For every compact subgroup H of G, the underlying H-spectrum of X is cofibrant by assumption. So the G-spectrum

$$X \wedge G/H_+ \ \cong \ G \ltimes_H \mathrm{res}^G_H(X)$$

is G-cofibrant. Since the stable model structure on Sp_G is G-topological, the morphism

$$X \wedge (G/H \times i^k)_+ \ : \ X \wedge (G/H \times \partial D^k)_+ \ \longrightarrow \ X \wedge (G/H \times D^k)_+$$

is a cofibration of G-spectra. This shows that the generating $\mathcal{C}om$-cofibrations belong to the class K. Because cofibrations are closed under cobase change, coproducts, sequential colimits, and retracts, and because $X \wedge (-)_+$ preserves colimits, the class K is closed under cobase change, coproducts, sequential colimits and retracts. So all $\mathcal{C}om$-cofibrations belong to the class K. $\square$

The following theorem shows that restriction of equivariant spectra along a continuous homomorphism between Lie groups has a total left derived functor, and it collects many important properties of the left derived functor. Among other things, the derived functor 'commutes with suspension spectra'. To make this precise we observe that the suspension spectrum functor $\Sigma^\infty_+ : G\mathbf{T} \longrightarrow \mathrm{Sp}_G$ is fully homotopical, i.e., it takes $\mathcal{C}om$-weak equivalences to π_*-isomorphisms, for example by [**60**, Prop. 3.1.44]. The space level restriction functor $\alpha^* : G\mathbf{T} \longrightarrow K\mathbf{T}$ is also fully homotopical for $\mathcal{C}om$-equivalences, compare Proposition 1.1.6 (i). Hence the functor

$$\Sigma^\infty_+ \circ \alpha^* \ = \ \alpha^* \circ \Sigma^\infty_+ \ : \ G\mathbf{T} \ \longrightarrow \ \mathrm{Sp}_K$$

is homotopical. So there is a unique functor

$$\mathrm{Ho}(\Sigma^\infty_+ \circ \alpha^*) \ : \ \mathrm{Ho}^{\mathcal{C}om}(G\mathbf{T}) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_K)$$

such that

$$\mathrm{Ho}(\Sigma^\infty_+ \circ \alpha^*) \circ \gamma_G^{\mathrm{un}} \ = \ \gamma_K \circ \Sigma^\infty_+ \circ \alpha^* \ : \ G\mathbf{T} \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_K) \ ,$$

where $\gamma_G^{\mathrm{un}} : G\mathbf{T} \longrightarrow \mathrm{Ho}^{\mathcal{C}om}(G\mathbf{T})$ is the localization functor.

THEOREM 1.4.17. *Let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups.*

- (i) *The restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ takes quasi-cofibrant orthogonal G-spectra to quasi-cofibrant orthogonal K-spectra, and it takes π_*-isomorphisms between quasi-cofibrant orthogonal G-spectra to π_*-isomorphisms of orthogonal K-spectra.*
- (ii) *The restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ has a total left derived functor $(L\alpha^*, \alpha_!)$. For every quasi-cofibrant G-spectrum X, the morphism $\alpha_! : (L\alpha^*)(X) \longrightarrow \gamma_K(\alpha^*(X))$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$.*
- (iii) *The derived functor $L\alpha^*$ preserves sums and has a right adjoint.*
- (iv) *There is a unique natural transformation*

$$\sigma \ : \ (L\alpha^*) \circ [1] \ \Longrightarrow \ [1] \circ (L\alpha^*)$$

of functors $\mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ such that

$$([1] \star \alpha_!) \circ (\sigma \star \gamma_G) \ = \alpha_! \star (- \wedge S^1) \ : \qquad (1.4.18)$$

$$(L\alpha^*) \circ [1] \circ \gamma_G \ \longrightarrow \ \gamma_K \circ \alpha^* \circ (- \wedge S^1) \ .$$

Moreover, the transformation σ is a natural isomorphism and the pair $(L\alpha^*, \sigma)$ is an exact functor of triangulated categories.
- (v) *There is a unique natural transformation*

$$\nu \ : \ (L\alpha^*) \circ \mathrm{Ho}(\Sigma^\infty_+) \ \Longrightarrow \ \mathrm{Ho}(\Sigma^\infty_+ \circ \alpha^*)$$

of functors $\mathrm{Ho}^{\mathcal{C}om}(G\mathbf{T}) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ such that

$$\nu \star \gamma_G^{\mathrm{un}} \ = \ \alpha_! \star \Sigma^\infty_+ \ : \ (L\alpha^*) \circ \mathrm{Ho}(\Sigma^\infty_+) \circ \gamma_G^{\mathrm{un}} \ \longrightarrow \ \mathrm{Ho}(\Sigma^\infty_+ \circ \alpha^*) \circ \gamma_G^{\mathrm{un}} \ .$$

Moreover, ν is a natural isomorphism.
- (vi) *If α is quasi-injective, then the universal natural transformation $\alpha_! : (L\alpha^*) \circ \gamma_G \Longrightarrow \gamma_K \circ \alpha^*$ is an isomorphism, and $L\alpha^*$ preserves products and has a left adjoint.*

PROOF. (i) For the first claim we let X be a quasi-cofibrant orthogonal G-spectrum and L a compact subgroup of K. Then the L-spectrum $\mathrm{res}^K_L(\alpha^*(X))$ is the same as $(\alpha|_L)^*(X)$, where $\alpha|_L : L \longrightarrow G$ is the restricted homomorphism. Since L is compact, $\alpha|_L$ has closed image and compact kernel, so $(\alpha|_L)^*(X)$ is a cofibrant L-spectrum by Theorem 1.4.1 (ii). Since L was any compact subgroup of K, this proves that $\alpha^*(X)$ is quasi-cofibrant.

Now we let $f : X \longrightarrow Y$ be a π_*-isomorphism between quasi-cofibrant orthogonal G-spectra. We let L be a compact subgroup of K. We factor the restriction $\alpha|_L : L \longrightarrow G$ as

$$L \ \xrightarrow{\ \beta\ } \ H \ \xrightarrow{\ \mathrm{incl}\ } \ G \ ,$$

where $H = \alpha(L)$ is the image of α, and β is the same map as $\alpha|_L$, but with target H. The group H is compact since L is; since X and Y are quasi-cofibrant, their underlying H-spectra are cofibrant. Since f is a π_*-isomorphism of quasi-cofibrant G-spectra, $\mathrm{res}^G_H(f)$ is a π_*-isomorphism between cofibrant H-spectra. The continuous epimorphism $\beta : L \longrightarrow H$ satisfies the hypotheses of Theorem 1.4.1 (ii) because L is compact; so $\beta^* : \mathrm{Sp}_H \longrightarrow \mathrm{Sp}_L$ is a left Quillen functor for the stable model structures. In particular, β^* takes π_*-isomorphisms between cofibrant H-spectra to π_*-isomorphisms, by Ken Brown's lemma [**25**, Lemma 1.1.12]. So the morphism

$$\mathrm{res}^K_L(\alpha^*(f)) \ = \ \beta^*(\mathrm{res}^G_H(f))$$

is a π_*-isomorphism of orthogonal L-spectra. Since L was an arbitrary compact subgroup of K, this proves the last claim.

(ii) Part (i) shows that the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ takes π_*-isomorphisms between cofibrant orthogonal G-spectra to π_*-isomorphisms of orthogonal K-spectra. Given this, Quillen's result [**57**, I.4, Prop. 1] provides the left derived functor and shows that $\alpha_! : (L\alpha^*)(X) \longrightarrow \alpha^*(X)$ is an isomorphism whenever X is cofibrant, compare also Theorem 1.4.12.

If X is quasi-cofibrant, we choose a π_*-isomorphism $f : Y \longrightarrow X$ from a cofibrant orthogonal G-spectrum. Then $\gamma_G(f)$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$, so the upper horizontal morphism in the commutative square

$$\begin{tikzcd}
(L\alpha^*)(Y) \arrow[r, "(L\alpha^*)(\gamma_G(f))"] \arrow[d, "\alpha_!^Y"'] & (L\alpha^*)(X) \arrow[d, "\alpha_!^X"] \\
\gamma_K(\alpha^*(Y)) \arrow[r, "\gamma_K(\alpha^*(f))"'] & \gamma_K(\alpha^*(X))
\end{tikzcd}$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$. The morphism $\alpha_!^Y$ is an isomorphism because Y is cofibrant. The morphism $\gamma_K(\alpha^*(f))$ is an isomorphism because α^* preserves π_*-isomorphisms between quasi-cofibrant spectra, by part (i). So $\alpha_!^X$ is an isomorphism.

(iii) We exploit that coproducts in $\mathrm{Ho}(\mathrm{Sp}_G)$ and $\mathrm{Ho}(\mathrm{Sp}_K)$ are modeled by wedges of equivariant spectra, because formation of wedges is fully homotopical. We let $\{X_i\}_{i\in I}$ be a family of cofibrant orthogonal G-spectra; then the wedge $\bigvee X_i$ is also cofibrant. So the vertical morphisms in the commutative square

$$\begin{tikzcd}
\bigvee (L\alpha^*)(X_i) \arrow[r, "\kappa"] \arrow[d, "\bigvee \alpha_!^{X_i}"'] & (L\alpha^*)(\bigvee X_i) \arrow[d, "\alpha_!^{\bigvee X_i}"] \\
\bigvee \gamma_K(\alpha^*(X_i)) \arrow[r, "\cong"'] & \gamma_K(\alpha^*(\bigvee X_i))
\end{tikzcd}$$

are isomorphisms in $\mathrm{Ho}(\mathrm{Sp}_K)$ by part (i). Since α^* preserves colimits, the lower morphism is an isomorphism, and hence so is the canonical morphism κ. This proves that $L\alpha^*$ preserves sums. Corollary 1.3.12 (iii) then provides a right adjoint for $L\alpha^*$.

(iv) The suspension functor is fully homotopical, so Proposition 1.4.13 shows that the pair $([1]\circ L\alpha^*, [1]\star\alpha_!)$ is an absolute left derived functor of the functor

$$\alpha^*(-)\wedge S^1 \ = \ \alpha^*(-\wedge S^1) \ : \ \mathrm{Sp}_G \ \longrightarrow \ \mathrm{Sp}_K \ .$$

The universal property of $([1]\circ L\alpha^*, [1]\star\alpha_!)$ thus provides a unique natural transformation $\sigma : (L\alpha^*)\circ[1] \Longrightarrow [1]\circ(L\alpha^*)$ that satisfies the relation specified in the statement of the theorem. The diagram

$$\begin{tikzcd}
(L\alpha^*)(X[1]) \arrow[r, "\sigma_X"] \arrow[d, "\alpha_!^{X\wedge S^1}"'] & ((L\alpha^*)(X))[1] \arrow[d, "\alpha_!^X[1]"] \\
\gamma_K(\alpha^*(X\wedge S^1)) \arrow[r, equal] & \gamma_K(\alpha^*(X))[1]
\end{tikzcd}$$

commutes by construction of σ. If X is cofibrant, so is $X \wedge S^1$, so both vertical morphisms are isomorphisms in $\mathrm{Ho}(\mathrm{Sp}_K)$, by part (ii). Hence σ is an isomorphism for cofibrant G-spectra; in $\mathrm{Ho}(\mathrm{Sp}_G)$, every object is isomorphic to a cofibrant spectrum, so σ is a natural isomorphism.

Every distinguished triangle in $\mathrm{Ho}(\mathrm{Sp}_G)$ is isomorphic to the mapping cone triangle (1.3.4) associated with a morphism $f : X \longrightarrow Y$ between cofibrant G-spectra. So to show that the pair $(L\alpha^*, \sigma)$ preserves distinguished triangles, it suffices to show exactness for these special ones. We contemplate the commutative diagram:

$$\begin{tikzcd}
(L\alpha^*)(X) \arrow[r, "(L\alpha^*)(f)"] \arrow[d, "\alpha_!^X"'] & (L\alpha^*)(Y) \arrow[r, "(L\alpha^*)(i)"] \arrow[d, "\alpha_!^Y"'] & (L\alpha^*)(Cf) \arrow[r, "\sigma_X\circ(L\alpha^*)(p)"] \arrow[d, "\alpha_!^{Cf}"] & (L\alpha^*)(X)\wedge S^1 \arrow[d, "\alpha_!^X\wedge S^1"] \\
\gamma_K(\alpha^*(X)) \arrow[r, "\gamma_K(\alpha^*(f))"'] & \gamma_K(\alpha^*(Y)) \arrow[r, "\gamma_K(\alpha^*(i))"'] & \gamma_K(\alpha^*(Cf)) \arrow[r, "\gamma_K(\alpha^*(p))"'] & \gamma_K(\alpha^*(X))\wedge S^1
\end{tikzcd}$$

Since X and Y are cofibrant, so are Cf and $X \wedge S^1$; hence all vertical morphisms are isomorphisms by part (ii). The lower triangle is distinguished because the pointset level restriction functor α^* commutes with formation of mapping cones and suspension. So the upper triangle is distinguished.

(v) The existence and characterization of ν are just the universal property of the pair $(\mathrm{Ho}(\Sigma^\infty_+ \circ \alpha^*), \mathrm{Id})$ which is a total left derived functor of the homotopical functor $\Sigma^\infty_+ \circ \alpha^*$. The characterizing property means that when we specialize the transformation ν to a G-space A, we have

$$\nu_A \;=\; \alpha_!^{\Sigma^\infty_+ A} \;:\; (L\alpha^*)(\Sigma^\infty_+ A) \;\longrightarrow \gamma_K(\alpha^*(\Sigma^\infty_+ A)) \;.$$

If A is $\mathcal{C}om$-cofibrant as a G-space, then $\Sigma^\infty_+ A$ is cofibrant as a G-spectrum. So in that case, the morphism $\alpha_!^{\Sigma^\infty_+ A}$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$, by part (ii). Hence ν is an isomorphism for $\mathcal{C}om$-cofibrant G-spaces; in $\mathrm{Ho}^{\mathcal{C}om}(G\mathbf{T})$, every object is isomorphic to a $\mathcal{C}om$-cofibrant G-space, so ν is a natural isomorphism.

(vi) If α is quasi-injective, then the restriction functor α^* is fully homotopical and a right Quillen functor by Theorem 1.4.1 (i). The universal natural transformation $\alpha_!$ is an isomorphism because α^* is fully homotopical. The functor $L\alpha^*$ has a left adjoint because α^* is a right Quillen functor; the left adjoint is a total left derived functor of $G \ltimes_\alpha - : \mathrm{Sp}_K \longrightarrow \mathrm{Sp}_G$. □

Remark 1.4.19 (Derived inflation and products). As the previous theorem shows, the derived functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ of restriction along a continuous homomorphism $\alpha : K \longrightarrow G$ always has a right adjoint, and it has a left adjoint whenever α is quasi-injective. While the pointset level restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ preserves products, its left derived functor $L\alpha^*$ does *not* preserve products in general.

The simplest example is inflation along the unique homomorphism $p : C_2 \longrightarrow e$ from a group with two elements to a trivial group. In the non-equivariant stable homotopy category the canonical map

$$\bigoplus_{k<0} H\mathbb{F}_2[k] \ \longrightarrow \ \prod_{k<0} H\mathbb{F}_2[k]$$

from the coproduct to the product of infinitely many desuspended copies of the mod-2 Eilenberg-Mac Lane spectrum is an isomorphism. Since Lp^* and equivariant homotopy groups preserves coproducts, the canonical map

$$\bigoplus_{k<0} \pi_0^{C_2}\left((Lp^*)(H\mathbb{F}_2[k])\right) \ \longrightarrow \ \pi_0^{C_2}\left((Lp^*)\left(\bigoplus_{k<0} H\mathbb{F}_2[k]\right)\right) \tag{1.4.20}$$

is an isomorphism of abelian groups. For every non-equivariant spectrum X, the C_2-spectrum $Lp^*(X)$ has 'constant geometric fixed points' and the geometric fixed point map $\Phi : \pi_0^{C_2}(Lp^*(X)) \longrightarrow \Phi_0^{C_2}(Lp^*(X)) \cong \pi_0^e(X)$ has a section, compare [**60**, Ex. 4.5.10]; hence the isotropy separation sequence (see [**60**, (3.3.9)]) splits. So the C_2-equivariant homotopy groups decompose as

$$\begin{aligned}\pi_0^{C_2}(Lp^*(X)) \ &\cong \ \pi_0^{C_2}(Lp^*(X) \wedge (EC_2)_+) \oplus \Phi_0^{C_2}(LX) \\ &\cong \ \pi_0^e(X \wedge (BC_2)_+) \oplus \ \pi_0^e(X) \ .\end{aligned}$$

The second step uses the Adams isomorphism and the fact that $Lp^*(X)$ has trivial C_2-action as a naive C_2-spectrum. When $X = H\mathbb{F}_2[k]$ for negative k, then the second summand is trivial and hence

$$\pi_0^{C_2}(Lp^*(H\mathbb{F}_2[k])) \ \cong \ \pi_0^e(H\mathbb{F}_2[k] \wedge (BC_2)_+) \ \cong \ H_{-k}(BC_2, \mathbb{F}_2) \ .$$

So the group (1.4.20) is a countably infinite sum of copies of $\mathbb{F}_2$. On the other hand,

$$\begin{aligned}\pi_0^{C_2}\left(\prod_{k<0} Lp^*(H\mathbb{F}_2[k])\right) \ &\cong \ \prod_{k<0} \pi_0^{C_2}(Lp^*(H\mathbb{F}_2[k])) \\ &\cong \ \prod_{k<0} \pi_0^e\left(H\mathbb{F}_2[k] \wedge (BC_2)_+\right) \ \cong \ \prod_{k<0} H_{-k}(BC_2, H\mathbb{F}_2) \ ,\end{aligned}$$

again by the split isotropy separation sequence. This is an infinite product of copies of $\mathbb{F}_2$, so the canonical map

$$Lp^*\left(\prod_{k<0} H\mathbb{F}_2[k]\right) \ \longrightarrow \ \prod_{k<0} Lp^*(H\mathbb{F}_2[k])$$

is not a π_*-isomorphism of C_2-spectra.

EXAMPLE 1.4.21. We give a rigorous formulation of the idea that 'inner automorphisms act as the identity'. For a Lie group G and $g \in G$, we let

$$c_g^* \ : \ \mathrm{Sp}_G \ \longrightarrow \ \mathrm{Sp}_G$$

be restriction along the inner automorphism $c_g : G \longrightarrow G$, $c_g(\gamma) = g^{-1}\gamma g$. Since the restriction functor c_g^* is fully homotopical, the induced functor

$$\mathrm{Ho}(c_g^*) \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G)$$

is also a total left derived functor, with respect to the identity natural transformation. We exhibit a specific natural isomorphism between $\mathrm{Ho}(c_g^*) = Lc_g^*$ and the identity functor. We let

$$l_g \ : \ c_g^* \ \Longrightarrow \ \mathrm{Id}_{\mathrm{Sp}_G}$$

denote the natural isomorphism of functors whose value $l_g^X : c_g^*(X) \longrightarrow X$ at an orthogonal G-spectrum X is left multiplication by g. This induces a natural isomorphism of functors on $\mathrm{Ho}(\mathrm{Sp}_G)$

$$L(l_g) \ : \ \mathrm{Ho}(c_g^*) \implies \mathrm{Id}_{\mathrm{Ho}(\mathrm{Sp}_G)} \ .$$

Our next aim is to show that total left derived functors organize themselves into a 'lax functor' (whenever they exist). Loosely speaking this means that while $LG \circ LF$ need not be isomorphic to $L(GF)$, there is a preferred natural transformation $\langle G, F\rangle : LG \circ LF \implies L(GF)$, and these natural transformations satisfy certain coherence conditions. This is surely well known among experts, but we were unable to find a complete reference. There is something to show here because:

- Not all functors between relative categories are left derivable.
- If two composable functors between relative categories are left derivable, the composite need not be left derivable.
- If $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ are left derivable functors such that GF is also left derivable, then $L(GF)$ need not be isomorphic to the composite $LG \circ LF$.

Construction 1.4.22. For every relative category $(\mathcal{C}, w)$, we choose a homotopy category $\gamma_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$. We also choose a total left derived functor (LF, τ_F) for every left derivable functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between relative categories. As we shall now explain, these choices determine all the remaining coherence data, without the need to make any further choices.

We let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ be left derivable functors between relative categories. If the composite $GF : \mathcal{C} \longrightarrow \mathcal{E}$ is also left derivable, then the universal property of $(L(GF), \tau_{GF})$ provides a unique natural transformation

$$\langle G, F\rangle \ : \ LG \circ LF \implies L(GF)$$

such that

$$\tau_{GF} \circ (\langle G, F\rangle \star \gamma_{\mathcal{C}}) \ = \ (\tau_G \star F) \circ (LG \star \tau_F) \ : \ LG \circ LF \circ \gamma_{\mathcal{C}} \ \longrightarrow \ \gamma_{\mathcal{E}} \circ G \circ F \ . \tag{1.4.23}$$

We apply the previous construction to restriction functors along continuous group homomorphisms. In a nutshell, the ultimate outcome is that the assignment $\alpha \mapsto L\alpha^*$ extends to a contravariant pseudo-functor from the category of Lie groups and continuous homomorphisms to the 2-category of triangulated categories, exact functors, and exact transformations.

Construction 1.4.24. We let $\alpha : K \longrightarrow G$ and $\beta : J \longrightarrow K$ be two composable continuous homomorphisms between Lie groups. We let $(L\alpha^*, \alpha_!)$, $(L\beta^*, \beta_!)$ and $(L(\alpha\beta)^*, (\alpha\beta)_!)$ be total left derived functors of α^*, β^* and $(\alpha\beta)^*$, respectively. The universal property of $(L(\alpha\beta)^*, (\alpha\beta)_!)$ provides a unique natural transformation

$$\langle \alpha, \beta\rangle \ : \ L\beta^* \circ L\alpha^* \implies L(\alpha\beta)^*$$

such that

$$\begin{aligned} (\beta_! \star \alpha^*) \circ (L\beta^* \star \alpha_!) \ &= \ (\alpha\beta)_! \circ (\langle \alpha, \beta\rangle \star \gamma_G) \ : \\ &L\beta^* \circ L\alpha^* \circ \gamma_G \implies \gamma_J \circ \beta^* \circ \alpha^* \ = \ \gamma_J \circ (\alpha\beta)^* \ . \end{aligned} \tag{1.4.25}$$

The natural transformations so obtained satisfy a coherence condition: if $\gamma : M \longrightarrow J$ is yet another continuous homomorphism, then the following square of natural transformations commutes:

$$\begin{tikzcd} L\gamma^* \circ L\beta^* \circ L\alpha^* \arrow[r, Rightarrow, "L\gamma^*\star\langle\alpha,\beta\rangle"] \arrow[d, Rightarrow, "\langle\beta,\gamma\rangle\star L\alpha^*"'] & L\gamma^* \circ L(\alpha\beta)^* \arrow[d, Rightarrow, "\langle\alpha\beta,\gamma\rangle"] \\ L(\beta\gamma) \circ L\alpha^* \arrow[r, Rightarrow, "\langle\alpha,\beta\gamma\rangle"'] & L(\alpha\beta\gamma)^* \end{tikzcd}$$

Indeed, this is an instance of the general coherence for total left derived functors that we spell out in the following Proposition 1.4.28, applied to the left derivable functors $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$, $\beta^* : \mathrm{Sp}_K \longrightarrow \mathrm{Sp}_J$, and $\gamma^* : \mathrm{Sp}_J \longrightarrow \mathrm{Sp}_M$.

PROPOSITION 1.4.26. *For all composable continuous homomorphisms $\alpha : K \longrightarrow G$ and $\beta : J \longrightarrow K$ between Lie groups, the natural transformation $\langle\alpha, \beta\rangle : L\beta^* \circ L\alpha^* \Longrightarrow L(\alpha\beta)^*$ is exact and a natural isomorphism.*

PROOF. Exactness of the transformation $\langle\alpha, \beta\rangle$ means that the following diagram of natural transformations commutes:

$$\begin{tikzcd} L\beta^* \circ L\alpha^* \circ [1] \arrow[r, Rightarrow, "L\beta^*\star\sigma^\alpha"] \arrow[d, Rightarrow, "\langle\alpha,\beta\rangle\star[1]"'] & L\beta^* \circ [1] \circ L\alpha^* \arrow[r, Rightarrow, "\sigma^\beta\star L\alpha^*"] & {[1]} \circ L\beta^* \circ L\alpha^* \arrow[d, Rightarrow, "{[1]}\star\langle\alpha,\beta\rangle"] \\ L(\alpha\beta)^* \circ [1] \arrow[rr, Rightarrow, "\sigma^{\alpha\beta}"'] & & {[1]} \circ L(\alpha\beta)^* \end{tikzcd}$$

Since suspension is fully homotopical, the pair $([1] \circ L(\alpha\beta)^*, [1] \star (\alpha\beta)_!)$ is a total left derived functor of $(- \wedge S^1) \circ (\alpha\beta)^*$, by Proposition 1.4.13. The universal property allows us to check the commutativity of the diagram after precomposition with the localization functor $\gamma_G : \mathrm{Sp}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ and postcomposition with the natural transformation $[1]\star(\alpha\beta)_!$. This is a straightforward, but somewhat lengthy, calculation with the various defining properties. We start with the observation:

$$\begin{aligned} ([1] \star \beta_! \star \alpha^*) \circ ([1] \star L\beta^* \star \alpha_!) &= [1] \star ((\beta_! \star \alpha^*) \circ (L\beta^* \star \alpha_!)) \\ \overset{(1.4.25)}{=} & \ [1] \star ((\alpha\beta)_! \circ (\langle\alpha, \beta\rangle \star \gamma_G)) \\ &= ([1] \star (\alpha\beta)_!) \circ ([1] \star \langle\alpha, \beta\rangle \star \gamma_G) \end{aligned} \tag{1.4.27}$$

The final relation is then obtained as follows:

$$\begin{aligned}
&([1] \star (\alpha\beta)_!) \circ \big((\sigma^{\alpha\beta} \circ (\langle \alpha, \beta\rangle \star [1])) \star \gamma_G\big) \\
&\qquad = \ ([1] \star (\alpha\beta)_!) \circ (\sigma^{\alpha\beta} \star \gamma_G) \circ (\langle \alpha, \beta\rangle \star [1] \star \gamma_G) \\
&\qquad \overset{(1.4.18)}{=} \ ((\alpha\beta)_! \star (- \wedge S^1)) \circ (\langle \alpha, \beta\rangle \star \gamma_G \star (- \wedge S^1)) \\
&\qquad = \ ((\alpha\beta)_! \circ (\langle \alpha, \beta\rangle \star \gamma_G)) \star (- \wedge S^1) \\
&\qquad \overset{(1.4.25)}{=} \ ((\beta_! \star \alpha^*) \circ (L\beta^* \star \alpha_!)) \star (- \wedge S^1) \\
&\qquad = \ (\beta_! \star \alpha^* \star (- \wedge S^1)) \circ (L\beta^* \star \alpha_! \star (- \wedge S^1)) \\
&\qquad = \ (\beta_! \star (- \wedge S^1) \star \alpha^*) \circ (L\beta^* \star \alpha_! \star (- \wedge S^1)) \\
&\qquad \overset{(1.4.18)}{=} \ ((([1] \star \beta_!) \circ (\sigma^\beta \star \gamma_K)) \star \alpha^*) \circ (L\beta^* \star (([1] \star \alpha_!) \circ (\sigma^\alpha \star \gamma_G))) \\
&\qquad = \ ([1] \star \beta_! \star \alpha^*) \circ (\sigma^\beta \star \gamma_K \star \alpha^*) \circ (L\beta^* \star [1] \star \alpha_!) \circ (L\beta^* \star \sigma^\alpha \star \gamma_G) \\
&\qquad \overset{(1.4.9)}{=} \ ([1] \star \beta_! \star \alpha^*) \circ ([1] \star L\beta^* \star \alpha_!) \circ (\sigma^\beta \star L\alpha^* \star \gamma_G) \circ (L\beta^* \star \sigma^\alpha \star \gamma_G) \\
&\qquad \overset{(1.4.27)}{=} \ ([1] \star (\alpha\beta)_!) \circ ([1] \star \langle \alpha, \beta\rangle \star \gamma_G) \circ (\sigma^\beta \star L\alpha^* \star \gamma_G) \circ (L\beta^* \star \sigma^\alpha \star \gamma_G) \\
&\qquad = \ ([1] \star (\alpha\beta)_!) \circ \big((([1] \star \langle \alpha, \beta\rangle) \circ (\sigma^\beta \star L\alpha^*) \circ (L\beta^* \star \sigma^\alpha)) \star \gamma_G\big)
\end{aligned}$$

To prove that $\langle \alpha, \beta\rangle$ is an isomorphism, we consider a quasi-cofibrant orthogonal G-spectrum X and contemplate the following commutative diagram in $\mathrm{Ho}(\mathrm{Sp}_L)$:

$$\begin{tikzcd}
(L\beta^*)((L\alpha^*)(X)) \arrow[r, "(L\beta^*)(\alpha_!^X)", "\cong"'] \arrow[d, "\langle \alpha,\beta\rangle^X"'] & (L\beta^*)(\alpha^*(X)) \arrow[r, equal] & (L\beta^*)(\alpha^*(X)) \arrow[d, "\cong"', "\beta_!^{\alpha^*(X)}"] \\
L(\alpha\beta)^*(X) \arrow[r, "\cong", "(\alpha\beta)_!^X"'] & (\alpha\beta)^*(X) \arrow[r, equal] & \beta^*(\alpha^*(X))
\end{tikzcd}$$

The orthogonal K-spectrum $\alpha^*(X)$ is quasi-cofibrant by Theorem 1.4.17 (i). Since X and $\alpha^*(X)$ are quasi-cofibrant, the morphisms $\alpha_!^X$, $(\alpha\beta)_!^X$ and $\beta_!^{\alpha^*(X)}$ are isomorphisms by Theorem 1.4.17 (ii). So the morphism $\langle \alpha, \beta\rangle^X$ is also an isomorphism. Every orthogonal G-spectrum is isomorphic in $\mathrm{Ho}(\mathrm{Sp}_G)$ to a quasi-cofibrant spectrum, so this proves that $\langle \alpha, \beta\rangle$ is a natural isomorphism. $\square$

The following proposition records the coherence property that the natural transformations $\langle G, F\rangle$ enjoy; the condition is essentially saying that these transformations make the assignment $F \mapsto LF$ into a lax functor from the 'category' of relative categories and left derivable functors to the 2-category of categories. The caveat is that the composite of left derivable functors need not be left derivable, so we don't really have a category of these. As we indicate in Remark 1.4.29 below, this implies that all further coherence conditions between tuples of composable derivable functors are automatically satisfied.

Proposition 1.4.28. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{E}$ and $H : \mathcal{E} \longrightarrow \mathcal{F}$ be composable functors between relative categories such that F, G, H and GF, HG and HGF are left derivable. Then*

$$\langle HG, F\rangle \circ (\langle H, G\rangle \star LF) \ = \ \langle H, GF\rangle \circ (LH \star \langle G, F\rangle)$$

as natural transformations $LH \circ LG \circ LF \Longrightarrow L(HGF)$.

Proof. The various defining relations and the interchange relation provide the following equalities of natural transformations between functors $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{F})$:

$$\begin{aligned}
&\tau_{HGF} \circ (((\langle HG, F\rangle \circ (\langle H, G\rangle \star LF)) \star \gamma_{\mathcal{C}}) \\
&\qquad = \tau_{HGF} \circ (\langle HG, F\rangle \star \gamma_{\mathcal{C}}) \circ (\langle H, G\rangle \star (LF \circ \gamma_{\mathcal{C}})) \\
&(1.4.23) = (\tau_{HG} \star F) \circ (L(HG) \star \tau_F) \circ (\langle H, G\rangle \star (LF \circ \gamma_{\mathcal{C}})) \\
&(1.4.9) = (\tau_{HG} \star F) \circ (\langle H, G\rangle \star (\gamma_{\mathcal{D}} \circ F)) \circ ((LH \circ LG) \star \tau_F) \\
&\qquad = (\tau_{HG} \star F) \circ (((\langle H, G\rangle \star \gamma_{\mathcal{D}}) \star F) \circ ((LH \circ LG) \star \tau_F) \\
&\qquad = ((\tau_{HG} \circ (\langle H, G\rangle \star \gamma_{\mathcal{D}})) \star F) \circ ((LH \circ LG) \star \tau_F) \\
&(1.4.23) = (((\tau_H \star G) \circ (LH \star \tau_G)) \star F) \circ ((LH \circ LG) \star \tau_F) \\
&\qquad = (\tau_H \star (GF)) \circ (LH \star \tau_G \star F) \circ ((LH \circ LG) \star \tau_F) \\
&\qquad = (\tau_H \star (GF)) \circ (LH \star ((\tau_G \star F) \circ (LG \star \tau_F))) \\
&(1.4.23) = (\tau_H \star (GF)) \circ (LH \star (\tau_{GF} \circ (\langle G, F\rangle \star \gamma_{\mathcal{C}}))) \\
&\qquad = (\tau_H \star (GF)) \circ (LH \star \tau_{GF}) \circ (LH \star \langle G, F\rangle \star \gamma_{\mathcal{C}}) \\
&(1.4.23) = \tau_{HGF} \circ (\langle H, GF\rangle \star \gamma_{\mathcal{C}}) \circ (LH \star \langle G, F\rangle \star \gamma_{\mathcal{C}}) \\
&\qquad = \tau_{HGF} \circ (((\langle H, GF\rangle \circ (LH \star \langle G, F\rangle)) \star \gamma_{\mathcal{C}}) \ .
\end{aligned}$$

The uniqueness clause in the universal property of the pair $(L(HGF), \tau_{HGF})$ then implies the desired relation. □

Remark 1.4.29. Proposition 1.4.28 implies that all coherence relations with respect to iterated composition of derivable functors are automatically satisfied. Given n composable, left derivable functors $F_1, \dots, F_n$, for $n \geq 3$, we define a natural transformation

$$\langle F_n, \dots, F_1\rangle \ : \ LF_n \circ \dots \circ LF_1 \implies L(F_n \circ \dots \circ F_1)$$

inductively by setting

$$\langle F_n, F_{n-1}, \dots, F_1\rangle \ = \ \langle F_n F_{n-1}, F_{n-2}, \dots, F_1\rangle \circ (\langle F_n, F_{n-1}\rangle \star (LF_{n-1} \circ \dots \circ LF_1)) \ ,$$

assuming that all iterated composites of adjacent functors are left derivable. Proposition 1.4.28 implies that we could have instead 'spliced' at any other intermediate spot in the composition; more generally, we could have 'collected adjacent factors' in any way we like, and get the same result. More precisely, if for $i \leq j$ we write

$$F_{[j,i]} \ = \ F_j \circ F_{j-1} \circ \dots \circ F_i \ ,$$

and we set $\langle F_k\rangle = \mathrm{Id}_{LF_k}$, then for all sequences of numbers $1 \leq m_1 < m_2 < \dots < m_k < n$, we have

$$\begin{aligned}
&\langle F_n, \dots, F_1\rangle \ = \\
&\quad \langle F_{[n,m_k+1]}, \dots, F_{[m_2,m_1+1]}, F_{[m_1,1]}\rangle \circ (\langle F_n, \dots, F_{m_k+1}\rangle \star \dots \star \langle F_{m_1}, \dots, F_1\rangle) \ .
\end{aligned}$$

The next Theorem 1.4.31 is a homotopy invariance statement for proper equivariant stable homotopy theory; it says, roughly speaking, that this homotopy theory only depends on the Lie group 'up to multiplicative weak equivalence'. In the context of *compact* Lie groups, this statement does not have much content: as we recall in the next proposition, every multiplicative weak equivalence between compact Lie groups is already an isomorphism.

PROPOSITION 1.4.30. *Let $\alpha : K \longrightarrow H$ be a continuous homomorphism between compact Lie groups. If α is a weak homotopy equivalence of underlying spaces, then α is a diffeomorphism, and hence an isomorphism of Lie groups.*

PROOF. This result should be well-known, but we have not found a reference. We owe this proof to George Raptis. Since α is a weak homotopy equivalence, it induces an isomorphism $\pi_0(\alpha) : \pi_0(K) \longrightarrow \pi_0(H)$ of component groups, and a weak homotopy equivalence $\alpha^\circ : K^\circ \longrightarrow H^\circ$ on the connected components of the identity elements. So by restriction to identity components, it suffices to treat the special case where K and H are path connected.

Since α is a weak homotopy equivalence, it induces an isomorphism on mod-2 homology. Since K and H are closed connected manifolds, their geometric dimension can be recovered as the largest dimension in which the mod-2 homology is non-trivial. So K and H have the same dimension.

Since α induces an isomorphism on the top dimensional mod-2 homology groups, it must thus be surjective. A continuous homomorphism between Lie groups is automatically smooth. Since K and H have the same dimension, the kernel of α is finite, and so α is a covering space projection. Since α induces an isomorphism of fundamental groups, this covering space projection must be a homeomorphism. Hence α is an isomorphism of Lie groups. □

When we drop the compactness hypothesis, Proposition 1.4.30 ceases to hold. For example, for every Lie group G, the projection $G \times \mathbb{R} \longrightarrow G$ is a continuous homomorphism and weak equivalence.

THEOREM 1.4.31. *Let $\alpha : K \longrightarrow G$ be a continuous homomorphism between Lie groups that is also a weak equivalence of underlying spaces. Then the following hold:*

(i) *The homomorphism α is quasi-injective.*
(ii) *For every compact subgroup H of G there is a compact subgroup L of K such that H is conjugate to $\alpha(L)$.*
(iii) *The restriction functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ is an equivalence of triangulated categories.*

PROOF. We start with the special case when the groups K and G are almost connected. We choose a maximal compact subgroup M of K and a maximal compact subgroup N of G that contains the compact group $\alpha(M)$. In the commutative diagram of continuous group homomorphisms

$$\begin{tikzcd} M \arrow[r, "\alpha|_M"] \arrow[d] & N \arrow[d] \\ K \arrow[r, "\alpha"'] & G \end{tikzcd}$$

the two vertical inclusions are then weak equivalence of underlying spaces, compare [**1**, Thm. A.5]. Since α is also a weak equivalence, so is $\alpha|_M : M \longrightarrow N$. But M and N are compact, so $\alpha|_M : M \longrightarrow N$ is an isomorphism by Proposition 1.4.30. In particular, α is injective on M; since every compact subgroup of K is contained in a maximal compact subgroup, α is quasi-injective. This proves claim (i) in the special case. Moreover, if H is a compact subgroup of G, then there is an element

$g \in G$ such that ${}^gH \subset N$. Then $L = \alpha^{-1}({}^gH) \cap M$ is a compact subgroup of K such that $\alpha(L) = {}^gH$. This proves claim (ii) in the special case.

Now we contemplate the square of triangulated categories and exact functors

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{Sp}_M) & \mathrm{Ho}(\mathrm{Sp}_N) \arrow[l, "L(\alpha|_M^*)"'] \\ \mathrm{Ho}(\mathrm{Sp}_K) \arrow[u, "\mathrm{res}^K_M"] & \mathrm{Ho}(\mathrm{Sp}_G) \arrow[l, "L\alpha^*"] \arrow[u, "\mathrm{res}^G_N"'] \end{tikzcd}$$

The two vertical functors are equivalences by Theorem 1.4.4, and the upper functor is an equivalence because $\alpha|_M$ is an isomorphism. Since the square commutes up to natural isomorphism, the functor $L\alpha^*$ is also an equivalence. This proves claim (iii) in the special case.

Now we treat the general case, i.e., K and G are allowed to have infinitely many components. For a compact subgroup L of K, we write

$$\langle L \rangle \ = \ L \cdot K^\circ$$

for the subgroup generated by L and the connected component K° of the identity. Another way to say this is that $\langle L \rangle$ is the union of all path components of K that have a non-empty intersection with L. By construction, the inclusion $L \longrightarrow \langle L \rangle$ induces a surjection $\pi_0(L) \longrightarrow \pi_0(\langle L \rangle)$ on path components. In particular, the component group of $\langle L \rangle$ is finite because L is compact. Similarly, the group $\langle \alpha(L) \rangle = \alpha(L) \cdot G^\circ$ is a closed almost connected subgroup of G. Then $\langle L \rangle$ is a union of finitely many of the path components of K, and $\langle \alpha(L) \rangle$ is the union of the corresponding path components of G; the restriction $\bar{\alpha} : \langle L \rangle \longrightarrow \langle \alpha(L) \rangle$ of the homomorphism α is thus again a weak equivalence of underlying spaces. Property (i) for the restriction $\bar{\alpha} : \langle L \rangle \longrightarrow \langle \alpha(L) \rangle$ shows that α is injective on L. Since L was an arbitrary compact subgroup of K, the homomorphism α is quasi-injective. This proves claim (i) in the general case.

Now we let H be a compact subgroup of G, and we consider the almost connected subgroup $\langle H \rangle = H \cdot G^\circ$ of G. Because $\pi_0(\alpha) : \pi_0(K) \longrightarrow \pi_0(G)$ is an isomorphism, $\alpha^{-1}(\langle H \rangle) = \langle \alpha^{-1}(H) \rangle = \alpha^{-1}(H) \cdot K^\circ$ is almost connected, and the restriction $\bar{\alpha} : \alpha^{-1}(\langle H \rangle) \longrightarrow \langle H \rangle$ of α is another weak equivalence of underlying spaces. Property (ii) for the restriction $\bar{\alpha}$ provides a compact subgroup L of $\alpha^{-1}(\langle H \rangle)$ and an element $g \in \langle H \rangle \subset G$ such that $\alpha(L) = {}^gH$. This proves claim (ii) in the general case.

It remains to show that the functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ is an equivalence. In a first step we show that it detects isomorphisms. Since $L\alpha^*$ is an exact functor of triangulated categories, it suffices to show for every G-spectrum X such that $L\alpha^*(X)$ is a zero object in $\mathrm{Ho}(\mathrm{Sp}_K)$, already $X = 0$. To see this we let H be any compact subgroup of G. Part (ii) provides a compact subgroup L of K and an element $g \in G$ such that $\alpha(L) = {}^gH$. Since α is quasi-injective by part (i), it restricts to an isomorphism $\alpha|_L : L \cong {}^gH$. Since $L\alpha^*(X)$ is a zero object, we conclude that

$$\pi_*^H(X) \ \xrightarrow[\cong]{c_g^*} \pi_*^{{}^gH}(X) \ \xrightarrow[\cong]{L(\alpha|_L)^*} \ \pi_*^L(L(\alpha|_L)^*(\mathrm{res}^G_{{}^gH}(X))) \ \cong \ \pi_*^L(L\alpha^*(X)) \ = \ 0 \ .$$

Since H was an arbitrary compact subgroup of G, this proves that X is a zero object in $\mathrm{Ho}(\mathrm{Sp}_G)$. We have thus shown that $L\alpha^*$ detects isomorphisms.

The homomorphism α is quasi-injective by part (i), so the derived functor $L\alpha^*$ has a left adjoint $G \ltimes^L_\alpha - : \mathrm{Ho}(\mathrm{Sp}_K) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ by Theorem 1.4.17 (vi), which is a total left derived functor of $G \ltimes_\alpha - : \mathrm{Sp}_K \longrightarrow \mathrm{Sp}_G$. We let $\mathcal{Y}$ denote the class of orthogonal K-spectra Y such that the adjunction unit

$$\eta_Y \;:\; Y \;\longrightarrow\; L\alpha^*(G \ltimes^L_\alpha Y)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$. The adjunction unit is a natural transformation between two exact functors that preserve arbitrary sums, so the class $\mathcal{Y}$ is a localizing subcategory of $\mathrm{Ho}(\mathrm{Sp}_K)$.

Now we consider any almost connected closed subgroup $\bar{K}$ of K with $(\bar{K})^\circ = K^\circ$; in other words, $\bar{K}$ is a finite union of path components of K. We show that for every orthogonal $\bar{K}$-spectrum Z, the induced spectrum $K \ltimes_{\bar{K}} Z$ belongs to the class $\mathcal{Y}$. We let $\bar{G} = \langle \alpha(\bar{K}) \rangle = \alpha(\bar{K}) \cdot G^\circ$ be the union of those path components of G that correspond to $\bar{K}$ under the isomorphism $\pi_0(\alpha) : \pi_0(K) \longrightarrow \pi_0(G)$. Then $\bar{\alpha} = \alpha|_{\bar{K}} : \bar{K} \longrightarrow \bar{G}$ is a continuous homomorphism and weak equivalence between almost connected Lie groups. So property (iii) for the homomorphism $\bar{\alpha}$ shows that the adjunction unit

$$\eta_Z \;:\; Z \;\longrightarrow\; L\bar{\alpha}^*(\bar{G} \ltimes^L_{\bar{\alpha}} Z)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_{\bar{K}})$. Hence the left vertical morphism in the following commutative square is an isomorphism:

$$\begin{tikzcd}
K \ltimes_{\bar{K}} Z \arrow[r, "\eta_{K \ltimes_{\bar{K}} Z}"] \arrow[d, "K \ltimes_{\bar{K}} \eta_Z"', "\cong"] & L\alpha^*(G \ltimes^L_\alpha (K \ltimes_{\bar{K}} Z)) \arrow[d, "L\alpha^*(\mu)", "\cong"'] \\
K \ltimes_{\bar{K}} (L\bar{\alpha}^*(\bar{G} \ltimes^L_{\bar{\alpha}} Z)) \arrow[r, "\cong", "\mathrm{Ho}(\lambda)(\bar{G} \ltimes^L_{\bar{\alpha}} Z)"'] & L\alpha^*(G \ltimes_{\bar{G}} (\bar{G} \ltimes^L_{\bar{\alpha}} Z))
\end{tikzcd} \tag{1.4.32}$$

Here $\mu : G \ltimes^L_\alpha (K \ltimes_{\bar{K}} Z) \longrightarrow G \ltimes_{\bar{G}} (\bar{G} \ltimes^L_{\bar{\alpha}} Z)$ is the mate (adjoint) of the isomorphism

$$L\bar{\alpha}^* \circ \mathrm{res}^G_{\bar{G}} \xrightarrow[\cong]{\langle \mathrm{incl}^G_{\bar{G}}, \bar{\alpha} \rangle} L(\alpha|_{\bar{K}})^* \xleftarrow[\cong]{\langle \alpha, \mathrm{incl}^K_{\bar{K}} \rangle} \mathrm{res}^K_{\bar{K}} \circ L\alpha^* \ .$$

Hence μ and $L\alpha^*(\mu)$ are isomorphisms. The lower horizontal isomorphism needs some explanation. On the pointset level, we can define a natural isomorphism of orthogonal K-spectra

$$\lambda \;:\; K \ltimes_{\bar{K}} \bar{\alpha}^*(W) \xrightarrow{\cong} \alpha^*(G \ltimes_{\bar{G}} W) \quad \text{by} \quad \lambda(k \wedge w) \;=\; \alpha(k) \wedge w \ ,$$

where W is any orthogonal $\bar{G}$-spectrum. To see that λ is indeed an isomorphism, we observe that the underlying non-equivariant spectrum of $K \ltimes_{\bar{K}} \alpha^*(W)$ is a wedge of copies of W indexed by $K/\bar{K}$, that $\alpha^*(G \ltimes_{\bar{G}} W)$ is a wedge of copies of W indexed by $G/\bar{G}$, and that α induces a bijection of sets $K/\bar{K} \cong G/\bar{G}$ because $\pi_0(\alpha) : \pi_0(K) \longrightarrow \pi_0(G)$ is a group isomorphism. The induction functors $K \ltimes_{\bar{K}} -$ and $G \ltimes_{\bar{G}} -$ are fully homotopical by Corollary 1.2.10; the restriction functors α^* and $\bar{\alpha}^*$ are fully homotopical by Theorem 1.4.1 (i) because α is quasi-injective by part (i). So λ is a natural isomorphism between homotopical functors, and hence it descends to a natural isomorphism

$$\mathrm{Ho}(\lambda) \;:\; (K \ltimes_{\bar{K}} -) \circ L\bar{\alpha}^* \;\cong\; L\alpha^* \circ (G \ltimes_{\bar{G}} -)$$

on the level of homotopy categories. Now we can wrap up: since the other three morphisms in the commutative square (1.4.32) are isomorphisms, so is the adjunction unit $\eta_{K\ltimes_{\bar{K}} Z}$. In other words, all K-spectra that are induced from subgroups of the form $\bar{K}$ are in the class $\mathcal{Y}$.

Now we let L be any compact subgroup of K. Then $\bar{K} = \langle L\rangle = L\cdot K^\circ$ is an almost connected closed subgroup of K of the kind considered in the previous paragraph. Since the suspension spectrum $\Sigma^\infty_+ K/L$ is isomorphic to $K\ltimes_{\bar{K}}(\Sigma^\infty_+\bar{K}/L)$, it is contained in the class $\mathcal{Y}$. So the class $\mathcal{Y}$ contains all the preferred compact generators of the triangulated category $\mathrm{Ho}(\mathrm{Sp}_K)$. Since $\mathcal{Y}$ is a localizing subcategory of $\mathrm{Ho}(\mathrm{Sp}_K)$, it contains all orthogonal K-spectra. In other words, the adjunction unit $\eta_Y : Y\longrightarrow L\alpha^*(G\ltimes^L_\alpha Y)$ is an isomorphism in complete generality.

Since the adjunction unit of the adjoint pair $(G\ltimes^L_\alpha -, L\alpha^*)$ is a natural isomorphism and the right adjoint functor $L\alpha^*$ detects isomorphisms, we finally conclude that the adjunction is an adjoint equivalence of categories. This proves property (iii) in general. □

Now we discuss another aspect of the homotopy invariance of proper equivariant stable homotopy theory, namely homotopy invariance of derived restriction functors: we will show that a homotopy through continuous homomorphisms provides an isomorphism of derived restriction functors. In the realm of *compact* Lie groups, the homotopy invariance is a direct consequence of the conjugation invariance of Example 1.4.21: a celebrated theorem of Montgomery and Zippin [**52**, Thm. 1 and Corollary] says that in a Lie group 'nearby compact subgroups are conjugate', and this implies that two homotopic continuous homomorphisms from a compact Lie group to a Lie group are already conjugate, compare [**12**, III, Lemma 38.1]. If we drop the compactness hypothesis, this statement need not hold anymore: the identity of the additive Lie group $\mathbb{R}$ is homotopic, through continuous homomorphisms, to the zero homomorphism. So the homotopy invariance of derived restriction functors requires an additional argument.

Construction 1.4.33 (Homotopy invariance of derived restriction). We let $\alpha,\beta : K\longrightarrow G$ be two continuous homomorphisms between Lie groups. We let

$$\omega \ :\ K\times[0,1]\ \longrightarrow\ G$$

be a homotopy from α to β through continuous homomorphisms, i.e., such that $\omega(-,t) : K\longrightarrow G$ is a homomorphism for every $t\in[0,1]$. We define a functor

$$\omega^*\ :\ \mathrm{Sp}_G\ \longrightarrow\ \mathrm{Sp}_K \qquad\text{by}\qquad \omega^*(X)\ =\ X\wedge[0,1]_+$$

equipped with K-action by

$$k\cdot(x\wedge t)\ =\ (\omega(k,t)\cdot x)\wedge t\ .$$

By hypothesis we have $\omega(k,0)=\alpha(k)$ and $\omega(k,1)=\beta(k)$, so the two assignments

$$a\ :\ \alpha^*(X)\ \longrightarrow\ \omega^*(X)\ ,\quad a(x)\ =\ x\wedge 0\quad\text{and}$$
$$b\ :\ \beta^*(X)\ \longrightarrow\ \omega^*(X)\ ,\quad b(x)\ =\ x\wedge 1$$

define natural morphisms of orthogonal K-spectra.

Theorem 1.4.34. *Let $\omega : K\times[0,1]\longrightarrow G$ be a homotopy of continuous homomorphisms from $\alpha=\omega(-,0)$ to $\beta=\omega(-,1)$. Then for every orthogonal G-spectrum X, the morphisms*

$$a\ :\ \alpha^*(X)\ \longrightarrow\ \omega^*(X)\qquad\textit{and}\qquad b\ :\ \beta^*(X)\ \longrightarrow\ \omega^*(X)$$

are π_*-isomorphisms of orthogonal K-spectra. Hence a and b induce natural isomorphisms of total left derived functors

$$L\alpha^* \xrightarrow[\cong]{La} L\omega^* \xleftarrow[\cong]{Lb} L\beta^* .$$

Proof. We show that $a : \alpha^*(X) \longrightarrow \omega^*(X)$ is a π_*-isomorphism; the argument for b is analogous. We let L be any compact subgroup of K. We let $\Gamma = \alpha(L) \cdot G^\circ$ be the union of those path components of G that are in the image of $\pi_0(\alpha|_L) : \pi_0(L) \longrightarrow \pi_0(G)$. Then $\alpha|_L$ has image in Γ, by construction, and $\pi_0(\Gamma)$ is finite. By continuity, each of the homomorphisms $\omega(-,t) : K \longrightarrow G$ also takes L to Γ. We can thus restrict ω to a path

$$\bar{\omega} \ = \ \omega|_{L\times[0,1]} \ : \ L \times [0,1] \ \longrightarrow \ \Gamma$$

of continuous homomorphisms from $\alpha|_L$ to $\beta|_L$.

We let $\hom(L,\Gamma)$ denote the space of continuous homomorphisms with the subspace topology of the compact-open topology (which coincides with the function space topology in the category $\mathbf{T}$). We let Γ° denote the identity path component of the group Γ; then Γ° is also the identity path component of G. Since L is compact, the image of the continuous map

$$\Gamma^\circ \ \longrightarrow \ \hom(L,\Gamma) \ , \quad g \ \longmapsto \ c_g \circ \alpha|_L$$

is the entire path component $\hom(L,\Gamma;\alpha)$ of $\alpha|_L$, see [**60**, Prop. A.25]. So the map factors over a continuous bijection

$$\Gamma^\circ/C \ \cong \ \hom(L,\Gamma;\alpha) \ ,$$

where $C = \Gamma^\circ \cap C_\Gamma(\alpha(L))$ is the centralizer in Γ of $\alpha(L)$, intersected with Γ°. Moreover, this map is a homeomorphism, for example by [**44**, Thm. B.2]. The projection $\Gamma^\circ \longrightarrow \Gamma^\circ/C$ is a locally trivial fiber bundle, see for example [**8**, I Thm. 4.3]. So the path

$$\omega|_L \ : \ [0,1] \ \longrightarrow \ \hom(L,\Gamma;\alpha)$$

admits a continuous lift

$$\lambda \ : \ [0,1] \ \longrightarrow \ \Gamma^\circ$$

such that $\lambda(0) = 1$ and $\omega|_L(t) = c_{\lambda(t)} \circ \alpha|_L$. The map

$$\omega^*(X) \ \longrightarrow \ (\alpha|_L)^*(X) \wedge [0,1]_+ \ , \quad x \wedge t \ \longmapsto \ \lambda(t) \cdot x \wedge t$$

is then an L-equivariant isomorphism of orthogonal L-spectra. Moreover, under this isomorphism, the restriction to L of the morphism $a : \alpha^*(X) \longrightarrow \omega^*(X)$ becomes the morphism $- \wedge 0 : (\alpha|_L)^*(X) \longrightarrow (\alpha|_L)^*(X) \wedge [0,1]_+$. This proves that a is an L-equivariant homotopy equivalence. Since L was an arbitrary compact subgroup of K, we have altogether shown that the morphism a is a π_*-isomorphism of orthogonal K-spectra. $\square$

Remark 1.4.35. The left derived functor $L\alpha^* : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ associated to a continuous homomorphism between Lie groups is also compatible with derived smash products, i.e., it can be given a preferred strong symmetric monoidal structure. The essential ingredient for this is the fact that the pointset level restriction functor α^* preserves quasi-cofibrant spectra and π_*-isomorphisms between quasi-cofibrant spectra, and that quasi-cofibrant spectra are quasi-flat, i.e., the smash product is fully homotopical on quasi-cofibrant spectra. We will not go into more details about multiplicative aspects of $L\alpha^*$.

CHAPTER 2

Equivariant homotopy groups

2.1. G-equivariant homotopy groups

For a *compact* Lie group H, the H-equivariant homotopy group $\pi_0^H(X)$ is defined as a colimit over all finite-dimensional H-subrepresentations V of a complete H-universe, of the sets $[S^V, X(V)]^H$. In this section we propose a generalization of these equivariant homotopy groups to arbitrary Lie groups, not necessarily compact, as the morphisms from the G-sphere spectrum $\mathbb{S}_G$ in the triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, see Definition 2.1.1. In contrast to the classical case of *compact* Lie groups, in our context the G-equivariant homotopy groups need not send wedges to direct sums. Equivalently, the G-sphere spectrum need not be a small object in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$. The question of whether or not $\mathbb{S}_G$ is small turns out to be related to finiteness properties of G. As an example we show in Proposition 2.1.4 that $\mathbb{S}_G$ is small if the group admits a finite G-CW-model for the universal G-space $\underline{E}G$ for proper actions, i.e., one with only finitely many equivariant cells (or equivalently, with compact orbit space). Example 2.1.5 illustrates that for this particular purpose, the existence of a *finite-dimensional* model for $\underline{E}G$ is not sufficient. A precise characterization of when $\mathbb{S}_G$ is a small object of $\mathrm{Ho}(\mathrm{Sp}_G)$ appears in the work of Bárcenas and the first and fourth author, see [**4**, Thm 5.1 and Thm. 5.4].

We also discuss the natural structure that relates the G-equivariant stable homotopy groups, specifically restriction homomorphisms associated with continuous homomorphisms between Lie groups (see Construction 2.1.7) and transfer maps for finite index inclusions (see Construction 2.1.12). Transfer maps are closely related to the Wirthmüller isomorphism, which we formulate and prove for finite index inclusions in Theorem 2.1.10. In the last part of this section we verify that the standard relations between restriction homomorphisms and transfer maps generalize from the classical context of compact Lie groups to our more general situation.

We refer to Remark 1.3.5 for our convention about the shifts $X[k]$ of an orthogonal G-spectrum, for $k \in \mathbb{Z}$.

Definition 2.1.1. Let G be a Lie group, X an orthogonal G-spectrum and $k \in \mathbb{Z}$. The k-th *G-homotopy group* is defined as

$$\pi_k^G(X) \ = \ [\mathbb{S}_G, X[-k]]^G$$

the group of morphisms, in the stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, from the G-sphere spectrum to the $(-k)$-fold shift of X.

If G is compact, then the new definition of π_*^G agrees with the old one, up to a specific natural isomorphism. Indeed, for compact G, the representability result of Proposition 1.3.10 provides a natural isomorphism

$$[\mathbb{S}_G, X[0]] \cong [\Sigma^\infty_+ G/G, X]^G \cong \pi_0^G(X) , \quad [f] \longmapsto f_*(1) .$$

Here we identified $\Sigma^\infty_+ G/G$ with $\mathbb{S}_G$, so that the tautological class $u_G \in \pi_0^G(\Sigma^\infty_+ G/G)$ becomes the class $1 \in \pi_0^G(\mathbb{S}_G)$ represented by the identity of $S^0 = (\mathbb{S}_G)(0)$. For $k > 0$, we combine this with the iterated loop isomorphism (1.2.6) into a natural isomorphism

$$[\mathbb{S}_G, X[-k]] = [\mathbb{S}_G, \Omega^k X]^G \cong \pi_0^G(\Omega^k X) \cong \pi_k^G(X) .$$

For $k < 0$, we instead use the iterated suspension isomorphism (1.2.7) and obtain a natural isomorphism

$$[\mathbb{S}_G, X[-k]] = [\mathbb{S}_G, X \wedge S^{-k}]^G \cong \pi_0^G(X \wedge S^{-k}) \cong \pi_k^G(X) .$$

If G is discrete and admits a finite G-CW-model for $\underline{E}G$, then the definition for compact groups generalizes, provided we replace 'G-representations' by 'G-vector bundles over $\underline{E}G$'. Indeed, Theorem 3.1.34 (iv) below provides an isomorphism

$$\pi_0^G(X) \cong X_G(\underline{E}G) \xrightarrow[\cong]{\mu^X_{\underline{E}G}} X_G[\![\underline{E}G]\!] ;$$

the right hand side is defined via fiberwise G-homotopy classes of G-maps $S^\xi \longrightarrow X(\xi)$, for G-vector bundles ξ over $\underline{E}G$, see Construction 3.1.19 below.

Remark 2.1.2. Let G be a discrete group. In Example 2.2.8 below we introduce the graded homotopy group G-Mackey functor $\underline{\pi}_*(X)$ of an orthogonal G-spectrum X. This algebraic object records the values of the H-equivariant homotopy groups for all finite subgroups H of G, and the natural structure between them. If the group G itself is infinite, then the groups $\pi_*^G(X)$ are *not* encoded in the G-Mackey functor $\underline{\pi}_*(X)$.

In this situation, there is an Atiyah-Hirzebruch type spectral sequence converging to $\pi_*^G(X)$ whose E_2-term is the Bredon cohomology of $\underline{E}G$ with coefficients in the homotopy group Mackey functors of X. Indeed, the G-equivariant homotopy group $\pi_{-n}^G(X)$ is canonically isomorphic to the group $X_G^n(\underline{E}G)$. The Atiyah-Hirzebruch spectral sequence (3.2.15), for $X = \underline{E}G$ and for the proper cohomology theory X_G^* represented by X, thus takes the form

$$E_2^{p,q} = H^p_G(\underline{E}G, \underline{\pi}_{-q}(X)) \Longrightarrow \pi^G_{-p-q}(X) . \tag{2.1.3}$$

If G has an n-dimensional model for $\underline{E}G$, or – more generally – an n-dimensional *stable* model for $\underline{E}G$, then the Mackey functor cohomological dimension of G is at most n, see for example [**4**, Thm. 1.2]. So in this case, the E_2-term of the Atiyah-Hirzebruch spectral sequence (2.1.3) vanishes for $p > n$, and the spectral sequence collapses at E_{n+1}.

The 0-th Bredon cohomology group of $\underline{E}G$ is the inverse limit over the $\mathcal{F}in$-orbit category $\mathrm{Or}_G^{\mathcal{F}in}$, so the edge homomorphism of the spectral sequence can be viewed as a homomorphism

$$\pi_k^G(X) \longrightarrow H^0_G(\underline{E}G, \underline{\pi}_k(X)) \cong \lim_{\mathrm{Or}_G^{\mathcal{F}in}} \underline{\pi}_k(X) .$$

A more detailed analysis would reveal that this edge homomorphism is given by the restriction maps

$$\mathrm{res}^G_H : \pi_k^G(X) = [\mathbb{S}_G, X[-k]]^G \longrightarrow [\mathbb{S}_H, X[-k]]^H \cong \pi_k^H(X)$$

for all finite subgroups H of G. So a compatible system $\{x_H\}_{G/H\in \mathrm{Or}_G^{\mathcal{F}in}}$ of homotopy classes in $\pi_k^H(X)$ is the restriction of some class in $\pi_k^G(X)$ if any only if the corresponding element of $H^0_G(\underline{E}G, \underline{\pi}_k(X))$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence (2.1.3).

As a special case we consider a countable discrete group G that is *locally finite*, i.e., every finitely generated subgroup of G is finite. Such groups have a 1-dimensional model for $\underline{E}G$. For example, if G is locally finite and countable, then it is the union of an ascending sequence of finite subgroups, and we described such a model in Example 1.4.5. A general locally finite group G is the filtered union of its finite subgroups. Group homology commutes with filtered unions, so the group homology and group cohomology of G with coefficients in any $\mathbb{Q}G$-module vanish in positive dimensions. A theorem of Dunwoody [**15**, Thm. 1.1] then provides a 1-dimensional model for $\underline{E}G$.

If G has a 1-dimensional $\underline{E}G$, then the spectral sequence (2.1.3) collapses at E_2 and specializes to a short exact sequence

$$0 \longrightarrow \lim^1_{\mathrm{Or}_G^{\mathcal{F}in}} \underline{\pi}_{n+1}(X) \longrightarrow \pi_n^G(X) \longrightarrow \lim_{\mathrm{Or}_G^{\mathcal{F}in}} \underline{\pi}_n(X) \longrightarrow 0 .$$

If G is locally finite and countable, then this short exact sequence is a special case of Corollary 1.4.6.

Essentially by definition, the G-equivariant homotopy groups take distinguished triangles to long exact sequences, and products to products. One should beware, though, that infinite products of orthogonal G-spectra are *not* generally products in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$; they are if all factors are G-Ω-spectra. However, in our more general context not all the 'usual' properties of equivariant homotopy groups carry over from compact to general Lie groups. For example, the functor π_*^G does *not* in general take infinite wedges to direct sums, because the G-sphere spectrum $\mathbb{S}_G$ need not be small in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$.

PROPOSITION 2.1.4. *Let G be a Lie group that has a model for $\underline{E}G$ that admits a finite G-CW-structure.*

(i) *Let X be an orthogonal G-spectrum such that the H-spectrum $\mathrm{res}^G_H(X)$ is a small object in $\mathrm{Ho}(\mathrm{Sp}_H)$ for every compact subgroup H of G. Then the G-spectrum X is a small object in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$.*
(ii) *The G-sphere spectrum $\mathbb{S}_G$ is a small object in the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$, and the functor $\pi_k^G : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathcal{A}b$ preserves all sums.*

PROOF. (i) The restriction functor $\mathrm{res}^G_H : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_H)$ preserves coproducts, so its left adjoint $G \ltimes_H -$ preserves compact objects. So if $\mathrm{res}^G_H(X)$ is small in $\mathrm{Ho}(\mathrm{Sp}_H)$, then $X \wedge G/H_+ \cong G \ltimes_H \mathrm{res}^G_H(X)$ is small in $\mathrm{Ho}(\mathrm{Sp}_G)$. The class of small objects in a triangulated category is closed under 2-out-of-3 in distinguished triangles. So induction over the number of equivariant cells shows that $X \wedge A_+$ is small for every finite proper G-CW-complex A. Since $\underline{E}G$ has a finite proper G-CW-model, $X \wedge \underline{E}G_+$ is small. But X is isomorphic in $\mathrm{Ho}(\mathrm{Sp}_G)$ to $X \wedge \underline{E}G_+$, so X itself is small.

(ii) The restriction functor $\mathrm{res}^G_H : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_H)$ takes the G-sphere spectrum to the H-sphere spectrum, which is small if H is compact. So the G-sphere spectrum is small in $\mathrm{Ho}(\mathrm{Sp}_G)$. □

As we just saw, a finite model for $\underline{E}G$ implies that the G-sphere spectrum is small in the triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$. The example below illustrates that this is not true in general, and that even a finite-dimensional model for $\underline{E}G$ does not imply smallness of $\mathbb{S}_G$. In fact, [**4**, Thm. 5.4] shows that for a countable discrete group G, the G-sphere spectrum is small if and only if there exists a finite-dimensional stable model for $\underline{E}G$, and there exists a finite-type stable model for $\underline{E}G$. Moreover, by [**4**, Thm. 5.1], a finite-type stable model for $\underline{E}G$ exists if and only if there are only finitely many conjugacy classes of finite subgroups in G, and every Weyl group $W_G(H) = N_G(H)/H$ of a finite subgroup H of G is of homological type FP_∞.

Example 2.1.5 (The G-sphere spectrum need not be small). We let F be any non-trivial finite group, and we define

$$G \;=\; {\prod}'_{k\geq 1} F \ ,$$

an infinite weak product of copies of F, i.e., the subgroup of the product consisting of tuples with almost all coordinates the neutral element. We will now show that $\mathbb{S}_G$ is not small in $\mathrm{Ho}(\mathrm{Sp}_G)$. Example 1.4.5 exhibits a 1-dimensional G-CW-model for $\underline{E}G$; Proposition 2.1.4 shows that there cannot be a finite G-CW-model for $\underline{E}G$.

We set $H_n = \prod_{k=1}^n F$. Then G is the ascending union of its finite subgroups H_n, and we can apply Corollary 1.4.6. The inclusion $H_{n-1} \longrightarrow H_n$ has a retraction $r : H_n \longrightarrow H_{n-1}$ by a group homomorphism. Now we let X be an orthogonal spectrum, which we give the trivial G-action. The inflation homomorphism $r^* : \pi_k^{H_{n-1}}(X) \longrightarrow \pi_k^{H_n}(X)$ is defined in (2.1.8) below; it is a section to the restriction $\mathrm{res}^{H_n}_{H_{n-1}} : \pi_k^{H_n}(X) \longrightarrow \pi_k^{H_{n-1}}(X)$. Since the restriction maps are surjective, the $\lim^1$ terms in the short exact sequence of Corollary 1.4.6 vanish, and we conclude that the map

$$\pi_k^G(X) \;\longrightarrow\; \lim \pi_k^{H_n}(X)$$

induced by restriction is an isomorphism.

In the commutative square

$$\begin{tikzcd}
\bigoplus_{\mathbb{N}} \pi_0^G(\mathbb{S}_G) \arrow[r, "\cong"] \arrow[d] & \bigoplus_{\mathbb{N}} \lim \pi_0^{H_n}(\mathbb{S}_{H_n}) \arrow[d] \\
\pi_0^G(\bigoplus_{\mathbb{N}} \mathbb{S}_G) \arrow[r, "\cong"'] & \lim \pi_0^{H_n}(\bigoplus_{\mathbb{N}} \mathbb{S}_{H_n})
\end{tikzcd} \tag{2.1.6}$$

the two horizontal maps are thus isomorphisms. Since H_n is finite, the group $\pi_0^{H_n}(\mathbb{S}_{H_n})$ is isomorphic to the Burnside ring $A(H_n)$, and the map

$$\mathrm{res}^{H_n}_{H_{n-1}} \;:\; \mathbb{A}(H_n) \;\longrightarrow\; \mathbb{A}(H_{n-1})$$

is a split epimorphism between finitely generated free abelian groups. Since the group F is non-trivial, the kernel of this restriction map is non-trivial. The upper right corner of square (2.1.6) is thus a countably infinite sum of a countably infinite product of copies of $\mathbb{Z}$, and the right vertical map is not surjective. So the left vertical map is not surjective, and hence the G-sphere spectrum is not small for the particular group under consideration.

Construction 2.1.7 (Restriction homomorphisms). We let $\alpha : K \longrightarrow G$ be a continuous homomorphism between Lie groups. As we shall now explain, such a homomorphism induces a *restriction homomorphism*

$$\alpha^* \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_k^K((L\alpha^*)(X)) \ , \tag{2.1.8}$$

natural for morphisms of orthogonal G-spectra, where $L\alpha^*$ is the total left derived functor of $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$, compare Theorem 1.4.17. The construction exploits the two isomorphisms

$$((L\alpha^*)(\mathbb{S}_G)) \ \xrightarrow{\alpha_!^{\mathbb{S}_G}} \ \alpha^*(\mathbb{S}_G) \ = \ \mathbb{S}_K \quad \text{and} \quad (L\alpha^*)(X)[-k] \ \cong \ (L\alpha^*)(X[-k])$$

in $\mathrm{Ho}(\mathrm{Sp}_K)$; the first one is an isomorphism because $\mathbb{S}_G$ is quasi-cofibrant, and the second one is provided by part (iv) of Theorem 1.4.17. So we define the restriction homomorphism as the composite

$$[\mathbb{S}_G, X[-k]]^G \ \xrightarrow{L\alpha^*} \ [(L\alpha^*)(\mathbb{S}_G), (L\alpha^*)(X[-k])]^G \ \xrightarrow{\cong} \ [\mathbb{S}_K, (L\alpha^*)(X)[-k]]^G \ .$$

Now we consider two composable continuous homomorphisms $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$. In (1.4.25) we exhibited a natural isomorphism $\langle \alpha, \beta\rangle : (L\beta^*) \circ (L\alpha^*) \Longrightarrow L(\alpha\beta)^*$ that relates the three derived functors.

Proposition 2.1.9. *Let $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$ be composable continuous homomorphisms between Lie groups. Then for every orthogonal G-spectrum X, the composite*

$$\pi_k^G(X) \ \xrightarrow{\alpha^*} \pi_k^K((L\alpha)^*(X)) \ \xrightarrow{\beta^*}$$
$$\pi_k^L((L\beta)^*((L\alpha)^*(X))) \ \xrightarrow{\langle \alpha,\beta\rangle_*^X} \ \pi_k^L(L(\alpha\beta)^*(X))$$

coincides with the restriction homomorphism $(\alpha\beta)^*$.

Proof. All maps are natural for G-maps in X and compatible with the suspension isomorphisms. So by naturality it suffices to prove the claim for the universal example, the identity of $\mathbb{S}_G$. After unraveling all definitions, the universal example then comes down to the relation

$$\beta_!^{\mathbb{S}_K} \circ (L\beta)^*(\alpha_!^{\mathbb{S}_G}) = \ (\alpha\beta)_!^{\mathbb{S}_G} \circ \langle \alpha, \beta\rangle^{\mathbb{S}_G} \ : \ (L\beta^*)((L\alpha^*)(\mathbb{S}_G)) \ \longrightarrow \ \mathbb{S}_L \ ,$$

which is an instance of the defining property (1.4.25) of the transformation $\langle \alpha, \beta\rangle$. $\square$

Now we discuss the Wirthmüller isomorphism for finite index inclusions of Lie groups, and the transfer maps that it gives rise to. We consider a closed subgroup Γ of G and we write $\mathrm{res}^G_\Gamma : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_\Gamma$ for the restriction functor. This restriction functor is fully homotopical, i.e., it takes π_*-isomorphisms of orthogonal G-spectra to π_*-isomorphisms of orthogonal Γ-spectra, simply because every compact subgroup of Γ is also a compact subgroup of G. So we get an induced restriction functor on the homotopy categories

$$\mathrm{res}^G_\Gamma \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_\Gamma)$$

from the universal property of localizations, for which we use the same name. This functor satisfies $\mathrm{res}^G_\Gamma \circ \gamma_G \ = \ \gamma_\Gamma \circ \mathrm{res}^G_\Gamma$, so it is in particular a total left derived functor of restriction. The restriction functor res^G_Γ is both a left Quillen functor and a right Quillen functor for the two stable model structures, by Corollary 1.4.3. We write

$$\mathrm{coind}_\Gamma^G \colon \mathrm{Ho}(\mathrm{Sp}_\Gamma) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$$

for the right adjoint of the derived restriction functor, which is a total right derived functor of the functor $\mathrm{map}^\Gamma(G,-) : \mathrm{Sp}_\Gamma \longrightarrow \mathrm{Sp}_G$. Moreover, the left adjoint $G \ltimes_\Gamma -$ is also fully homotopical, so it, too, passes to a functor on the homotopy categories

$$G \ltimes_\Gamma \ : \ \mathrm{Ho}(\mathrm{Sp}_\Gamma) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$$

by the universal property of localizations, for which we also use the same name.

If we also assume that Γ has finite index in G, then the derived left and right adjoint to the restriction functor are in fact isomorphic; this generalizes the classical 'Wirthmüller isomorphism' in equivariant homotopy theory of finite groups [**78**]. Indeed, in this situation the group G is the disjoint union of finitely many Γ-cosets. If X is a based Γ-space, we can define a natural G-map

$$w_X \ : \ G \ltimes_\Gamma X \longrightarrow \mathrm{map}^\Gamma(G,X)$$

by sending $[g,x]$ to the Γ-equivariant map

$$G \longrightarrow X \ , \quad g' \longmapsto \begin{cases} g'gx & \text{for } g'g \in \Gamma, \text{ and} \\ * & \text{for } g'g \notin \Gamma. \end{cases}$$

For an orthogonal Γ-spectrum Y these maps are defined levelwise, and they form a morphism of orthogonal G-spectra

$$w_Y \ : \ G \ltimes_\Gamma Y \longrightarrow \mathrm{map}^\Gamma(G,Y) \ .$$

THEOREM 2.1.10. *Let Γ be a closed subgroup of finite index of a Lie group G. For every orthogonal Γ-spectrum Y the morphism $w_Y : G \ltimes_\Gamma Y \longrightarrow \mathrm{map}^\Gamma(G,Y)$ is a π_*-isomorphism. Hence w_Y descends to a natural isomorphism between the functors*

$$G \ltimes_\Gamma \ , \ \mathrm{coind}_\Gamma^G \ : \ \mathrm{Ho}(\mathrm{Sp}_\Gamma) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G) \ .$$

PROOF. We let H be any compact subgroup of G. By our hypothesis, G/Γ is a finite set and for every $g \in G$ the subgroup $H \cap {}^g\Gamma$ has finite index in H. So for every orthogonal $(H \cap {}^g\Gamma)$-spectrum X the morphism

$$w_X \ : \ H \ltimes_{H \cap {}^g\Gamma} X \longrightarrow \mathrm{map}^{H \cap {}^g\Gamma}(H,X)$$

is a π_*-isomorphism of H-spectra by the classical Wirthmüller isomorphism for the finite index pair $(H, H \cap {}^g\Gamma)$, see [**78**] or [**60**, Thm. 3.2.15]. Moreover, there are double coset decompositions

$$\mathrm{res}_H^G(G \ltimes_\Gamma X) \ \cong \ \bigvee_{[g] \in H\backslash G/\Gamma} H \ltimes_{H \cap {}^g\Gamma} \left(c_g^*\left(\mathrm{res}^\Gamma_{H^g \cap \Gamma}(X)\right)\right)$$

and

$$\mathrm{res}_H^G(\mathrm{map}^\Gamma(G,X)) \ \cong \ \prod_{[g] \in H\backslash G/\Gamma} \mathrm{map}^{H \cap {}^g\Gamma}\left(H, c_g^*\left(\mathrm{res}^\Gamma_{H^g \cap \Gamma}(X)\right)\right) \ .$$

The morphism w_X respects these decomposition. Since finite wedges are π_*-isomorphic to finite products, we are done. $\square$

An immediate consequence of Theorem 2.1.10 is the isomorphism between the group $\pi_k^G(G \ltimes_\Gamma Y)$ and the group $\pi_k^\Gamma(Y)$, defined as the composite

$$\pi_k^G(G \ltimes_\Gamma Y) \xrightarrow[\cong]{(\omega_Y)_*} \pi_k^G(\mathrm{coind}_\Gamma^G(Y)) \xrightarrow[\cong]{\text{adjunction}} \pi_k^\Gamma(Y) \ .$$

This map is also the composite

$$\mathrm{Wirth}_\Gamma^G \ : \ \pi_k^G(G \ltimes_\Gamma Y) \xrightarrow{\mathrm{res}_\Gamma^G} \pi_k^\Gamma(G \ltimes_\Gamma Y) \xrightarrow{(\mathrm{pr}_\Gamma^G)_*} \pi_k^\Gamma(Y) \ . \tag{2.1.11}$$

In the special case where G (and hence Γ) are compact, this isomorphism specializes to the *Wirthmüller isomorphism* [**78**], see also [**60**, Thm. 3.2.15]. In our more general context, we also refer to the isomorphism (2.1.11) as the Wirthmüller isomorphism.

CONSTRUCTION 2.1.12 (Transfer). We continue to let Γ be a finite index subgroup of a Lie group G. If X is an orthogonal G-spectrum, then we can define a *transfer homomorphism* as the composite

$$\mathrm{tr}_\Gamma^G \ : \ \pi_k^\Gamma(X) \xrightarrow[\cong]{(\mathrm{Wirth}_\Gamma^G)^{-1}} \pi_k^G(G \ltimes_\Gamma \mathrm{res}_\Gamma^G(X)) \xrightarrow{(\mathrm{act}_\Gamma^G)_*} \pi_k^G(X) \ , \tag{2.1.13}$$

where $\mathrm{act}_\Gamma^G : G \ltimes_\Gamma \mathrm{res}_\Gamma^G(X) \longrightarrow X$ is the action morphism (the counit of the adjunction).

Now we prove that the transfer maps satisfy the 'usual properties'. We start by studying how transfer maps interact with inflation, i.e., the restriction homomorphism along a continuous epimorphism $\alpha : K \longrightarrow G$. We let Γ be any closed subgroup of the Lie group G, and we let $\Delta = \alpha^{-1}(\Gamma)$ be the inverse image, a closed subgroup of K. On the pointset level, the relation

$$\mathrm{res}_\Delta^K \circ \alpha^* \ = \ (\alpha|_\Delta)^* \circ \mathrm{res}_\Gamma^G$$

holds as functors from Sp_G to Sp_Δ. On the level of homotopy categories, this relation becomes a natural isomorphism between derived functors. Indeed, the isomorphisms

$$\langle \mathrm{incl}_\Gamma^G, \alpha|_\Delta \rangle \ : \ L(\alpha|_\Delta)^* \circ \mathrm{res}_\Gamma^G \implies L(\mathrm{incl}_\Gamma^G \circ \alpha|_\Delta)^*$$

and

$$\langle \alpha, \mathrm{incl}_\Delta^K \rangle \ : \ \mathrm{res}_\Delta^K \circ L\alpha^* \implies L(\alpha \circ \mathrm{incl}_\Delta^K)^*$$

combine into a composite natural isomorphism

$$\begin{aligned}[\alpha, \Gamma] \ : \ L(\alpha|_\Delta)^* \circ \mathrm{res}_\Gamma^G & \xrightarrow[\cong]{\langle \mathrm{incl}_\Gamma^G, \alpha|_\Delta \rangle} L(\mathrm{incl}_\Gamma^G \circ \alpha|_\Delta)^* \\ & = L(\alpha \circ \mathrm{incl}_\Delta^K)^* \xrightarrow[\cong]{\langle \alpha, \mathrm{incl}_\Delta^K \rangle^{-1}} \mathrm{res}_\Delta^K \circ L\alpha^* \ .\end{aligned}$$

PROPOSITION 2.1.14. *Let $\alpha : K \longrightarrow G$ be a continuous epimorphism between Lie groups, let Γ be a closed subgroup of G of finite index, and set $\Delta = \alpha^{-1}(\Gamma)$. Then the following square commutes*

$$\begin{tikzcd}
\pi_*^\Gamma(X) \arrow[rr, "\mathrm{tr}_\Gamma^G"] \arrow[d, "(\alpha|_\Delta)^*"'] & & \pi_*^G(X) \arrow[d, "\alpha^*"] \\
\pi_*^\Delta(L(\alpha|_\Delta)^*(\mathrm{res}_\Gamma^G(X))) \arrow[r, "\cong", "{[\alpha,\Gamma]_*}"'] & \pi_*^\Delta((L\alpha^*)(X)) \arrow[r, "\mathrm{tr}_\Delta^K"'] & \pi_*^K((L\alpha^*)(X))
\end{tikzcd}$$

for every orthogonal G-spectrum X.

Proof. We first consider a cofibrant orthogonal Γ-spectrum Y. An isomorphism of orthogonal K-spectra

$$u \ : \ K \ltimes_\Delta (\alpha|_\Delta)^*(Y) \xrightarrow{\ \cong\ } \alpha^*(G \ltimes_\Gamma Y)$$

is defined levelwise by $u[k,y] = [\alpha(k), y]$, for $k \in K$ and $y \in Y(V)$. Moreover, the composite

$$K \ltimes_\Delta (\alpha|_\Delta)^*(Y) \xrightarrow{\ u\ } \alpha^*(G \ltimes_\Gamma Y) \xrightarrow{(\alpha|_\Delta)^*(\mathrm{pr}^G_\Gamma)} (\alpha|_\Delta)^*(Y)$$

coincides with the morphism $\mathrm{pr}^K_\Delta : K \ltimes_\Delta (\alpha|_\Delta)^*(Y) \longrightarrow (\alpha|_\Delta)^*(Y)$. Since Y is cofibrant, we can calculate $(L\alpha^*)(Y)$ as $\alpha^*(Y)$. Also, $G \ltimes_\Gamma Y$ is cofibrant as an orthogonal G-spectrum, and we can calculate $(L\alpha^*)(G \ltimes_\Gamma Y)$ as $\alpha^*(G \ltimes_\Gamma Y)$. Similarly, the underlying Γ-spectrum of $G \ltimes_\Gamma Y$ is cofibrant, so we can calculate $(L\alpha|^*_\Delta)(G \ltimes_\Gamma Y)$ as $(\alpha|_\Delta)^*(G \ltimes_\Gamma Y)$. The following diagram commutes by naturality and transitivity of restriction maps:

$$\begin{tikzcd}
\pi_*^G(G \ltimes_\Gamma Y) \arrow[r, "\mathrm{res}^G_\Gamma"'] \arrow[d, "\alpha^*"'] \arrow[rr, bend left, "\mathrm{Wirth}^G_\Gamma"] & \pi_*^\Gamma(G \ltimes_\Gamma Y) \arrow[r, "(\mathrm{pr}^G_\Gamma)_*"'] \arrow[d, "(\alpha|_\Delta)^*"'] & \pi_*^\Gamma(Y) \arrow[dd, "(\alpha|_\Delta)^*"] \\
\pi_*^K(\alpha^*(G \ltimes_\Gamma Y)) \arrow[r, "\mathrm{res}^K_\Delta"] & \pi_*^\Delta((\alpha|_\Delta)^*(G \ltimes_\Gamma Y)) \arrow[dr, "((\alpha|_\Delta)^*(\mathrm{pr}^G_\Gamma))_*"] & \\
\pi_*^K(K \ltimes_\Delta (\alpha|_\Delta)^*(Y)) \arrow[u, "u_*"', "\cong"] \arrow[r, "\mathrm{res}^K_\Delta"] \arrow[rr, bend right, "\mathrm{Wirth}^K_\Delta"'] & \pi_*^\Delta(K \ltimes_\Delta (\alpha|_\Delta)^*(Y)) \arrow[u, "u_*"', "\cong"] \arrow[r, "(\mathrm{pr}^K_\Delta)_*"] & \pi_*^\Delta((\alpha|_\Delta)^*(Y))
\end{tikzcd}$$

In formulas:

$$(\alpha|_\Delta)^* \circ \mathrm{Wirth}^G_\Gamma \ = \ \mathrm{Wirth}^K_\Delta \circ u_*^{-1} \circ \alpha^* \ . \tag{2.1.15}$$

Now we let X be a cofibrant orthogonal G-spectrum. Then the following square commutes:

$$\begin{tikzcd}
K \ltimes_\Delta (\alpha|_\Delta)^*(\mathrm{res}^G_\Gamma(X)) \arrow[r, equal] \arrow[d, "u"'] & K \ltimes_\Delta \mathrm{res}^K_\Delta(\alpha^*(X)) \arrow[d, "\mathrm{act}^K_\Delta"] \\
\alpha^*(G \ltimes_\Gamma X) \arrow[r, "\alpha^*(\mathrm{act}^G_\Gamma)"'] & \alpha^*(X)
\end{tikzcd}$$

The Wirthmüller maps are isomorphisms, so we can deduce

$$\begin{aligned}
\mathrm{tr}^K_\Delta \circ (\alpha|_\Delta)^* \ &= \ (\mathrm{act}^K_\Delta)_* \circ (\mathrm{Wirth}^K_\Delta)^{-1} \circ (\alpha|_\Delta)^* \\
\overset{(2.1.15)}{=} \ & (\mathrm{act}^K_\Delta)_* \circ u_*^{-1} \circ \alpha^* \circ (\mathrm{Wirth}^G_\Gamma)^{-1} \\
&= \ (\alpha^*(\mathrm{act}^G_\Gamma))_* \circ \alpha^* \circ (\mathrm{Wirth}^G_\Gamma)^{-1} \\
&= \ \alpha^* \circ (\mathrm{act}^G_\Gamma)_* \circ (\mathrm{Wirth}^G_\Gamma)^{-1} \ = \ \alpha^* \circ \mathrm{tr}^G_\Gamma \ .
\end{aligned}$$

This proves the claim for cofibrant orthogonal G-spectra. In $\mathrm{Ho}(\mathrm{Sp}_G)$, every object is isomorphic to a cofibrant G-spectrum, so naturality concludes the argument. □

Now we spell out how transfers interact with the conjugation homomorphism. For this purpose we let Γ be any closed subgroup of a Lie group G and $g \in G$. We let $\Gamma^g = g^{-1}\Gamma g$ be the conjugate subgroup and denote by

$$c_g \ : \ \Gamma \ \longrightarrow \ \Gamma^g \ , \quad c_g(\gamma) \ = \ g^{-1}\gamma g$$

the conjugation homomorphism. Restriction of group actions along c_g is fully homotopical; we abuse notation and write

$$c_g^* \ = \ \mathrm{Ho}(c_g^*) \ : \ \mathrm{Ho}(\mathrm{Sp}_{\Gamma^g}) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_\Gamma)$$

for the induced functor on homotopy categories. This induced functor is then also a total left derived functor of c_g^*, relative to the identity transformation. As a special case of the restriction homomorphism (2.1.8), the restriction functor thus induces an isomorphism

$$c_g^* \ : \ \pi_k^{\Gamma^g}(Y) \ \longrightarrow \ \pi_k^\Gamma(c_g^*(Y))$$

for every orthogonal Γ^g-spectrum Y. We call this the *conjugation isomorphism.*

Now we let X be an orthogonal G-spectrum. Then left multiplication by g is an isomorphism

$$l_g \ : \ c_g^*(X) \ \longrightarrow \ X$$

of orthogonal G-spectra, which induces an isomorphism on $\pi_k^\Gamma(-)$. The composite

$$\pi_k^{\Gamma^g}(X) \ \xrightarrow{\ c_g^*\ } \ \pi_k^\Gamma(c_g^*(X)) \ \xrightarrow{\ (l_g)_*\ } \ \pi_k^\Gamma(X)$$

is an 'internal' conjugation isomorphism which we denote by

$$g_\star \ : \ \pi_k^{\Gamma^g}(X) \ \longrightarrow \ \pi_k^\Gamma(X) \ . \tag{2.1.16}$$

REMARK 2.1.17. The conjugation isomorphism has another interpretation as follows. The map

$$l_g \ : \ G/\Gamma \ \longrightarrow \ G/\Gamma^g \ : \ k\Gamma \ \longmapsto \ kg\Gamma^g$$

is an isomorphism of G-spaces, and it induces an isomorphism of G-equivariant suspension spectra

$$\Sigma^\infty_+ l_g \ : \ \Sigma^\infty_+ G/\Gamma \ \longrightarrow \ \Sigma^\infty_+ G/\Gamma^g \ .$$

For every orthogonal G-spectrum, the derived adjunctions provide natural isomorphisms

$$\pi_0^{\Gamma^g}(X) \ = \ [\mathbb{S}_{\Gamma^g}, \mathrm{res}^G_{\Gamma^g}(X)]^{\Gamma^g} \ \xrightarrow{\ \cong\ } \ [\Sigma^\infty_+ G/\Gamma^g, X]^G \quad \text{and}$$
$$\pi_0^{\Gamma}(X) \ = \ [\mathbb{S}_{\Gamma}, \mathrm{res}^G_{\Gamma}(X)]^{\Gamma} \ \xrightarrow{\ \cong\ } \ [\Sigma^\infty_+ G/\Gamma, X]^G \ .$$

We omit the verification that under these isomorphisms, the conjugation map $g_\star : \pi_k^{\Gamma^g}(X) \longrightarrow \pi_k^\Gamma(X)$ corresponds to precomposition with $\Sigma^\infty_+ l_g$.

PROPOSITION 2.1.18. *Let G be a Lie group and $g \in G$.*

(i) *Let $\Delta \subset \Gamma$ be nested closed subgroups of G, such that Δ has finite index in Γ. Then*

$$\mathrm{tr}_\Delta^\Gamma \circ g_\star \ = \ g_\star \circ \mathrm{tr}_{\Delta^g}^{\Gamma^g} \ : \ \pi_*^{\Delta^g}(X) \ \longrightarrow \ \pi_*^\Gamma(X)$$

for every orthogonal G-spectrum X.

(ii) *The conjugation map $g_\star : \pi_*^G(X) \longrightarrow \pi_*^G(X)$ is the identity.*

Proof. (i) In the special case $\alpha = c_g : \Gamma \longrightarrow \Gamma^g$, applied to the finite index subgroup Δ^g of Γ^g, Proposition 2.1.14 says that the following square commutes:

$$\begin{tikzcd} \pi_*^{\Delta^g}(X) \arrow[rr, "\mathrm{tr}_{\Delta^g}^{\Gamma^g}"] \arrow[d, "c_g^*"'] & & \pi_*^{\Gamma^g}(X) \arrow[d, "c_g^*"] \\ \pi_*^{\Delta}(c_g^*(\mathrm{res}^G_{\Delta^g}(X))) \arrow[r, equal] & \pi_*^{\Delta}(\mathrm{res}^G_{\Delta}(c_g^*(X))) \arrow[r, "\mathrm{tr}_{\Delta}^{\Gamma}"'] & \pi_*^{\Gamma}(c_g^*(X)) \end{tikzcd}$$

Naturality of restriction and transfer for the morphism of orthogonal G-spectra $l_g : c_g^*(X) \longrightarrow X$ then yields the desired relation:

$$g_\star \circ \mathrm{tr}_{\Delta^g}^{\Gamma^g} \ = \ (l_g)_* \circ c_g^* \circ \mathrm{tr}_{\Delta^g}^{\Gamma^g} \ = \ (l_g)_* \circ \mathrm{tr}_{\Delta}^{\Gamma} \circ c_g^* \ = \ \mathrm{tr}_{\Delta}^{\Gamma} \circ (l_g)_* \circ c_g^* \ = \ \mathrm{tr}_{\Delta}^{\Gamma} \circ g_\star$$

For claim (ii) we exploit that the map $g_\star$ is natural for morphisms in $\mathrm{Ho}(\mathrm{Sp}_G)$ and commutes with the suspension isomorphism. So it suffices to prove the claim in the universal example, the identity of $\mathbb{S}_G$. Since G acts trivially on $\mathbb{S}_G$, we have $c_g^*(\mathbb{S}_G) = \mathbb{S}_G$ and $l_g^{\mathbb{S}_G} = \mathrm{Id}$. So $c_g^*(\mathrm{Id}) = \mathrm{Id}$. $\square$

Now we prove transitivity with respect to a nested triple of finite index subgroups $\Gamma \leq \Delta \leq G$, and the double coset formula.

Proposition 2.1.19. *Let Γ be a closed finite index subgroup of a Lie group G.*

(i) *Let $\Delta \leq G$ be another closed subgroup with $\Gamma \leq \Delta$. Then the transfer maps are transitive, i.e.,*

$$\mathrm{tr}_{\Delta}^{G} \circ \mathrm{tr}_{\Gamma}^{\Delta} \ = \ \mathrm{tr}_{\Gamma}^{G} \ : \ \pi_*^{\Gamma}(X) \ \longrightarrow \ \pi_*^{G}(X)$$

for every orthogonal G-spectrum X.

(ii) *Let K be another closed subgroup of G. Then for every orthogonal G-spectrum X the relation*

$$\mathrm{res}^G_K \circ \mathrm{tr}_{\Gamma}^{G} \ = \ \sum_{[g]\in K\backslash G/\Gamma} \mathrm{tr}^K_{K\cap {}^g\Gamma} \circ g_\star \circ \mathrm{res}^{\Gamma}_{K^g\cap\Gamma}$$

holds as maps $\pi_*^{\Gamma}(X) \longrightarrow \pi_*^{K}(X)$. Here the sum is indexed over a set of representatives of the finite set of K-Γ-double cosets in G.

Proof. We reduce both properties to the special case of *finite* groups. We set

$$N \ = \ \bigcap_{g\in G} \Gamma^g \ ,$$

the intersection of all G-conjugates of Γ. Then N is the largest normal subgroup of G that is contained in Γ, and it is the kernel of the translation action of G on G/Γ. Hence the quotient group $H = G/N$ acts faithfully on the finite set G/Γ; in particular, the group H is finite. We let

$$q \ : \ G \ \longrightarrow \ G/N \ = \ H$$

denote the quotient map, which is a continuous epimorphism. The morphism q induces an isomorphism of finite G-sets $G/\Gamma \cong q^*(H/I)$, where $I = q(\Gamma) = \Gamma/N$, and hence an isomorphism of orthogonal G-spectra from $G \ltimes_\Gamma \mathbb{S}_\Gamma$ to $q^*(\Sigma^\infty_+ H/I)$. The natural bijections

$$\pi_0^{\Gamma}(X) \ = \ [\mathbb{S}_\Gamma, \mathrm{res}^G_\Gamma(X)]^\Gamma \ \cong \ [G \ltimes_\Gamma \mathbb{S}_\Gamma, X]^G \ \cong \ [q^*(\Sigma^\infty_+ H/I), X]^G$$

witness that the functor $\pi_0^\Gamma : \mathrm{Ho}(\mathrm{Sp}_G) \longrightarrow (\mathrm{sets})$ is represented by the orthogonal G-spectrum $q^*(\Sigma_+^\infty H/I)$; the universal element is the class $q|_\Gamma^*(u_I)$ in the group $\pi_0^\Gamma(q^*(\Sigma_+^\infty H/I))$, where $u_I \in \pi_0^I(\Sigma_+^\infty H/I)$ is the tautological class (1.3.9). The formulas of parts (i) and (ii) are relations between natural transformations of functors on $\mathrm{Ho}(\mathrm{Sp}_G)$ with source the representable functor π_0^Γ. By the Yoneda lemma, it thus suffices to prove the two formulas applied to the universal example $(q^*(\Sigma_+^\infty H/I), q|_\Gamma^*(u_I))$.

(i) We set $J = q(\Delta)$, another subgroup of the finite group H. For the universal example we can then argue:

$$\begin{aligned}\mathrm{tr}_\Delta^G(\mathrm{tr}_\Gamma^\Delta(q|_\Gamma^*(u_I))) \ &= \ \mathrm{tr}_\Delta^G(q|_\Delta^*(\mathrm{tr}_I^J(u_I))) \\ &= \ q^*(\mathrm{tr}_J^H(\mathrm{tr}_I^J(u_I))) \ = \ q^*(\mathrm{tr}_I^H(u_I)) \ = \ \mathrm{tr}_\Gamma^G(q|_\Gamma^*(u_I)) \ .\end{aligned}$$

The third equation is the transitivity property for transfers in the realm of finite groups, see for example [**60**, Prop. 3.2.9]. The other three equalities are instances of the fact that transfers and inflations commute, see Proposition 2.1.14.

(ii) We set $J = q(K)$, another subgroup of the finite group H. For every $g \in G$ the following relation holds:

(2.1.20)
$$\begin{aligned}q|_K^*(\mathrm{tr}_{J\cap^{q(g)}I}^J(q(g)_\star(\mathrm{res}^I_{J^{q(g)}\cap I}(u_I)))) \ &= \ \mathrm{tr}^K_{K\cap^g\Gamma}(q|^*_{K\cap^g\Gamma}(q(g)_\star(\mathrm{res}^I_{J^{q(g)}\cap I}(u_I)))) \\ &= \ \mathrm{tr}^K_{K\cap^g\Gamma}(g_\star(q|^*_{K^g\cap\Gamma}(\mathrm{res}^I_{J^{q(g)}\cap I}(u_I)))) \\ &= \ \mathrm{tr}^K_{K\cap^g\Gamma}(g_\star(\mathrm{res}^\Gamma_{K^g\cap\Gamma}(q|^*_\Gamma(u_I)))) \ .\end{aligned}$$

The first equation is Proposition 2.1.14, i.e., the fact that transfers and inflations commute. The second and third equations are transitivity of restriction maps. Now we deduce the double coset formula for the universal example:

$$\begin{aligned}\mathrm{res}^G_K(\mathrm{tr}^G_\Gamma(q|^*_\Gamma(u_I))) \ &= \ \mathrm{res}^G_K(q^*(\mathrm{tr}^H_I(u_I))) \ = \ q|^*_K(\mathrm{res}^H_J(\mathrm{tr}^H_I(u_I))) \\ &= \sum_{[h]\in J\backslash H/I} q|^*_K(\mathrm{tr}^J_{J\cap^h I}(h_\star(\mathrm{res}^I_{J^h\cap I}(u_I)))) \\ &\overset{(2.1.20)}{=} \sum_{[g]\in K\backslash G/\Gamma} \mathrm{tr}^K_{K\cap^g\Gamma}(g_\star(\mathrm{res}^\Gamma_{K^g\cap\Gamma}(q|^*_\Gamma(u_I)))) \ .\end{aligned}$$

The first equation is Proposition 2.1.14, i.e., the fact that transfers and inflations commute. The third equation is the classical double coset formula for the subgroups $J = q(K)$ and $I = q(\Gamma)$ of the finite group H, see for example [**60**, Ex. 3.4.11]. The fourth equation exploits that the epimorphism q induces a bijection from the set $K\backslash G/\Gamma$ to the set $J\backslash H/I$. $\square$

2.2. Equivariant homotopy groups as Mackey functors

If G is a finite group and X is an orthogonal G-spectrum, then the H-equivariant homotopy groups $\pi_0^H(X)$, for all subgroups H of G, form a G-Mackey functor, see for example [**35**, V.9] or [**60**, Sec. 3.4]; in this section we generalize this well-known fact to arbitrary discrete groups. In Definition 2.2.2 we recall the notion of a G-Mackey functor for discrete groups G, and in Example 2.2.8 we show that the collection of equivariant homotopy groups of an orthogonal G-spectrum forms a graded G-Mackey functor. Finally, in Theorem 2.2.9 we identify the heart of the preferred t-structure on the equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ with the abelian category of G-Mackey functors. A consequence is that every G-Mackey functor has an Eilenberg-Mac Lane spectrum, i.e., an orthogonal G-spectrum with equivariant homotopy groups concentrated in dimension 0, where they realize the given G-Mackey functor, compare Remark 2.2.11.

Construction 2.2.1 (G-Mackey category). For a discrete group G, the pre-additive *Mackey category* $\mathbf{A}_G$ has as objects all finite subgroups of G. A G-set is *finitely generated* if it is generated by finitely many of its element, or – equivalently– if it has finitely many orbits. For two finite subgroups H and K of G, a *span* is a triple (S, α, β) consisting of a finitely generated G-set S and G-maps

$$G/H \xleftarrow{\ \alpha\ } S \xrightarrow{\ \beta\ } G/K \ .$$

An isomorphism of spans is an isomorphism of G-sets $\psi : S \longrightarrow S'$ such that $\alpha' \circ \psi = \alpha$ and $\beta' \circ \psi = \beta$. The isomorphism classes of spans form an abelian monoid under disjoint union, and the morphism group $\mathbf{A}_G(H, K)$ is defined as the Grothendieck group of isomorphism classes of spans from H to K. Composition

$$\circ \ : \ \mathbf{A}_G(K, L) \times \mathbf{A}_G(H, K) \ \longrightarrow \ \mathbf{A}_G(H, L)$$

is induced by pullback of spans over the intermediate G-set G/K.

The following definition is taken from [**49**, Sec. 3].

Definition 2.2.2. Let G be a discrete group. A *G-Mackey functor* is an additive functor from the Mackey category $\mathbf{A}_G$ to the category of abelian groups. A morphism of G-Mackey functors is a natural transformation. We denote the category of G-Mackey functors by $\mathcal{M}_G$.

As a category of additive functors, $\mathcal{M}_G$ is an abelian category with enough projectives and injectives. Monomorphisms, epimorphisms and exactness are detected objectwise.

As in the case of finite groups, G-Mackey functors also have a description via transfer, restriction and conjugation maps as follows. Every G-set is the disjoint union of transitive G-sets, so the group $\mathbf{A}_G(H, K)$ is a free abelian group with basis the classes of those spans (S, α, β) where G acts transitively on S. Up to isomorphism, every such 'transitive span' is of the form

$$G/H \xleftarrow{\ g\gamma H \leftarrow\!\shortmid gL\ } G/L \xrightarrow{\ gL \mapsto gK\ } G/K \tag{2.2.3}$$

for some pair (L, γ) consisting of a subgroup L of K and an element $\gamma \in G$ such that $L \leq {}^{\gamma}H$. Two such pairs (L, γ) and (L', γ') define isomorphic spans if and only if there is an element $k \in K$ such that $L' = L^k$ and $\gamma^{-1} k \gamma' \in H$. A different way to say the same thing is as an isomorphism

$$\mathbf{A}_G(H, K) \ \cong \ \bigoplus_{K\gamma H \in K\backslash G/H} A(K \cap {}^{\gamma}H) \ ,$$

where on the right hand side $A(-)$ is the Burnside ring functor for finite groups.

The bases of the morphism groups of $\mathbf{A}_G$ lead to a more computational description of G-Mackey functors by 'generators and relations'. To specify a G-Mackey functor M, one has to give the following data:

- an abelian group $M(H)$ for every finite subgroup H of G,

- a *restriction homomorphism* $\mathrm{res}^H_K \colon M(H) \longrightarrow M(K)$ and a *transfer homomorphism* $\mathrm{tr}^H_K \colon M(K) \longrightarrow M(H)$ for every pair of nested finite subgroups $K \leq H$ of G, and
- *conjugation homomorphisms* $\gamma_\star \colon M(H^\gamma) \longrightarrow M(H)$ for all $\gamma \in G$ and all finite subgroups H of G.

These data must satisfy certain conditions which we do not recall here in details, but refer to [**49**]. We only summarize them briefly: restrictions, transfers and conjugations are transitive; conjugations commute with the restriction and transfers; inner automorphisms act as the identity; and finally the double coset formula holds. In the 'generators-and-relations' description of Mackey functors, the image of a basic transitive span (2.2.3) under a G-Mackey functor $M : \mathbf{A}_G \longrightarrow \mathcal{A}b$ is the composite

$$M(H) \xrightarrow{\mathrm{res}^H_{L^\gamma}} M(L^\gamma) \xrightarrow{\gamma_\star} M(L) \xrightarrow{\mathrm{tr}^K_L} M(K)$$

of the restriction map to L^γ, the conjugation by the element γ and the transfer map to K.

EXAMPLE 2.2.4. (i) The *Burnside ring Mackey functor* is the G-Mackey functor $\mathbb{A}$ given by

$$\mathbb{A}(H) \;=\; A(H) \;,$$

the Burnside ring of the finite subgroup H of G. If G is finite, then $\mathbb{A}$ is represented by the group G itself, hence $\mathbb{A}$ is then projective as a G-Mackey functor. If G is infinite, however, $\mathbb{A}$ is neither representable nor projective.

(ii) Given an abelian group B, the *constant G-Mackey functor $\underline{B}$* is given by $\underline{B}(H) = B$, and all restriction and conjugation maps are identity maps. The transfer $\mathrm{tr}^H_K : \underline{B}(K) \longrightarrow \underline{B}(H)$ is multiplication by the index $[H : K]$.

There is a well-known point set level model of an Eilenberg-Mac Lane spectrum $B[\mathbb{S}]$ that we recall in Example 2.2.12; for discrete groups, $B[\mathbb{S}]$ is an Eilenberg-Mac Lane spectrum for the constant G-Mackey functor $\underline{B}$.

(iii) The *representation ring G-Mackey functor R* assigns to a finite subgroup H of G the unitary representation ring $R(H)$, i.e., the Grothendieck group of finite-dimensional complex H-representations, with product induced by tensor product of representations. The restriction maps are induced by restriction of representations. The transfer maps are induced by induction of representations.

(iv) Given any generalized cohomology theory E (in the non-equivariant sense), we can define a G-Mackey functor $\underline{E}$ by setting

$$\underline{E}(H) \;=\; E^0(BH) \;,$$

the 0-th E-cohomology of a classifying space of the finite group H. Restriction and conjugation maps come from the contravariant functoriality of classifying spaces in group homomorphisms. The transfer map for a subgroup inclusion $K \leq H$ comes from the stable transfer map associated with the finite covering

$$BK \;\simeq\; (EH)/K \;\longrightarrow\; (EH)/H \;=\; BH \;.$$

As we will discuss in Examples 3.3.10 and 3.3.13, the G-Mackey functor $\underline{E}$ is realized by the 0th equivariant homotopy groups of a specific orthogonal G-spectrum, the 'G-Borel theory' associated with E.

Now we link the purely algebraic concept of a G-Mackey functor to the equivariant homotopy groups of orthogonal G-spectra.

CONSTRUCTION 2.2.5. We define an additive functor

$$\Phi \;:\; \mathbf{A}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)^{\mathrm{op}}$$

from the G-Mackey category to the opposite of the triangulated homotopy category of orthogonal G-spectra. On objects we set $\Phi(H) = \Sigma^\infty_+ G/H$.

We let $L \leq K$ be two nested finite subgroups of the discrete group G. The preferred coset eL is an L-fixed point of G/L, so it defines an equivariant homotopy class

$$u_L \;\in\; \pi_0^L(\Sigma^\infty_+ G/L)\;,$$

compare (1.3.9). By Proposition 1.3.10 there is a unique morphism

$$t_L^K \;:\; \Sigma^\infty_+ G/K \longrightarrow \Sigma^\infty_+ G/L$$

in the stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, characterized by the property

$$(t_L^K)_*(u_K) \;=\; \mathrm{tr}_L^K(u_L)$$

in the group $\pi_0^K(\Sigma^\infty_+ G/L)$. In other words, the morphism t_L^K represents the transfer homomorphism $\mathrm{tr}_L^K : \pi_0^L(X) \longrightarrow \pi_0^K(X)$ defined in (2.1.13). We emphasize that for $L \neq K$, the transfer morphism does *not* arise from an unstable G-map. For finite subgroups H and K of G we can now define

$$\Phi \;:\; \mathbf{A}_G(H,K) \longrightarrow [\Sigma^\infty_+ G/K, \Sigma^\infty_+ G/H]^G \tag{2.2.6}$$

as the homomorphism that sends the basis element (2.2.3) indexed by a pair (L,γ) to the composite morphism

$$\Sigma^\infty_+ G/K \xrightarrow{t_L^K} \Sigma^\infty_+ G/L \xrightarrow{\Sigma^\infty_+ \pi} \Sigma^\infty_+ G/H\;.$$

Here $\pi : G/L \longrightarrow G/H$ is the G-map defined by

$$\pi(gL) \;=\; g\gamma H\;.$$

The various properties of the transfer homomorphisms translate into corresponding properties of the representing morphisms: the normalization $t_K^K = \mathrm{Id}$, transitivity

$$t_L^K \circ t_K^J \;=\; t_L^J$$

for nested triples of finite subgroups $L \leq K \leq J$, and compatibility with conjugation

$$(\Sigma^\infty_+ l_\gamma) \circ t_L^K \;=\; t_{L^\gamma}^{K^\gamma} \circ (\Sigma^\infty_+ l_\gamma)\;.$$

If H and L are both subgroups of K, we write $\rho_H^K : G/H \longrightarrow G/K$ for the quotient map, which satisfies $(\Sigma^\infty_+ \rho_H^K)_*(u_H) = \mathrm{res}_H^K(u_K)$. The double coset formula Proposition 2.1.19 (ii) for the orthogonal K-spectrum $\Sigma^\infty_+ G/L$ yields

$$\begin{aligned}
(t_L^K \circ \Sigma^\infty_+ \rho_H^K)_*(u_H) \;&=\; (t_L^K)_*((\Sigma^\infty_+ \rho_H^K)_*(u_H)) \;=\; (t_L^K)_*(\mathrm{res}_H^K(u_K))\\
&=\; \mathrm{res}_H^K((t_L^K)_*(u_K)) \;=\; \mathrm{res}_H^K(\mathrm{tr}_L^K(u_L))\\
&=\; \sum_{K\gamma L \in H\backslash K/L} \mathrm{tr}^H_{H\cap {}^\gamma L}(\gamma_\star(\mathrm{res}^L_{H^\gamma \cap L}(u_L)))\\
&=\; \sum_{K\gamma L \in H\backslash K/L} \big((\Sigma^\infty_+ \rho^L_{H^\gamma\cap L}) \circ (\Sigma^\infty_+ l_\gamma) \circ t^H_{H\cap {}^\gamma L}\big)_*(u_H)\;.
\end{aligned} \tag{2.2.7}$$

The last equality exploits the relations

$$\begin{aligned}
\mathrm{tr}^H_{H\cap{}^\gamma L}(\gamma_\star(\mathrm{res}^L_{H^\gamma\cap L}(u_L))) &= \mathrm{tr}^H_{H\cap{}^\gamma L}(\gamma_\star((\Sigma^\infty_+\rho^L_{H^\gamma\cap L})_*(u_{H^\gamma\cap L}))) \\
&= \mathrm{tr}^H_{H\cap{}^\gamma L}((\Sigma^\infty_+\rho^L_{H^\gamma\cap L})_*(\gamma_\star(u_{H^\gamma\cap L}))) \\
&= \mathrm{tr}^H_{H\cap{}^\gamma L}((\Sigma^\infty_+\rho^L_{H^\gamma\cap L})_*((\Sigma^\infty_+ l_\gamma)_*(u_{H\cap{}^\gamma L}))) \\
&= (\Sigma^\infty_+\rho^L_{H^\gamma\cap L})_*((\Sigma^\infty_+ l_\gamma)_*(\mathrm{tr}^H_{H\cap{}^\gamma L}(u_{H\cap{}^\gamma L}))) \\
&= (\Sigma^\infty_+\rho^L_{H^\gamma\cap L})_*((\Sigma^\infty_+ l_\gamma)_*((t^H_{H\cap{}^\gamma L})_*(u_H))) \\
&= \big((\Sigma^\infty_+\rho^L_{H^\gamma\cap L})\circ(\Sigma^\infty_+ l_\gamma)\circ t^H_{H\cap{}^\gamma L}\big)_*(u_H)\ .
\end{aligned}$$

By the representability property of Proposition 1.3.10, the relation (2.2.7) implies the relation

$$t^K_L\circ\rho^K_H \;=\; \sum_{K\gamma L\in H\backslash K/L}(\Sigma^\infty_+\rho^L_{H^\gamma\cap L})\circ(\Sigma^\infty_+ l_\gamma)\circ t^H_{H\cap{}^\gamma L}$$

as morphisms $\Sigma^\infty_+ G/H\longrightarrow\Sigma^\infty_+ G/L$. Altogether, these properties imply functoriality of the homomorphisms Φ.

EXAMPLE 2.2.8 (G-Mackey functor of an orthogonal G-spectrum). We can now associate a G-Mackey functor $\underline{\pi}_0(X)$ to every orthogonal G-spectrum X, namely as the composite functor

$$\mathbf{A}_G \xrightarrow{\ \Phi\ } \mathrm{Ho}(\mathrm{Sp}_G)^{\mathrm{op}} \xrightarrow{[-,X]^G} \mathcal{A}b\ ,$$

where Φ was introduced in Construction 2.2.5. We take the time to translate this definition into the 'explicit' description of Mackey functors in terms of restriction, conjugation and transfer homomorphisms. For every finite subgroup H of G, evaluation at the class $u_H\in\pi^H_0(\Sigma^\infty_+ G/H)$ is an isomorphism

$$\underline{\pi}_0(X)(H) \;=\; [\Sigma^\infty_+ G/H, X]^G \;\cong\; \pi^H_0(X)\ ,$$

see Proposition 1.3.10. Now we let $K\leq H$ be nested finite subgroups of G. Under the above identification, the restriction map $\mathrm{res}^H_K:\pi^H_0(X)\longrightarrow\pi^K_0(X)$ becomes a special case of the restriction homomorphism (2.1.8) for the inclusion $K\longrightarrow H$. The transfer map $\mathrm{tr}^H_K:\pi^K_0(X)\longrightarrow\pi^H_0(X)$ becomes the one defined in Construction 2.1.12; the conjugation homomorphism $\gamma_\star:\pi^{H^\gamma}_0(X)\longrightarrow\pi^H_0(X)$ was defined in (2.1.16). Since H and K are finite, the groups $\pi^H_0(X)$ and $\pi^K_0(X)$ have the explicit colimit descriptions (1.2.3), and in this picture, restriction, conjugation and transfer are the 'classical' ones in the context of equivariant homotopy theory of finite groups, see for example Constructions 3.1.5 and 3.2.7 of [**60**].

In Corollary 1.3.16 above we established a non-degenerate t-structure on the G-equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, in which the classes of connective and coconnective objects are detected by equivariant homotopy groups at all compact subgroups of G. The heart $\mathcal{H}$ of this t-structure consist of those orthogonal G-spectra X such that $\pi^H_n(X)=0$ for all compact subgroups H of G and all $n\neq 0$. In the special case of discrete groups, we will now identify the heart with the abelian category of G-Mackey functors.

Part (i) of the following Theorem 2.2.9 says that for every finite subgroup $H\leq G$, the G-Mackey functor $\underline{\pi}_0(\Sigma^\infty_+ G/H)$ is a *free G-Mackey functor* represented by the object H of $\mathbf{A}_G$. Part (ii) implies that every G-Mackey functor arises from an orthogonal G-spectrum, see also Remark 2.2.11.

THEOREM 2.2.9. *Let G be a discrete group.*

(i) *The maps* (2.2.6) *define a fully faithful functor* $\Phi : \mathbf{A}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)^{\mathrm{op}}$.

(ii) *The functor*

$$\underline{\pi}_0 \ : \ \mathcal{H} \ \longrightarrow \ \mathcal{M}_G$$

is an equivalence of categories from the heart of the t-structure on $\mathrm{Ho}(\mathrm{Sp}_G)$ *to the category of G-Mackey functors.*

PROOF. (i) The argument is essentially the same as for finite groups, so we will be brief. The maps $\Phi : \mathbf{A}_G(H,K) \longrightarrow [\Sigma^\infty_+ G/K, \Sigma^\infty_+ G/H]^G$ are additive, by definition, and they send the identity of H to the identity of $\Sigma^\infty_+ G/H$.

To see that Φ is fully faithful it suffices, by Proposition 1.3.10, to show that the map

$$\mathbf{A}_G(H,K) \ \longrightarrow \ \pi_0^K(\Sigma^\infty_+ G/H)$$

sending the basis element (2.2.3) to the class $\mathrm{tr}_L^K(\gamma_\star(\mathrm{res}^H_{L^\gamma}(u_H)))$ is an isomorphism. By [**60**, Thm. 3.3.15 (i)], the group $\pi_0^K(\Sigma^\infty_+ G/H)$ is free abelian, with a basis given by the classes $\mathrm{tr}_L^K(\sigma^L(\gamma H))$, where L runs through conjugacy classes of subgroups of K, and γH runs through $W_K L$-orbits of the set $(G/H)^L$, and $\sigma^L(\gamma H)$ is the class in $\pi_0^L(\Sigma^\infty_+ G/H)$ represented by the L-map $S^0 \longrightarrow G/H_+ = (\Sigma^\infty_+ G/H)(0)$ sending 0 to γH. The fact that γH is an L-fixed point of G/H precisely means that $L^\gamma \leq H$, and in our present notation we have

$$\sigma^L(\gamma H) \ = \ \gamma_\star(\sigma^{L^\gamma}(eH)) \ = \ \gamma_\star(\mathrm{res}^H_{L^\gamma}(u_H)) \ .$$

So our claim follows from the fact that sending $\gamma H \in (G/H)^L$ to the equivalence class of the span (2.2.3) passes to a bijection between the $W_K L$-orbits of $(G/H)^L$ and the equivalence classes of transitive span in which the middle term is isomorphic to G/L. Altogether, this shows that the functor Φ takes the preferred basis of $\mathbf{A}_G(H,K)$ given by 'transitive spans' to a basis of $\pi_0^K(\Sigma^\infty_+ G/H)$, so it is an isomorphism.

(ii) We denote by End the 'endomorphism category' of the preferred small generators, i.e., the full pre-additive subcategory of $\mathrm{Ho}(\mathrm{Sp}_G)$ with objects $\Sigma^\infty_+ G/H$ for all finite subgroups H of G. By an End-module we mean an additive functor

$$\mathrm{End}^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

from the opposite category of End. The tautological functor

$$\mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \text{mod-}\,\mathrm{End} \tag{2.2.10}$$

takes an object X to the restriction of the contravariant Hom-functor $[-,X]^G$ to the full subcategory End. By Proposition 1.3.11, the spectra $\Sigma^\infty_+ G/H$ form a set of small weak generators for the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G)$; moreover, the group of maps from a generator to a positive shift of any other generator is trivial, compare (1.3.13). So [**6**, Thm. III.3.4] applies and shows that the restriction of the tautological functor (2.2.10) to the heart of the t-structure is an equivalence of categories

$$\mathcal{H} \ \xrightarrow{\cong} \ \text{mod-}\,\mathrm{End} \ .$$

Part (i) shows that the functor $\Phi : \mathbf{A}_G \longrightarrow \mathrm{End}^{\mathrm{op}}$ is an isomorphism of pre-additive categories, so it induces an isomorphism between the category of End-modules and the category of G-Mackey functors. This equivalence turns the End-module $[-,X]^G$ into the G-Mackey functor $\underline{\pi}_0(X)$. This completes the proof. $\square$

Remark 2.2.11 (Eilenberg-Mac Lane spectra for G-Mackey functors). For discrete groups G, part (ii) of Theorem 2.2.9 in particular provides an Eilenberg-Mac Lane spectrum for every G-Mackey functor M, i.e., an orthogonal G-spectrum HM such that $\underline{\pi}_k(HM) = 0$ for all $k \neq 0$ and such that the G-Mackey functor $\underline{\pi}_0(HM)$ is isomorphic to M; and these properties characterize HM up to preferred isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$. Indeed, a choice of inverse to the equivalence $\underline{\pi}_0$ of Theorem 2.2.9 (ii), composed with the inclusion of the heart, provides an Eilenberg-Mac Lane functor

$$H \ : \ \mathcal{M}_G \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_G)$$

to the stable G-homotopy category.

The previous remark constructs Eilenberg-Mac Lane spectra associated to G-Mackey functors; the stable G-homotopy type is determined by the algebraic input data up to preferred isomorphism, but the construction is an abstract version of 'killing homotopy groups' and does not yield an explicit pointset level model. In the next example we recall a well-known pointset level construction that yields an Eilenberg-Mac Lane spectrum for the constant G-Mackey functor, compare Example 2.2.4 (ii).

Example 2.2.12 (Eilenberg-Mac Lane spectra for constant Mackey functors). Let B be an abelian group. The orthogonal Eilenberg-Mac Lane spectrum $B[\mathbb{S}]$ is defined at an inner product space V by

$$B[\mathbb{S}](V) \ = \ B[S^V] \ ,$$

the reduced B-linearization of the V-sphere. The underlying set of this space consists of finite linear combinations of elements of S^V with coefficients in B, modulo the subgroup of B-multiples of the basepoint. The topology is as a quotient space of $\amalg_{n\geq 0} B^n \times (S^V)^n$.

The orthogonal group $O(V)$ acts through the action on S^V and the structure map $\sigma_{V,W} : S^V \wedge B[\mathbb{S}](W) \longrightarrow B[\mathbb{S}](V \oplus W)$ is given by

$$S^V \wedge B[S^W] \ \longrightarrow \ B[S^{V\oplus W}] \ , \quad v \wedge \left(\sum\nolimits_i b_i \cdot w_i\right) \ \longmapsto \ \sum\nolimits_i b_i \cdot (v \wedge w_i) \ .$$

The underlying non-equivariant space of $B[S^V]$ is an Eilenberg-Mac Lane space of type (B,n), where $n = \dim(V)$. Hence the underlying non-equivariant homotopy type of $B[\mathbb{S}]$ is that of an Eilenberg-Mac Lane spectrum for B. If G is any Lie group, then $B[\mathbb{S}]$ becomes an orthogonal G-spectrum by letting G act trivially. We warn the reader that for compact Lie groups of positive dimension, the equivariant homotopy groups of $B[\mathbb{S}]$ are *not* generally concentrated in dimension zero; for example, the group $\pi_1^{U(1)}(\mathbb{Z}[\mathbb{S}])$ is isomorphic to $\mathbb{Q}$ by [**60**, Thm. 5.3.16]. Also, the group $\pi_0^G(B[\mathbb{S}])$ may not be isomorphic to B; for example, the group $\pi_0^{SU(2)}(\mathbb{Z}[\mathbb{S}])$ has rank 2 by [**59**, Ex. 4.16].

However, for *discrete* groups G, the equivariant behavior of $B[\mathbb{S}]$ is as expected, and the orthogonal G-spectrum $B[\mathbb{S}]$ is an Eilenberg-Mac Lane spectrum of the constant G-Mackey functor $\underline{B}$. Indeed, $B[\mathbb{S}]$ is obtained from a Γ-space $B[-]$ by evaluation on spheres. For every finite subgroup K of G, we can view this Γ-space as a Γ-K-space by letting K act trivially. For every finite K-set S, the map

$$P_S \ : \ B[S] \ \longrightarrow \ B[1_+]^S \ = \ B^S$$

is then a homeomorphism, so in particular a K-homotopy equivalence, and $B[-]$ is a very special Γ-K-space in the sense of Shimakawa [**69**, Def. 1.3]. Since $\pi_0(B[1^+])$

is a group (as opposed to a monoid only), Shimakawa's Theorem B proves that the adjoint structure maps $\tilde{\sigma}_{V,W} : B[S^V] \longrightarrow \mathrm{map}(S^W, B[S^{V\oplus W}])$ are K-weak equivalences, see also [**60**, Thm. B.61]. Since K was an arbitrary finite subgroup of G, this shows that $B[\mathbb{S}]$ is a G-Ω-spectrum, and an Eilenberg-Mac Lane spectrum for $\underline{B}$.

2.3. Rational proper stable homotopy theory

The purpose of this section is to give an algebraic model for the rational proper G-equivariant stable homotopy category of a discrete group. We call an orthogonal G-spectrum X *rational* if the equivariant homotopy groups $\pi_k^H(X)$ are uniquely divisible (i.e., $\mathbb{Q}$-vector spaces) for all compact subgroups H of G. For discrete groups G, Theorem 2.3.4 below shows that the full triangulated subcategory of rational G-spectra inside $\mathrm{Ho}(\mathrm{Sp}_G)$ is equivalent to the unbounded derived category of rational G-Mackey functors. As in the case of finite groups, the abelian category of rational G-Mackey functors can be simplified by 'dividing out transfers', see Proposition 2.3.5. However – in contrast to the case of finite groups – this abelian category is in general *not* semisimple, see Remark 2.3.7.

Remark 2.3.1. Let G be a Lie group and X a rational orthogonal G-spectrum. For $n \in \mathbb{Z}$ we let $n \cdot X \in [X, X]^G$ denote the n-fold sum of the identity morphism of X. For every compact subgroup H of G, the morphism $n \cdot X$ induces multiplication by n on $\pi_*^H(X)$, which is invertible since X is rational. This means that $n \cdot X$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$. Hence the endomorphism ring $[X, X]^G$ of X in $\mathrm{Ho}(\mathrm{Sp}_G)$ is a $\mathbb{Q}$-algebra. So all morphism groups in the full subcategory $\mathrm{Ho}^{\mathbb{Q}}(\mathrm{Sp}_G)$ of rational spectra are uniquely divisible, i.e., $\mathrm{Ho}^{\mathbb{Q}}(\mathrm{Sp}_G)$ is a $\mathbb{Q}$-linear category.

Proposition 2.3.2. *Let H and K be finite subgroups of a discrete group G. Then the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty G/H)$ is torsion for every $k > 0$.*

Proof. The underlying K-space of G/H is the disjoint union of its K-orbits $K(gH)$; this becomes a wedge decomposition after passing to unreduced suspension spectra. Since equivariant homotopy groups takes wedges to direct sums, $\pi_k^K(\Sigma_+^\infty G/H)$ is isomorphic to the direct sum, indexed by K-H-double coset, of the groups $\pi_k^K(\Sigma_+^\infty K(gH))$. The K-set $K(gH)$ is isomorphic to $K/(K \cap {}^gH)$, so the Wirthmüller isomorphism [**78**], see also (2.1.11) or [**60**, Thm. 3.2.15], provides an identification

$$\pi_*^K(\Sigma_+^\infty K(gH)) \cong \pi_*^K(\Sigma_+^\infty K/(K \cap {}^gH)) \cong \pi_*^{K\cap {}^gH}(\mathbb{S}_G) .$$

The claim thus follows because for every finite group L, the L-equivariant stable stems are finite in positive degrees. To see this, we can exploit the fact that the groups $\pi_k^L(\mathbb{S}_G)$ can rationally be recovered as the product of the W_LJ-fixed subgroup of the geometric fixed point homotopy groups $\Phi_k^J(\mathbb{S}_G)$, see for example [**60**, Cor. 3.4.28]; the latter groups are stable homotopy groups of spheres, which are finite in positive degrees. □

Before establishing an algebraic model for the rational stable G-homotopy category, we first recall the two rational model structures to be compared. We let $\mathcal{A}$ be a pre-additive category, such as the Mackey category $\mathbf{A}_G$. We denote by $\mathcal{A}$-mod the category of additive functors from $\mathcal{A}$ to the category of $\mathbb{Q}$-vector spaces. This is an abelian category, and the rationalized represented functors $\mathbb{Q} \otimes \mathcal{A}(a, -)$, for all objects a of $\mathcal{A}$, form a set of finitely presented projective generators of $\mathcal{A}$-mod. The category of $\mathbb{Z}$-graded chain complexes in the abelian category $\mathcal{A}$-mod then admits the *projective model structure* with the quasi-isomorphisms as weak equivalences. The fibrations in the projective model structure are those chain morphisms that are surjective in every chain complex degree and at every object of $\mathcal{A}$. This projective model structure for complexes of $\mathcal{A}$-modules is a special case of [**11**, Thm. 5.1].

We also need the rational version of the stable model structure on orthogonal G-spectra established in Theorem 1.2.22. We call a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra a *rational equivalence* if the map

$$\mathbb{Q} \otimes \pi_k^H(f) \ : \ \mathbb{Q} \otimes \pi_k^H(X) \ \longrightarrow \ \mathbb{Q} \otimes \pi_k^H(Y)$$

is an isomorphism for all integers k and all compact subgroups H of G.

THEOREM 2.3.3 (Rational stable model structure). *Let G be a Lie group.*

(i) *The rational equivalences and the cofibrations are part of a model structure on the category of orthogonal G-spectra, the* rational stable model structure.
(ii) *The fibrant objects in the rational stable model structure are the rational G-Ω-spectra.*
(iii) *The rational stable model structure is cofibrantly generated, proper and topological.*

Theorem 2.3.3 is obtained by Bousfield localization of the stable model structure on orthogonal G-spectra, and one can use a similar proof as for the rational stable model structure on sequential spectra in [**63**, Lemma 4.1]. We omit the details.

THEOREM 2.3.4. *Let G be a discrete group. There is a chain of Quillen equivalences between the category of orthogonal G-spectra with the rational stable model structure and the category of chain complexes of rational G-Mackey functors. In particular, this induces an equivalence of triangulated categories*

$$\mathrm{Ho}^{\mathbb{Q}}(\mathrm{Sp}_G) \ \xrightarrow{\ \cong\ } \ \mathcal{D}\big(\mathcal{M}_G^{\mathbb{Q}}\big) \ .$$

The equivalence can be chosen so that the homotopy G-Mackey functor on the left hand side corresponds to the homology G-Mackey functor on the right hand side.

PROOF. We prove this as a special case of the 'generalized tilting theorem' of Brooke Shipley and the fifth author. Indeed, by Corollary 1.3.11 the unreduced suspension spectra of the G-sets G/H are small weak generators of the stable G-homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ as H varies through all finite subgroups of G. So the rationalizations $(\Sigma^\infty_+ G/H)_{\mathbb{Q}}$ are small weak generators of the rational stable G-homotopy category $\mathrm{Ho}^{\mathbb{Q}}(\mathrm{Sp}_G)$.

By Proposition 1.3.10 the evaluation map

$$[\Sigma^\infty_+ G/H, X]^G_* \ \longrightarrow \ \pi^H_*(X) \ , \quad [f] \longmapsto f_*(u_H)$$

is an isomorphism, where $u_H \in \pi_0^H(\Sigma^\infty_+ G/H)$ is the tautological class. So the graded morphism groups between the small generators are given by

$$\begin{aligned} [(\Sigma^\infty_+ G/K)_{\mathbb{Q}}[k], \ (\Sigma^\infty_+ G/H)_{\mathbb{Q}}] \ &\cong \ \pi_k^K((\Sigma^\infty_+ G/H)_{\mathbb{Q}}) \ \cong \ \mathbb{Q} \otimes \pi_k^K(\Sigma^\infty_+ G/H) \\ &\cong \ \begin{cases} \mathbb{Q} \otimes \mathbf{A}_G(H,K) & \text{for } k = 0, \text{ and} \\ \quad 0 & \text{for } k \neq 0. \end{cases} \end{aligned}$$

Here we have used Theorem 2.2.9, Proposition 2.3.2, and the fact that the equivariant homotopy groups of unreduced suspension spectra vanish in negative dimensions, see for example [**60**, Prop. 3.1.44]. The rational stable model structure on orthogonal G-spectra is topological (hence simplicial), cofibrantly generated, proper and stable; so we can apply the 'generalized tilting theorem' [**64**, Thm. 5.1.1]. This theorem yields a chain of Quillen equivalences between orthogonal G-spectra in the rational stable model structure and the category of chain complexes of $\mathbb{Q} \otimes \mathbf{A}_G$-modules, i.e., additive functors from the rationalized G-Mackey category $\mathbb{Q} \otimes \mathbf{A}_G$ to abelian groups. This functor category is equivalent to the category of additive functors from $\mathbf{A}_G$ to $\mathbb{Q}$-vector spaces, and this proves the theorem. □

There is a further algebraic simplification of the category of rational G-Mackey functors: Given a G-Mackey functor M and a finite subgroup H of G, we let $\tau(M)(H)$ denote the quotient of $M(H)$ by all transfers from proper subgroups of H. The values $\tau(M)(H)$ no longer assemble into a G-Mackey functor, but they inherit induced conjugation maps $g_\star : \tau(M)(H^g) \longrightarrow \tau(M)(H)$ since conjugations and transfers of G-Mackey functors commute. We let Conj_G denote the conjugation category of G, i.e., the category with objects the finite subgroups of G and morphisms $\mathrm{Conj}_G(H,K) = \{g \in G \mid {}^gH = K\}/K$, the set of elements of G which conjugate H onto K, modulo conjugation by elements of K. Then $\tau(M)$ naturally forms a covariant functor from Conj_G to abelian groups.

It turns out that every *rational* G-Mackey functor M can be reconstructed uniquely from the Conj_G-functor $\tau(M)$:

PROPOSITION 2.3.5. *For every discrete group G, the functor $\tau : \mathcal{M}_G^{\mathbb{Q}} \longrightarrow \mathcal{F}(\mathrm{Conj}_G, \mathbb{Q})$ is an equivalence of abelian categories.*

PROOF. We explain how to deduce the claim from the finite group case, which can be found in [**22**, App. A] or [**60**, Thm. 3.4.22]. When comparing to [**22**, App. A], one must use that quotienting $M(H)$ by all proper transfers can be identified with inverting the idempotent called e_H in [**22**, App. A].

Since τ commutes with colimits and $\mathcal{M}_G^{\mathbb{Q}}$ is a functor category, τ has a right adjoint $R : \mathcal{F}(\mathrm{Conj}_G, \mathbb{Q}) \longrightarrow \mathcal{M}_G^{\mathbb{Q}}$. The value of the right adjoint at a Conj_G-functor N is given by

$$R(N)(H) \ = \ \mathrm{Nat}_{\mathrm{Conj}_G}(\tau(\mathbb{Q} \otimes \mathbf{A}_G(H,-)), N) \ ,$$

where H is a finite subgroup of G and $\mathbf{A}_G(H,-)$ denotes the represented G-Mackey functor. We claim that, as a $\mathbb{Q}$-vector space, $R(N)(H)$ only depends on the underlying Conj_H-functor of N. For this we note that $\tau(\mathbb{Q} \otimes \mathbf{A}_G(H,-))(K)$ can be identified with the $\mathbb{Q}$-linearization of the set of G-equivariant maps from G/K to G/H. This set corresponds to the subset of elements $g \in G$ for which the conjugate K^g is a subgroup of H, modulo the right H-action. Stated differently, it is the disjoint union of the sets of all g such that ${}^gJ$ is equal to K, where J ranges through all subgroups of H, again modulo H. This yields an isomorphism of Conj_G-modules

$$\tau(\mathbb{Q} \otimes \mathbf{A}_G(H,-)) \ \cong \ (\bigoplus_{J \subseteq H} \mathbb{Q}[\mathrm{Conj}_G(J,-)])/H \ .$$

Thus, for every rational Conj_G-functor N the morphism group $\mathrm{Nat}_{\mathrm{Conj}_G}(\tau(\mathbb{Q} \otimes \mathbf{A}_G(H,-)), N)$ is naturally isomorphic to $(\bigoplus_{J \subseteq H} N(J))^H$. In particular, it only depends on the underlying Conj_H-functor of N, which proves the claim.

By definition, the value $\tau(M)(H)$ also only depends on the underlying H-Mackey functor of M. Both in G-Mackey functors and in Conj_G-functors isomorphisms are tested levelwise, so we can reduce to the finite group case to see that unit and counit of the adjunction are isomorphisms, compare [**60**, Thm. 3.4.22]. This finishes the proof. □

The category Conj_G is a groupoid and equivalent to the disjoint union of Weyl groups $W_G H = (N_G H)/H$, where H ranges through a system of representatives of conjugacy classes of finite subgroups. Hence the category of Conj_G-functors is equivalent to the product of the $\mathbb{Q}[W_G H]$-module categories. Since forming derived categories commutes with products of abelian categories, we get:

COROLLARY 2.3.6. *Let G be a discrete group. The rational stable G-homotopy category is equivalent to the product of the derived categories of $\mathbb{Q}[W_G H]$-modules, where H ranges through a system of representatives of conjugacy classes of finite subgroups of G.*

The equivalence of the previous corollary is implemented by 'geometric fixed points', see [**60**, Prop. 3.4.26] for the precise statement.

REMARK 2.3.7. There is an important homological difference between rational G-Mackey functors for finite groups versus infinite discrete groups. If G is finite, all Weyl groups of subgroups are also finite and hence the abelian category of rational G-Mackey functors is semisimple. So every object is projective and injective and the derived category is equivalent, by taking homology, to the category of graded rational Mackey functors over G.

This does not generalize to rational G-Mackey functors for infinite discrete groups. Indeed, already the simplest case $G = \mathbb{Z}$ illustrates this. Since the trivial subgroup is the only finite subgroup of $\mathbb{Z}$, the category of $\mathbb{Z}$-Mackey functors is equivalent to the category of abelian groups with a $\mathbb{Z}$-action; this category in turn is equivalent to the category of modules over the Laurent series ring $\mathbb{Z}[t, t^{-1}]$. So $\mathrm{Ho}^{\mathbb{Q}}(\mathrm{Sp}_{\mathbb{Z}})$ is equivalent to the derived category of the ring $\mathbb{Q}[t, t^{-1}]$ which has global dimension 1. For example, $\mathbb{Q}$, with t acting as the identity, is not projective.

EXAMPLE 2.3.8. As an example we consider the G-sphere spectrum $\mathbb{S}_G$. For every finite subgroup H of G, the group $\pi_k^H(\mathbb{S}_G)$ is the k-th H-equivariant stable stem. So this group is trivial for negative k, finite for positive k, and isomorphic to the Burnside ring of H for $k = 0$. Hence

$$\mathbb{Q} \otimes \underline{\pi}_k(\mathbb{S}_G) \;\cong\; \begin{cases} \mathbb{Q} \otimes \mathbb{A} & \text{for } k = 0, \text{ and} \\ \quad 0 & \text{for } k \neq 0. \end{cases}$$

Since the rationalized homotopy group G-Mackey functors are concentrated in a single degree, the equivalence of categories of Theorem 2.3.4 takes the G-sphere spectrum to the G-Mackey functor $\mathbb{Q} \otimes \mathbb{A}$, considered as a complex in degree 0.

The equivalence of categories in particular induces isomorphisms of the graded endomorphism rings of corresponding objects. The graded endomorphism ring of a G-Mackey functor M in the derived category is its Ext algebra, i.e., the graded abelian group $\mathrm{Ext}^*_{\mathcal{M}_G}(M, M)$ with multiplication by Yoneda product (splicing of exact sequences). So we conclude that

$$\mathbb{Q} \otimes [\mathbb{S}_G[k], \mathbb{S}_G]^G \;\cong\; \mathcal{D}(\mathcal{M}_G)(\mathbb{Q} \otimes \mathbb{A}[k], \mathbb{Q} \otimes \mathbb{A}) \;\cong\; \mathrm{Ext}^{-k}_{\mathcal{M}_G}(\mathbb{Q} \otimes \mathbb{A}, \mathbb{Q} \otimes \mathbb{A}) \ .$$

For infinite groups G, the Burnside ring G-Mackey functor is typically neither projective nor injective, and has non-trivial Ext groups in non-zero degrees. So for infinite groups G, the G-sphere spectrum typically has non-trivial stable self-maps of *negative* degrees.

Again, the simplest case $G = \mathbb{Z}$ already illustrates this phenomenon. As we explained in the previous remark, the category of $\mathbb{Z}$-Mackey functors is equivalent, by evaluation at the trivial subgroup, to the category of $\mathbb{Z}[t, t^{-1}]$-modules, and the Burnside ring $\mathbb{Z}$-Mackey functor corresponds to $\mathbb{Z}$ with t acting as the identity. The Ext algebra of this $\mathbb{Z}[t, t^{-1}]$-module is an exterior algebra on a class in $\mathrm{Ext}^1_{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z}, \mathbb{Z})$. So the rationalized algebra is an exterior algebra over $\mathbb{Q}$ on one generator of (cohomological) degree 1.

The exterior generator in $\mathbb{Q} \otimes [\mathbb{S}_{\mathbb{Z}}, \mathbb{S}_{\mathbb{Z}}[1]]^{\mathbb{Z}}$ is realized by the universal cover of the circle, in the following sense. The real line $\mathbb{R}$ is a $\mathbb{Z}$-space by translation:

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R} \ , \quad (n, x) \longmapsto n + x \ .$$

In fact, this action makes $\mathbb{R}$ into a universal space for the group $\mathbb{Z}$ (for proper actions or, equivalently, for free actions). Since $\mathbb{R}$ is non-equivariantly contractible, the unique map $\mathbb{R} \longrightarrow *$ is a $\mathcal{C}om$-weak equivalence, so we obtain a weak $\mathbb{Z}$-map from a point to S^1 as the composite

$$* \xleftarrow{\simeq} \mathbb{R} \longrightarrow S^1 \ ,$$

where the right map is a universal cover. Adding disjoint basepoints to $*$ and $\mathbb{R}$, passing to suspension spectra and going into the stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_{\mathbb{Z}})$ produces a non-trivial self map of $\mathbb{S}_{\mathbb{Z}}$ of degree -1.

CHAPTER 3

Proper equivariant cohomology theories

3.1. Excisive functors from G-spectra

In this section we discuss 'excisive functors', i.e., contravariant homotopy functors defined on finite proper G-CW-complexes that satisfy excision for certain pushouts, see Definition 3.1.1. Excisive functors are the components of proper G-cohomology theories, to be studied in Section 3.2 below. In Construction 3.1.9 we recall the classical procedure to define an excisive functor from a sequential G-spectrum. Remark 3.1.12 explains why the excisive functors represented by sequential G-spectra are precisely the ones represented by 'G-orbit spectra' in the sense of Davis and the third author [**13**, Def. 4.1], i.e., by contravariant functors from the $\mathcal{F}in$-orbit category of G to spectra.

In Definition 3.1.13 we explain that orthogonal G-spectra also define excisive functors by taking morphism groups in the triangulated stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ from unreduced suspension spectra. As we show in the proof of Proposition 3.1.15, every such 'represented' cohomology theory is also represented by a sequential G-spectrum, namely the underlying sequential G-spectrum of a π_*-isomorphic orthogonal G-Ω-spectrum. While the represented functor $[\Sigma^\infty_+(-), E]^G$ is easily seen to extend to a proper G-cohomology theory, it does not come with explicit 'cycles' that represent cohomology classes. This makes it difficult to compare the represented G-cohomology theory with other theories, such as equivariant cohomotopy or equivariant K-theory. To remedy this, Construction 3.1.19 introduces a more down-to-earth description, in the case of discrete groups, based on parameterized equivariant homotopy theory, of the excisive functor represented by an orthogonal G-spectrum E. The construction generalizes the equivariant cohomotopy groups of the third author [**38**, Sec. 6], which is the special case $E = \mathbb{S}_G$ of the equivariant sphere spectrum; many of the arguments are inspired by that special case. We then show in Theorem 3.1.34 that for discrete groups, the new theory agrees with the represented theory.

Definition 3.1.1. Let G be a Lie group. A functor

$$\mathcal{H} \ : \ (\text{finite proper } G\text{-CW-complexes})^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

is *excisive* if it satisfies the following conditions:

(i) (Homotopy invariance) Let $f, f' : Y \longrightarrow X$ be continuous G-maps between finite proper G-CW-complexes that are equivariantly homotopic. Then $\mathcal{H}(f) = \mathcal{H}(f')$.

(ii) (Additivity) For all finite proper G-CW-complexes X and Y, the map

$$(i_X^*, i_Y^*) \ : \ \mathcal{H}(X \amalg Y) \ \longrightarrow \ \mathcal{H}(X) \times \mathcal{H}(Y)$$

is bijective, where $i_X : X \longrightarrow X \amalg Y$ and $i_Y : Y \longrightarrow X \amalg Y$ are the summand inclusions.

(iii) (Excision) Let (X, A) and (Y, B) be two finite proper G-CW-pairs, and let

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] & Y \end{tikzcd} \tag{3.1.2}$$

be a pushout square of G-spaces, where the horizontal maps are inclusions, and f and g are cellular maps. Then for all $(b, x) \in \mathcal{H}(B) \times \mathcal{H}(X)$ such that $f^*(b) = i^*(x)$ in $\mathcal{H}(A)$, there is an element $y \in \mathcal{H}(Y)$ such that $j^*(y) = b$ and $g^*(y) = x$.

Now we develop some general features of excisive functors, in particular a 5-term Mayer-Vietoris sequence, see Proposition 3.1.5. For a finite G-CW-pair (X, A) we use the notation

$$\mathcal{H}(X|A) \;=\; \ker(i^* : \mathcal{H}(X) \longrightarrow \mathcal{H}(A)) ,$$

where $i : A \longrightarrow X$ is the inclusion.

Proposition 3.1.3. *Let G be a Lie group and let $f : A \longrightarrow Y$ be a cellular G-map between finite proper G-CW-complexes. Then for every excisive functor $\mathcal{H}$, the canonical map $A \times S^1 \longrightarrow Y \cup_{A\times\infty, f} A \times S^1$ induces an isomorphism*

$$\mathcal{H}(Y \cup_{A\times\infty} A \times S^1|Y) \;\xrightarrow{\cong}\; \mathcal{H}(A \times S^1|A \times \infty) .$$

Proof. We start with the special case where f is the inclusion of a subcomplex. Then both squares in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "{(-,\infty)}"] \arrow[d] & A \times S^1 \arrow[r, "\mathrm{proj}"] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & Y \cup_{A\times\infty} A \times S^1 \arrow[r, "\mathrm{proj}"'] & Y \end{tikzcd}$$

are pushouts, where all vertical maps are inclusions. Excision for the left square is the surjectivity of the map in question.

For injectivity we consider a class $w \in \mathcal{H}(Y \cup_{A\times\infty} A \times S^1|Y)$ that restricts to 0 on $A \times S^1$. Excision for the right pushout square provides a class $y \in \mathcal{H}(Y)$ such that $\mathrm{proj}^*(y) = w$. Since the projection restricts to the identity on Y, we obtain the relation

$$y \;=\; \mathrm{proj}^*(y)|_Y \;=\; w|_Y \;=\; 0 ,$$

and hence also $w = \mathrm{proj}^*(y) = 0$.

Now we treat the general case where $f : A \longrightarrow Y$ is an arbitrary cellular G-map. We let $Z = A \times [0, 1] \cup_{A\times 1, f} Y$ be the mapping cylinder of f. The map $(-, 0) : A \longrightarrow Z$ is the inclusion of a subcomplex, and the map $q : Z \longrightarrow Y$ that projects $A \times [0, 1]$ to A and is the identity on Y is an equivariant homotopy equivalence. So the induced map

$$q \cup (A \times S^1) \;:\; Z \cup_{A\times\infty} A \times S^1 \;\longrightarrow\; Y \cup_{A\times\infty} A \times S^1$$

is also an equivariant homotopy equivalence, by the gluing lemma. The two induced maps

$$\mathcal{H}(Y \cup_{A\times\infty} A \times S^1 | Y) \xrightarrow{(q\cup(A\times S^1))^*} \mathcal{H}(Z \cup_{A\times\infty} A \times S^1 | Z) \xrightarrow{(-,0)^*} \mathcal{H}(A \times S^1 | A \times \infty) \ .$$

are then isomorphisms by homotopy invariance and the previous paragraph, respectively. So the composite is an isomorphism, which proves the claim. □

As we shall now explain, the excision property extends to a Mayer-Vietoris sequence for a pushout square of G-spaces (3.1.2), where (X, A) and (Y, B) are finite proper G-CW-pairs. We define a *connecting homomorphism*

$$\partial \ : \ \mathcal{H}(A \times S^1 | A \times \infty) \longrightarrow \mathcal{H}(Y) \tag{3.1.4}$$

as the composite

$$\mathcal{H}(A \times S^1 | A \times \infty) \xrightarrow{\cong} \mathcal{H}(Y \cup_{A\times\infty} A \times S^1 | Y) \xrightarrow{\text{incl}} \mathcal{H}(Y \cup_{A\times\infty} A \times S^1)$$
$$\xrightarrow{(j\cup(A\times t)\cup g)^*} \mathcal{H}(B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X) \xrightarrow[\cong]{(q^*)^{-1}} \mathcal{H}(Y) \ .$$

The first isomorphism is the one provided by Proposition 3.1.3. The quotient map $t : [0,1] \longrightarrow S^1$ was defined in (1.3.3). The map

$$q \ = \ j \cup (g|_A \circ \text{proj}) \cup g \ : \ B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X \ \longrightarrow \ Y$$

is an equivariant homotopy equivalence, so it induces an isomorphism in the homotopy functor $\mathcal{H}$.

PROPOSITION 3.1.5. *Let G be a Lie group and $\mathcal{H}$ an excisive functor. Let (X, A) and (Y, B) be two finite proper G-CW-pairs, and let* (3.1.2) *be a pushout square of G-spaces, where f and g are cellular maps. Then the following sequence is exact:*

$$\mathcal{H}(B \times S^1 | B \times \infty) \times \mathcal{H}(X \times S^1 | X \times \infty) \xrightarrow{(f\times S^1)^* - (i\times S^1)^*} \mathcal{H}(A \times S^1 | A \times \infty)$$
$$\xrightarrow{\partial} \mathcal{H}(Y) \xrightarrow{(j^*, g^*)} \mathcal{H}(B) \times \mathcal{H}(X) \xrightarrow{f^* - i^*} \mathcal{H}(A)$$

PROOF. Exactness at $\mathcal{H}(B) \times \mathcal{H}(X)$ is the excision property for the functor $\mathcal{H}$. For exactness at $\mathcal{H}(Y)$ we consider the pushout square:

$$\begin{tikzcd} B \amalg X \arrow[r, "\iota"] \arrow[d, "j+g"'] & B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X \arrow[d, "j\cup(A\times t)\cup g"] \\ Y \arrow[r, "\text{incl}"'] & Y \cup_{A\times\infty} A \times S^1 \end{tikzcd} \tag{3.1.6}$$

Excision for this square provides an exact sequence

$$\mathcal{H}(Y \cup_{A\times\infty} A \times S^1 | Y) \xrightarrow{(j\cup(A\times t)\cup g)^*} \mathcal{H}(B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X) \xrightarrow{\iota^*} \mathcal{H}(B \amalg X) \ .$$

Proposition 3.1.3 identifies the group $\mathcal{H}(Y \cup_{A\times\infty} A \times S^1 | Y)$ with $\mathcal{H}(A \times S^1 | A \times \infty)$; the equivariant homotopy equivalence $q : B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X \longrightarrow Y$ induces an isomorphism from $\mathcal{H}(Y)$ to $\mathcal{H}(B \cup_{A\times 0,f} A \times [0,1] \cup_{A\times 1} X)$; and additivity identifies the group $\mathcal{H}(B \amalg X)$ with $\mathcal{H}(B) \times \mathcal{H}(X)$. These substitutions prove exactness of the original sequence at $\mathcal{H}(Y)$.

To establish exactness at $\mathcal{H}(A\times S^1|A\times\infty)$, we employ exactness at the target of the connecting homomorphism, but for the pushout square (3.1.6) instead of the original square. The result is an exact sequence

$$\mathcal{H}((B\amalg X)\times S^1|(B\amalg X)\times\infty) \xrightarrow{\partial} \mathcal{H}(Y\cup_{A\times\infty} A\times S^1) \xrightarrow{(\mathrm{incl}^*,(j\cup(A\times t)\cup g)^*)} \mathcal{H}(Y)\times\mathcal{H}(B\cup_{A\times 0,f} A\times[0,1]\cup_{A\times 1} X)\ .$$

The map $\mathrm{incl}^* : \mathcal{H}(Y\cup_{A\times\infty} A\times S^1) \longrightarrow \mathcal{H}(Y)$ is a split epimorphism, so passing to kernels gives another exact sequence

$$\mathcal{H}((B\amalg X)\times S^1|(B\amalg X)\times\infty) \xrightarrow{\partial} \mathcal{H}(Y\cup_{A\times\infty} A\times S^1|Y) \xrightarrow{(j\cup(A\times t)\cup g)^*} \mathcal{H}(B\cup_{A\times 0,f} A\times[0,1]\cup_{A\times 1} X)\ .$$

We use additivity to identify the first group with the product of $\mathcal{H}(B\times S^1|B\times\infty)$ and $\mathcal{H}(X\times S^1|X\times\infty)$; we use Proposition 3.1.3 to identify the middle group with $\mathcal{H}(A\times S^1|A\times\infty)$; and we use the equivariant homotopy equivalence $q : B\cup_{A\times 0,f} A\times[0,1]\cup_{A\times 1} X \longrightarrow Y$ to identify the third group with $\mathcal{H}(Y)$. Because the following square commutes

$$\begin{tikzcd}
\mathcal{H}((B\amalg X)\times S^1|(B\amalg X)\times\infty) \arrow[r, "\partial"] \arrow[d, "((i_B\times S^1)^*,(i_X\times S^1)^*)"'] & \mathcal{H}(Y\cup_{A\times\infty} A\times S^1|Y) \arrow[d, "\cong"] \\
\mathcal{H}(B\times S^1|B\times\infty)\times\mathcal{H}(X\times S^1|X\times\infty) \arrow[r, "(f\times S^1)^*-(i\times S^1)^*"'] & \mathcal{H}(A\times S^1|A\times\infty)
\end{tikzcd}$$

these substitutions result in the desired exactness at $\mathcal{H}(A\times S^1|A\times\infty)$ of the original sequence. □

The Mayer-Vietoris sequence yields a convenient criterion for checking that a natural transformation between excisive functors is an isomorphism.

Proposition 3.1.7. *Let G be a Lie group and let $\Psi : \mathcal{H} \longrightarrow \mathcal{H}'$ be a natural transformation between excisive functors. Suppose that for every compact subgroup H of G and every non-equivariant finite CW-complex X, the homomorphism $\Psi_{G/H\times X} : \mathcal{H}(G/H\times X) \longrightarrow \mathcal{H}'(G/H\times X)$ is an isomorphism. Then $\Psi_Y : \mathcal{H}(Y) \longrightarrow \mathcal{H}'(Y)$ is an isomorphism for every finite proper G-CW-complex Y.*

Proof. We show by induction over the number of cells in an equivariant CW-structure on Y that for every non-equivariant finite CW-complex L, the map $\Psi_{Y\times L} : \mathcal{H}(Y\times L) \longrightarrow \mathcal{H}'(Y\times L)$ is an isomorphism. Taking L to be a point proves the claim.

If there are no equivariant cells, then Y and $Y\times L$ are empty, and hence $\mathcal{H}(Y\times L) = \mathcal{H}'(Y\times L) = 0$. If Y is non-empty we choose a pushout square of G-spaces:

$$\begin{tikzcd}
G/H\times S^{n-1} \arrow[r, "i"] \arrow[d, "f"'] & G/H\times D^n \arrow[d, "g"] \\
B \arrow[r, "j"'] & Y
\end{tikzcd}$$

Here H is a compact subgroup of G, and B is a G-subcomplex of Y with one fewer cell. Taking product with L yields another pushout square. We also know by induction that Ψ is an isomorphism for $B \times L$ and $B \times L \times S^1$, and by hypothesis for $G/H \times S^{n-1} \times L \times S^1$, $G/H \times S^{n-1} \times L$, $G/H \times D^n \times L \times S^1$ and $G/H \times D^n \times L$. So the natural exact sequence

$$\begin{aligned} &\mathcal{H}(B \times L \times S^1 | B \times L \times \infty) \times \mathcal{H}(G/H \times D^n \times L \times S^1 | G/H \times D^n \times L \times \infty) \\ &\qquad \longrightarrow \mathcal{H}(G/H \times S^{n-1} \times L \times S^1 | G/H \times S^{n-1} \times L \times \infty) \ \longrightarrow \ \mathcal{H}(Y \times L) \\ &\qquad \longrightarrow \mathcal{H}(B \times L) \times \mathcal{H}(G/H \times D^n \times L) \ \longrightarrow \ \mathcal{H}(G/H \times S^{n-1} \times L) \end{aligned}$$

provided by Proposition 3.1.5 and the five lemma show that the map $\Psi_{Y \times L}$: $\mathcal{H}(Y \times L) \longrightarrow \mathcal{H}'(Y \times L)$ is an isomorphism. $\square$

Now we discuss three different ways to define an excisive functor on finite proper G-CW-complexes from an orthogonal G-spectrum E.

(i) The functor $E_G\langle X \rangle$ is defined as the colimit, over $n \geq 0$, of the sets $[S^n \wedge X_+, E(\mathbb{R}^n)]^G$, compare (3.1.10).
(ii) The functor $E_G(X) = [\Sigma^\infty_+ X, E]^G$ is represented by E in the triangulated stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, compare (3.1.14).
(iii) The functor $E_G[\![X]\!]$ is defined via parameterized homotopy classes indexed by G-vector bundles over X, see Construction 3.1.19.

All three constructions can be extended to $\mathbb{Z}$-graded proper cohomology theories by replacing E by its shifts $E[k]$, for $k \in \mathbb{Z}$, as defined in Remark 1.3.5. The first functor $E_G\langle - \rangle$ only depends on the underlying sequential G-spectrum of E, and can also be defined via 'G-orbit spectra' in the sense of [**13**], see Remark 3.1.12. The second construction $E_G(-)$ defines 'genuine' proper cohomology theories in complete generality, for all Lie groups; we refer to Remark 3.2.10 below for an explanation of the adjective 'genuine' in this context. The third functor $E_G[\![-]\!]$ is excisive for all discrete groups G, but *not* generally for positive dimensional Lie groups with infinitely many components. We explain in Remark 3.1.31 why the restriction to discrete groups arises here. For discrete groups, $E_G[\![X]\!]$ is naturally isomorphic to the represented theory $E_G(X)$, see Theorem 3.1.34; this isomorphism is the link to comparing the represented cohomology theories with other theories, such as equivariant cohomotopy or equivariant K-theory.

DEFINITION 3.1.8. Let G be a Lie group. A *sequential G-spectrum* E consists of a sequence of based G-spaces E_n, for $n \geq 0$, and based continuous G-maps $\sigma_n : S^1 \wedge E_n \longrightarrow E_{1+n}$.

Every orthogonal G-spectrum X has an *underlying sequential G-spectrum* with terms $X_n = X(\mathbb{R}^n)$ and structure maps $\sigma_n = \sigma_{\mathbb{R},\mathbb{R}^n} : S^1 \wedge X_n \longrightarrow X_{1+n}$.

CONSTRUCTION 3.1.9. Let G be a Lie group and let E be a sequential G-spectrum. For every based G-space Y, we define the G-equivariant E-cohomology group as

$$\tilde{E}_G\langle Y \rangle \ = \ \mathrm{colim}_n \, [S^n \wedge Y, E_n]^G_* \ ,$$

where $[-,-]^G_*$ denotes the set of equivariant homotopy classes of based G-maps. The colimit is taken over the poset of natural numbers, along the maps

$$[S^n \wedge Y, E_n]^G_* \ \xrightarrow{S^1 \wedge -} \ [S^{1+n} \wedge Y, S^1 \wedge E_n]^G_* \ \xrightarrow{(\sigma_n)_*} \ [S^{1+n} \wedge Y, E_{1+n}]^G_* \ .$$

For $n \geq 2$, the set $[S^n \wedge Y, E_n]^G_*$ is an abelian group under the pinch sum. The stabilization maps are group homomorphisms, so the colimit inherits an abelian group structure. The group $\tilde{E}_G\langle - \rangle$ is contravariantly functorial, by precomposition, for continuous based G-maps in Y.

If X is a finite proper G-CW-complex (without a basepoint), we define the (unreduced) G-equivariant E-cohomology group as

$$E_G\langle X \rangle \ = \ \tilde{E}_G\langle X_+ \rangle \ . \tag{3.1.10}$$

PROPOSITION 3.1.11. *Let G be a Lie group and E a sequential G-spectrum. Then the functor $E_G\langle - \rangle$ is excisive.*

PROOF. (i) Homotopy invariance is clear by the very definition, since each of the functors $[S^n \wedge X_+, E_n]^G$ sends G-homotopic maps in X to the same map.

(ii) The universal property of a disjoint union, applied to G-maps and G-equivariant homotopies, shows that for fixed n, the map

$$(i_X^*, i_Y^*) \ : \ [S^n \wedge (X \amalg Y)_+, E_n]_*^G \ \longrightarrow \ [S^n \wedge X_+, E_n]_*^G \times [S^n \wedge Y_+, E_n]_*^G$$

is bijective. Sequential colimits commute with finite products, so additivity follows by passing to colimits over n.

(iii) For the excision property we consider two finite proper G-CW-pairs (X, A) and (Y, B) and a pushout square of G-spaces (3.1.2). We consider two classes $b \in E_G\langle B \rangle$ and $x \in E_G\langle X \rangle$ such that $f^*(b) = i^*(x)$ in $E_G\langle A \rangle$. Then b and x can be represented by continuous based G-maps $\bar{b} : S^n \wedge B_+ \longrightarrow E_n$ and $\bar{x} : S^n \wedge X_+ \longrightarrow E_n$, for some $n \geq 0$. The hypothesis $f^*(b) = i^*(x)$ means that, possibly after increasing n, the two based G-maps $\bar{b} \circ (S^n \wedge f_+), \bar{x} \circ (S^n \wedge i_+) : S^n \wedge A_+ \longrightarrow E_n$ are equivariantly homotopic. Since (X, A) is a G-CW-pair, the inclusion $S^n \wedge A_+ \longrightarrow S^n \wedge X_+$ has the G-equivariant homotopy extension property for continuous based G-maps. So we can modify $\bar{x}$ into an equivariantly homotopic based G-map $\tilde{x} : S^n \wedge X_+ \longrightarrow E_n$ such that $\bar{b} \circ (S^n \wedge f_+) = \tilde{x} \circ (S^n \wedge i_+)$. Then $\bar{b}$ and $\tilde{x}$ glue to a continuous based G-map $S^n \wedge Y_+ \longrightarrow E_n$; this G-map represents a class $y \in E_G\langle Y \rangle$ such that $j^*(y) = b$ and $g^*(y) = x$. □

REMARK 3.1.12 (Cohomology theories from spectra over the orbit category). We let G be a discrete group. We denote by $\mathrm{Or}_G^{\mathcal{F}in}$ the $\mathcal{F}in$-orbit category of G, i.e., the full subcategory of the category of G-sets with objects G/H for all finite subgroups H of G. Davis and the third author explain in [**13**, Def. 4.1] how to construct a proper cohomology theory from a 'G-orbit spectrum', i.e., a functor

$$\mathbf{E} \ : \ \left(\mathrm{Or}_G^{\mathcal{F}in}\right)^{\mathrm{op}} \ \longrightarrow \ \mathrm{Sp}^{\mathbb{N}}$$

to the category of non-equivariant sequential spectra. We claim that the proper cohomology theories represented by G-orbit spectra are precisely the ones represented by sequential G-spectra as in (3.1.10); we sketch this comparison without giving complete details.

We recall the construction from [**13**]. For a G-space X, we denote by

$$\Phi(X) \ : \ \left(\mathrm{Or}_G^{\mathcal{F}in}\right)^{\mathrm{op}} \ \longrightarrow \ \mathbf{T}$$

the fixed point functor, i.e.,

$$\Phi(X)(G/H) \ = \ \mathrm{map}^G(G/H, X) \ .$$

Evaluation at the preferred coset eH is a homeomorphism $\Phi(X)(G/H) \cong X^H$ to the H-fixed point space. The $\mathbf{E}$-cohomology groups are then defined as

$$\mathbf{E}_G^k(X) \ = \ \pi_{-k}(\mathrm{map}^{\mathrm{Or}_G^{\mathcal{F}in}}(\Phi(X), \mathbf{E})) \ ,$$

the $(-k)$-th homotopy group of the spectrum $\mathrm{map}^{\mathrm{Or}_G^{\mathcal{F}in}}(\Phi(X), \mathbf{E})$ of natural transformations from $\Phi(X)$ to $\mathbf{E}$. By [**13**, Lemma 4.4], this indeed defines a proper G-cohomology theory on finite proper G-CW-complexes. Moreover, if $\mathbf{E}$ happens to take values in sequential Ω-spectra, then the cohomology theory also takes arbitrary disjoint unions to products.

We shall now explain how sequential G-spectra give rise to G-orbit spectra in such a way that the Davis-Lück cohomology theory recovers the cohomology theory as in (3.1.10). As in the case of G-spaces, a sequential G-spectrum E gives rise to a fixed point diagram of sequential spectra

$$\Phi(E) \ : \ \left(\mathrm{Or}_G^{\mathcal{F}in}\right)^{\mathrm{op}} \ \longrightarrow \ \mathrm{Sp}^{\mathbb{N}}$$

by setting

$$\Phi(E)(G/H) \ = \ \mathrm{map}^G(G/H, E) \ \cong \ E^H \ .$$

The fixed point functor Φ from G-spaces to spaces over the $\mathcal{F}in$-orbit category is fully faithful, so it induces a bijection

$$[S^n \wedge X_+, E_n]^G_* \ \xrightarrow[\cong]{\Phi} \ [S^n \wedge \Phi(X)_+, \Phi(E_n)]_*^{\mathrm{Or}_G^{\mathcal{F}in}} \ = \ \pi_n(\mathrm{map}^{\mathrm{Or}_G^{\mathcal{F}in}}(\Phi(X), \Phi(E_n))) \ .$$

Passing to colimits over n yields an isomorphism

$$E_G\langle X\rangle \ \cong \ \Phi(E)^0_G(X) \ .$$

So every sequential G-spectrum gives rise to a G-orbit spectrum that represents the same proper cohomology theory. The converse is also true. A G-orbit spectrum can be viewed as a sequential spectrum internal to the category of based spaces over the $\mathcal{F}in$-orbit category. Elmendorf's theorem [**19**] can be adapted to a Quillen equivalence between the $\mathcal{C}om$-model structure on the category of based G-spaces and the category of based spaces over the $\mathcal{F}in$-orbit category (with the objectwise, or projective, model structure). So every spectrum of based $\mathrm{Or}_G^{\mathcal{F}in}$-spaces is levelwise equivalent to $\Phi(E)$ for some sequential G-spectrum E.

The next definition is based on the triangulated stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$, so it makes essential use of our entire theory.

DEFINITION 3.1.13. Let G be a Lie group and E an orthogonal G-spectrum. For every G-space X, we define the represented G-equivariant E-cohomology group as

$$E_G(X) \ = \ [\Sigma^\infty_+ X, E]^G \ , \tag{3.1.14}$$

the group of morphisms in $\mathrm{Ho}(\mathrm{Sp}_G)$ from the unreduced suspension spectrum of X to E.

The group $E_G(X)$ is contravariantly functorial for continuous G-maps in X. For the one-point G-space $X = *$, the cohomology group already has another name:

$$E_G(*) \ = \ [\mathbb{S}_G, E]^G \ \cong \ \pi_0^G(E) \ ,$$

the 0-th G-equivariant homotopy group of E.

PROPOSITION 3.1.15. *Let G be a Lie group and E an orthogonal G-spectrum. Then the functor $E_G(-)$ is excisive.*

Proof. We start with the special case where E is an orthogonal G-Ω-spectrum. Since E is fibrant in the stable model structure on Sp_G, the derived adjunction isomorphism stemming from the Quillen adjoint functor pair

$$\Sigma^\infty_+ \ : \ G\mathbf{T} \rightleftarrows \mathrm{Sp}_G \ : \ (-)(0)$$

provides a bijection

$$[X, E(0)]^{G\mathbf{T}} \ \cong \ [\Sigma^\infty_+ X, E]^G = E_G(X) \ .$$

An orthogonal G-Ω-spectrum is in particular a naive Ω-spectrum, i.e., the adjoint structure map

$$\tilde{\sigma}_{\mathbb{R},\mathbb{R}^n} \ : \ E(\mathbb{R}^n) \ \longrightarrow \ \Omega E(\mathbb{R}^{1+n})$$

is a $\mathcal{C}om$-equivalence for every $n \geq 0$. For every finite proper G-CW-complex X, the based G-space $S^n \wedge X_+$ is $\mathcal{C}om$-cofibrant, so the map

$$(\tilde{\sigma}_{\mathbb{R},\mathbb{R}^n})_* \ : \ [S^n \wedge X_+, E(\mathbb{R}^n)]^G_* \ \longrightarrow \ [S^n \wedge X_+, \Omega E(\mathbb{R}^{1+n})]^G_*$$

is bijective. Hence also the stabilization maps in the colimit system defining $E_G\langle X\rangle$ are bijective. So the canonical map

$$[X, E(0)]^{G\mathbf{T}} \cong [X_+, E(0)]^G_* \ \longrightarrow \ \mathrm{colim}_{n\geq 0}\, [S^n \wedge X_+, E(\mathbb{R}^n)]^G_* \ = \ E_G\langle X\rangle$$

is bijective. Altogether we have exhibited a bijection between $E_G(X)$ and $E_G\langle X\rangle$ that is natural for G-maps in X. The functor $E_G\langle -\rangle$ is excisive by Proposition 3.1.11. Since the homotopy invariance, additivity and excision properties do not use the abelian group structure and only refer to the underlying set-valued functor, we conclude that the functor $E_G(-)$ is also excisive.

In the general case the stable model structure of Theorem 1.2.22 provides a π_*-isomorphism of orthogonal G-spectra $q : E \longrightarrow F$ whose target F is an orthogonal G-Ω-spectrum. Then $\gamma_G(q)$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$, and hence induces a natural isomorphism $E_G(-) \cong F_G(-)$. The latter functor is excisive by the previous paragraph, hence so is the former. $\square$

In order to compare the 'genuine' equivariant cohomology theories represented by orthogonal G-spectra with other theories, such as equivariant cohomotopy [**38**, Sec. 6] or equivariant K-theory [**42**, Sec. 3], we provide another description of the excisive functor represented by a G-spectrum E. This alternative description $E_G[\![X]\!]$ is in terms of parameterized equivariant homotopy theory over X, see Construction 3.1.19. The construction generalizes the equivariant cohomotopy groups of the third author [**38**, Sec. 6], which is the special case $E = \mathbb{S}_G$ of the equivariant sphere spectrum; many of the arguments are inspired by that special case.

While the definition of the group $E_G[\![X]\!]$ makes sense for all Lie groups, the excision property established in Theorem 3.1.29 below does not hold in that generality. Consequently, various results that depend on excision for the theory $E_G[\![-]\!]$ are only formulated for *discrete* groups; we explain in Remark 3.1.31 why the restriction to discrete groups arises. The new theory $E_G[\![X]\!]$ calculates the represented theory $E_G(X)$ for *discrete* groups and finite proper G-CW-complexes X, compare Theorem 3.1.34 (iv).

Definition 3.1.16. Let G be a Lie group and X a G-space. A *retractive G-space* over X is a triple (E, p, s), where E is a G-space, and

$$p \ : \ E \ \longrightarrow \ X \qquad \text{and} \qquad s \ : \ X \ \longrightarrow \ E$$

are continuous G-maps that satisfy $p \circ s = \mathrm{Id}_X$. If (E', p', s') is another retractive G-space over X, then a *morphism* from (E, p, s) to (E', p', s') is a continuous G-map $f : E \longrightarrow E'$ such that $p' \circ f = p$ and $s' = f \circ s$. A *parameterized homotopy* between two such morphisms is a continuous G-map $H \colon E \times I \longrightarrow E'$ such that $H(-, t)$ is a morphism of retractive G-spaces over X for every $t \in [0, 1]$.

CONSTRUCTION 3.1.17. We let G be a Lie group, E an orthogonal G-spectrum and $\xi : B \longrightarrow X$ a euclidean G-vector bundle over a G-space X. We define a G-space $E(\xi)$ as follows. If ξ has constant rank n, then we denote by $\mathcal{F}_n(\xi)$ the frame bundle of ξ, i.e., the principal $O(n)$-bundle whose fiber over $x \in X$ is the space of n-frames (orthonormal bases) of ξ_x. We now form the space

$$E(\xi) \;=\; \mathcal{F}_n(\xi) \times_{O(n)} E(\mathbb{R}^n)$$

endowed with the diagonal G-action from the action on ξ and on $E(\mathbb{R}^n)$. If the bundle does not have constant rank, then we let $X_{(n)}$ be the subspace of those $x \in X$ such that $\dim(\xi_x) = n$. The subspaces $X_{(n)}$ are open by local triviality, and they are G-invariant. We define

$$E(\xi) \;=\; \coprod_{n \geq 0} E(\xi|_{X_{(n)}}) \ .$$

The space $E(\xi)$ comes with a projection to X which is a locally trivial fiber bundle, with fiber $E(\mathbb{R}^n)$ over $X_{(n)}$. The $(G \times O(n))$-fixed basepoint of $E(\mathbb{R}^n)$ gives a preferred section

$$s \;:\; X \;\longrightarrow\; E(\xi) \ .$$

The projection and section are G-equivariant; so we can – and will – consider $E(\xi)$ as a retractive G-space over X.

An important special case of this construction is when $E = \mathbb{S}_G$ is the sphere spectrum. Here we write

$$S^\xi \;=\; \mathbb{S}_G(\xi) \;=\; \mathcal{F}_n(\xi) \times_{O(n)} S^n$$

for the locally trivial bundle with fiber S^n over $X_{(n)}$. In this situation, the section $s \colon X \longrightarrow S^\xi$ is a $\mathcal{C}om$-cofibration of G-spaces by Proposition 1.2.31. The quotient space $\mathrm{Th}(\xi) = S^\xi / s(X)$ is the Thom space of ξ as defined in (1.2.32) above.

The structure maps of the orthogonal G-spectrum E can be used to relate the spaces defined from different vector bundles. Given another G-vector bundle η over X of dimension m, a frame in η_x and a frame in ξ_x concatenate into a frame in $\eta_x \oplus \xi_x = (\eta \oplus \xi)_x$, and the resulting map

$$\mathcal{F}_m(\eta) \times_X \mathcal{F}_n(\xi) \;\longrightarrow\; \mathcal{F}_{m+n}(\eta \oplus \xi)$$

is $(G \times O(m) \times O(n))$-equivariant. Using the $(O(m) \times O(n))$-equivariant structure map $\sigma_{m,n} : S^m \wedge E(\mathbb{R}^n) \longrightarrow E(\mathbb{R}^{m+n})$ we obtain a continuous G-map

$$\begin{aligned} \sigma_{E,\eta,\xi} \;:\; S^\eta \wedge_X E(\xi) \;&= (\mathcal{F}_m(\eta) \times_{O(m)} S^m) \wedge_X (\mathcal{F}_n(\xi) \times_{O(n)} E_n) \\ &\cong (\mathcal{F}_m(\eta) \times_X \mathcal{F}_n(\xi)) \times_{O(m) \times O(n)} (S^m \wedge E(\mathbb{R}^n)) \\ &\xrightarrow{(\sigma_{m,n})_*} (\mathcal{F}_m(\eta) \times_X \mathcal{F}_n(\xi)) \times_{O(m) \times O(n)} E(\mathbb{R}^{m+n}) \\ &\longrightarrow \mathcal{F}_{m+n}(\eta \oplus \xi) \times_{O(m+n)} E(\mathbb{R}^{m+n}) \;=\; E(\eta \oplus \xi) \end{aligned}$$

of retractive G-spaces over X. If the bundles do not have constant rank, we perform these constructions separately over the components $X_{(n)}$. If $E = \mathbb{S}_G$ is the sphere spectrum, then these structure maps are isomorphisms

$$S^\eta \wedge_X S^\xi \;\cong\; S^{\eta\oplus\xi}$$

of retractive G-spaces over X.

Definition 3.1.18. We call a morphism $\psi : \xi \longrightarrow \eta$ of euclidean G-vector bundles an *isometric embedding* if ψ is fiberwise a linear isometric embedding of inner product spaces.

Construction 3.1.19. We let G be a Lie group, E an orthogonal G-spectrum, and X a finite, proper G-CW-complex. We define an abelian group $E_G[\![X]\!]$ as follows. Elements of $E_G[\![X]\!]$ are equivalence classes of pairs (ξ, u), where

(i) ξ is a euclidean G-vector bundle over X,
(ii) $u : S^\xi \longrightarrow E(\xi)$ is a map of retractive G-spaces over X.

To explain the equivalence relation we let $\psi : \xi \longrightarrow \eta$ be an isometric embedding of euclidean G-vector bundles over X. We write γ for the orthogonal complement of the image of ψ in η. Then γ is another G-vector bundle over X, and the map

$$\gamma\oplus\xi \;\longrightarrow\; \eta\;, \quad (x,y) \;\longmapsto\; x+\psi(y) \tag{3.1.20}$$

is an isomorphism. If $u : S^\xi \longrightarrow E(\xi)$ is a map of retractive G-spaces over X, we write $\psi_*(u)$ for the map of retractive G-spaces

$$S^\eta \;\cong\; S^\gamma\wedge_X S^\xi \xrightarrow{S^\gamma\wedge_X u} S^\gamma\wedge_X E(\xi) \xrightarrow{\sigma_{E,\gamma,\xi}} E(\gamma\oplus\xi) \;\cong\; E(\eta)\;.$$

We will refer to $\psi_*(u)$ as the *stabilization* of u along ψ. The two isomorphisms are induced by the bundle isomorphism (3.1.20). We call two pairs (ξ, u) and (ξ', v) *equivalent* if there is a G-vector bundle η over X and isometric embeddings $\psi : \xi \longrightarrow \eta$ and $\psi' : \xi' \longrightarrow \eta$ such that the two maps of retractive G-spaces

$$\psi_*(u)\;,\;\psi'_*(v) \;:\; S^\eta \;\longrightarrow\; E(\eta)$$

are parameterized G-equivariantly homotopic. We omit the verification that this relation is reflexive, symmetric and transitive, and hence an equivalence relation.

We suppose that $\psi : \xi \longrightarrow \eta$ is an equivariant isometric isomorphism of euclidean G-vector bundles over X. Then ψ is in particular an isometric embedding. So for every map of retractive G-spaces $u : S^\xi \longrightarrow E(\xi)$ over X, the pair (ξ, u) is equivalent to the pair

$$(\eta,\psi_*(u)) \;=\; (\eta, E(\psi)\circ u\circ S^{\psi^{-1}})\;.$$

Informally speaking, this says that conjugation by an isometric isomorphism does not change the class in $E_G[\![X]\!]$.

The isomorphism classes of G-vector bundles over X form a set, hence so do the equivalence classes. We write

$$[\xi, u] \;\in\; E_G[\![X]\!]$$

for the equivalence class of a pair (ξ, u). This finishes the definition of the set $E_G[\![X]\!]$.

Proposition 3.1.21. *Let G be a Lie group, X a finite proper G-CW-complex and ξ a euclidean G-vector bundle over X.*

(i) *Let $a, b : \xi \longrightarrow \nu$ be two equivariant isometric embeddings of euclidean G-vector bundles over X. Then $i_1 \circ a, i_1 \circ b : \xi \longrightarrow \nu \oplus \nu$ are homotopic through G-equivariant isometric embeddings, where $i_1 : \nu \longrightarrow \nu \oplus \nu$ is the embedding as the first summand.*

(ii) *Let $u, v : S^\xi \longrightarrow E(\xi)$ be two maps of retractive G-spaces. Then $[\xi, u] = [\xi, v]$ in $E_G[\![X]\!]$ if and only if there is an isometric embedding $\psi : \xi \longrightarrow \eta$ of euclidean G-vector bundles over X such that $\psi_*(u), \psi_*(v) : S^\eta \longrightarrow E(\eta)$ are parameterized G-homotopic.*

Proof. (i) We let $i_1, i_2 : \nu \longrightarrow \nu \oplus \nu$ be the embeddings as the first and second summand, respectively. Then the images of the isometric embeddings $i_1 \circ a$, $i_2 \circ b : \xi \longrightarrow \nu \oplus \nu$ are orthogonal. So there is an equivariant homotopy $H : \xi \times [0, 1] \longrightarrow \nu \oplus \nu$ through isometric embeddings from $i_1 \circ a$ to $i_2 \circ b$, for example

$$H(x, t) \;=\; (\sqrt{1 - t^2} \cdot a(x),\; t \cdot b(x)) \;.$$

For $a = b$ this in particular shows that $i_1 \circ b$ and $i_2 \circ b$ are G-homotopic through equivariant isometric embeddings. Altogether, $i_1 \circ a$ and $i_1 \circ b$ are G-homotopic through isometric embeddings.

(ii) The 'if' part of the claim holds by definition of the equivalence relation that defines $E_G[\![X]\!]$. Now we suppose that conversely, $[\xi, u] = [\xi, v]$ in $E_G[\![X]\!]$. Then there are two isometric embeddings $a, b : \xi \longrightarrow \nu$ of euclidean G-vector bundles over X such that $a_*(u)$ and $b_*(v)$ are parameterized equivariantly homotopic. Part (i) provides a homotopy between $i_1 \circ a$ and $i_1 \circ b$ through G-equivariant isometric embeddings. The homotopy induces a parameterized equivariant homotopy between the two maps $(i_1 \circ a)_*(v) : S^{\nu \oplus \nu} \longrightarrow E(\nu \oplus \nu)$ and $(i_1 \circ b)_*(v)$. So we obtain a chain of parameterized equivariant homotopies

$$(i_1 \circ a)_*(u) \;=\; (i_1)_*(a_*(u)) \;\simeq\; (i_1)_*(b_*(v)) \;=\; (i_1 \circ b)_*(v) \;\simeq\; (i_1 \circ a)_*(v) \;.$$

So the isometric G-embedding $\psi = i_1 \circ a : \xi \longrightarrow \nu \oplus \nu = \eta$ has the desired property. □

Now we define an abelian group structure on the set $E_G[\![X]\!]$. We let $\nabla : S^1 \longrightarrow S^1 \vee S^1$ be a pinch map; for definiteness, we take the same map as in [**38**, 6.2], namely

$$\nabla(x) \;=\; \begin{cases} \ \ \ln(x) \text{ in the first copy of } S^1 & \text{if } x \in (0, \infty), \\ -\ln(-x) \text{ in the second copy of } S^1 & \text{if } x \in (-\infty, 0), \text{ and} \\ \ \ \infty & \text{if } x \in \{0, \infty\}. \end{cases}$$

By Proposition 3.1.21 we can represent any two given classes of $E_G[\![X]\!]$ by pairs (ξ, u) and (ξ, v), defined on the same G-vector bundle ξ over X. To add the classes we stabilize the representative by the trivial line bundle $\underline{\mathbb{R}}$ and then form the 'pinch sum', i.e., the composite

$$\begin{aligned} \nabla(u, v) \;:\; S^{\underline{\mathbb{R}} \oplus \xi} \;&\cong\; S^1 \wedge_X S^\xi \xrightarrow{\nabla \wedge_X S^\xi} (S^1 \vee S^1) \wedge_X S^\xi \\ &\cong\; (S^1 \wedge_X S^\xi) \vee_X (S^1 \wedge_X S^\xi) \xrightarrow{(S^1 \wedge u) +_X (S^1 \wedge v)} \\ &\cong\; S^1 \wedge_X E(\xi) \xrightarrow{\sigma_{E, \underline{\mathbb{R}}, \xi}} E(\underline{\mathbb{R}} \oplus \xi) \;. \end{aligned}$$

This way of adding representatives is compatible with parameterized homotopy and stabilization along isometric embeddings, so we get a well-defined map

$$+ \ : \ E_G[\![X]\!] \times E_G[\![X]\!] \longrightarrow E_G[\![X]\!] \ , \quad [\xi, u] + [\xi, v] \ = \ [\xi \oplus \underline{\mathbb{R}}, \nabla(u, v)] \ .$$

The pinch map is coassociative and counital up to homotopy, and has an inverse up to homotopy; this implies that the binary operation thus defined is associative and unital, with the class of the trivial map (with values the respective basepoints) as unit, and that inverses exist. After stabilizing one additional time by $\underline{\mathbb{R}}$, the Eckmann-Hilton argument shows that the binary operation is commutative. So we have indeed defined an abelian group structure on the set $E_G[\![X]\!]$.

The groups $E_G[\![X]\!]$ are clearly covariantly functorial for morphisms of orthogonal G-spectra in E. A contravariant functoriality in the G-space X arises from pullback of vector bundles. We let $f : Y \longrightarrow X$ be a continuous G-map between two finite proper G-CW-complexes. We let ξ be a euclidean G-vector bundle over X and $u : S^\xi \longrightarrow E(\xi)$ a map of retractive G-spaces over X. Then $f^*(\xi)$ is a euclidean G-vector bundle over Y, and

$$f^*(u) \ : \ S^{f^*\xi} \ = \ f^*(S^\xi) \ \longrightarrow \ f^*(E(\xi)) \ = \ E(f^*\xi)$$

a map of retractive G-spaces over Y. The pullback construction respects parameterized homotopies and is compatible with stabilization along isometric embeddings. So we can define

$$f^* \ = \ E_G[\![f]\!] \ : \ E_G[\![X]\!] \ \longrightarrow \ E_G[\![Y]\!] \quad \text{by} \quad f^*[\xi, u] \ = \ [f^*(\xi), f^*(u)] \ .$$

Example 3.1.22. We consider $X = G/H$, the homogeneous G-space for a compact subgroup H of the Lie group G. Every orthogonal H-representation V gives rise to a euclidean G-vector bundle

$$\xi_V \ : \ G \times_H V \ \longrightarrow \ G/H \ , \quad [g, v] \ \longmapsto \ gH \ .$$

Moreover, every euclidean G-vector bundle over G/H is isomorphic to a bundle of this form.

We have $\mathcal{F}_n(\xi_V) = G \times_H \mathbf{L}(\mathbb{R}^n, V)$, where $n = \dim(V)$. So

$$E(\xi_V) \ = \ G \times_H E(V)$$

for every orthogonal G-spectrum E. In particular, $S^{\xi_V} = G \times_H S^V$. Every map of retractive G-spaces over G/H

$$S^{\xi_V} \ = \ G \times_H S^V \ \longrightarrow \ G \times_H E(V) \ = \ E(\xi_V)$$

is of the form $G \times_H f$ for a unique based H-map $f : S^V \longrightarrow E(V)$, and this correspondence passes to a bijection of homotopy classes. If we let V exhaust the finite-dimensional subrepresentations of a complete H-universe, these bijections assemble into an isomorphism

$$\begin{aligned} \pi_0^H(E) \ = \ \operatorname{colim}_V [S^V, E(V)]_*^G \ &\xrightarrow{\cong} \ E_G[\![G/H]\!] \\ [f : S^V \longrightarrow E(V)] \ &\longmapsto \ [G \times_H V, G \times_H f] \ . \end{aligned}$$

Example 3.1.23 (Compact Lie groups). We let H be a compact Lie group, X a finite H-CW-complex, and E an orthogonal H-spectrum. We define an isomorphism

$$\omega \ : \ \operatorname{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge X_+, E(V)]_*^H \ \longrightarrow \ E_H[\![X]\!] \quad \text{by} \quad [f] \ \longmapsto \ [X \times V, f^\sharp] \ .$$

Here $\mathcal{U}_H$ is a complete H-universe, and $s(\mathcal{U}_H)$ is the poset, under inclusion, of finite-dimensional H-subrepresentations of $\mathcal{U}_H$. Moreover, for a continuous based H-map $f : S^V \wedge X_+ \longrightarrow E(V)$, we write $X \times V$ for the trivial H-vector bundle over X with fiber V; a map of retractive H-spaces over X

$$f^\sharp \ : \ S^{X\times V} \ = \ X \times S^V \ \longrightarrow \ X \times E(V) \ = \ E(X \times V) \tag{3.1.24}$$

is defined by $f^\sharp(x, v) = (x, f(v \wedge x))$. Since H is compact Lie, every euclidean H-vector bundle over the compact H-space X embeds into a trivial bundle of the form $X \times V$, for some H-representation V, for example by [**65**, Prop. 2.4]; we can suppose that V is a subrepresentation of the complete H-universe $\mathcal{U}_H$. Moreover, every map of retractive H-spaces $X \times S^V \longrightarrow X \times E(V)$ is of the form $f^\sharp$ for a unique based H-map $f : S^V \wedge X_+ \longrightarrow E(V)$. So the map ω is surjective.

For injectivity we exploit that ω is a group homomorphism for the group structure on the source arising from the identification with $\pi_0^H(\mathrm{map}(X, E))$. We suppose that $\omega[f] = [X\times V, f^\sharp] = 0$. There is then an isometric embedding $\psi : X\times V \longrightarrow \eta$ of euclidean H-vector bundles over X such that $\psi_*(f^\sharp)$ is parameterized equivariantly null-homotopic. We let γ be the orthogonal complement of the image of $X \times V$ in η. We can embed γ into the trivial H-vector bundle $X\times W$ associated with another H-representation W, and we can then embed $V \oplus W$ into the complete H-universe $\mathcal{U}_H$ in a way that extends the inclusion $V \longrightarrow \mathcal{U}_H$. So we can altogether assume that $\psi = \mathrm{incl} : X \times V \longrightarrow X \times \bar{V}$ for $V \subset \bar{V}$ in the poset $s(\mathcal{U}_H)$. After stabilizing $f : S^V \wedge X_+ \longrightarrow E(V)$ along the inclusion of V into $\bar{V}$, we can assume without loss of generality that the map $f^\sharp$ is parameterized H-null-homotopic. Maps of retractive H-spaces $S^{X\times V} \longrightarrow E(X \times V)$ biject with continuous based H-maps $S^V \wedge X_+ \longrightarrow E(V)$, so we conclude that the map f is based H-null-homotopic. Thus f represents the zero element in the source of ω, and we have shown that the map ω is also injective.

Example 3.1.25. We let Γ be a closed subgroup of a Lie group G, and we let Y be a finite proper Γ-CW-complex. We let E be an orthogonal G-spectrum. We define an induction homomorphism

$$\mathrm{ind} \ : \ E_\Gamma[\![Y]\!] \ \longrightarrow \ E_G[\![G \times_\Gamma Y]\!] \tag{3.1.26}$$

as follows. We let (ξ, u) represent a class in $E_\Gamma[\![Y]\!]$. Then $G \times_\Gamma \xi$ is a G-vector bundle over $G \times_\Gamma Y$, with frame bundle

$$\mathcal{F}_n(G \times_\Gamma \xi) \ = \ G \times_\Gamma \mathcal{F}_n(\xi) \ ,$$

where $n = \dim(\xi)$. Hence

$$\begin{aligned} E(G \times_\Gamma \xi) \ &= \ \mathcal{F}_n(G \times_\Gamma \xi) \times_{O(n)} E(\mathbb{R}^n) \\ &= \ G \times_\Gamma \mathcal{F}_n(\xi) \times_{O(n)} E(\mathbb{R}^n) \ = \ G \times_\Gamma E(\xi) \end{aligned}$$

as retractive G-spaces over $G \times_\Gamma Y$. Moreover,

$$G \times_\Gamma u \ : \ S^{G\times_\Gamma \xi} \ = \ G \times_\Gamma S^\xi \ \longrightarrow \ G \times_\Gamma E(\xi) \ = \ E(G \times_\Gamma \xi)$$

is a map of retractive G-spaces. We can thus define the induction homomorphism by

$$\mathrm{ind}[\xi, u] \ = \ [G \times_\Gamma \xi, G \times_\Gamma u] \ .$$

Every euclidean G-vector bundle η over $G \times_\Gamma Y$ is isomorphic to a bundle of the form $G \times_\Gamma \xi$: we can take ξ as the restriction of η along the Γ-equivariant map

$$Y \ \longrightarrow \ G \times_\Gamma Y \ , \quad y \ \longmapsto \ [1, y] \ .$$

Similarly, every G-map $v : G \times_\Gamma S^\xi \longrightarrow G \times_\Gamma E(\xi)$ is of the form $G \times_\Gamma u$ for a unique Γ-map $u : S^\xi \longrightarrow E(\xi)$. Hence the induction homomorphism (3.1.26) is an isomorphism.

Remark 3.1.27 (Vector bundles versus representations). Now is a good time to explain why we build the theory $E_G[\![X]\!]$ using G-vector bundles, as opposed to just G-representations. One could contemplate a variation $\hat{E}_G[\![X]\!]$ where elements are represented by classes (V, v), where V is a G-representation and $v : S^V \wedge X_+ \longrightarrow E(V)$ is a based G-map. Two pairs (U, u) and (V, v) represent the same class in $\hat{E}_G[\![X]\!]$ if and only if there is a G-representation W and G-equivariant linear isometric embeddings $\psi : U \longrightarrow W$ and $\psi' : V \longrightarrow W$ such that the two stabilizations $\psi_*(u), \psi'_*(v) : S^W \wedge X_+ \longrightarrow E(W)$ are based G-homotopic. This construction provides a homotopy functor from the category of based G-spaces to abelian groups. A G-representation V gives rise to the trivial G-vector bundle $X \times V$ over any G-space X, and a based G-map $v : S^V \wedge X_+ \longrightarrow E(V)$ gives rise to a map of retractive G-spaces $v^\sharp : S^{X\times V} \longrightarrow E(X \times V)$, compare (3.1.24). This assignment is compatible with the equivalence relations, and provides a natural group homomorphism $\hat{E}_G[\![X]\!] \longrightarrow E_G[\![X]\!]$. Example 3.1.23 can be rephrased as saying that for *compact* Lie groups, this homomorphism is an isomorphism.

However, the construction $\hat{E}_G[\![-]\!]$ based on G-representations (as opposed to G-vector bundles) does not in general have induction isomorphisms. Our represented equivariant cohomology theories support induction isomorphisms, so this shows that the functor $\hat{E}_G[\![-]\!]$ is not in general represented by an orthogonal G-spectrum. The case $E = \mathbb{S}_G$ of stable cohomotopy, already considered in [**38**, Rk. 6.17], can serve to illustrate the lack of induction isomorphisms. We let G be any discrete group with the following two properties:

(a) Every finite-dimensional G-representation is trivial, and
(b) the group G has a non-trivial finite subgroup H.

As explained in [**38**, Rk. 6.17], for every finite G-CW-complex X, the cohomotopy group $\hat{\mathbb{S}}_G[\![X]\!]$ based on G-representations (as opposed to G-vector bundles over X) is isomorphic to the non-equivariant cohomotopy group $\pi^0_e(X/G)$ of the G-orbit space. In particular,

$$\hat{\mathbb{S}}_G[\![G/H]\!] \cong \pi^0_e(*) \cong \mathbb{Z} .$$

On the other other hand, $\hat{\mathbb{S}}_H[\![*]\!] = \pi^0_H(*)$ is isomorphic to the Burnside ring of the finite group H, which has rank bigger than one since H is non-trivial.

An explicit example of a group satisfying (a) and (b) is Thompson's group T, see for example [**10**] and the references given therein. The group T is infinite, finitely presented, and simple (i.e., the only normal subgroups are $\{e\}$ and T). As explained in [**38**, Sec. 2.5], an infinite, finitely generated simple group does not have non-trivial finite-dimensional representations over any field; in particular, every finite-dimensional $\mathbb{R}$-linear representation of Thompson's group T is trivial. On the other hand, T has plenty of finite subgroups.

The following proposition is a slight refinement of [**42**, Lemma 3.7].

Proposition 3.1.28. *Let G be a discrete group. Let $h : A \longrightarrow Y$ be a continuous G-map between finite proper G-CW-complexes. Let ζ be a euclidean G-vector bundle over Y and $\psi : h^*(\zeta) \longrightarrow \kappa$ an isometric embedding of euclidean G-vector bundles over A. Then there is an isometric embedding $\varphi : \zeta \longrightarrow \omega$ of euclidean*

G-vector bundles over Y and an isometric embedding $\lambda : \kappa \longrightarrow h^*(\omega)$ of euclidean G-vector bundles over A such that the composite

$$h^*(\zeta) \xrightarrow{\psi} \kappa \xrightarrow{\lambda} h^*(\omega)$$

coincides with $h^*(\varphi)$.

PROOF. We let γ be the orthogonal complement of the image of $h^*(\zeta)$ under ψ; this is a euclidean G-vector subbundle of κ. By [**42**, Lemma 3.7], there is a G-vector bundle ν over Y and an isometric embedding $j : \gamma \longrightarrow h^*(\nu)$ of G-vector bundles over A. We let $\varphi : \zeta \longrightarrow \nu \oplus \zeta = \omega$ be the embedding of the second summand. We let $\lambda : \kappa = \gamma \oplus \psi(h^*(\zeta)) \longrightarrow h^*(\nu) \oplus h^*(\zeta) = h^*(\omega)$ be the internal direct sum of j and the inverse of the isomorphism $\psi : h^*(\zeta) \cong \psi(h^*(\zeta))$. □

Now we can prove the main result about the functor $E_G[\![-]\!]$.

THEOREM 3.1.29. *Let G be a discrete group. For every orthogonal G-spectrum E, the functor $E_G[\![-]\!]$ is excisive.*

PROOF. Homotopy invariance of the functor $E_G[\![-]\!]$ can be proved in the same way as [**38**, Lemma 6.6], which is the special case $E = \mathbb{S}_G$. For excision we consider two finite proper G-CW-pairs (X, A) and (Y, B) and a pushout square of G-spaces:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] & Y \end{tikzcd} \tag{3.1.30}$$

We consider $(b, x) \in E_G[\![B]\!] \times E_G[\![X]\!]$ such that $f^*(b) = i^*(x)$ in $E_G[\![A]\!]$.

Case 1: We suppose that there is a G-vector bundle ζ over Y such that the given classes can be represented as $b = [\zeta|_B, u]$ and $x = [g^*(\zeta), v]$. We observe that $f^*(\zeta|_B) = g^*(\zeta)|_A$ as G-vector bundles over A. We assume moreover that the maps

$$f^*(u)\ ,\ v|_A \ :\ S^{g^*(\zeta)|_A} \ \longrightarrow\ E(g^*(\zeta)|_A)$$

are parameterized G-homotopic. We let

$$D \ =\ s(X) \cup S^{g^*(\zeta)|_A}$$

be the G-subspace of $S^{g^*(\zeta)}$ given by the union of the image of the section at infinity $s : X \longrightarrow S^{g^*(\zeta)}$ and the part sitting over A. Proposition 1.2.31 implies that the inclusion $D \longrightarrow S^{g^*(\zeta)}$ is a $\mathcal{C}om$-cofibration of G-spaces. The bundle projection $p : E(g^*(\zeta)) \longrightarrow X$ is locally trivial in the equivariant sense, and hence a $\mathcal{C}om$-fibration of G-spaces. So the inclusion

$$S^{g^*(\zeta)} \times 0 \cup_{D\times 0} D \times [0,1] \longrightarrow S^{g^*(\zeta)} \times [0,1]$$

has the left lifting property with respect to the bundle projection $p : E(g^*(\zeta)) \longrightarrow X$. We can thus replace v by a map of retractive G-spaces $\bar v : S^{g^*(\zeta)} \longrightarrow E(g^*(\zeta))$ over X that is equivariantly parameterized homotopic to v, and such that

$$f^*(u) \ =\ \bar v|_A \ .$$

The two maps

$$S^{\zeta|_B} \ \xrightarrow{u}\ E(\zeta|_B) \ \xrightarrow{\text{incl}}\ E(\zeta) \quad \text{and} \quad S^{g^*(\zeta)} \ \xrightarrow{\bar v}\ E(g^*(\zeta)) \ \xrightarrow{E(\bar g)}\ E(\zeta)$$

are then compatible over $S^{f^*(\zeta|_B)} = S^{g^*(\zeta)|_A}$, where $\bar{g} : g^*(\zeta) \longrightarrow \zeta$ is the bundle morphism covering $g : X \longrightarrow Y$. So these maps glue to a map of retractive G-spaces over Y

$$w \;=\; (\mathrm{incl}\circ u) \cup (E(\bar{g}) \circ \bar{v}) \;:\; S^{\zeta} \;=\; S^{\zeta|_B} \cup_{S^{g^*(\zeta)|_A}} S^{g^*(\zeta)} \;\longrightarrow\; E(\zeta)\ .$$

The pair (ζ, w) then represents a class in $E_G[\![Y]\!]$ that satisfies $j^*[\zeta, w] = [\zeta|_B, u] = b$ and $g^*[\zeta, w] = [g^*(\zeta), \bar{v}] = [g^*(\zeta), v] = x$.

Case 2: We suppose that there is a G-vector bundle ζ over Y such that the given classes can be represented as $b = [\zeta|_B, u]$ and $x = [g^*(\zeta), v]$. In contrast to the previous Case 1, we make no further assumptions on the maps u and v. The hypothesis yields the relation

$$[f^*(\zeta|_B), f^*(u)] \;=\; f^*(b) \;=\; i^*(x) \;=\; [g^*(\zeta)|_A, v|_A] \;=\; [f^*(\zeta|_B), v|_A]$$

in $E_G[\![A]\!]$. So Proposition 3.1.21 provides an isometric embedding $\psi : f^*(\zeta|_B) \longrightarrow \kappa$ of euclidean G-vector bundles over A such that the maps $\psi_*(f^*(u))$ and $\psi_*(v|_A)$ are parameterized G-homotopic. Proposition 3.1.28 for $h = gi = jf$ provides an isometric embedding $\varphi : \zeta \longrightarrow \omega$ of G-vector bundles over Y and an isometric embedding $\lambda : \kappa \longrightarrow f^*(\omega|_B)$ of G-vector bundles over A such that the composite

$$f^*(\zeta|_B) \;\xrightarrow{\psi}\; \kappa \;\xrightarrow{\lambda}\; f^*(\omega|_B)$$

coincides with $f^*(\varphi|_B)$. So the maps

$$\begin{aligned} f^*((\varphi|_B)_*(u)) \;&=\; (f^*(\varphi|_B))_*(f^*(u)) \;=\; \lambda_*(\psi_*(f^*(u))) \quad \text{and} \\ ((g^*(\varphi))_*(v))|_A \;&=\; (f^*(\varphi|_B))_*(v|_A) \;=\; \lambda_*(\psi_*(v|_A)) \end{aligned}$$

are parameterized G-homotopic. We have

$$b \;=\; [\zeta|_B, u] \;=\; [\omega|_B, (\varphi|_B)_*(u)] \quad \text{and} \quad x \;=\; [g^*(\zeta), v] \;=\; [g^*(\omega), (g^*(\varphi))_*(v)]\ ,$$

and the new representatives $(\omega|_B, (\varphi|_B)_*(u))$ and $(g^*(\omega), (g^*(\varphi))_*(v))$ satisfy the hypotheses of Case 1, so we are done.

Case 3: Now we treat the general case. We consider two pairs (ξ, u) and (η, v) that represent classes in $E_G[\![B]\!]$ and $E_G[\![X]\!]$, respectively, and such that $f^*[\xi, u] = i^*[\eta, v]$ in $E_G[\![A]\!]$. By [**42**, Lemma 3.7] there are G-vector bundles ω, ω' over Y such that ξ is a direct summand in $\omega|_B$, and η is a direct summand in $g^*(\omega')$. We set $\zeta = \omega \oplus \omega'$. Then there are isometric embeddings

$$a :\; \xi \;\longrightarrow\; \zeta|_B \qquad \text{and} \qquad b \;:\; \eta \;\longrightarrow\; g^*(\zeta)$$

of G-vector bundles over B and X, respectively. Hence

$$b \;=\; [\xi, u] \;=\; [\zeta|_B, a_*(u)] \qquad \text{and} \qquad x \;=\; [\eta, v] \;=\; [g^*(\zeta), b_*(v)]\ .$$

The new representatives satisfy the hypotheses of Case 2, so we are done. This completes the proof of excision for the functor $E_G[\![-]\!]$.

It remains to prove additivity. Excision for a pushout square (3.1.30) with $A = \emptyset$ and $Y = B \amalg X$ shows that the map

$$(i_B^*, i_X^*) \;:\; E_G[\![B \amalg X]\!] \;\longrightarrow\; E_G[\![B]\!] \times E_G[\![X]\!]$$

is surjective. For injectivity we let (ξ, u) represent a class in $E_G[\![B \amalg X]\!]$ such that $i_B^*[\![\xi, u]\!] = i_X^*[\![\xi, u]\!] = 0$. Then after stabilizing along some isometric embedding, if necessary, we can assume that the restriction of u to B is parameterized G-null-homotopic, and the restriction of u to X is parameterized G-null-homotopic. The total space of the sphere bundle S^{ξ} over $B \amalg X$ is the disjoint union of the total spaces of $S^{\xi|_B}$ and $S^{\xi|_X}$, so the two null-homotopies combine into a parameterized G-null-homotopy of u. $\square$

Remark 3.1.31 (Hilbert bundles versus vector bundles). We would like to clarify where the restriction to discrete groups in Theorem 3.1.29, and hence in all subsequent results regarding the functor $E_G[\![-]\!]$, comes from. While the definition of $E_G[\![X]\!]$ in Construction 3.1.19 makes sense for all Lie groups G, the proof of excision needs a crucial fact, proved in [**42**, Lemma 3.7]: when G is discrete and $\varphi : X \longrightarrow Y$ is a continuous G-map between finite proper G-CW-complexes, then every G-vector bundle over X is a summand of $\varphi^*(\xi)$ for some G-vector bundle ξ over Y. As explained in Section 5 of [**42**], this fact does *not* generalize from discrete groups to Lie groups. For *compact* Lie groups, excision still holds, as a consequence of Example 3.1.23.

In the larger generality, Phillips [**56**] has defined equivariant K-theory for proper actions by using suitable Hilbert G-bundles instead of finite-dimensional G-vector bundles. One can speculate whether Phillips' approach can be adapted to generalize our results about the functor $E_G[\![-]\!]$ from discrete groups to non-compact Lie groups, but we will not go down that avenue in this monograph.

Proposition 3.1.32. *Let G be a discrete group and X a finite proper G-CW-complex.*

(i) *For every π_*-isomorphism $f : E \longrightarrow F$ of orthogonal G-spectra, the induced homomorphism $f_* : E_G[\![X]\!] \longrightarrow F_G[\![X]\!]$ is an isomorphism.*
(ii) *For all orthogonal G-spectra E and F, the maps*

$$(E \vee F)_G[\![X]\!] \xrightarrow{\kappa_*} (E \times F)_G[\![X]\!] \xrightarrow{(p^E_*, p^F_*)} E_G[\![X]\!] \times F_G[\![X]\!]$$

are isomorphisms, where $\kappa : E \vee F \longrightarrow E \times F$ is the canonical map and $p^E : E \times F \longrightarrow E$ and $p^F : E \times F \longrightarrow F$ are the projections.

Proof. (i) We start with the special case when G is finite, in which case we denote it by H instead. The horizontal maps in the commutative square

$$\begin{tikzcd}
\mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge X_+, E(V)]^H_* \arrow[r, "\omega", "\cong"'] \arrow[d, "f_*"'] & E_H[\![X]\!] \arrow[d, "f_*"] \\
\mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge X_+, F(V)]^H_* \arrow[r, "\cong", "\omega"'] & F_H[\![X]\!]
\end{tikzcd}$$

are isomorphisms by Example 3.1.23. Adjointness identifies $[S^V \wedge X_+, E(V)]^H_*$ with $[S^V, \mathrm{map}(X, E(V))]^H_*$. So the upper left group is isomorphic to

$$\mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, \mathrm{map}(X, E(V))]^H_* \ = \ \pi_0^H(\mathrm{map}(X, E)) \ ,$$

and hence invariant under π_*-isomorphisms in E, by [**60**, Prop. 3.1.40]. This proves the proposition for finite groups.

Now we let G be any discrete group, and we suppose that $X = G/H \times K$ for a finite subgroup H of G and a finite non-equivariant CW-complex K. Since f is a π_*-isomorphism of orthogonal G-spectra, the underlying morphism of orthogonal H-spectra is a π_*-isomorphism. The induction isomorphisms (3.1.26) participate in a commutative diagram:

$$\begin{tikzcd} E_H[\![K]\!] \arrow[r, "\mathrm{ind}", "\cong"'] \arrow[d, "f_*"'] & E_G[\![G/H \times K]\!] \arrow[d, "f_*"] \\ F_H[\![K]\!] \arrow[r, "\cong", "\mathrm{ind}"'] & F_G[\![G/H \times K]\!] \end{tikzcd}$$

The left vertical map is an isomorphism by the previous paragraph, hence so is the right vertical map. This proves the special case $X = G/H \times K$. Since $E_G[\![-]\!]$ and $F_G[\![-]\!]$ are excisive by Theorem 3.1.29, the general case is now taken care of by Proposition 3.1.7.

(ii) The morphism $\kappa : E \vee F \longrightarrow E \times F$ is a π_*-isomorphism, for example by [**60**, Cor. 3.1.37 (iii)]; so the first map κ_* is an isomorphism by part (i).

The morphisms of orthogonal G-spectra $i^E = (\mathrm{Id}_E, *) : E \longrightarrow E \times F$ and $i^F = (*, \mathrm{Id}_F) : F \longrightarrow E \times F$ induce a homomorphism

$$E_G[\![X]\!] \times F_G[\![X]\!] \longrightarrow (E \times F)_G[\![X]\!] , \quad (x, y) \longmapsto i^E_*(x) + i^F_*(y) .$$

This homomorphism splits the homomorphism (p^E_*, p^F_*), which is thus surjective. Now we consider a pair (ξ, u) that represents a class in the kernel of (p^E_*, p^F_*) : $(E \times F)_G[\![X]\!] \longrightarrow E_G[\![X]\!] \times F_G[\![X]\!]$. Then after stabilizing the representative along an isometric embedding, if necessary, we can assume that the composites

$$S^\xi \xrightarrow{u} (E \times F)(\xi) \xrightarrow{p^E(\xi)} E(\xi) \quad \text{and} \quad S^\xi \xrightarrow{u} (E \times F)(\xi) \xrightarrow{p^F(\xi)} F(\xi)$$

are parameterized equivariantly null-homotopic. The canonical map $(E \times F)(\xi) \longrightarrow E(\xi) \times_X F(\xi)$ is an isomorphism, where the target is the fiber product over X; so the map u itself is parameterized equivariantly null-homotopic. Hence $[\xi, u] = 0$, and the homomorphism (p^E_*, p^F_*) is also injective. □

We have completed the construction of the excisive functor $E_G[\![-]\!]$. Now we compare it to the functor that is represented by E in the stable G-homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$.

CONSTRUCTION 3.1.33. We consider a morphism of orthogonal G-spectra $f : \Sigma^\infty_+ X \longrightarrow E$. Such a morphism represents a class $\gamma_G(f)$ in $E_G(X) = [\Sigma^\infty_+ X, E]^G$, where $\gamma_G : \mathrm{Sp}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ is the localization functor. The value of f at the zero vector space is a map of based G-spaces $f(0) : X_+ = (\Sigma^\infty_+ X)(0) \longrightarrow E(0)$, and we let $f^\flat$ be the map of retractive G-spaces over X

$$f^\flat \ : \ S^{\underline{0}} \ = \ S^0 \times X \ \longrightarrow \ E(0) \times X \ = \ E(\underline{0})$$

that sends $(0, x)$ to $(f(0)(x), x)$. The pair $(\underline{0}, f^\flat)$ then represents a class in the group $E_G[\![X]\!]$.

By Proposition 3.1.32 (i), the functor

$$(-)_G[\![X]\!] \ : \ \mathrm{Sp}_G \ \longrightarrow \ (\text{sets}) \ , \quad E \ \longmapsto \ E_G[\![X]\!]$$

takes π_*-isomorphisms of orthogonal G-spectra to bijections. So the functor factors uniquely through the localization $\gamma_G : \mathrm{Sp}_G \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$. We abuse notation and also write $E_G[\![X]\!]$ for the resulting functor defined on the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$.

THEOREM 3.1.34. *Let G be a discrete group and X a finite proper G-CW-complex.*

(i) *There is a unique natural transformation*

$$\mu_X^E \; : \; E_G(X) \longrightarrow E_G[\![X]\!]$$

of covariant functors in E from $\mathrm{Ho}(\mathrm{Sp}_G)$ to abelian groups with the following property: for every morphism of orthogonal G-spectra $f : \Sigma_+^\infty X \longrightarrow E$, the relation

$$\mu_X^E(\gamma_G(f)) \; = \; [\underline{0}, f^\flat]$$

holds in $E_G[\![X]\!]$.

(ii) *For every orthogonal G-spectrum E, the maps μ_X^E are natural in continuous G-maps $\varphi : Y \longrightarrow X$ between finite proper G-CW-complexes.*

(iii) *Let Γ be a subgroup of G, and let Y be a finite proper Γ-CW-complex. Then the composite*

$$E_G(G \times_\Gamma Y) \xrightarrow{\text{adjunction}} E_\Gamma(Y) \xrightarrow{\mu_Y^E} E_\Gamma[\![Y]\!] \xrightarrow[(3.1.26)]{\text{ind}} E_G[\![G \times_\Gamma Y]\!]$$

coincides with $\mu_{G\times_\Gamma Y}^E$.

(iv) *For every orthogonal G-spectrum E, the map μ_X^E is an isomorphism of abelian groups.*

PROOF. (i) We can apply Construction 3.1.33 to the identity of the orthogonal G-spectrum $\Sigma_+^\infty X$; it yields a class

$$[\underline{0}, \mathrm{Id}^\flat] \; \in \; (\Sigma_+^\infty X)_G[\![X]\!] \; .$$

Since $\Sigma_+^\infty X$ represents the functor $(-)_G(X)$ on $\mathrm{Ho}(\mathrm{Sp}_G)$, the Yoneda lemma provides a unique natural transformation $\mu_X : (-)_G(X) = [\Sigma_+^\infty X, -]^G \longrightarrow (-)_G[\![X]\!]$ such that

$$\mu_X^{\Sigma_+^\infty X}(\mathrm{Id}_{\Sigma_+^\infty X}) \; = \; [\underline{0}, \mathrm{Id}^\flat] \; .$$

This transformation then satisfies the relation stated in part (i), by naturality:

$$\mu_X^E(\gamma_G(f)) \; = \; (\gamma_G(f))_*(\mu_X^E(\mathrm{Id}_{\Sigma_+^\infty X})) = \; (\gamma_G(f))_*[\underline{0}, \mathrm{Id}^\flat] \; = \; [\underline{0}, f^\flat] \; .$$

The natural transformation $\mu_X : (-)_G(X) \longrightarrow (-)_G[\![X]\!]$ is a priori only set-valued, and we must prove that it is additive. The two functors $(-)_G(X) = [\Sigma_+^\infty X, -]^G$ and $(-)_G[\![X]\!]$ are *reduced*, i.e., they send the trivial orthogonal G-spectrum to the trivial abelian group. Proposition 3.1.32 (ii) says that the target functor is also additive in E. As shown in [**60**, Prop. 2.2.12], every set-valued natural transformation between reduced additive functors is automatically additive, so this proves that μ_X^E is a homomorphism of abelian groups.

(ii) We must prove the commutativity of the following square:

$$\begin{tikzcd}
{[\Sigma_+^\infty X, E]^G} \arrow[r, "\mu_X^E"] \arrow[d, "\varphi^*"'] & E_G[\![X]\!] \arrow[d, "\varphi^*"] \\
{[\Sigma_+^\infty Y, E]^G} \arrow[r, "\mu_Y^E"'] & E_G[\![Y]\!]
\end{tikzcd}$$

We let E vary in $\mathrm{Ho}(\mathrm{Sp}_G)$. Then the Yoneda lemma reduces the claim to the universal example, the identity of $E = \Sigma^\infty_+ X$. The universal case is straightforward:

$$\begin{aligned}\varphi^*(\mu_X^{\Sigma^\infty_+ X}(\mathrm{Id}_{\Sigma^\infty_+ X})) \ &= \ \varphi^*[\underline{0}, \mathrm{Id}^\flat] \ = \ [\underline{0}, \varphi^*(\mathrm{Id}^\flat)] \ = \ [\underline{0}, (\Sigma^\infty_+ \varphi)^\flat] \\ &= \ \mu_Y^E(\gamma_G(\Sigma^\infty_+ \varphi)) \ = \ \mu_Y^E(\varphi^*(\mathrm{Id}_{\Sigma^\infty_+ X})) \ .\end{aligned}$$

(iii) If we let E vary in $\mathrm{Ho}(\mathrm{Sp}_G)$, the Yoneda lemma reduces the claim to the universal example, the identity of $E = \Sigma^\infty_+(G \times_\Gamma Y)$. We consider the Γ-equivariant map $[1, -] : Y \longrightarrow G \times_\Gamma Y$, the unit of the adjunction between restriction and extension of scalars. Then

$$\begin{aligned}\mathrm{ind}(\mu_Y^{\Sigma^\infty_+(G\times_\Gamma Y)}(\mathrm{adj}(\mathrm{Id}_{\Sigma^\infty_+(G\times_\Gamma Y)}))) \ &= \ \mathrm{ind}(\mu_Y^{\Sigma^\infty_+(G\times_\Gamma Y)}(\gamma_\Gamma(\Sigma^\infty_+[1,-]))) \\ &= \ \mathrm{ind}[\underline{0}, (\Sigma^\infty_+[1,-])^\flat] \\ &= \ [\underline{0}, \mathrm{Id}^\flat_{\Sigma^\infty_+(G\times_\Gamma Y)}] \\ &= \ \mu_{G\times_\Gamma Y}^{\Sigma^\infty_+(G\times_\Gamma Y)}(\mathrm{Id}_{\Sigma^\infty_+(G\times_\Gamma Y)}) \ .\end{aligned}$$

The third equation exploits that extending the zero Γ-vector bundle from Y to $G \times_\Gamma Y$ yields the zero G-vector bundle, and that the two maps of retractive G-spaces over $G \times_\Gamma Y$

$$G \times_\Gamma (\Sigma^\infty_+[1,-])^\flat \ , \mathrm{Id}^\flat_{\Sigma^\infty_+(G\times_\Gamma Y)} \ : \ S^0 \times (G \times_\Gamma Y) \ \longrightarrow \ (G \times_\Gamma Y)_+ \times (G \times_\Gamma Y)$$

are equal. This proves the universal case, and hence the claim.

(iv) Source and target of the transformation μ_X^E send π_*-isomorphisms in E to isomorphisms of abelian groups, by Proposition 3.1.32 and by construction, respectively. So by appeal to the stable model structure of Theorem 1.2.22 we can assume without loss of generality that E is a G-Ω-spectrum.

We start with the special case of a *finite* group, which we denote by H (instead of G). We let X be a finite H-CW-complex, and E an orthogonal H-spectrum. We define a homomorphism

$$\nu \ : \ \mathrm{colim}_{W \in s(\mathcal{U}_H)} [S^W \wedge X_+, E(W)]^H_* \ \longrightarrow \ [\Sigma^\infty_+ X, E]^H = E_H(X) \ .$$

The construction uses the shift $\mathrm{sh}^W X$ of an orthogonal H-spectrum X by an H-representation W, defined by

$$(\mathrm{sh}^W X)(V) \ = \ X(V \oplus W)$$

with structure maps $\sigma_{U,V}^{\mathrm{sh}^W X} = \sigma^X_{U,V\oplus W}$. Here the H-action on $X(V \oplus W)$ is diagonal, from the H-actions on X and on W. A natural morphism of orthogonal H-spectra

$$\lambda_X^W \ : \ X \wedge S^W \ \longrightarrow \ \mathrm{sh}^W X$$

is defined at V as the composite

$$X(V) \wedge S^W \ \xrightarrow{\text{twist}} \ S^W \wedge X(V) \ \xrightarrow{\sigma^X_{W,V}} \ X(W \oplus V) \ \xrightarrow{X(\tau_{W,V})} \ X(V \oplus W) \ .$$

The morphism λ_X^W and its adjoint $\tilde{\lambda}_X^W : X \longrightarrow \Omega^W \mathrm{sh}^W X$ are π_*-isomorphisms by [**60**, Prop. 3.1.25]. The transformation ν now takes the class represented by a continuous based H-map $x : S^W \wedge X_+ \longrightarrow E(W)$ to the composite morphism

$$\Sigma^\infty_+ X \ \xrightarrow{\gamma_H(\tilde{x})} \ \Omega^W \mathrm{sh}^W E \ \xrightarrow[\cong]{\gamma_H(\tilde{\lambda}_E^W)^{-1}} \ E$$

in $\mathrm{Ho}(\mathrm{Sp}_H)$. Here $\tilde{x} : \Sigma^\infty_+ X \longrightarrow \Omega^W \mathrm{sh}^W E$ is the morphism of orthogonal H-spectra adjoint to x.

We must argue that ν is well-defined. If we vary the map $x : S^W \wedge X_+ \longrightarrow E(W)$ by a based equivariant homotopy, then $\tilde{x}$ changes by a homotopy of morphisms of orthogonal H-spectra, and its image $\gamma_H(\tilde{x})$ in $\mathrm{Ho}(\mathrm{Sp}_H)$ remains unchanged. Now we stabilize $x : S^W \wedge X_+ \longrightarrow E(W)$ along an inclusion $\iota : W \longrightarrow \bar{W}$ in the poset $s(\mathcal{U}_H)$ to a new representative $\iota_*(x) : S^{\bar{W}} \wedge X_+ \longrightarrow E(\bar{W})$, the composite

$$\begin{aligned} S^{\bar{W}} \wedge X_+ \ = \ S^{\bar{W}-W} \wedge S^W \wedge X_+ \ &\xrightarrow{S^{\bar{W}-W} \wedge x} \ S^{\bar{W}-W} \wedge E(W) \\ &\xrightarrow{\sigma_{\bar{W}-W,W}} \ E((\bar{W}-W) \oplus W) \ = \ E(\bar{W}) \ . \end{aligned}$$

We let $\Psi : \Omega^W \mathrm{sh}^W E \longrightarrow \Omega^{\bar{W}} \mathrm{sh}^{\bar{W}} E$ be the morphism of orthogonal H-spectra defined at an inner product space V as

$$\Psi(V) \ : \ \mathrm{map}_*(S^W, E(V \oplus W)) \ \longrightarrow \ \mathrm{map}_*(S^{\bar{W}}, E(V \oplus \bar{W})) \ , \quad f \longmapsto \ \iota_*(f) \ .$$

Then the various morphisms of orthogonal H-spectra participate in a commutative diagram:

$$\begin{tikzcd} \Sigma^\infty_+ X \arrow[r, "\tilde{x}"] \arrow[dr, "\widetilde{\iota_*(x)}"'] & \Omega^W \mathrm{sh}^W E \arrow[d, "\Psi"] & E \arrow[l, "\tilde{\lambda}^W_E"'] \arrow[dl, "\tilde{\lambda}^{\bar{W}}_E"] \\ & \Omega^{\bar{W}} \mathrm{sh}^{\bar{W}} E & \end{tikzcd}$$

Hence we conclude that

$$\begin{aligned} \gamma_H(\tilde{\lambda}^W_E)^{-1} \circ \gamma_H(\tilde{x}) \ &= \ \gamma_H(\tilde{\lambda}^{\bar{W}}_E)^{-1} \circ \gamma_H(\Psi) \circ \gamma_H(\tilde{x}) \\ &= \ \gamma_H(\tilde{\lambda}^{\bar{W}}_E)^{-1} \circ \gamma_H(\widetilde{\iota_*(x)}) \ . \end{aligned}$$

So the homomorphism ν is well-defined.

Now we contemplate three composable maps:

$$[X_+, E(0)]^H_* \longrightarrow \mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge X_+, E(V)]^H_* \xrightarrow{\ \nu\ } E_H(X) \xrightarrow{\mu^E_X} E_H[\![X]\!] \ .$$

The first map is the canonical map to the colimit, and it is an isomorphism because E is an H-Ω-spectrum. The composite map from $[X_+, E(0)]^H$ to $E_H(X)$ is the derived adjunction for the Quillen adjoint functor pair

$$\Sigma^\infty_+ \ : \ H\mathbf{T} \rightleftarrows \mathrm{Sp}_H \ : \ (-)(0) \ ;$$

it is thus bijective. So the map ν is also an isomorphism. The composite $\mu^E_X \circ \nu : \mathrm{colim}_V [S^V \wedge X_+, E(V)]^H_* \longrightarrow E_H[\![X]\!]$ is the homomorphism ω discussed in Example 3.1.23, which is thus an isomorphism. Since ν and ω are isomorphisms, so is the map μ^E_X. This completes the proof of part (iv) for finite groups.

Now we let G be any discrete group, H a finite subgroup of G, and we consider $X = G/H \times K$, for a non-equivariant finite CW-complex K. By part (iii), the map $\mu^E_{G/H \times K}$ factors as the composite

$$E_G(G/H \times K) \ \xrightarrow{\text{adjunction}} \ E_H(K) \ \xrightarrow{\mu^E_K} \ E_H[\![K]\!] \ \xrightarrow{\text{ind}} \ E_G[\![G/H \times K]\!] \ .$$

All three maps are isomorphisms, by the derived adjunction (see Corollary 1.4.3), the special case of the previous paragraph, and by Example 3.1.25, respectively. So

$\mu^E_{G/H\times K}$ is an isomorphism. Since $E_G(-)$ and $E_G[\![-]\!]$ are both excisive functors, Proposition 3.1.7 then finishes the argument. □

3.2. Proper cohomology theories from G-spectra

In this section we discuss how orthogonal G-spectra give rise to 'proper equivariant cohomology theories' for G-spaces, where G is any Lie group. For our purposes, a proper G-cohomology theory is a $\mathbb{Z}$-indexed family of excisive functors, linked by suspension isomorphisms, compare Definition 3.2.1. Our main example is the proper G-cohomology theory represented by an orthogonal G-spectrum E, defined by taking morphism groups in the triangulated stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ into the shifts of E, see Construction 3.2.3. One of the main points of this book is that the equivariant cohomology theories represented by orthogonal G-spectra have additional structure not generally present in the equivariant cohomology theories arising from sequential G-spectra or G-orbit spectra. This additional structure manifests itself in different forms, such as a '$KO_G(\underline{E}G)$-grading' or transfer maps that extend the homotopy group coefficient system to a G-Mackey functor, compare Remark 3.2.10.

For discrete groups, we also discuss how two prominent equivariant cohomology theories are represented by orthogonal G-spectra: equivariant stable cohomotopy in the sense of the third author [**38**] is represented by the G-sphere spectrum, see Example 3.2.9; and Bredon cohomology with coefficients in a G-Mackey functor is represented by an Eilenberg-MacLane spectrum, see Example 3.2.16.

Definition 3.2.1. Let G be a Lie group. A *proper G-cohomology theory* consists of a collection of excisive functors

$$\mathcal{H}^k \ : \ (\text{finite proper } G\text{-CW-complexes})^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

and a collection of natural isomorphisms

$$\sigma \ : \ \mathcal{H}^{k-1}(X) \ \xrightarrow{\cong} \ \mathcal{H}^k(X\times S^1|X\times\infty) \ ,$$

for $k\in\mathbb{Z}$.

Some previously studied proper G-cohomology theories are Bredon cohomology, Borel cohomology, equivariant stable cohomotopy [**38**], and equivariant K-theory [**42, 56**]. One of the main motivations for the present work was to provide a general method for constructing proper G-cohomology theories from G-spectra, see Theorem 3.2.7 below. As reality check, we will show that for discrete groups G, Bredon cohomology, Borel cohomology, equivariant stable cohomotopy, and equivariant K-theory are indeed represented by orthogonal G-spectra, see Example 3.2.16, Example 3.3.10, Example 3.2.9, and Theorem 3.4.22, respectively.

We let $(\mathcal{H}^k,\sigma)_{k\in\mathbb{Z}}$ be a proper G-cohomology theory, and we consider a pushout square of G-spaces

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] & Y \end{tikzcd}$$

where (X, A) and (Y, B) are finite proper G-CW-pairs, and f and g are cellular maps. Then the 5-term Mayer-Vietoris sequence established in Proposition 3.1.5 extends to a long exact sequence as follows. We define a *connecting homomorphism*

$$\delta^{k-1} \ : \ \mathcal{H}^{k-1}(A) \ \longrightarrow \ \mathcal{H}^k(Y)$$

as the composite

$$\mathcal{H}^{k-1}(A) \ \xrightarrow[\cong]{\sigma} \ \mathcal{H}^k(A \times S^1 | A \times \infty) \ \xrightarrow{\partial} \ \mathcal{H}^k(Y) \ ,$$

where the homomorphism ∂ was defined in (3.1.4). Then we can splice the various exact sequences for the functor $\mathcal{H}^k$ into an exact sequence

$$(3.2.2) \qquad \cdots \longrightarrow \mathcal{H}^{k-1}(A) \ \xrightarrow{\delta^{k-1}} \ \mathcal{H}^k(Y) \ \xrightarrow{(j^*, g^*)}$$
$$\mathcal{H}^k(B) \times \mathcal{H}^k(X) \ \xrightarrow{f^* - i^*} \ \mathcal{H}^k(A) \ \xrightarrow{\delta^k} \ \mathcal{H}^{k+1}(Y) \ \longrightarrow \ \cdots \ .$$

In Section 3.1 we used orthogonal G-spectra E to define three excisive functors $E_G\langle - \rangle$, $E_G(-)$ and $E_G[\![-]\!]$. The functor $E_G\langle - \rangle$ is less relevant for us, among other things because it does not in general extend to a 'genuine' cohomology theory. We now upgrade the two constructions $E_G(-)$ and $E_G[\![-]\!]$ to proper G-cohomology theories by using the shifts of E.

Construction 3.2.3 (Proper cohomology theories from orthogonal G-spectra). We let G be a Lie group and E an orthogonal G-spectrum. If k is any integer, we recall from Remark 1.3.5 that the k-fold shift $E[k]$ is defined as

$$E[k] \ = \ \begin{cases} E \wedge S^k & \text{for } k \geq 0, \text{ and} \\ \Omega^{-k} E & \text{for } k < 0. \end{cases}$$

We define functors $E^k_G(-)$ and $E^k_G[\![-]\!]$ by

$$E^k_G(X) \ = \ E[k]_G(X) \ = \ [\Sigma^\infty_+ X, E[k]]^G \qquad \text{and} \qquad E^k_G[\![X]\!] \ = \ E[k]_G[\![X]\!] \ .$$

The functor $E^k_G(-)$ is excisive by Proposition 3.1.15. If G is discrete, then the functor $E^k_G[\![-]\!]$ is excisive by Theorem 3.1.29.

Now we link these excisive functors by suspension isomorphisms. We define a suspension homomorphism

$$(3.2.4) \qquad \sigma \ : \ E^k_G(X) \ \longrightarrow \ E^{k+1}_G(X \times S^1 | X \times \infty)$$

as the composite

$$[\Sigma^\infty_+ X, E[k]]^G \ \xrightarrow{[1]} \ [\Sigma^\infty X_+ \wedge S^1, E[k][1]]^G \ \xrightarrow{[\Sigma^\infty q, s_k]} \ [\Sigma^\infty_+ (X \times S^1), E[k+1]]^G \ ,$$

where $q : (X \times S^1)_+ \longrightarrow X_+ \wedge S^1$ is the projection, and the isomorphism $s_k : E[k][1] \cong E[k+1]$ was defined in (1.3.6).

We define a suspension homomorphism

$$(3.2.5) \qquad \Sigma^E \ : \ E_G[\![X]\!] \ \longrightarrow \ (E \wedge S^1)_G[\![X \times S^1 | X \times \infty]\!]$$

as follows. We let (ξ, u) represent a class in $E_G[\![X]\!]$. We pull back the vector bundle ξ along the projection $X \times S^1 \longrightarrow X$ to obtain a vector bundle $\xi \times S^1$ over $X \times S^1$. We define

$$\Sigma(u) \ : \ S^{\xi \times S^1} \ = \ S^\xi \times S^1 \ \longrightarrow \ (E(\xi) \wedge_X S^1) \times S^1 \ = \ (E \wedge S^1)(\xi \times S^1)$$

by $\Sigma(u)(z,x) = (u(z)\wedge x, x)$. We can then define the suspension homomorphism (3.2.5) by

$$\Sigma^E[\xi,u] \ = \ [\xi\times S^1, \Sigma(u)]\ .$$

By construction, the map $\Sigma(u)$ is trivial over $X\times\infty$, so the image of Σ^E indeed lies in the kernel of the restriction map $i^*:(E\wedge S^1)_G[\![X\times S^1]\!]\longrightarrow (E\wedge S^1)_G[\![X\times\infty]\!]$. The suspension homomorphisms

$$\sigma\ :\ E^k_G[\![X]\!] \xrightarrow{\ \cong\ } E^{k+1}_G[\![X\times S^1|X\times\infty]\!] \tag{3.2.6}$$

for the theory $E^*_G[\![-]\!]$ are then defined as the composite

$$\begin{aligned} E^k_G[\![X]\!] \ &= \ E[k]_G[\![X]\!] \ \underset{\cong}{\xrightarrow{\Sigma^{E[k]}}} \ (E[k][1])_G[\![X\times S^1|X\times\infty]\!] \\ &\xrightarrow{(s_k)_*} E[k+1]_G[\![X\times S^1|X\times\infty]\!] \ = \ E^{k+1}_G[\![X\times S^1|X\times\infty]\!]\ . \end{aligned}$$

Theorem 3.2.7. *Let G be a Lie group and E an orthogonal G-spectrum.*

(i) *The functors $E^k_G(-)$ and the suspension homomorphisms* (3.2.4) *form a proper G-cohomology theory.*
(ii) *If G is discrete, then the functors $E^k_G[\![-]\!]$ and the suspension homomorphisms* (3.2.6) *form a proper G-cohomology theory.*
(iii) *If G is discrete, then the natural transformations $\mu^{E[k]} : E^k_G(-)\longrightarrow E^k_G[\![-]\!]$ form a natural isomorphism of proper G-cohomology theories.*

Proof. (i) The functors $E^k_G(-)$ are excisive by Proposition 3.1.15. It remains to show that the suspension homomorphism (3.2.4) is an isomorphism. This is a direct consequence of the fact that the shift functor in a triangulated category is fully faithful, and that precomposition with the projection $q:(X\times S^1)_+\longrightarrow X_+\wedge S^1$ is an isomorphism from the group $[\Sigma^\infty X_+\wedge S^1, E[1]]^G$ to the kernel of the split epimorphism

$$i^*\ :\ [\Sigma^\infty_+(X\times S^1), E[1]]^G \ \longrightarrow\ [\Sigma^\infty_+ X, E[1]]^G\ .$$

(ii) We show first that the transformations μ^E_X commute with the suspension homomorphisms. This amounts to checking that the following square commutes for all orthogonal G-spectra E and all finite proper G-CW-complexes X:

$$\begin{tikzcd} E_G(X) \arrow[r, "\mu^E_X"] \arrow[d, "\sigma"'] & E_G[\![X]\!] \arrow[d, "\Sigma^E"] \\ (E\wedge S^1)_G(X\times S^1|X\times\infty) \arrow[r, "\mu^{E\wedge S^1}_{X\times S^1}"'] & (E\wedge S^1)_G[\![X\times S^1|X\times\infty]\!] \end{tikzcd} \tag{3.2.8}$$

We start with a class of the form $\gamma_G(f)$, for some morphism of orthogonal G-spectra $f:\Sigma^\infty_+X\longrightarrow E$. Then

$$\begin{aligned} \Sigma^E(\mu^E_X(\gamma_G(f))) \ &= \ \Sigma^E[\underline{0}, f^\flat] \ = \ [\underline{0},\Sigma(f^\flat)] \ = \ [\underline{0}, ((f\wedge S^1)\circ\Sigma^\infty q)^\flat] \\ &= \ \mu^{E\wedge S^1}_{X\times S^1}(\gamma_G((f\wedge S^1)\circ\Sigma^\infty q)) \ = \ \mu^{E\wedge S^1}_{X\times S^1}(\sigma(\gamma_G(f)))\ . \end{aligned}$$

The third equation exploits that pulling back the zero vector bundle yields the zero vector bundle, and that the functions $\Sigma(f^\flat)$ and $((f\wedge S^1)\circ\Sigma^\infty q)^\flat$ are equal on the nose.

Now we let E vary in $\mathrm{Ho}(\mathrm{Sp}_G)$; the Yoneda lemma reduces the commutativity of the square (3.2.8) to the universal example, the identity of $E = \Sigma^\infty_+ X$. This universal class is of the form considered in the previous paragraph, so we are done.

The suspension homomorphism (3.2.4) for the theory $E^*_G(-)$ is an isomorphism by part (i), and the maps μ^E_X and $\mu^{E\wedge S^1}_{X\times S^1}$ are isomorphisms by Theorem 3.1.34 (iv). So the commutative square (3.2.8) shows that the suspension homomorphism (3.2.6) for the theory $E^*_G[\![-]\!]$ is an isomorphism. The functors $E^k_G[\![-]\!]$ are excisive by Theorem 3.1.29, so this shows part (ii).

We have now shown that the transformations $\mu^{E[k]}_X$ commute with the suspension homomorphisms, so they form a morphism of proper G-cohomology theories. Since $\mu^{E[k]}$ is a natural isomorphism, this proves claim (iii). $\square$

Example 3.2.9 (Equivariant stable cohomotopy). Let G be a discrete group. In [**38**, Sec. 6], the third author introduced an equivariant cohomology theory $\pi^*_G(X)$ on finite proper G-CW-complexes X, called *equivariant stable cohomotopy*. By the main result of [**41**], this particular equivariant version of stable cohomotopy satisfies a completion theorem, generalizing Carlsson's theorem (previously known as the *Segal conjecture*) for finite groups.

The definition of the group $\pi^k_G(X)$ is precisely the special case of $E^k_G[\![X]\!]$ for $E = \mathbb{S}_G$ the G-sphere spectrum:

$$\pi^k_G(X) \ = \ \mathbb{S}^k_G[\![X]\!] \ .$$

In fact, Section 6 of [**38**] is a blueprint for much of what we do here, and the definitions and proofs involving $E^*_G[\![X]\!]$ were inspired by [**38**].

Anyhow, for $E = \mathbb{S}_G$, Theorem 3.2.7 (iii) shows that equivariant stable cohomotopy is represented in $\mathrm{Ho}(\mathrm{Sp}_G)$ by the G-sphere spectrum, i.e., the map

$$\mu^{\mathbb{S}_G[k]}_X \ : \ \mathbb{S}^k_G(X) \ = \ [\Sigma^\infty_+ X, \mathbb{S}_G[k]]^G \ \longrightarrow \ \pi^k_G(X)$$

is an isomorphism for every finite proper G-CW-complex X and every integer k.

Remark 3.2.10 (Genuine versus naive cohomology theories). As we argued in the proof of Proposition 3.1.15, every proper equivariant cohomology theory arising from an orthogonal G-spectrum as in Definition 3.1.13 is also represented by a sequential G-spectrum as in (3.1.10). Hence these cohomology theories are also proper equivariant cohomology theories in the sense of Davis and the third author [**13**, Def. 4.1], compare Remark 3.1.12.

As we already indicated a number of times, the equivariant cohomology theories represented by orthogonal G-spectra are a lot richer than those arising from sequential G-spectra or G-orbit spectra. When G is *compact*, then these special cohomology theories are known under the names *genuine* or *$RO(G)$-graded* cohomology theories. In our more general context, one could informally refer to the extra structure as a '$KO_G(\underline{E}G)$-grading', where $KO_G(\underline{E}G)$ is the Grothendieck group of isomorphism classes of real G-equivariant vector bundles over $\underline{E}G$. If G happens to be compact, then $\underline{E}G$ can be taken to be a point, so an equivariant vector bundle is just a representation, and then the group $KO_G(\underline{E}G)$ is isomorphic to the real representation ring $RO(G)$.

The $KO_G(\underline{E}G)$-grading can be encoded as follows. Given a G-vector bundle ξ over $\underline{E}G$, we continue to denote by $\mathrm{Th}(\xi) = S^\xi/s_\infty(\underline{E}G)$ its Thom space. We define the ξ-th G-equivariant E-cohomology group as

$$E^\xi_G(X) \;=\; [\Sigma^\infty_+ X, E \wedge \mathrm{Th}(\xi)]^G \ .$$

If ξ is the trivial vector bundle of rank k, its Thom space is $\underline{E}G_+ \wedge S^k$, which is $\mathcal{C}om$-equivalent to S^k; so in that case we recover the group $E^k_G(X)$. If η is another such G-vector bundle, a suspension homomorphism

$$\sigma^\eta \colon E^\xi_G(X) \;\longrightarrow\; E^{\xi\oplus\eta}_G(X_+ \wedge \mathrm{Th}(\eta))$$

is essentially defined by derived smash product with the G-space $\mathrm{Th}(\eta)$. Since smashing with the Thom space $\mathrm{Th}(\eta)$ is invertible (compare Proposition 1.2.33 (iv)), this suspension homomorphism is an isomorphism. So up to a non-canonical isomorphism, the group $E^\xi_G(X_+ \wedge \mathrm{Th}(\eta))$ only depends on the class $[\xi] - [\eta]$ in the Grothendieck group $KO_G(\underline{E}G)$.

As a concrete example, let us consider the infinite dihedral group $D_\infty = \mathbb{Z} \ltimes \mathbb{Z}/2$. In Example 1.1.4 we exhibited a D_∞-CW-structure on the real line $\mathbb{R}$ that is a model for $\underline{E}D_\infty$. There are 2 equivariant 0-cells whose isotropy groups the two-element subgroups H_1 generated by $(0, 1 + 2\mathbb{Z})$ and H_2 generated by $(1, 1 + 2\mathbb{Z})$, respectively; and there is one free equivariant 1-cell. The resulting Mayer-Vietoris sequence shows that the restriction homomorphisms combine into a monomorphism of rings

$$(\mathrm{res}^{D_\infty}_{H_1}, \mathrm{res}^{D_\infty}_{H_2}) \ : \ KO_{D_\infty}(\underline{E}D_\infty) \;\longrightarrow\; RO(H_1) \times RO(H_2)$$

whose image consists of pairs of virtual representations with the same virtual dimension. The rings $RO(H_1)$ and $RO(H_2)$ are isomorphic to $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$, where σ is the class of the sign representation. We let σ_1, σ_2 denote the classes in $KO_{D_\infty}(\underline{E}D_\infty)$ that restrict to $(\sigma, 1)$ and to $(1, \sigma)$, respectively. Then the elements $1, \sigma_1$ and σ_2 form an additive basis of $KO_{D_\infty}(\underline{E}D_\infty)$, and the multiplicative structure is determined by $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma_1 + \sigma_2 - 1$. In this special case, the three generators $1, \sigma_1$ and σ_2 are actually all represented by 1-dimensional representations of the group D_∞. For a general infinite discrete group G, not all classes in $KO_G(\underline{E}G)$ will be virtual G-representations.

Another piece of structure consists of transfers. Every proper cohomology theory gives rise to a coefficient system by restriction to orbits. If the cohomology theory arises from a sequential G-spectrum E, the coefficient system is simply the composite

$$\left(\mathrm{Or}^{\mathcal{F}in}_G\right)^{\mathrm{op}} \ \xrightarrow{\Phi(E)}\ \mathrm{Sp}^{\mathbb{N}} \ \xrightarrow{\pi_0}\ \mathcal{A}b \ ,$$

where $\Phi(E)(G/H) = E^H$ is the spectrum of H-fixed points. Every coefficient system arises in this way from a G-orbit spectrum, by postcomposition with a suitable pointset Eilenberg-Mac Lane functor $H : \mathcal{A}b \longrightarrow \mathrm{Sp}^{\mathbb{N}}$, and then assembling the resulting G-orbit spectrum into a sequential G-spectrum as indicated in Remark 3.1.12.

If the cohomology theory arises from a genuine stable G-homotopy type E (i.e., an object in $\mathrm{Ho}(\mathrm{Sp}_G)$), then the coefficient system is given by the equivariant homotopy groups $\pi^H_0(E)$ for finite subgroups H of G; the transfers discussed in Construction 2.1.12 then extend the coefficient system to a G-Mackey functor, see Example 2.2.8. Moreover, every Mackey functor arises in this way from an orthogonal G-spectrum, by Theorem 2.2.9. In [**37**, Sec. 5], the presence of transfers is exploited to construct rational splittings of proper cohomology theories.

We found it convenient to formulate our results in terms of *absolute* equivariant cohomology theories. As we shall now explain, every proper G-cohomology theory can be extended in a specific way to a *relative* theory defined on finite proper G-CW-pairs.

DEFINITION 3.2.11. (Relative cohomology groups) We let G be a Lie group and $\{\mathcal{H}^k, \sigma\}_{k\in\mathbb{Z}}$ a proper G-cohomology theory. For every finite proper G-CW pair (X, A) we define the *relative $\mathcal{H}$-cohomology groups* by

$$\mathcal{H}^k(X, A) \ = \ \ker(i_2^* \ : \ \mathcal{H}^k(X \cup_A X) \ \longrightarrow \ \mathcal{H}^k(X)) \ ,$$

where $i_2 : X \longrightarrow X \cup_A X$ is the embedding of the second copy of X.

The relative groups are contravariantly functorial for continuous G-maps of pairs. We define a natural homomorphism $r : \mathcal{H}^k(X, A) \longrightarrow \mathcal{H}^k(X)$ as the composite

$$\mathcal{H}^k(X, A) \ \xrightarrow{\text{incl}} \ \mathcal{H}^k(X \cup_A X) \ \xrightarrow{i_1^*} \ \mathcal{H}^k(X) \ ,$$

where $i_1 : X \longrightarrow X \cup_A X$ is the embedding of the first copy of X. The Mayer-Vietoris sequence (3.2.2) for the pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X \cup_A X \end{tikzcd}$$

has the form

$$\cdots \ \longrightarrow \ \mathcal{H}^{k-1}(A) \ \xrightarrow{\delta^{k-1}} \ \mathcal{H}^k(X \cup_A X) \ \xrightarrow{(i_1^*, i_2^*)}$$
$$\mathcal{H}^k(X) \times \mathcal{H}^k(X) \ \xrightarrow{(x,x') \mapsto i^*(x) - i^*(x')} \ \mathcal{H}^k(A) \ \xrightarrow{\delta^k} \ \cdots \ .$$

So the connecting homomorphism $\delta^{k-1} : \mathcal{H}^{k-1}(A) \longrightarrow \mathcal{H}^k(X \cup_A X)$ lands in the subgroup $\mathcal{H}^k(X, A)$, and the exact sequence splits off a long exact sequence (3.2.12)

$$\cdots \ \longrightarrow \ \mathcal{H}^{k-1}(A) \ \xrightarrow{\delta^{k-1}} \ \mathcal{H}^k(X, A) \ \xrightarrow{r} \ \mathcal{H}^k(X) \ \xrightarrow{i^*} \ \mathcal{H}^k(A) \ \longrightarrow \ \cdots \ .$$

REMARK 3.2.13. For every orthogonal G-spectrum E, the functor $E_G(-) = [-, E]^G \circ \Sigma_+^\infty$ is excisive by Proposition 3.1.15. For every finite proper G-CW-pair (X, A), the relative group $E_G(X, A)$ is defined as in Definition 3.2.11. This relative group can in fact be described more directly as a morphism group in $\mathrm{Ho}(\mathrm{Sp}_G)$: we let $q : X \cup_A X \longrightarrow X/A$ denote the map that sends the second copy of X to the basepoint and that is the projection $X \longrightarrow X/A$ on the first copy. Since the inclusion of the second summand has a retraction, the cofiber sequence

$$X_+ \ \xrightarrow{i_2} \ (X \cup_A X)_+ \ \xrightarrow{q} \ X/A$$

induces a short exact sequence

$$0 \ \longrightarrow \ [\Sigma^\infty X/A, E]^G \ \xrightarrow{q^*} \ E_G(X \cup_A X) \ \xrightarrow{i_2^*} \ E_G(X) \ \longrightarrow \ 0$$

and hence an isomorphism

$$[\Sigma^\infty X/A, E]^G \ \cong \ E_G(X, A) \ .$$

With a little more work one can show that under this identification, the long exact sequence (3.2.12)

$$\dots \longrightarrow E_G^{k-1}(A) \xrightarrow{\delta^{k-1}} E_G^k(X,A) \longrightarrow E_G^k(X) \longrightarrow E_G^k(A) \longrightarrow \dots$$

is the effect of applying $[-, E[k]]^G$ to the distinguished triangle in $\mathrm{Ho}(\mathrm{Sp}_G)$

$$\Sigma_+^\infty A \xrightarrow{\Sigma^\infty \mathrm{incl}} \Sigma_+^\infty X \xrightarrow{\Sigma^\infty \mathrm{proj}} \Sigma^\infty X/A \xrightarrow{(\Sigma^\infty p)\circ(\Sigma^\infty q)^{-1}} (\Sigma_+^\infty A) \wedge S^1$$

and its rotations. Here $q : CA \cup_A X \longrightarrow X/A$ is the projection from the mapping cone of the inclusion, and $p : CA \cup_A X \longrightarrow A_+ \wedge S^1$ was defined in Construction 1.3.2.

For discrete groups G, the functor $E_G[\![-]\!]$ defined in Construction 3.1.19 via parameterized equivariant homotopy theory is excisive by Theorem 3.1.29. The relative groups $E_G[\![X, A]\!]$ defined as in Definition 3.2.11 can also be described more directly by relative parameterized homotopy classes, in much the same way as for $E = \mathbb{S}_G$ in [**38**, Sec. 6.2]; we won't dwell on this any further.

A formal consequence of the properties of an equivariant cohomology theory is an Atiyah-Hirzebruch type spectral sequence that starts from Bredon cohomology. This spectral sequence is a useful calculational tool and a systematic generalization of various previous statements, so we take the time to spell out the details. The following discussion is a special case of the Atiyah-Hirzebruch spectral sequence of [**13**, Thm. 4.7], for the $\mathcal{F}in$-orbit category of a discrete group.

Construction 3.2.14 (Atiyah-Hirzebruch spectral sequence). We let G be a discrete group, and we let $(\mathcal{H}^k, \sigma)_{k\in\mathbb{Z}}$ be a proper G-cohomology theory. We consider a proper G-CW-complex X with equivariant skeleton filtration

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X^n \subset \dots \ .$$

If the cohomology theory is only defined on *finite* proper G-CW-complexes as in our original Definition 3.2.1, then we must insist that X has only finitely many cells. However, the following arguments apply without this size restriction if $\mathcal{H}^k$ is defined on all G-CW-complexes (i.e., possibly infinite), satisfies conditions (i), (ii) and (iii) of Definition 3.1.1 in this larger category, and additionally, $\mathcal{H}^k$ takes coproducts (possibly infinite) to products.

From the cohomology theory and the CW-structure we define an exact couple in the standard way by setting

$$D_1^{p,q} \ = \ \mathcal{H}^{p+q}(X^p) \qquad \text{and} \qquad E_1^{p,q} \ = \ \mathcal{H}^{p+q}(X^p, X^{p-1}) \ ,$$

see for example [**50**, Sec. 2.2, p. 37ff]. These groups are linked by the homomorphisms

$$\begin{aligned}
\mathcal{H}^{p+q}(\mathrm{incl}) \ = \ i \ &: \ D_1^{p,q} \ = \ \mathcal{H}^{p+q}(X^p) \ \longrightarrow \ \mathcal{H}^{p+q}(X^{p-1}) \ = \ D_1^{p-1,q+1} \\
\delta \ = \ j \ &: \ D_1^{p,q} \ = \ \mathcal{H}^{p+q}(X^p) \ \longrightarrow \ \mathcal{H}^{p+q+1}(X^{p+1}, X^p) \ = \ E_1^{p+1,q} \\
r \ = \ k \ &: \ E_1^{p,q} \ = \ \mathcal{H}^{p+q}(X^p, X^{p-1}) \ \longrightarrow \ \mathcal{H}^{p+q}(X^p) \ = \ D_1^{p,q} \ .
\end{aligned}$$

Now we identify the E_1-term with the Bredon cohomology complex, and hence the E_2-term with Bredon cohomology. Bredon cohomology is defined for *G-coefficient systems*, i.e., functors

$$M \ : \ \left(\mathrm{Or}_G^{\mathcal{F}in}\right)^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b \ .$$

Here $\mathrm{Or}_G^{\mathcal{F}in}$ is the G-orbit category with finite stabilizers: the objects are the G-sets G/H for all finite subgroups H of G, and morphisms are G-maps. The G-CW-structure gives rise to a cellular cochain complex $C^*(X,M)$ as follows. For every finite subgroup K of X, the fixed point space X^K is a non-equivariant CW-complex with respect to the skeleton filtration

$$(X^0)^K \subset (X^1)^K \subset \ldots \subset (X^n)^K \subset \ldots .$$

So X^K has a cellular chain complex $C_*(X^K)$ with n-th chain group

$$C_n(X^K) = H_n\left((X^n)^K,(X^{n-1})^K,\mathbb{Z}\right) ,$$

the relative integral homology of the pair $((X^n)^K,(X^{n-1})^K)$. Since X^K is the space of G-maps from G/K to X, every morphism $f : G/K \longrightarrow G/H$ in the orbit category $\mathrm{Or}_G^{\mathcal{F}in}$ induces a cellular map $f^* : X^H \longrightarrow X^K$, and hence a morphism of cellular chain complexes

$$f^* : C_*(X^H) \longrightarrow C_*(X^K) .$$

These maps make the complexes $\{C_*(X^K)\}_{K\in\mathcal{F}in}$ into a contravariant functor from $\mathrm{Or}_G^{\mathcal{F}in}$ to the category of chain complexes. Equivalently, we can consider $C_*(X^\bullet) = \{C_*(X^K)\}_{K\in\mathcal{F}in}$ as a chain complex of coefficient systems. Thus we can define a cochain complex of abelian groups by mapping into the given coefficient system, i.e., we set

$$C^n(X,M) = \mathrm{Hom}_{G\text{-coeff}}(C_n(X^\bullet),M) ,$$

the group of natural transformations of coefficient systems. The cellular differential $C_{n+1}(X^\bullet) \longrightarrow C_n(X^\bullet)$ induces a differential $C^n(X,M) \longrightarrow C^{n+1}(X,M)$. The *Bredon cohomology* of X with coefficients in M is then given by

$$H^n_G(X,M) = H^n(C^*(X,M)) .$$

The G-spaces G/H for finite subgroups of G are finite proper G-CW-complexes. So the proper G-cohomology theory gives rise to a G-coefficient system $\underline{\mathcal{H}}^k$, namely the composite

$$\left(\mathrm{Or}_G^{\mathcal{F}in}\right)^{\mathrm{op}} \xrightarrow{\text{incl}} (\text{finite proper } G\text{-CW-complexes})^{\mathrm{op}} \xrightarrow{\mathcal{H}^k} \mathcal{A}b .$$

We describe an isomorphism of abelian groups

$$C^p(X,\underline{\mathcal{H}}^q) = \mathrm{Hom}_{G\text{-coeff}}(C_p(X^\bullet),\underline{\mathcal{H}}^q) \cong \mathcal{H}^{p+q}(X^p,X^{p-1}) = E_1^{p,q} .$$

We choose a presentation of how X^p is obtained by attaching equivariant p-cells, in the form of a pushout of G-spaces:

$$\begin{array}{ccc} \coprod_I G/H_i \times \partial D^p & \longrightarrow & X^{p-1} \\ \downarrow & & \downarrow \\ \coprod_I G/H_i \times D^p & \longrightarrow & X^p \end{array}$$

These data induce an isomorphism

$$\bigoplus_{i\in I} H_p((G/H_i)^K\times D^p,(G/H_i)^K\times\partial D^p,\mathbb{Z}) \cong H_p((X^p)^K,(X^{p-1})^K,\mathbb{Z}) = C_p(X^K) .$$

Moreover,

$$H_p((G/H_i)^K \times D^p,(G/H_i)^K \times \partial D^p,\mathbb{Z}) \cong \mathbb{Z}[\mathrm{Or}_G^{\mathcal{F}in}(G/K,G/H_i)]$$

is the coefficient system represented by the coset G/H_i, so

$$C_p(X^\bullet) \;\cong\; \bigoplus_{i\in I} \mathbb{Z}[\mathrm{Or}_G^{\mathcal{F}in}(-, G/H_i)] \ .$$

This isomorphism induces an isomorphism of Bredon cochain groups

$$\begin{aligned} C^p(X, \underline{\mathcal{H}}^q) \;&=\; \mathrm{Hom}_{G\text{-coeff}}(C_p(X^\bullet), \underline{\mathcal{H}}^q) \;\cong\; \prod_{i\in I} \mathcal{H}^q(G/H_i) \\ &\cong\; \prod_{i\in I} \mathcal{H}^{p+q}(G/H \times D^p, G/H \times \partial D^p) \;\cong\; \mathcal{H}^{p+q}(X^p, X^{p-1}) \;=\; E_1^{p,q} \ . \end{aligned}$$

The same argument as for the classical Atiyah-Hirzebruch spectral sequence identifies the Bredon cohomology differential

$$d \;:\; C^p(X, \underline{\mathcal{H}}^q) \longrightarrow C^{p+1}(X, \underline{\mathcal{H}}^q)$$

with the d_1-differential

$$d_1 = j \circ k \;:\; E_1^{p,q} \;\longrightarrow\; E_1^{p+1,q} \ .$$

The $E_2^{p,q}$-term of the exact couple is thus the p-th Bredon cohomology group of X with coefficients in $\underline{\mathcal{H}}^q$. So the exact couple gives rise to a conditionally convergent half plane spectral sequence

$$E_2^{p,q} \;=\; H^p_G(X, \underline{\mathcal{H}}^q) \;\Longrightarrow\; \mathcal{H}^{p+q}(X) \ . \tag{3.2.15}$$

The d_r-differential has bidegree $(r, 1-r)$.

In the case where $X = G/H$ is a single orbit for a finite subgroup H of G, the E_2-term of the spectral sequence (3.2.15) is concentrated in bidegrees $(0, q)$. So the spectral sequence collapses at E_2 and recovers the isomorphism between $\mathcal{H}^q(G/H)$ and $H^0_G(G/H, \underline{\mathcal{H}}^q)$.

Example 3.2.16 (Eilenberg-Mac Lane spectra represent Bredon cohomology). As before we let G be a discrete group. Every G-Mackey functor M has an associated Eilenberg-Mac Lane G-spectrum HM, compare Remark 2.2.11. Since the homotopy group Mackey functors of HM are concentrated in degree zero, the E_2-term of the Atiyah-Hirzebruch spectral sequence (3.2.15) for the proper G-cohomology represented by HM is concentrated in bidegrees $(p, 0)$. So the spectral sequence collapses at E_2 and yields an isomorphism

$$H^*_G(X, M) \;\cong\; (HM)^*_G(X) \ .$$

In this sense, Bredon cohomology is represented by an Eilenberg-Mac Lane spectrum.

The reader should beware, however, that Bredon cohomology is defined for G-coefficients systems, whereas the construction of an Eilenberg-Mac Lane spectrum requires a full-fledged G-Mackey functor. Not every G-coefficient system can be extended to a G-Mackey functor, and if an extension exists, it need not be unique. Different extensions of a G-coefficient system to a G-Mackey functor give orthogonal G-spectra that are non-isomorphic in $\mathrm{Ho}(\mathrm{Sp}_G)$. As we just argued, the $\mathbb{Z}$-graded cohomology theory on G-CW-complexes only depends on the underlying coefficient system, and so it does not 'see' the extension to a G-Mackey functor. The extension is visible, however, if we extend the grading for the cohomology theory. Indeed, the cohomology theory represented by an orthogonal G-spectrum can be indexed on equivariant vector bundles over $\underline{E}G$, see Remark 3.2.10. Different extensions to a Mackey functor will typically lead to non-isomorphic '$KO_G(\underline{E}G)$-graded' cohomology theories.

3.3. Global versus proper stable homotopy types

In this section we explain how global stable homotopy types give rise to proper G-homotopy types, and we identify some extra structure that is present in this case. Several specific examples of such global equivariant theories feature throughout this monograph, for example equivariant stable cohomotopy (Example 3.2.9, arising from the global sphere spectrum), Borel cohomology (Example 3.3.13), and equivariant K-theory (Section 3.4, arising from the periodic global K-theory spectrum **KU**).

In [**60**] the fifth author developed a framework for *global stable homotopy theory*, i.e., equivariant stable homotopy theory where all compact Lie groups act simultaneously and in a compatible way. The technical realization of this slogan is via a certain *global model structure* on the category of orthogonal spectra in which the weak equivalences are the 'global equivalences' of [**60**, Def. 4.1.3], see also Definition 3.3.1 below. A direct consequence of the definition is a 'forgetful functor' from the global stable homotopy category to $\mathrm{Ho}(\mathrm{Sp}_G)$ for every Lie group G, see Theorem 3.3.3. This forgetful functor is an exact functor of triangulated categories that admits a left adjoint and a right adjoint. In a sense made precise by Theorem 3.3.5, the forgetful functors for different Lie groups are compatible with derived restriction along a continuous homomorphism introduced in Section 1.4. One specific benefit of coming from a global stable homotopy type is that the associated equivariant cohomology theories satisfy an induction isomorphism, see Proposition 3.3.8. At the end of this section, we introduce the Borel G-spectrum associated with a non-equivariant spectrum (see Example 3.3.10), and we show that the Borel spectra in fact underlie a global homotopy type (see Example 3.3.13).

For every Lie group G we can consider the functor

$$(-)_G \;:\; \mathrm{Sp} \longrightarrow \mathrm{Sp}_G \;, \quad X \longmapsto X_G$$

from orthogonal spectra to orthogonal G-spectra given by endowing an orthogonal spectrum with the trivial G-action.

Definition 3.3.1. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the map

$$\pi_k^H(f_H) \;:\; \pi_k^H(X_H) \longrightarrow \pi_k^H(Y_H)$$

is an isomorphism for every compact Lie group H and every integer k.

We denote by $\mathcal{GH} = \mathrm{Ho}^{\mathrm{gl}}(\mathrm{Sp})$ the category obtained by formally inverting the global equivalences of orthogonal spectra, and we refer to this as the *global stable homotopy category*. We write

$$\gamma_{\mathrm{gl}} \;:\; \mathrm{Sp} \longrightarrow \mathrm{Ho}^{\mathrm{gl}}(\mathrm{Sp}) \;=\; \mathcal{GH}$$

for the localization functor. By [**60**, Thm. 4.3.18], the global equivalences are part of a stable model structure. The global stable homotopy category is a compactly generated triangulated category, and a specific set of compact generators is given by the suspension spectra of the 'global classifying spaces' of all compact Lie groups, see [**60**, Thm. 4.4.3].

By the very definition, the functor $(-)_G$ takes global equivalences of orthogonal spectra to π_*-isomorphisms of orthogonal G-spectra. So we obtain a 'forgetful' functor on the homotopy categories

$$U_G \;=\; \mathrm{Ho}((-)_G) \;:\; \mathcal{GH} \;\longrightarrow\; \mathrm{Ho}(\mathrm{Sp}_G) \tag{3.3.2}$$

from the universal property of localizations. In other words, U_G is the unique functor that satisfies

$$U_G \circ \gamma_{\mathrm{gl}} \;=\; \gamma_G \circ (-)_G \;.$$

Moreover, U_G is canonically an exact functor of triangulated categories: the pointset level equality

$$X_G \wedge S^1 \;=\; (X \wedge S^1)_G$$

of functors $\mathrm{Sp} \longrightarrow \mathrm{Sp}_G$ descends to an equality

$$\begin{aligned} U_G \circ [1] \;&=\; \mathrm{Ho}((-)_G) \circ \mathrm{Ho}(- \wedge S^1) \;=\; \mathrm{Ho}((-)_G \circ (- \wedge S^1)) \\ &=\; \mathrm{Ho}((- \wedge S^1) \circ (-)_G) \;=\; \mathrm{Ho}(- \wedge S^1) \circ \mathrm{Ho}((-)_G) \;=\; [1] \circ U_G \;. \end{aligned}$$

Since distinguished triangles are defined in exactly the same way in $\mathcal{GH}$ and $\mathrm{Ho}(\mathrm{Sp}_G)$, the functor U_G preserves them.

The functor $X \mapsto X_G$ is fully faithful on the pointset level, but its derived functor U_G is typically *not* fully faithful. A hint is the fact that the equivariant homotopy groups of a global homotopy type, restricted to G and its subgroups, have more structure than is available for a general G-homotopy type, and satisfy certain relations that do not hold for general orthogonal G-spectra.

THEOREM 3.3.3. *For every Lie group G the forgetful functor*

$$U_G \;:\; \mathcal{GH} \;\longrightarrow\; \mathrm{Ho}(\mathrm{Sp}_G)$$

preserves all set-indexed sums and products, and it has a left adjoint and a right adjoint.

PROOF. Sums in $\mathcal{GH}$ and $\mathrm{Ho}(\mathrm{Sp}_G)$ are represented in both cases by the pointset level wedge. On the pointset level, the forgetful functor preserves wedges, so the derived forgetful functor preserves sums. The existence of the right adjoint is an abstract consequence of the fact that $\mathcal{GH}$ is compactly generated and that functor U preserves sums, compare [**60**, Cor. 4.4.5 (iv)].

The forgetful functor also preserves infinite products, but the argument here is slightly more subtle because products in $\mathcal{GH}$ are not generally represented by the pointset level product, and because equivariant homotopy groups do not in general commute with infinite pointset level products. We let $\{X_i\}_{i\in I}$ be a set of orthogonal spectra. By replacing each factor by a globally equivalent spectrum, if necessary, we can assume without loss of generality that each X_i is a global Ω-spectrum in the sense of [**60**, Def. 4.3.8]. Since global Ω-spectra are the fibrant objects in a model structure underlying $\mathcal{GH}$, the pointset level product $\prod_{i\in I} X_i$ then represents the product in $\mathcal{GH}$.

Even though X_i is a global Ω-spectrum, the underlying orthogonal G-spectrum $(X_i)_G$ need *not* be a G-Ω-spectrum. However, as spelled out in the proof of [**60**, Prop. 4.3.22 (ii)], the natural map

$$\pi_k^H\left(\prod\nolimits_{i\in I} X_i\right) \;\longrightarrow\; \prod\nolimits_{i\in I} \pi_k^H(X_i)$$

is an isomorphism for all compact Lie groups H and all integers k. This implies that in this situation, the pointset level product is also a product in $\mathrm{Ho}(\mathrm{Sp}_G)$. So the derived forgetful functor preserves products.

The existence of the left adjoint is then again an abstract consequence of the fact that $\mathcal{GH}$ is compactly generated and that the functor U preserves products, compare [**60**, Cor. 4.4.5 (v)]. □

We let $\alpha\colon K \longrightarrow G$ be a continuous homomorphism between Lie groups. In Theorem 1.4.17 we discussed various properties of the total left derived functor

$$L\alpha^* \ : \ \mathrm{Ho}(\mathrm{Sp}_G) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Sp}_K)$$

of the restriction functor $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$, with $\alpha_! : L\alpha^* \circ \gamma_G \Longrightarrow \gamma_K \circ \alpha^*$ the universal natural transformation. For example, the functor $L\alpha$ is exact and has a right adjoint. For another homomorphism $\beta : J \longrightarrow K$, we constructed a specific exact natural isomorphism

$$\langle \alpha, \beta \rangle \ : \ (L\beta^*) \circ (L\alpha^*) \ \Longrightarrow \ L(\alpha\beta)^*$$

in (1.4.25). The data of the functors $L\alpha^*$ and the transformations $\langle \alpha, \beta \rangle$ form a pseudo-functor from the category of Lie groups and continuous homomorphisms to the 2-category of triangulated categories, exact functors, and exact transformations.

Now we discuss how the derived restriction functors interact with the passage from global to proper homotopy theory. If X is any orthogonal spectrum, then on the pointset level, we have $\alpha^*(X_G) = X_K$, because K acts trivially on both sides. However, X_G will typically not be cofibrant as an orthogonal G-spectrum, so the relationship between the derived functors is more subtle: the universal property of the derived functor U_K provides a unique natural transformation

$$\alpha^\sharp \ : \ L\alpha^* \circ U_G \ \Longrightarrow \ U_K \tag{3.3.4}$$

of functors $\mathcal{GH} \longrightarrow \mathrm{Ho}(\mathrm{Sp}_K)$ that satisfies the relation

$$\alpha^\sharp \star \gamma_{\mathrm{gl}} \ = \ \alpha_! \star (-)_G$$

as transformations from the functor $L\alpha^* \circ U_G \circ \gamma_{\mathrm{gl}} = L\alpha^* \circ \gamma_G \circ (-)_G$ to the functor $U_K \circ \gamma_{\mathrm{gl}} = \gamma_K \circ \alpha^* \circ (-)_G$.

We recall that a continuous homomorphism between Lie groups is *quasi-injective* if the restriction to every compact subgroup of the source is injective.

THEOREM 3.3.5. *Let $\alpha : K \longrightarrow G$ be a continuous homomorphism between Lie groups.*

(i) *If α is quasi-injective, then the natural transformation $\alpha^\sharp : L\alpha^* \circ U_G \Longrightarrow U_K$ is an isomorphism.*
(ii) *If $\beta : J \longrightarrow K$ is another continuous homomorphism, then*

$$(\alpha\beta)^\sharp \circ (\langle \alpha, \beta \rangle \star U_G) \ = \ \beta^\sharp \circ (L\beta^* \star \alpha^\sharp)$$

as natural transformations $L\beta^* \circ L\alpha^* \circ U_G \Longrightarrow U_J$.

Proof. (i) We let X be any orthogonal spectrum. We choose a π_*-isomorphism of orthogonal G-spectra $\psi : Y \longrightarrow X_G$ whose source is cofibrant. We obtain a commutative diagram in $\mathrm{Ho}(\mathrm{Sp}_K)$:

$$\begin{tikzcd}
(L\alpha^*)(Y) \arrow[r, "(L\alpha^*)(\gamma_G(\psi))", "\cong"'] \arrow[d, "\alpha_!"', "\cong"] & (L\alpha^*)(X_G) \arrow[r, "\alpha^\sharp"] \arrow[dr, "\alpha_!"] & X_K \arrow[d, equal] \\
\alpha^*(Y) \arrow[rr, "\cong", "\gamma_K(\alpha^*(\psi))"'] & & \alpha^*(X_G)
\end{tikzcd}$$

The left vertical morphism $\alpha_!$ is an isomorphism by Theorem 1.4.17 (ii). Since α is quasi-injective, the functor α^* is homotopical by Theorem 1.4.1 (i), so the lower horizontal morphism is an isomorphism as well. This shows that $\alpha^\sharp : (L\alpha^*)(X_G) \longrightarrow X_K$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$.

Part (ii) is Proposition 1.4.28, applied to the left derivable functors $(-)_G : \mathrm{Sp} \longrightarrow \mathrm{Sp}_G$, $\alpha^* : \mathrm{Sp}_G \longrightarrow \mathrm{Sp}_K$ and $\beta^* : \mathrm{Sp}_K \longrightarrow \mathrm{Sp}_J$. $\square$

As we just explained, every global homotopy type gives rise to a G-homotopy type for every Lie group G. The 'global' nature is also reflected in the G-equivariant cohomology theories represented by the G-spectra. The following proposition says that for every orthogonal spectrum E, the collection of equivariant cohomology theories E^*_G for varying G form an 'equivariant cohomology theory' in the sense of [**38**, 5.2].

Construction 3.3.6 (Restriction maps for global homotopy types). If all the equivariant cohomology theories E^*_G arise from a global homotopy type (i.e., from a single orthogonal spectrum), then there is extra structure in the form of *restriction homomorphisms*

$$\alpha^* \ : \ E^*_G(X) \ \longrightarrow \ E^*_K(\alpha^*(X))$$

associated with every continuous homomorphism $\alpha : K \longrightarrow G$ between Lie groups. Here X is any $\mathcal{C}om$-cofibrant G-space, so that $\Sigma^\infty_+ X$ is a cofibrant orthogonal G-spectrum; hence the morphism

$$\alpha_! \ : \ (L\alpha^*)(\Sigma^\infty_+ X) \ \longrightarrow \ \alpha^*(\Sigma^\infty_+ X) \ = \ \Sigma^\infty_+ \alpha^*(X)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_K)$, compare Theorem 1.4.17 (ii). We then define α^* as the composite

$$\begin{aligned} E^0_G(X) \ = \ [\Sigma^\infty_+ X, E_G]^G \ &\xrightarrow{L\alpha^*} \ [(L\alpha)^*(\Sigma^\infty_+ X), (L\alpha^*)(E_G)]^K \\ &\xrightarrow{[\alpha_!^{-1}, \alpha^\sharp]} \ [\Sigma^\infty_+ \alpha^*(X), E_K]^K \ = \ E^0_K(\alpha^*(X)) \ . \end{aligned}$$

The natural transformation $\alpha^\sharp : L\alpha^* \circ U_G \Longrightarrow U_K$ was defined in (3.3.4).

We let $\alpha : K \longrightarrow G$ be a continuous homomorphism between Lie groups. As before, for a K-space X we denote the *induced G-space* by

$$G \times_\alpha X \ = \ (G \times X)/(g \cdot \alpha(k), x) \sim (g, k \cdot x) \ .$$

The functor $G \times_\alpha -$ is left adjoint to restriction along α, and the map

$$\eta_X \ : \ X \ \longrightarrow \ \alpha^*(G \times_\alpha X) \ , \quad x \ \longmapsto \ [1, x]$$

is the unit of the adjunction.

Now we let E be an orthogonal spectrum. If X is a $\mathcal{C}om$-cofibrant K-space, then $G \times_\alpha X$ is a $\mathcal{C}om$-cofibrant G-space, by Proposition 1.1.6 (ii). We define the *induction map*

$$\mathrm{ind}_\alpha \ : \ E^*_G(G \times_\alpha X) \ \longrightarrow \ E^*_K(X) \tag{3.3.7}$$

as the composite

$$E^*_G(G \times_\alpha X) \ \xrightarrow{\alpha^*} \ E^*_K(\alpha^*(G \times_\alpha X)) \ \xrightarrow{\eta^*_X} \ E^*_K(X) \ .$$

PROPOSITION 3.3.8. *Let E be an orthogonal spectrum and $\alpha : K \longrightarrow G$ a continuous homomorphism between Lie groups. Let X be a proper K-CW-complex on which the kernel of α acts freely. Then the induction map* (3.3.7) *is an isomorphism.*

PROOF. The functor $G \times_\alpha -$ preserves equivariant homotopies and commutes with wedges and mapping cones. So the functor $E^*_G(G \times_\alpha -)$ from the category of K-spaces to graded abelian groups is a proper cohomology theory. The induction maps form a transformation of cohomology theories, so it suffices to check the claim on orbits of the form $X = K/L$, for all compact subgroups L of K, on which the kernel of α acts freely. The freeness condition precisely means that the restriction

$$\bar\alpha = \alpha|_L \ : \ L \ \longrightarrow \ G$$

of α to L is injective. The G-map

$$\psi \ : \ G \times_{\bar\alpha} (L/L) \ \longrightarrow \ G \times_\alpha (K/L) \ , \quad [g, eL] \ \longmapsto \ [g, eL]$$

is a homeomorphism, and it makes the following square commute:

$$\begin{tikzcd}
L/L \arrow[r, "\mathrm{incl}"] \arrow[d, "\eta_{L/L}"'] & K/L \arrow[d, "\eta_{K/L}"] \\
G \times_{\bar\alpha} (L/L) \arrow[r, "\psi"'] & G \times_\alpha (K/L)
\end{tikzcd}$$

So the following diagram of equivariant cohomology groups commutes as well:

$$\begin{tikzcd}
 & E^*_G(G \times_\alpha (K/L)) \arrow[r, "\psi^*", "\cong"'] \arrow[d, "\bar\alpha^*"] \arrow[dl, bend right, "\mathrm{ind}_\alpha"'] & E^*_G(G \times_{\bar\alpha} L/L) \arrow[d, "\bar\alpha^*"'] \arrow[dd, bend left=60, "\mathrm{adj}", "\cong"'] \\
 & E^*_L(\bar\alpha^*(G \times_\alpha (K/L))) \arrow[r, "\psi^*", "\cong"'] \arrow[d, "\eta^*_{K/L}"] & E^*_L(\bar\alpha^*(G \times_{\bar\alpha} L/L)) \arrow[d, "\eta^*_{L/L}"'] \\
E^*_K(K/L) \arrow[r, "\mathrm{res}^K_L"] \arrow[rr, bend right, "\cong", "\mathrm{adj}"'] & E^*_L(K/L) \arrow[r, "\mathrm{incl}^*"] & E^*_L(L/L)
\end{tikzcd}$$

Commutativity of the left part uses the relation $\mathrm{res}^K_L \circ \alpha^* = \bar\alpha^*$ and the naturality of restriction from K to L. The lower horizontal composite and the right vertical composite are adjunction bijections. This proves that the induction map is an isomorphism for $X = K/L$. $\square$

The following corollary is the special case of Proposition 3.3.8 for the unique homomorphism $K \longrightarrow e$. Part (ii) also uses that whenever K has no non-trivial compact subgroups, then K acts freely on the universal proper K-space $\underline{E}K$.

Corollary 3.3.9. *Let E be an orthogonal spectrum and K a Lie group.*

(i) *For every free K-CW-complex X, the induction map* (3.3.7)
$$\mathrm{ind} \ : \ E_e^*(X/K) \ \longrightarrow \ E_K^*(X)$$
for the unique homomorphism $K \longrightarrow e$ is an isomorphism.

(ii) *If K has no non-trivial compact subgroups, then the induction map* (3.3.7)
$$\mathrm{ind} \ : \ E_e^*(BK) \ \longrightarrow \ E_K^*(\underline{E}K) \ \cong \ \pi_{-*}^K(E)$$
for the unique homomorphism $K \longrightarrow e$ is an isomorphism.

Example 3.3.10 (Borel cohomology). We let F be a non-equivariant generalized cohomology theory. For a G-space A, the associated *Borel cohomology theory* is given
$$F^*(EG \times_G A) \ ,$$
the F-cohomology of the Borel construction. This Borel cohomology theory is realized by an orthogonal G-spectrum. For this purpose we represent the given cohomology theory by an orthogonal Ω-spectrum X (in the non-equivariant sense). We claim that then the orthogonal G-spectrum
$$bX \ = \ \mathrm{map}(EG, X) \ ,$$
obtained by taking the space of unbased maps from EG levelwise (see Construction 1.1.14), represents Borel cohomology.

Proposition 3.3.11. *Let G be a Lie group and X an orthogonal Ω-spectrum.*

(i) *The orthogonal G-spectrum* $\mathrm{map}(EG, X)$ *is a G-Ω-spectrum.*

(ii) *For every $\mathcal{C}om$-cofibrant G-space A, there is an isomorphism*
$$\mathrm{map}(EG, X)_G^k(A) \ \cong \ X^k(EG \times_G A)$$
that is natural for G-maps in A. In particular,
$$\pi_{-k}^G(\mathrm{map}(EG, X)) \ \cong \ X^k(BG) \ .$$

Proof. (i) We let H be any compact subgroup of G, and let V and W be two H-representations. Then the adjoint structure map
$$\tilde{\sigma}_{V,W}^X \ : \ X(W) \ \longrightarrow \ \mathrm{map}_*(S^V, X(V \oplus W))$$
is H-equivariant and a weak equivalence on underlying non-equivariant spaces. The underlying H-space of EG is a free cofibrant H-space, so applying $\mathrm{map}^H(EG, -)$ to $\tilde{\sigma}_{V,W}^X$ returns a weak equivalence
$$\begin{aligned} \mathrm{map}^H(EG, \tilde{\sigma}_{V,W}^X) \ : \ (\mathrm{map}(EG, X)(W))^H \ &= \ \mathrm{map}^H(EG, X(W)) \\ &\longrightarrow \ \mathrm{map}^H(EG, \mathrm{map}_*(S^V, X(V \oplus W))) \ . \end{aligned}$$
The target of this map is homeomorphic to
$$\mathrm{map}_*^H(S^V, \mathrm{map}(EG, X(V \oplus W))) \ = \ \mathrm{map}_*^H(S^V, \mathrm{map}(EG, X)(V \oplus W))$$
in such a way that $\mathrm{map}^H(EG, \tilde{\sigma}_{V,W}^X)$ becomes the H-fixed points of the adjoint structure map of $\mathrm{map}(EG, X)$. So $\mathrm{map}(EG, X)$ is a G-Ω-spectrum.

(ii) Because A is $\mathcal{C}om$-cofibrant, its unreduced suspension spectrum $\Sigma_+^\infty A$ is cofibrant in the stable model structure of orthogonal G-spectra. On the other hand, the spectrum $\mathrm{map}(EG, X)$ is a G-Ω-spectrum by part (i), hence it is fibrant in the stable model structure of orthogonal G-spectra. So morphisms from $\Sigma_+^\infty A$

to $\mathrm{map}(EG, X)$ in $\mathrm{Ho}(\mathrm{Sp}_G)$ can be calculated as homotopy classes of morphisms of orthogonal G-spectra. Combining this with various adjunction bijections yields the desired isomorphism for $k = 0$:

$$\begin{aligned} \mathrm{map}(EG, X)^0_G(A) \ &= \ [\Sigma^\infty_+ A, \mathrm{map}(EG, X)]^G \\ &\cong \ \mathrm{Sp}_G(\Sigma^\infty_+ A, \mathrm{map}(EG, X))/\sim \\ &\cong \ \pi_0(\mathrm{map}^G(A, \mathrm{map}(EG, X)(0))) \\ &\cong \ \pi_0(\mathrm{map}(EG \times_G A, X(0))) \ \cong \ X^0(EG \times_G A) \ . \end{aligned} \tag{3.3.12}$$

Here the symbol '$\sim$' stands for the homotopy relation. For $k > 0$ we exploit that the shifted spectrum $\mathrm{sh}^k X$ is again an Ω-spectrum. Proposition 3.1.25 of [**60**] provides a π_*-isomorphism

$$\lambda^k_{\mathrm{map}(EG,X)} \ : \ \mathrm{map}(EG, X) \wedge S^k \ \longrightarrow \ \mathrm{sh}^k(\mathrm{map}(EG, X)) \ = \ \mathrm{map}(EG, \mathrm{sh}^k X)$$

which induces a natural isomorphism

$$\begin{aligned} \mathrm{map}(EG, X)^k_G(A) \ &= \ [\Sigma^\infty_+ A, \mathrm{map}(EG, X) \wedge S^k]^G \\ &\cong \ [\Sigma^\infty_+ A, \mathrm{map}(EG, \mathrm{sh}^k X)]^G \\ &= \ \mathrm{map}(EG, \mathrm{sh}^k X)^0_G(A) \\ &\overset{(3.3.12)}{\cong} \ (\mathrm{sh}^k X)^0_G(EG \times_G A) = \ X^k_G(EG \times_G A) \ . \end{aligned}$$

Similarly, the looped spectrum $\Omega^k X$ is another Ω-spectrum. So we get natural isomorphisms

$$\begin{aligned} \mathrm{map}(EG, X)^{-k}_G(A) \ &= \ [\Sigma^\infty_+ A, \Omega^k \mathrm{map}(EG, X)]^G \\ &\cong \ [\Sigma^\infty_+ A, \mathrm{map}(EG, \Omega^k X)]^G \\ &= \ \mathrm{map}(EG, \Omega^k X)^0_G(A) \\ &\overset{(3.3.12)}{\cong} \ (\Omega^k X)^0_G(EG \times_G A) \ = \ X^{-k}_G(EG \times_G A) \ . \end{aligned}$$

The last claim is the special case where $A = \underline{E}G$, in which case the $\mathcal{C}om$-equivalence $\underline{E}G \longrightarrow *$ induces an isomorphism

$$\begin{aligned} \mathrm{map}(EG, X)^k_G(\underline{E}G) \ &= \ [\Sigma^\infty_+ \underline{E}G, \mathrm{map}(EG, X)[k]]^G \\ &\cong \ [\Sigma^\infty_+ \mathbb{S}_G, \mathrm{map}(EG, X)[k]]^G \ = \ \pi^G_{-k}(\mathrm{map}(EG, X)) \ . \end{aligned}$$

On the other hand, the projection $EG \times \underline{E}G \longrightarrow EG$ is a G-equivariant homotopy equivalence, so the induced map on orbits

$$EG \times_G \underline{E}G \ \longrightarrow \ EG/G \ = \ BG$$

is a homotopy equivalence. □

Example 3.3.13 (Borel cohomology is global). In Proposition 3.3.11 we showed that the Borel cohomology theory associated with a (non-equivariant) cohomology theory is represented by an orthogonal G-spectrum. We will now argue that the Borel cohomology theories are in fact 'global', i.e., can be represented by an orthogonal G-spectrum with trivial G-action. The global version of the Borel construction actually models the right adjoint to the forgetful functor $U : \mathcal{GH} \longrightarrow \mathrm{Ho}(\mathrm{Sp})$ from the global to the non-equivariant stable homotopy category, see [**60**, Prop. 4.5.22]. The following construction is taken from [**60**, Con. 4.5.21].

We start with an orthogonal spectrum X (in the non-equivariant sense) that represents a (non-equivariant) cohomology theory $X^*(-)$. We define a new orthogonal spectrum bX as follows. For an inner product space V, we write $\mathbf{L}(V,\mathbb{R}^\infty)$ for the contractible space of linear isometric embeddings of V into $\mathbb{R}^\infty$. We set

$$(bX)(V) \;=\; \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty), X(V))\;,$$

the space of all continuous maps from $\mathbf{L}(V,\mathbb{R}^\infty)$ to $X(V)$. The orthogonal group $O(V)$ acts on this mapping space by conjugation, through its actions on $\mathbf{L}(V,\mathbb{R}^\infty)$ and on $X(V)$. We define structure maps $\sigma_{V,W} : S^V \wedge (bX)(W) \longrightarrow (bX)(V\oplus W)$ as the composite

$$\begin{aligned} S^V \wedge \mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), X(W)) \quad &\xrightarrow{\text{assembly}} \quad \mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), S^V\wedge X(W)) \\ &\xrightarrow{\mathrm{map}(\mathrm{res}_W,\sigma^X_{V,W})} \mathrm{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty), X(V\oplus W)) \end{aligned}$$

where $\mathrm{res}_W : \mathbf{L}(V\oplus W,\mathbb{R}^\infty) \longrightarrow \mathbf{L}(W,\mathbb{R}^\infty)$ is the map that restricts an isometric embedding from $V\oplus W$ to W.

The endofunctor b on the category of orthogonal spectra comes with a natural transformation

$$i_X \;:\; X \;\longrightarrow\; bX$$

whose value at an inner product space V sends a point $x\in X(V)$ to the constant map $\mathbf{L}(V,\mathbb{R}^\infty)\longrightarrow X(V)$ with value x. With the help of the morphism i_X we can now compare the spectrum bX to the G-spectrum $\mathrm{map}(EG,X)$ defined in Example 3.3.10 via the two natural morphisms of orthogonal G-spectra

$$\mathrm{map}(EG,X) \xrightarrow{\mathrm{map}(EG,i_X)} \mathrm{map}(EG,bX) \xleftarrow{\text{const}} bX\;. \tag{3.3.14}$$

Both maps are morphism of orthogonal G-spectra, where bX is endowed with trivial G-action.

Proposition 3.3.15. *For every orthogonal Ω-spectrum X the two morphisms (3.3.14) are π_*-isomorphisms of orthogonal G-spectra. So the orthogonal spectrum bX, endowed with trivial G-action, represents the Borel G-cohomology theory associated with X.*

Proof. Since the space $\mathbf{L}(V,\mathbb{R}^\infty)$ is contractible, the morphism $i_X : X\longrightarrow bX$ is a non-equivariant level equivalence. So applying $\mathrm{map}(EG,-)$ takes it to a level equivalence of orthogonal G-spectra. Since level equivalences are in particular π_*-isomorphisms, this takes care of the morphism $\mathrm{map}(EG,i_X)$.

For the second morphism we consider a compact subgroup H of G and a *faithful* H-representation V. Then the H-space $\mathbf{L}(V,\mathbb{R}^\infty)$ is cofibrant, non-equivariantly contractible and has a free H-action, i.e., it is a model for EH, see for example [**60**, Prop. 1.1.26]. The underlying H-space of EG is also a model for EH, so the projection

$$\mathrm{proj} \;:\; EG\times\mathbf{L}(V,\mathbb{R}^\infty) \;\longrightarrow\; \mathbf{L}(V,\mathbb{R}^\infty)$$

is an H-equivariant homotopy equivalence. The induced map

$$\begin{aligned} \mathrm{map}^H(\mathrm{proj},X(V)) \;:\; (bX(V))^H \;&=\; \mathrm{map}^H(\mathbf{L}(V,\mathbb{R}^\infty),X(V)) \\ &\longrightarrow\; \mathrm{map}^H(EG\times\mathbf{L}(V,\mathbb{R}^\infty),X(V)) \end{aligned}$$

is thus a homotopy equivalence. The target of this map is isomorphic to the space $\mathrm{map}^H(EG, bX)(V)$, and under this isomorphism the map $\mathrm{map}^H(\mathrm{proj}, X(V))$ becomes the 'constant' map. So the second morphism is a level equivalence of orthogonal H-spectra when restricted to *faithful* H-representations. Since faithful representations are cofinal in all H-representations, the second morphism induces an isomorphism of H-homotopy groups. Since H is any compact subgroups of G, this shows that the second morphism is a π_*-isomorphism of orthogonal G-spectra. □

3.4. Equivariant K-theory

In this final section we show that for discrete groups, the equivariant K-theory defined from G-vector bundles is representable by the global equivariant K-theory spectrum $\mathbf{KU}$. Before going into more details, we give a brief overview of the main players. In [**42**, Thm. 3.2], the third author and Oliver show that the classical way to construct equivariant K-theory still works for discrete groups G and finite proper G-CW-complexes X: the group $K_G(X)$, defined as the Grothendieck group of G-vector bundles over X, is excisive in X. Moreover, the functor $K_G(X)$ is Bott periodic, and can thus be extended to a $\mathbb{Z}$-graded theory. Also, the theory $K^*_G(X)$ supports Thom isomorphisms for hermitian G-vector bundles [**42**, Thm. 3.14], and it satisfies a version of the Atiyah-Segal completion theorem, see [**42**, Thm. 4.4].

In [**30**, Def. 3.6], Joachim introduced a model $\mathbf{KU}$ for periodic global K-theory that is based on spaces of homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras, see also [**60**, Sec. 6.4]. This is a commutative orthogonal ring spectrum, and the underlying orthogonal G-spectrum $\mathbf{KU}_G$ represents equivariant K-theory for all compact Lie groups, see [**30**, Thm. 4.4] or [**60**, Cor. 6.4.13]. For a general Lie group G, the underlying G-spectrum $\mathbf{KU}_G$ represents a proper G-cohomology theory by Theorem 3.2.7.

The purpose of this final section is to establish a natural isomorphism between the two proper G-cohomology theories K^*_G and $\mathbf{KU}^*_G$ for discrete groups G, stated in Theorem 3.4.22. Since the two theories are defined very differently, the main issue is to construct a natural transformation of cohomology theories in one direction, and most of our work goes into this. Our strategy is to first compare the excisive functors $K_G(X)$ and $\mathbf{KU}_G(X)$, see Theorem 3.4.16. This comparison passes through the theory $\mathbf{ku}_G[\![X]\!]$, represented by the *connective global K-theory spectrum* $\mathbf{ku}$ in the sense of the fifth author [**60**, Con. 6.3.9], a variation of Segal's configuration space model for K-homology [**68**]. We use $\mathbf{ku}_G[\![X]\!]$ as a convenient target of an explicit map $\langle - \rangle : \mathrm{Vect}_G(X) \longrightarrow \mathbf{ku}_G[\![X]\!]$ that turns a G-vector bundle into an equivariant homotopy class, see (3.4.4). The fact that the map $\langle - \rangle$ is additive and multiplicative is not entirely obvious, and this verification involves some work, see Propositions 3.4.8 and 3.4.15. The connective and periodic global K-theory spectra are related by a homomorphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ of commutative orthogonal ring spectra, defined in [**60**, Con. 6.4.13]. The induced transformation of excisive functors $j_* : \mathbf{ku}_G[\![X]\!] \longrightarrow \mathbf{KU}_G[\![X]\!]$ is then additive and multiplicative. Because the isomorphism $K_G(X) \cong \mathbf{KU}_G[\![X]\!]$ is suitably multiplicative, it matches the two incarnations of Bott periodicity, and can thus be extended to the $\mathbb{Z}$-graded periodic theories in a relatively formal (but somewhat tedious) way, see the final Theorem 3.4.22.

The main results in this section require the group G to be discrete (as opposed to allowing general Lie groups). The restriction arises from the vector bundle side of the story: Example 5.2 of [**42**] shows that the theory made from isomorphism classes of G-vector bundles is *not* in general excisive in the context of non-discrete Lie groups. We think of the represented theory $\mathbf{KU}_G^*(X)$ as the 'correct' equivariant K-theory in general; indeed, it restricts to equivariant K-theory for compact Lie groups by [**30**, Thm. 4.4] or [**60**, Cor. 6.4.23], and the point of this book is precisely that proper G-cohomology theories represented by orthogonal G-spectra always have the desired formal properties, for all Lie groups. Phillips [**56**] has defined equivariant K-theory for second countable locally compact topological groups G, defined on proper locally compact G-spaces. His construction is based on Hilbert G-bundles instead of finite-dimensional G-vector bundles. It seems plausible that in the common realm of Lie groups, our represented theory $\mathbf{KU}_G^*(X)$ ought to be isomorphic to Phillips' theory, but we have not attempted to define an isomorphism.

Construction 3.4.1 (Connective global K-theory). We recall the definition of the orthogonal spectrum $\mathbf{ku}$ from [**60**, Sec. 6.3]. For a real inner product space V, we let $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ denote the complexification which inherits a unique hermitian inner product $(-,-)$ characterized by

$$(1 \otimes v, 1 \otimes w) \ = \ \langle v, w\rangle$$

for all $v, w \in V$. The symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification inherits a preferred hermitian inner product in such a way that the canonical algebra isomorphism

$$\mathrm{Sym}(V_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathrm{Sym}(W_{\mathbb{C}}) \ \cong \ \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$$

becomes an isometry, compare [**60**, Prop. 6.3.8]. The V-th space of the orthogonal spectrum $\mathbf{ku}$ is the value on S^V of the Γ-space of finite-dimensional, pairwise orthogonal subspaces of $\mathrm{Sym}(V_{\mathbb{C}})$. More explicitly, we define the V-th space of the orthogonal spectrum $\mathbf{ku}$ as the quotient space

$$\mathbf{ku}(V) = \Big(\coprod_{m \geq 0} \mathrm{Gr}_{\langle m\rangle}(\mathrm{Sym}(V_{\mathbb{C}})) \times (S^V)^m \Big) \Big/ {\sim_V}$$

where S^V is the one-point compactification of V and $\mathrm{Gr}_{\langle m\rangle}(\mathrm{Sym}(V_{\mathbb{C}}))$ is the space of m-tuples of pairwise orthogonal subspaces in $\mathrm{Sym}(V_{\mathbb{C}})$. Here, the equivalence relation $\sim_V$ makes the following identifications

(i) The tuple $(E_1, \ldots, E_m; v_1, \ldots, v_m)$, where $(E_1, \ldots, E_m)$ is an m-tuple of pairwise orthogonal subspaces of $\mathrm{Sym}(V_{\mathbb{C}})$ and $v_1, \ldots, v_m \in S^V$, is identified with $(E_{\sigma(1)}, \ldots, E_{\sigma(m)}; v_{\sigma(1)}, \ldots, v_{\sigma(m)})$ for any σ in the symmetric group Σ_m. This implies that we can represent equivalence classes as formal sums $\sum_{k=1}^m v_k E_k$.

(ii) If $v_i = v_j$ for some $i \neq j$, then

$$\sum_{k=1}^m v_k E_k \ = \ v_i(E_i \oplus E_j) + \sum_{\substack{k=1\\ k \neq i,j}}^m v_k E_k \ .$$

(iii) If $E_i = 0$ is the trivial subspace or $v_i = \infty$ is the basepoint of S^V at infinity, then

$$\sum_{k=1}^m v_k E_k = \sum_{\substack{k=1\\ k \neq i}}^m v_k E_k \ .$$

Hence, the topology of $\mathbf{ku}(V)$ is such that, informally speaking, the labels E_i and E_j add up inside $\mathrm{Sym}(V_{\mathbb{C}})$ whenever the two points v_i and v_j collide, and the label E_i disappears when v_i reaches the basepoint at infinity. The action of the orthogonal group $O(V)$ on S^V and $\mathrm{Sym}(V_{\mathbb{C}})$ induces a based continuous $O(V)$-action on $\mathbf{ku}(V)$. We define, for all inner product spaces V and W, an $O(V)\times O(W)$-equivariant multiplication map

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{ku}(V)\wedge\mathbf{ku}(W) \quad &\longrightarrow \quad \mathbf{ku}(V\oplus W) \\ \Big(\sum v_i E_i\Big)\wedge\Big(\sum w_j F_j\Big) \ &\longmapsto \ \sum (v_i\wedge w_j)\cdot(E_i\otimes F_j) \end{aligned}$$

where the canonical isometry $\mathrm{Sym}(V_{\mathbb{C}})\otimes\mathrm{Sym}(W_{\mathbb{C}})\cong\mathrm{Sym}((V\oplus W)_{\mathbb{C}})$ is used to interpret $E_i\otimes F_j$ as a subspace of $\mathrm{Sym}((V\oplus W)_{\mathbb{C}})$. Finally, we define the $O(V)$-equivariant unit map

$$\nu_V : S^V \longrightarrow \mathbf{ku}(V) \ , \quad v\mapsto v\mathbb{C} \ ,$$

where $\mathbb{C}$ refers to the 'constants' in the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$, i.e., the subspace spanned by the multiplicative unit 1. The maps $\{\mu_{V,W}\}$ together with the maps $\{\nu_V\}$ turn $\mathbf{ku}=\{\mathbf{ku}(V)\}$ into a commutative orthogonal ring spectrum.

For a Lie group G, the *connective G-equivariant K-theory spectrum* $\mathbf{ku}_G$ is the orthogonal spectrum $\mathbf{ku}$ equipped with trivial G-action. It is relevant for our purposes that $\mathbf{ku}_G$ arises from a global stable homotopy type, i.e., it is obtained by the forgetful functor of Section 3.3, applied to $\mathbf{ku}$.

CONSTRUCTION 3.4.2. For this construction, G is any Lie group. We write $\mathrm{Vect}_G(X)$ for the abelian monoid of isomorphism classes of hermitian G-vector bundles over a G-space X. We introduce a natural homomorphism of abelian monoids

$$\langle - \rangle \ : \ \mathrm{Vect}_G(X)\longrightarrow \mathbf{ku}_G[\![X]\!] \tag{3.4.3}$$

for any finite proper G-CW-complex X. The construction is a parameterized version of the construction in [**60**, Thm. 6.3.31], and proceeds as follows. For a hermitian G-vector bundle ξ over X, we let $u\xi$ denote the underlying euclidean vector bundle. We denote by $(u\xi)_{\mathbb{C}}$ the complexification of the latter. Then the maps

$$j_x \ : \ \xi_x \ \longrightarrow \ (u\xi_x)_{\mathbb{C}} \ , \quad v \ \longmapsto \ 1/\sqrt{2}\cdot(1\otimes v - i\otimes(iv))$$

are $\mathbb{C}$-linear isometric embeddings of each fiber that vary continuously with $x\in X$. Altogether, these define an isometric embedding of hermitian G-vector bundles

$$j \ : \ \xi \ \longrightarrow \ (u\xi)_{\mathbb{C}} \ .$$

By design, the fiber of the retractive G-space $\mathbf{ku}(u\xi)$ over $x\in X$ is $\mathbf{ku}(u\xi_x)$. So we can define a map of retractive G-spaces over X

$$\{\xi\} \ : \ S^{u\xi} \ \longrightarrow \ \mathbf{ku}(u\xi) \qquad \text{by} \qquad \{\xi\}(x,v) \ = \ [j_x(\xi_x);v].$$

In more detail: we view $j_x(\xi_x)$ as sitting in the linear summand in the symmetric algebra $\mathrm{Sym}((u\xi_x)_{\mathbb{C}})$, and $[j_x(\xi_x);v]$ as the configuration in $\mathbf{ku}(u\xi_x)$ consisting of the single vector v labeled by the vector space $j_x(\xi_x)$. The point $[j_x(\xi_x);v]\in\mathbf{ku}(u\xi)$ varies continuously with $(x,v)\in S^{u\xi}$, and altogether this defines the G-equivariant map $\{\xi\}$. If $\psi:\xi\longrightarrow\eta$ is an isomorphism of hermitian G-vector bundles over X, then the maps $\{\xi\}$ and $\{\eta\}$ are conjugate by ψ, and so they represent the same class in $\mathbf{ku}_G^0[\![X]\!]$. So we obtain a well-defined map

$$(3.4.4)\qquad \langle - \rangle \;:\; \mathrm{Vect}_G(X) \longrightarrow \mathbf{ku}_G[\![X]\!] \qquad \text{by} \qquad \langle \xi \rangle \;=\; [u\xi, \{\xi\}]\ .$$

For every continuous G-map $f : Y \longrightarrow X$ we have $f^*(u\xi) = u(f^*\xi)$ as euclidean G-vector bundles over Y. Moreover, the two maps

$$f^*\{\xi\} \;:\; S^{f^*(u\xi)} \longrightarrow \mathbf{ku}(f^*(u\xi)) \quad \text{and} \quad \{f^*\xi\} \;:\; S^{u(f^*\xi)} \longrightarrow \mathbf{ku}(u(f^*\xi))$$

are equal. So

$$\langle f^*[\xi] \rangle \;=\; [u(f^*\xi), \{f^*\xi\}] \;=\; [f^*(u\xi), f^*\{\xi\}] \;=\; f^*[u\xi, \{\xi\}] \;=\; f^*\langle \xi \rangle\ ,$$

i.e., for varying G-spaces X, the maps $\langle - \rangle$ constitute a natural transformation.

The following proposition provides additional freedom in the passage (3.4.4) from G-vector bundles to $\mathbf{ku}$-cohomology classes: it lets us replace the embedding

$$\xi \xrightarrow{\ j\ } (u\xi)_{\mathbb{C}} \xrightarrow{\text{linear summand}} \mathrm{Sym}((u\xi)_{\mathbb{C}})$$

used in the definition of $\langle \xi \rangle$ by any other equivariant isometric embedding of ξ into the complexified symmetric algebra of any other euclidean vector bundle.

Proposition 3.4.5. *Let G be a Lie group, X a G-space and ξ a hermitian G-vector bundle over X. Let μ be a euclidean G-vector bundle over X and*

$$J \;:\; \xi \longrightarrow \mathrm{Sym}(\mu_{\mathbb{C}})$$

a G-equivariant $\mathbb{C}$-linear isometric embedding. We define a map of retractive G-spaces

$$\lambda(J) \;:\; S^{\mu} \longrightarrow \mathbf{ku}(\mu) \qquad by \qquad \lambda(J)(x, v) \;=\; [J_x(\xi_x), v]\ .$$

Then $\langle \xi \rangle$ coincides with the class of $(\mu, \lambda(J))$.

Proof. The two composites around the (non-commutative!) square

$$\begin{tikzcd} \xi \arrow[r, "J"] \arrow[d, "i \circ j^{\xi}"'] & \mathrm{Sym}(\mu_{\mathbb{C}}) \arrow[d, "\mathrm{Sym}(i_2)"] \\ \mathrm{Sym}((u\xi)_{\mathbb{C}}) \arrow[r, "\mathrm{Sym}(i_1)"'] & \mathrm{Sym}((u\xi)_{\mathbb{C}} \oplus \mu_{\mathbb{C}}) \end{tikzcd}$$

are G-equivariant isometric embeddings whose images are orthogonal inside the hermitian vector bundle $\mathrm{Sym}((u\xi)_{\mathbb{C}} \oplus \mu_{\mathbb{C}})$. Here $i : (u\xi)_{\mathbb{C}} \longrightarrow \mathrm{Sym}((u\xi)_{\mathbb{C}})$ is the embedding of the linear summand. The diagram thus commutes up to equivariant homotopy of linear embeddings, fiberwise given by the formula

$$H(t, v) \;=\; t \cdot (\mathrm{Sym}(i_1) \circ i \circ j^{\xi})(v) \;+\; \sqrt{1 - t^2} \cdot (\mathrm{Sym}(i_2) \circ J)(v)\ .$$

Such a homotopy induces an equivariant homotopy of maps of retractive G-spaces over X between $\lambda(\mathrm{Sym}(i_1) \circ i \circ j^{\xi})$ and $\lambda(\mathrm{Sym}(i_2) \circ J)$. Hence

$$\begin{aligned} \langle \xi \rangle \;=\; [u\xi, \{\xi\}] \;=\; [u\xi, \lambda(i \circ j^{\xi})] \;&=\; [u\xi \oplus \mu, \lambda(i_1 \circ i \circ j^{\xi})] \\ &=\; [u\xi \oplus \mu, \lambda(i_2 \circ J)] \;=\; [\mu, \lambda(J)]\ . \qquad \square \end{aligned}$$

Our next aim is to establish additivity of the map $\langle - \rangle : \mathrm{Vect}_G(X) \longrightarrow \mathbf{ku}_G[\![X]\!]$ defined in (3.4.3). This relation is slightly subtle because the sum in $\mathbf{ku}_G[\![X]\!]$ is defined by addition via a fiberwise pinch map, which a priori has no connection to Whitney sum of vector bundles. The concept of 'ample bundle' we are about to introduce will serve as a tool in the proof of the additivity relation.

Definition 3.4.6. Let G be a Lie group and X a proper G-space. A hermitian G-vector bundle ζ over X is *ample* if the following property holds: for every point $x \in X$ the infinite-dimensional unitary G-representation $\mathrm{Sym}(\zeta_x)$ is a complete complex G_x-universe, where G_x is the stabilizer group of x. In other words, every finite-dimensional unitary G_x-representation admits a G_x-equivariant linear isometric embedding into $\mathrm{Sym}(\zeta_x)$.

Proposition 3.4.7. *Let G be a Lie group and X a finite proper G-CW-complex.*

(i) *Let ζ be an ample hermitian G-vector bundle over X. Let ξ be a hermitian G-vector bundle over X, of finite or countably infinite dimension. Then there is a G-equivariant linear isometric embedding of ξ into* $\mathrm{Sym}(\zeta)$ *over X.*

(ii) *If G is discrete, then X has an ample G-vector bundle.*

Proof. (i) We prove a more general relative version of the claim: given a G-subcomplex A of X, every G-equivariant linear isometric embedding of $\xi|_A$ into $\mathrm{Sym}(\zeta)|_A$ over A can be extended to a G-equivariant linear isometric embedding of ξ into $\mathrm{Sym}(\zeta)$ over X. The case $A = \emptyset$ then proves the proposition.

Induction over the number of relative G-cells reduces the claim to the case where X is obtained from A by attaching a single G-cell with compact isotropy group H. Hence we may assume that $X = G/H \times D^n$ and $A = G/H \times S^{n-1}$, for some $n \geq 0$. Since H is a compact Lie group, every hermitian G-vector bundle ζ over $G/H \times D^n$ is of the form $\zeta = (G \times_H W) \times D^n$, for some unitary H-representation W, projecting away from W, compare [**42**, Lemma 1.1 (a)] or [**65**, Prop. 1.3]. Since ζ is an ample bundle, W must be an ample H-representation, i.e., the symmetric algebra $\mathrm{Sym}(W)$ is a complete complex H-universe. Similarly, we may assume that $\xi = (G \times_H V) \times D^n$ where V is a unitary H-representation of finite or countably infinite dimension.

Every linear isometric embedding of the bundle $\xi|_{S^{n-1}} = (G \times_H V) \times S^{n-1}$ into the bundle $\mathrm{Sym}(\zeta|_{S^{n-1}}) = (G \times_H \mathrm{Sym}(W)) \times S^{n-1}$ is of the form

$$\begin{array}{ccc} (G \times_H V) \times S^{n-1} & \longrightarrow & (G \times_H \mathrm{Sym}(W)) \times S^{n-1} \\ ([g, v], x) & \longmapsto & ([g, \psi(x)(v)], x) \end{array}$$

for some continuous map $\psi : S^{n-1} \longrightarrow \mathbf{L}^H(V, \mathrm{Sym}(W))$ into the space of H-equivariant $\mathbb{C}$-linear isometric embeddings from V into $\mathrm{Sym}(W)$. Because $\mathrm{Sym}(W)$ is a complete complex H-universe, the space $\mathbf{L}^H(V, \mathrm{Sym}(W))$ is weakly contractible: when V is finite-dimensional, this is the complex analog of [**60**, Prop. 1.1.21]; Proposition A.10 of [**61**] (or rather its complex analog) reduces the infinite-dimensional case to the finite-dimensional case. Because $\mathbf{L}^H(V, \mathrm{Sym}(W))$ is weakly contractible, ψ admits a continuous extension to a map $D^n \longrightarrow \mathbf{L}^H(V, \mathrm{Sym}(W))$, which yields the desired linear isometric embedding $\xi \longrightarrow \mathrm{Sym}(\zeta)$ by the same formula as for ψ.

(ii) We let X be a finite proper G-CW-complex. Since X has only finitely many G-cells, there are only finitely many conjugacy classes of finite subgroups of G that occur as isotropy groups of points of X. In particular, the isotropy groups of X have bounded order. Since G is discrete, Corollary 2.7 of [**42**] thus provides a hermitian G-vector bundle ζ over X such that for every point $x \in X$, the fiber ζ_x is a multiple of the regular representation of the isotropy group G_x. In particular, the G_x-action on ζ_x is faithful, and hence $\mathrm{Sym}(\zeta_x)$ is a complete complex G_x-universe, compare [**60**, Rk. 6.3.22]. So the bundle ζ is ample. □

PROPOSITION 3.4.8. *Let G be a discrete group and X a finite proper G-CW-complex. Then the map $\langle - \rangle : \mathrm{Vect}_G(X) \longrightarrow \mathbf{ku}_G[\![X]\!]$ defined in* (3.4.4) *is additive.*

PROOF. The sum in the group $\mathbf{ku}_G[\![X]\!]$ is defined by addition via a fiberwise pinch map. We will relate the pinch sum to a different binary operation, the 'bundle sum', by an Eckmann-Hilton argument.

Proposition 3.4.7 (ii) provides an ample hermitian G-vector bundle ζ over X. We let $\mu = u\zeta$ denote the underlying euclidean vector bundle. Because ζ embeds into $(u\zeta)_{\mathbb{C}} = \mu_{\mathbb{C}}$, the hermitian G-vector bundle $\mu_{\mathbb{C}}$ is also ample. By adding a trivial complex line bundle to ζ, if necessary, we can moreover assume that there exists a G-equivariant linear isometric embedding $j : X \times \underline{\mathbb{R}} \longrightarrow \mu = u\zeta$ of the trivial $\mathbb{R}$-line bundle over X. The embedding j parameterizes a trivial 1-dimensional summand in μ, and hence a pinch map $p : S^\mu \longrightarrow S^\mu \vee_X S^\mu$. The 'pinch sum' on the set $[S^\mu, \mathbf{ku}_G(\mu)]^G_X$ of parameterized homotopy classes is given by

$$[f] \vee [g] \ = \ [(f+g) \circ p] \ , \tag{3.4.9}$$

where $f+g : S^\mu \vee_X S^\mu \longrightarrow \mathbf{ku}_G(\mu)$ is given by f and g on the respective summands.

Because $\mu_{\mathbb{C}}$ is ample, Proposition 3.4.7 (i) provides a G-equivariant linear isometric embedding

$$\theta \ : \ \mathrm{Sym}(\mu_{\mathbb{C}}) \oplus \mathrm{Sym}(\mu_{\mathbb{C}}) \ \longrightarrow \ \mathrm{Sym}(\mu_{\mathbb{C}})$$

of bundles over X. The embedding θ in turn yields a map of retractive G-spaces over X

$$\oplus \ : \ \mathbf{ku}_G(\mu) \times_X \mathbf{ku}_G(\mu) \ \longrightarrow \ \mathbf{ku}_G(\mu)$$

defined fiberwise by

$$\begin{aligned} &[E_1, \dots, E_k; v_1, \dots, v_k] \oplus [F_1, \dots, F_l; w_1, \dots, w_l] \\ &\quad = [\theta(E_1 \oplus 0), \dots, \theta(E_k \oplus 0), \theta(0 \oplus F_1), \dots, \theta(0 \oplus F_l); v_1, \dots, v_k, w_1, \dots, w_l] \\ &\quad = \sum_{i=1}^k v_i \theta(E_i \oplus 0) + \sum_{j=1}^l w_j \theta(0 \oplus F_j) \ . \end{aligned}$$

The 'bundle sum' on the set $[S^\mu, \mathbf{ku}_G(\mu)]^G_X$ is given by

$$[f] \oplus [g] \ = \ [\oplus \circ (f, g)] \ , \tag{3.4.10}$$

where $(f, g) : S^\mu \longrightarrow \mathbf{ku}_G(\mu) \times_X \mathbf{ku}_G(\mu)$ has components f and g, respectively. The pinch sum (3.4.9) and the bundle sum (3.4.10) share the same neutral element, and they satisfy the interchange relation

$$([f] \oplus [g]) \vee ([h] \oplus [k]) \ = ([f] \vee [h]) \oplus ([g] \vee [k]) \ .$$

The Eckmann-Hilton argument then applies: taking $[g]$ and $[h]$ as the common neutral element shows that the pinch sum and the bundle sum on the set $[S^\mu, \mathbf{ku}_G(\mu)]^G_X$ coincide.

Now we prove additivity. We let ξ and η be two hermitian G-vector bundles over X. Proposition 3.4.7 (i) provides G-equivariant linear isometric embeddings

$$\varphi \ : \ \xi \ \longrightarrow \ \mathrm{Sym}(\mu_{\mathbb{C}}) \qquad \text{and} \qquad \psi \ : \ \eta \ \longrightarrow \ \mathrm{Sym}(\mu_{\mathbb{C}})$$

of hermitian G-vector bundles over X. The map

$$\theta \circ (\varphi \oplus \psi) \ : \ \xi \oplus \eta \ \longrightarrow \ \mathrm{Sym}(\mu_{\mathbb{C}})$$

is another equivariant isometric embedding. Proposition 3.4.5 turns these bundle embeddings into maps of retractive G-spaces over X

$$\lambda(\varphi) \ , \ \lambda(\psi) \ , \ \lambda(\theta \circ (\varphi \oplus \psi)) \ : \ S^\mu \ \longrightarrow \ \mathbf{ku}_G(\mu) \ .$$

Moreover, the relation

$$\lambda(\theta \circ (\varphi \oplus \psi)) \ = \ \oplus \circ (\lambda(\varphi), \lambda(\psi))$$

holds by design. Proposition 3.4.5 then yields

$$\begin{aligned} \langle \xi \rangle + \langle \eta \rangle \ &= \ [\lambda(\varphi)] \vee [\lambda(\psi)] \ = \ [\lambda(\varphi)] \oplus [\lambda(\psi)] \\ &= \ [\oplus \circ (\lambda(\varphi), \lambda(\psi))] \ = \ [\lambda(\theta \circ (\varphi \oplus \psi))] \ = \ \langle \xi \oplus \eta \rangle \ . \end{aligned}$$ □

We have now constructed a well-defined monoid homomorphism

$$\langle - \rangle \ : \ \mathrm{Vect}_G(X) \ \longrightarrow \ \mathbf{ku}_G[\![X]\!] \ .$$

We write $K_G(X)$ for the group completion (Grothendieck group) of the abelian monoid $\mathrm{Vect}_G(X)$. The universal property of the Grothendieck group extends $\langle - \rangle$ to a unique group homomorphism

$$\kappa_X \ : \ K_G(X) \ \longrightarrow \ \mathbf{ku}_G[\![X]\!] \ . \tag{3.4.11}$$

Since the maps $\langle - \rangle$ are natural for G-maps in X, so are the extensions κ_X.

Equivariant K-groups admit products induced from tensor product of vector bundles. The cohomology groups represented by **ku** admit products arising from the ring spectrum structure. Our next aim is to show that the homomorphisms (3.4.11) are suitably multiplicative. We start by formally introducing the relevant pairings in the represented **ku**-cohomology, in somewhat larger generality.

Construction 3.4.12. We let E be an orthogonal ring spectrum and M a left E-module spectrum. Given a Lie group G, a finite proper G-CW-complex X and a finite CW-complex Y, we now construct natural pairings

$$\cup \ : \ E_G[\![X]\!] \times M[\![Y]\!] \ \longrightarrow \ M_G[\![X \times Y]\!] \ ; \tag{3.4.13}$$

here $M[\![Y]\!]$ is the *non-equivariant* special case of the Construction 3.1.19, i.e., for G a trivial group. We write

$$\alpha_{m,n} \ : \ E(\mathbb{R}^m) \wedge M(\mathbb{R}^n) \ \longrightarrow \ M(\mathbb{R}^{m+n})$$

for the $(O(m) \times O(n))$-equivariant component of the action morphism $\alpha : E \wedge M \longrightarrow M$. We let η and ξ be vector bundles over X and Y, of dimension m and n, respectively. We write $\eta \times \xi$ for the exterior product bundle over $X \times Y$, and $-\triangle-$ for the external smash product of retractive spaces, with fibers

$$(\eta \times \xi)_{(x,y)} \ = \ \eta_x \wedge \xi_y \ .$$

Then the multiplication maps give rise to a map of retractive spaces over $X \times Y$

$$\begin{tikzcd}
E_G(\eta)\triangle M(\xi) = \Big(\mathcal{F}_n(\eta) \times_{O(m)} E(\mathbb{R}^m)\Big)\triangle\Big(\mathcal{F}_n(\xi) \times_{O(n)} M(\mathbb{R}^n)\Big) \arrow[d, "\cong"] \\
\Big(\mathcal{F}_m(\eta) \times \mathcal{F}_n(\xi)\Big) \times_{O(m)\times O(n)} \Big(E(\mathbb{R}^m) \wedge M(\mathbb{R}^n)\Big) \arrow[d, "\psi\times\alpha_{m,n}"] \\
\mathcal{F}_{m+n}(\eta\times\xi) \times_{O(m+n)} M(\mathbb{R}^{m+n}) = E_G(\eta\times\xi)
\end{tikzcd}$$

that we denote by $\alpha_{\eta,\xi}$. The map $\psi : \mathcal{F}_m(\eta) \times \mathcal{F}_n(\xi) \longrightarrow \mathcal{F}_{m+n}(\eta \times \xi)$ takes direct products of frames, i.e., it is given by

$$\psi((x_1, \dots, x_m), (y_1, \dots, y_n)) \ = \ ((x_1, 0), \dots, (x_m, 0), (0, y_1), \dots, (0, y_n)) \ .$$

Now we let (η, u) represent a class in $E_G[\![X]\!]$, and we let (ξ, v) represent a class in $M[\![Y]\!]$. Then the composite

$$u \cup v \ : \ S^{\eta\times\xi} \ \cong \ S^{\eta}\triangle S^{\xi} \ \xrightarrow{u\triangle v} \ E_G(\eta)\triangle M(\xi) \ \xrightarrow{\alpha_{\eta,\xi}} \ M_G(\eta\times\xi)$$

is an equivariant map of retractive G-spaces over $X \times Y$, where G acts trivially on Y and on ξ. The construction passes to equivalence classes under fiberwise homotopy and stabilization, so we can define the pairing (3.4.13) by

$$[\eta, u] \cup [\xi, v] \ = \ [\eta\times\xi, u \cup v] \ .$$

The following naturality properties of the cup product construction are straightforward from the definitions, and we omit the formal proofs.

Proposition 3.4.14. *Let E be an orthogonal ring spectrum and G a Lie group. The pairing* (3.4.13) *is natural for morphisms in the global homotopy category of E-module spectra in the variable M, for continuous G-maps in X, and for continuous maps in Y.*

The cases we mostly care about are the orthogonal ring spectra **ku** and **KU**, each acting on itself by multiplication. We can now state and prove the multiplicativity property of the homomorphisms (3.4.11). In the next proposition, the upper horizontal pair is induced by exterior tensor product of vector bundles.

Proposition 3.4.15. *For every discrete group G, every proper finite G-CW-complex X and every finite CW-complex Y, the diagram*

$$\begin{tikzcd}
K_G(X) \times K(Y) \arrow[r, "\otimes"] \arrow[d, "\kappa_X\times\kappa_Y"'] & K_G(X\times Y) \arrow[d, "\kappa_{X\times Y}"] \\
\mathbf{ku}_G[\![X]\!] \times \mathbf{ku}[\![Y]\!] \arrow[r, "\cup"'] & \mathbf{ku}_G[\![X\times Y]\!]
\end{tikzcd}$$

commutes.

Proof. Since both pairings are biadditive, it suffices to check the commutativity for classes represented by actual vector bundles (as opposed to virtual vector bundles). We let η be a hermitian G-vector bundle over X, and ξ a hermitian vector bundle over Y. The class $\langle\eta\rangle \cup \langle\xi\rangle$ is represented by the map of retractive G-spaces over $X \times Y$

$$S^{u(\eta)\times u(\xi)} \cong S^{u\eta} \triangle S^{u\xi} \xrightarrow{\{\eta\}\triangle\{\xi\}} \mathbf{ku}(u\eta)\triangle\mathbf{ku}(u\xi) \xrightarrow{\mu_{\eta,\xi}} \mathbf{ku}((u\eta)\times(u\xi)) .$$

The multiplication in **ku** ultimately stems from the tensor product of hermitian vector spaces. Unraveling the definition of $\mu_{\eta,\xi}$ shows that the above composite coincides with the map of retractive G-spaces $\lambda(J) : S^{(u\eta)\times(u\xi)} \longrightarrow \mathbf{ku}((u\eta)\times(u\xi))$, associated with the isometric embedding

$$J \ : \ u(\eta\otimes\xi) \longrightarrow \mathrm{Sym}(((u\eta)\times(u\xi))_{\mathbb{C}}) , \quad J_{(x,y)}(v\otimes w) \ = \ (j_x^\eta(v), 0)\cdot(0, j_y^\xi(w)) .$$

So the image of J belongs to $\mathrm{Sym}^2(u(\eta)\times u(\xi))$, the quadratic summand in the complexified symmetric algebra of $u(\eta)\times u(\xi)$. We emphasize that J is *not* the isometric embedding used in the definition of the class $\langle\eta\otimes\xi\rangle$: the defining isometric embedding takes values in the linear summand of the symmetric algebra of the exterior tensor product $u(\eta\otimes\xi)$. However, Proposition 3.4.5 shows that the map $\lambda(J)$ also represents the class $\langle\eta\otimes\xi\rangle$. So we conclude that

$$\langle\eta\rangle \cup \langle\xi\rangle \ = \ \langle\lambda(J)\rangle \ = \ \langle\eta\otimes\xi\rangle . \qquad \square$$

Now we consider the periodic global K-theory spectrum **KU** introduced by Joachim in [**30**] and later studied in [**60**, Sec. 6.4]. The definition of **KU** is based on spaces of homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras, and can be found in [**30**, Sec. 4] and [**60**, Con. 6.4.9]. For our purposes, we can (and will) use **KU** as a black box; the main properties we use is that **KU** is a commutative orthogonal ring spectrum, that it receives a ring spectrum homomorphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ (see [**60**, Con. 6.4.13]), that the homomorphism j sends the Bott class in $\pi_2^e(\mathbf{ku})$ to a unit in the graded ring $\pi_*^e(\mathbf{KU})$ (see [**60**, Thm. 6.4.29]) and that **KU** represents equivariant K-theory for compact Lie groups (see [**30**, Thm. 4.4] or [**60**, Cor. 6.4.23]).

Now we let G be a discrete group. The functor $(-)_G$ from Section 3.3 yields a G-equivariant commutative orthogonal ring spectrum $\mathbf{KU}_G$. The morphism of commutative orthogonal ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ defined in [**60**, Con. 6.4.13] induces a morphism of commutative orthogonal G-ring spectra $j_G : \mathbf{ku}_G \longrightarrow \mathbf{KU}_G$. For X a finite proper G-CW-complex, we write $c_X : K_G(X) \longrightarrow \mathbf{KU}_G[\![X]\!]$ for the composite

$$K_G(X) \xrightarrow{\kappa_X} \mathbf{ku}_G[\![X]\!] \xrightarrow{(j_G)_*} \mathbf{KU}_G[\![X]\!] .$$

Source and target of this natural transformation are excisive functors in X by [**42**, Lemma 3.8] and by Theorem 3.1.29, respectively.

Theorem 3.4.16. *For every discrete group G and every finite proper G-CW-complex X, the homomorphism*

$$c_X \ : \ K_G(X) \longrightarrow \mathbf{KU}_G[\![X]\!]$$

is an isomorphism.

Proof. In the special case when the group G is finite, the map c_X factors as the composite

$$K_G(X) \xrightarrow{\cong} \mathbf{KU}_G(X) \xrightarrow{\mu_X^{\mathbf{KU}_G}} \mathbf{KU}_G[\![X]\!] ,$$

where the first map is the isomorphism established in [**60**, Cor. 6.4.23], and the second map is the isomorphism constructed in Theorem 3.1.34. This proves the claim for finite groups.

Now we let G be any discrete group, and we consider the special case $X = G/H \times K$ for a finite subgroup H of G and a finite non-equivariant CW-complex K. Lemma 3.4 of [**42**] and our Example 3.1.25 show that vertical induction maps in the commutative square

$$\begin{tikzcd} K_H(K) \arrow[r, "c_K"] \arrow[d, "\cong"', "\mathrm{ind}"] & \mathbf{KU}_H[\![K]\!] \arrow[d, "\mathrm{ind}"', "\cong"] \\ K_G(G/H \times K) \arrow[r, "c_{G/H\times K}"'] & \mathbf{KU}_G[\![G/H \times K]\!] \end{tikzcd}$$

are isomorphisms. Hence $c_{G/H\times K}$ is an isomorphism, and Proposition 3.1.7 concludes the proof. $\square$

The rest of this section is devoted to extending the natural isomorphism $c_X : K_G(X) \longrightarrow \mathbf{KU}_G[\![X]\!]$ to an isomorphism of $\mathbb{Z}$-graded proper G-cohomology theories, using different incarnations of Bott periodicity for source and target of c_X.

Construction 3.4.17. In [**42**, Sec. 3], the third author and Oliver use Bott periodicity to extend the excisive functor $K_G(X)$ to a $\mathbb{Z}$-graded proper G-cohomology theory. We quickly recall the relevant definitions. We let L and $\underline{\mathbb{C}}$ denote the tautological line bundle and the trivial line bundle, respectively, over the complex projective line $\mathbb{C}P^1$. Their formal difference is a reduced K-theory class

$$[L] - [\underline{\mathbb{C}}] \;\in\; K(\mathbb{C}P^1|\infty) \;=\; \mathrm{Ker}(K(\mathbb{C}P^1) \longrightarrow K(\{\infty\}))\ .$$

We identify $S^2 = \mathbb{C} \cup \{\infty\}$ with $\mathbb{C}P^1$ by sending $\lambda \in \mathbb{C}$ to the point $[\lambda : 1]$. The *Bott class* $b \in K(S^2|\infty)$ is the image of $[L] - [\underline{\mathbb{C}}]$ under the induced isomorphism $K(\mathbb{C}P^1|\infty) \cong K(S^2|\infty)$. The reduced K-group $K(S^2|\infty)$ is infinite cyclic, and the Bott class b is a generator.

Now we let G be a discrete group and X a finite proper G-CW-complex. Exterior tensor product of vector bundles induces the exterior product map

$$-\otimes b \;:\; K_G(X) \;\longrightarrow\; K_G(X \times S^2)\ ;$$

because b is a reduced K-theory class, this map takes values in the relative group $K_G(X \times S^2|X \times \infty)$. Equivariant K-theory is Bott periodic in the sense that this refined exterior product map

$$-\otimes b \;:\; K_G(X) \;\xrightarrow{\cong}\; K_G(X \times S^2|X \times \infty)$$

is an isomorphism for every finite proper G-CW-complex X, see [**42**, Thm. 3.12]. For an integer m, the third author and Oliver define

$$K_G^m(X) \;=\; \begin{cases} K_G(X) & \text{for } m \text{ even, and} \\ K_G(X \times S^1|X \times \infty) & \text{for } m \text{ odd.} \end{cases}$$

The suspension isomorphism

$$\sigma \;:\; K_G^m(X) \;\xrightarrow{\cong}\; K_G^{m+1}(X \times S^1|X \times \infty)$$

is the identity when m is odd. When m is even, the suspension isomorphism is the composite

$$K_G(X) \;\xrightarrow[\cong]{-\otimes b}\; K_G(X \times S^2|X \times \infty) \;\xrightarrow[\cong]{(X\times q)^*}\; K_G(X \times S^1 \times S^1|X \times (S^1 \vee S^1))\ ,$$

where $q : S^1 \times S^1 \longrightarrow S^1 \wedge S^1 \cong S^2$ is the composite of the projection and the canonical homeomorphism.

CONSTRUCTION 3.4.18. The Bott class $b \in K(S^2)$ is a generator of the reduced K-group $K(S^2|\infty)$, which is infinite cyclic. The homomorphisms $\kappa_{S^2} : K(S^2) \longrightarrow \mathbf{ku}[\![S^2]\!]$ and $\kappa_* : K(*) \longrightarrow \mathbf{ku}[\![*]\!]$ are isomorphisms by [**60**, Thm. 6.3.31 (iii)]. So the class $\kappa_{S^2}(b)$ is a generator of the infinite cyclic group

$$\mathbf{ku}[\![S^2|\infty]\!] \ \cong\ \pi_2^e(\mathbf{ku}) \ .$$

The composite $c_{S^2} = j_* \circ \kappa_{S^2} : K(S^2|\infty) \longrightarrow \mathbf{KU}[\![S^2|\infty]\!]$ is an isomorphism by Theorem 3.4.16, so the class

$$\beta \ = \ c_{S^2}(b) \ = \ j_*(\kappa_{S^2}(b)) \ \in \ \mathbf{KU}[\![S^2|\infty]\!]$$

is a generator. We represent β by a morphism $\hat{\beta} : \Sigma^\infty S^2 \longrightarrow \mathbf{KU}$ in the global stable homotopy category $\mathcal{GH}$, i.e., such that $1 \wedge S^2 \in \pi_2^e(\Sigma^\infty S^2)$ maps to the class corresponding to β under the isomorphism

$$\mathbf{KU}[\![S^2|\infty]\!] \ \cong\ \pi_2^e(\mathbf{KU}) \ .$$

We define $\bar{\beta} : \mathbf{KU} \wedge S^2 \longrightarrow \mathbf{KU}$ as the free extension to a morphism of $\mathbf{KU}$-module spectra, i.e., the composite in $\mathcal{GH}$

$$\mathbf{KU} \wedge S^2 \ \xrightarrow{\mathbf{KU}\wedge\hat{\beta}}\ \mathbf{KU} \wedge \mathbf{KU} \ \xrightarrow{\mu^{\mathbf{KU}}}\ \mathbf{KU} \ .$$

By [**60**, Thm. 6.4.29], the homomorphism of orthogonal ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ sends each of the additive generators of $\pi_2^e(\mathbf{ku})$ to a unit of degree 2 in the graded ring $\pi_*^e(\mathbf{KU})$. In particular, the class $u = \hat{\beta}_*(1 \wedge S^2)$ in $\pi_2^e(\mathbf{KU})$ corresponding to $\beta \in \mathbf{KU}[\![S^2|\infty]\!]$ is a graded unit.

For every compact Lie group G, the effect of $\bar{\beta}$ on G-equivariant homotopy groups is multiplication by the class $p^*(u) \in \pi_2^G(\mathbf{KU})$, where $p^* : \pi_*^e(\mathbf{KU}) \longrightarrow \pi_*^G(\mathbf{KU})$ is inflation along the unique group homomorphism $p : G \longrightarrow e$. Since p^* is a ring homomorphism, $p^*(u)$ is unit in the graded ring $\pi_*^G(\mathbf{KU})$. So $\bar{\beta}$ induces an isomorphism of G-equivariant stable homotopy groups. Since G was any compact Lie group, the morphism $\bar{\beta} : \mathbf{KU} \wedge S^2 \longrightarrow \mathbf{KU}$ is a global equivalence. We apply the 'forgetful' functor $U_G : \mathcal{GH} \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G)$ discussed in (3.3.2) to obtain an isomorphism

$$\tilde{\beta} \ = \ U_G(\bar{\beta}) \ : \ \mathbf{KU}_G \wedge S^2 \ \xrightarrow{\cong}\ \mathbf{KU}_G$$

in the homotopy category of orthogonal G-spectra.

In the following proposition, we write again $q : S^1 \times S^1 \longrightarrow S^1 \wedge S^1 \cong S^2$ for the composite of the projection and the canonical homeomorphism.

PROPOSITION 3.4.19. *Let G be a discrete group and X a finite proper G-CW-complex. The following square commutes:*

$$\begin{array}{ccc}
\mathbf{KU}_G[\![X]\!] & \xrightarrow{\quad -\cup\beta \quad} & \mathbf{KU}_G[\![X \times S^2 | X \times \infty]\!] \\
{\scriptstyle \Sigma^{(\mathbf{KU}\wedge S^1)_G} \circ \Sigma^{\mathbf{KU}_G}} \downarrow & & \downarrow {\scriptstyle (X\times q)^*} \\
(\mathbf{KU} \wedge S^2)_G[\![X \times S^1 \times S^1 | X \times (S^1 \vee S^1)]\!] & \xrightarrow[\tilde{\beta}_*]{} & \mathbf{KU}_G[\![X \times S^1 \times S^1 | X \times (S^1 \vee S^1)]\!]
\end{array}$$

Proof. We let $1 \in \mathbf{KU}[\![*]\!]$ denote the class represented by the trivial 0-dimensional vector bundle over a point, and the map of based spaces

$$\eta \; : \; S^0 \longrightarrow \mathbf{KU}(0) \; ,$$

the unit of the ring spectrum structure of $\mathbf{KU}$. We write $\iota_2 \in (\mathbf{KU} \wedge S^2)[\![S^2|\infty]\!]$ for the unique class satisfying

$$q^*(\iota_2) \; = \; \Sigma^{\mathbf{KU}\wedge S^1}(\Sigma^{\mathbf{KU}}(1)) \; \in \; (\mathbf{KU} \wedge S^2)[\![S^1 \times S^1|S^1 \vee S^1]\!] \; ,$$

where $\Sigma^{\mathbf{KU}}$ is the suspension homomorphism (3.2.5). Then for every class $x \in \mathbf{KU}_G[\![X]\!]$, the relation

$$\begin{aligned} \Sigma^{(\mathbf{KU}\wedge S^1)_G}(\Sigma^{\mathbf{KU}_G}(x)) \; &= \; x \cup (\Sigma^{\mathbf{KU}\wedge S^1}(\Sigma^{\mathbf{KU}}(1))) \\ &= \; x \cup q^*(\iota_2) \; = \; (X \times q)^*(x \cup \iota_2) \end{aligned}$$

holds in $(\mathbf{KU} \wedge S^2)_G[\![X \times S^1 \times S^1|X \times (S^1 \vee S^1)]\!]$. The relation

$$\beta \; = \; \tilde{\beta}_*(\iota_2)$$

holds in $\mathbf{KU}[\![S^2|\infty]\!]$, by construction of the morphism $\tilde{\beta} : \mathbf{KU} \wedge S^2 \longrightarrow \mathbf{KU}$. The naturality properties of the cup product pairing, recorded in Proposition 3.4.14, thus provide the relations

$$\begin{aligned} \tilde{\beta}_*(\Sigma^{\mathbf{KU}_G\wedge S^1}(\Sigma^{\mathbf{KU}_G}(x))) \; &= \; \tilde{\beta}_*((X \times q)^*(x \cup \iota_2)) \; = \; (X \times q)^*(\tilde{\beta}_*(x \cup \iota_2)) \\ &= \; (X \times q)^*(x \cup \tilde{\beta}_*(\iota_2)) \; = \; (X \times q)^*(x \cup \beta) \; . \end{aligned}$$

The third equality exploits that $\tilde{\beta}$ is underlying a morphism in the global homotopy category of left $\mathbf{KU}$-module spectra. □

Now we define the periodicity isomorphisms of the proper G-cohomology theory $\mathbf{KU}_G^*[\![-]\!]$, essentially as the effect of the π_*-isomorphism $\tilde{\beta} : \mathbf{KU}_G[2] = \mathbf{KU}_G \wedge S^2 \longrightarrow \mathbf{KU}_G$. We recall that Proposition 1.3.7 specifies a natural isomorphism

$$t_{2,m} \; : \; \mathbf{KU}_G[2][m] \xrightarrow{\cong} \mathbf{KU}_G[2+m]$$

in $\mathrm{Ho}(\mathrm{Sp}_G)$, for every integer m. We define a natural isomorphism

$$B^{[m]} \; = \; (\tilde{\beta}[m] \circ t_{2,m}^{-1})_* \; : \; \mathbf{KU}_G^{2+m}[\![X]\!] \xrightarrow{\cong} \mathbf{KU}_G^m[\![X]\!] \; , \tag{3.4.20}$$

the effect of the composite isomorphism

$$\mathbf{KU}_G[2+m] \xrightarrow[\cong]{t_{2,m}^{-1}} \mathbf{KU}_G[2][m] \xrightarrow[\cong]{\tilde{\beta}[m]} \mathbf{KU}_G[m] \; .$$

Proposition 3.4.21. *Let G be a discrete group and X a finite proper G-CW-complex. Then the square*

$$\begin{tikzcd} \mathbf{KU}_G^{2+m}[\![X]\!] \arrow[r, "B^{[m]}"] \arrow[d, "\sigma"'] & \mathbf{KU}_G^m[\![X]\!] \arrow[d, "\sigma"] \\ \mathbf{KU}_G^{2+m+1}[\![X \times S^1|X \times \infty]\!] \arrow[r, "B^{[m+1]}"'] & \mathbf{KU}_G^{m+1}[\![X \times S^1|X \times \infty]\!] \end{tikzcd}$$

commutes for every $m \in \mathbb{Z}$.

PROOF. The associativity and naturality property of the natural isomorphisms $t_{k,l}$ stated in Proposition 1.3.7 imply that the following two squares commute:

$$\begin{tikzcd}
\mathbf{KU}_G[2+m][1] \arrow[r, "t_{2,m}^{-1}[1]"] \arrow[d, "t_{2+m,1}"'] & \mathbf{KU}_G[2][m][1] \arrow[r, "\tilde{\beta}[m][1]"] \arrow[d, "t_{m,1}"'] & \mathbf{KU}_G[m][1] \arrow[d, "t_{m,1}"] \\
\mathbf{KU}_G[2+m+1] \arrow[r, "t_{2,m+1}^{-1}"'] & \mathbf{KU}_G[2][m+1] \arrow[r, "\tilde{\beta}[m+1]"'] & \mathbf{KU}_G[m+1]
\end{tikzcd}$$

The suspension isomorphisms $\sigma : E_G^k[\![X]\!] \longrightarrow E_G^{k+1}[\![X \times S^1 | X \times \infty]\!]$ defined in (3.2.6) are natural in the variable E for morphisms in $\mathrm{Ho}(\mathrm{Sp}_G)$. The two facts together provide the desired commutativity. $\square$

Now we can put all the ingredients together and prove the main result of this section, identifying vector bundle K-theory with the proper G-cohomology theory represented by the orthogonal G-spectrum underlying the global K-theory spectrum $\mathbf{KU}$. We let G be a discrete group. Theorems 3.1.34 and 3.4.16 together provide a natural isomorphism

$$d_X \;=\; (\mu_X^{\mathbf{KU}_G})^{-1} \circ c_X \;:\; K_G(X) \;\longrightarrow\; \mathbf{KU}_G(X)$$

of excisive functors on finite proper G-CW-complexes.

THEOREM 3.4.22. *Let G be a discrete group. The natural isomorphism of excisive functors*

$$d_X \;:\; K_G(X) \;\xrightarrow{\cong}\; \mathbf{KU}_G(X)$$

extends to an isomorphism $K_G^* \cong \mathbf{KU}_G^*$ of proper G-cohomology theories on finite proper G-CW-complexes from the equivariant K-theory in these sense of [**42**] *to the G-cohomology theory represented by the orthogonal G-spectrum $\mathbf{KU}_G$.*

PROOF. We define natural isomorphisms

$$\psi_{2k}^X \;:\; K_G(X) \;\xrightarrow{\cong}\; \mathbf{KU}_G^{2k}[\![X]\!]$$

and

$$\psi_{2k-1}^X \;:\; K_G(X \times S^1 | X \times \infty) \;\xrightarrow{\cong}\; \mathbf{KU}_G^{2k-1}[\![X]\!] \;,$$

for all integers k, compatible with the suspension isomorphisms. We start by setting $\psi_0^X = c_X : K_G(X) \longrightarrow \mathbf{KU}_G[\![X]\!]$. For $k < 0$, we define ψ_{2k} inductively as the composite $\psi_{2k} = B^{[2k]} \circ \psi_{2+2k}$, where $B^{[m]} : \mathbf{KU}_G^{2+m}[\![X]\!] \longrightarrow \mathbf{KU}_G^m[\![X]\!]$ is the natural isomorphism (3.4.20). For $k > 0$, we define ψ_{2k} inductively as the composite $\psi_{2k} = (B^{[2k-2]})^{-1} \circ \psi_{2k-2}$. In odd dimensions, we define ψ_{2k-1} as the composite

$$K_G(X \times S^1 | X \times \infty) \;\xrightarrow{\psi_{2k}^{X \times S^1}}\; \mathbf{KU}_G^{2k}[\![X \times S^1 | X \times \infty]\!] \;\xrightarrow{\sigma^{-1}}\; \mathbf{KU}_G^{2k-1}[\![X]\!] \;.$$

With these definitions, the relation

$$\psi_m \;=\; B^{[m]} \circ \psi_{2+m}$$

holds for all integers m, by definition in even dimension, and by Proposition 3.4.21 in odd dimensions.

Now we show that the isomorphisms ψ_m are compatible with suspension isomorphisms. We first observe that the following diagram commutes:

$$\begin{tikzcd}
K_G(X) \arrow[r, "-\otimes b"] \arrow[d, "\kappa_X"] \arrow[dd, bend right=60, "c_X"'] & K_G(X\times S^2) \arrow[d, "\kappa_{X\times S^2}"'] \arrow[dd, bend left=60, "c_{X\times S^2}"] \\
\mathbf{ku}_G\llbracket X\rrbracket \arrow[r, "-\cup\kappa_{S^2}(b)"'] \arrow[d, "(j_G)_*"] & \mathbf{ku}_G\llbracket X\times S^2\rrbracket \arrow[d, "(j_G)_*"'] \\
\mathbf{KU}_G\llbracket X\rrbracket \arrow[r, "-\cup\beta"'] & \mathbf{KU}_G\llbracket X\times S^2\rrbracket
\end{tikzcd}$$

Indeed, the upper square commutes by Proposition 3.4.15, and the lower square commutes because $j:\mathbf{ku}\longrightarrow\mathbf{KU}$ is a morphism of ultra-commutative ring spectra, and the class β was defined as $c_{S^2}(b) = j_*(\kappa_{S^2}(b))$. Now we contemplate the following diagram of isomorphisms:

$$\begin{tikzcd}
K_G(X) \arrow[d, "-\otimes b"] \arrow[dr, bend left, "\sigma"] \arrow[dddd, bend right=60, "\psi_0^X = c_X"'] & \\
K_G(X\times S^2|X\times\infty) \arrow[r, "(X\times q)^*"'] \arrow[d, "c_{X\times S^2}"] & K_G(X\times S^1\times S^1|X\times(S^1\vee S^1)) \arrow[d, "c_{X\times S^1\times S^1}"] \arrow[ddd, bend left=60, "\psi_1^X"] \\
\mathbf{KU}_G\llbracket X\times S^2|X\times\infty\rrbracket \arrow[r, "(X\times q)^*"'] \arrow[dd, "(-\cup\beta)^{-1}"] & \mathbf{KU}_G\llbracket X\times S^1\times S^1|X\times(S^1\vee S^1)\rrbracket \\
 & (\mathbf{KU}\wedge S^2)_G\llbracket X\times S^1\times S^1|X\times(S^1\vee S^1)\rrbracket \arrow[u, "\tilde\beta_*", "\cong"'] \\
\mathbf{KU}_G\llbracket X\rrbracket \arrow[r, "\Sigma^{\mathbf{KU}_G}"'] & (\mathbf{KU}\wedge S^1)_G\llbracket X\times S^1|X\times\infty\rrbracket \arrow[u, "\Sigma^{(\mathbf{KU}\wedge S^1)_G}", "\cong"']
\end{tikzcd}$$

As we argued above, the left vertical composite coincides with the map $\psi_0^X = c_X$. The middle square commutes by naturality of c_X; and the lower part of the diagram commutes by Proposition 3.4.19. This proves the relation $\sigma\circ\psi_0 = \psi_1\circ\sigma$, i.e., the degree 0 instance of compatibility with the suspension isomorphisms.

For compatibility in other even dimensions we consider the following diagram:

$$\begin{tikzcd}
K_G(X) \arrow[r, "-\otimes b"'] \arrow[rr, bend left, "\sigma"] \arrow[d, "\psi^X_{2+2k}"] \arrow[dd, bend right=60, "\psi^X_{2k}"'] & K_G(X\times S^2|X\times\infty) \arrow[r, "(X\times q)^*"'] & K_G(X\times S^1\times S^1|X\times(S^1\vee S^1)) \arrow[d, "\psi^{X\times S^1}_{2+2k+1}"'] \arrow[dd, bend left=60, "\psi^{X\times S^1}_{2k+1}"] \\
\mathbf{KU}_G^{2+2k}\llbracket X\rrbracket \arrow[rr, "\sigma"] \arrow[d, "B^{[2k]}"] & & \mathbf{KU}_G^{2+2k+1}\llbracket X\times S^1|X\times\infty\rrbracket \arrow[d, "B^{[2k+1]}"'] \\
\mathbf{KU}_G^{2k}\llbracket X\rrbracket \arrow[rr, "\sigma"'] & & \mathbf{KU}_G^{2k+1}\llbracket X\times S^1|X\times\infty\rrbracket
\end{tikzcd}$$

The lower square commutes by Proposition 3.4.21. So the outer diagram commutes if and only if the upper part commutes. In other words: compatibility with the suspension isomorphisms holds in dimension $2k$ if and only if it holds in dimension $2 + 2k$. We already showed compatibility in dimension 0, so we conclude that compatibility with the suspension isomorphisms holds in all even dimensions. In odd dimensions, compatibility with the suspension isomorphisms was built into the definition of suspension isomorphism in K_G^* and the maps ψ_{2k-1}. This completes the construction of the isomorphism from the vector bundle K-theory K_G^* to the theory $\mathbf{KU}_G^*[\![-]\!]$.

Theorem 3.2.7 provides another isomorphism of proper G-cohomology theories from the represented theory $\mathbf{KU}_G^*(-)$ to the theory $\mathbf{KU}_G^*[\![-]\!]$, given by $\mu_X^{\mathbf{KU}_G} : K_G(X) \longrightarrow \mathbf{KU}_G[\![X]\!]$ in dimension 0. This concludes the proof. $\square$

We leave it to the interested reader to verify that the isomorphisms of Theorem 3.4.22 are compatible with restriction to finite index subgroups, with the induction isomorphisms (3.1.26), and with graded products.

Bibliography

[1] H Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups.* Math. Ann. **212**, (1974/75), 1–19.

[2] P Baum, A Connes, *Geometric K-theory for Lie groups and foliations.* Preprint, Hautes Études Sci., Bures-sur-Yvette, 1982; published version in: Enseign. Math. (2) **46** (2000), no. 1-2, 3–42.

[3] P Baum, A Connes, N Higson, *Classifying space for proper actions and K-theory of group C^*-algebras.* C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[4] N Bárcenas, D Degrijse, I Patchkoria, *Stable finiteness properties of infinite discrete groups.* J. Topol. **10** (2017), 1169–1196.

[5] A A Beilinson, J Bernstein, P Deligne, *Faisceaux pervers.* Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[6] A Beligiannis, I Reiten, *Homological and homotopical aspects of torsion theories.* Mem. Amer. Math. Soc. **188** (2007), no. 883. viii+207 pp.

[7] A Borel, *Sous-groupes compacts maximaux des groupes de Lie.* Séminaire Bourbaki, Vol. 1, Exp. No. 33, 271–279, Soc. Math. France, Paris, 1995.

[8] T Bröcker, T tom Dieck, *Representations of compact Lie groups.* Graduate Texts in Mathematics, 98. Springer-Verlag, New York, 1985. x+313 pp.

[9] M Brun, B I Dundas, M Stolz, *Equivariant structure on smash powers.* `arXiv:1604.05939`

[10] J W Cannon, W J Floyd, *What is . . . Thompson's group?* Notices Amer. Math. Soc. **58** (2011), no. 8, 1112–1113.

[11] J D Christensen, M Hovey, *Quillen model structures for relative homological algebra.* Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 2, 261–293.

[12] P E Conner, E E Floyd, *Differentiable periodic maps.* Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33 Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1964. vii+148 pp.

[13] J Davis, W Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory.* K-Theory 15 (1998), no. 3, 201–252.

[14] B Day, *On closed categories of functors.* Reports of the Midwest Category Seminar, IV pp. 1–38. Lecture Notes in Mathematics, Vol. 137. Springer-Verlag, Berlin, 1970.

[15] M J Dunwoody, *Accessibility and groups of cohomological dimension one.* Proc. London Math. Soc. (3) **38** (1979), no. 2, 193–215.

[16] W G Dwyer, J Spalinski, *Homotopy theories and model categories.* In: Handbook of algebraic topology, ed. I M James, Elsevier (1995), 73–126.

[17] W G Dwyer, P S Hirschhorn, D M Kan, J H Smith, *Homotopy limit functors on model categories and homotopical categories.* Mathematical Surveys and Monographs, 113. Amer. Math. Soc., 2004. viii+181 pp.

[18] S Echterhoff, W Lück, N C Phillips, S Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$.* J. Reine Angew. Math. **639** (2010), 173–221.

[19] A D Elmendorf, *Systems of fixed point sets.* Trans. Amer. Math. Soc. **277** (1983), no. 1, 275–284.

[20] F T Farrell, L E Jones, *Isomorphism conjectures in algebraic K-theory.* J. Amer. Math. Soc. **6** (1993), no. 2, 249–297.

[21] H Fausk, *Equivariant homotopy theory for pro-spectra.* Geom. Topol. **12** (2008), 103–176.

[22] J P C Greenlees, J P May, *Generalized Tate cohomology.* Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178 pp.

[23] J P C Greenlees, J P May, *Localization and completion theorems for MU-module spectra.* Ann. of Math. (2) **146** (1997), 509–544.

[24] M Hill, M Hopkins, D Ravenel, *On the nonexistence of elements of Kervaire invariant one.* Ann. of Math. (2) **184** (2016), 1–262.

[25] M Hovey, *Model categories.* Mathematical Surveys and Monographs, **63**. Amer. Math. Soc., Providence, RI, 1999. xii+209 pp.

[26] M Hovey, B Shipley, J Smith, *Symmetric spectra.* J. Amer. Math. Soc. **13** (2000), 149–208.

[27] S Illman, *The equivariant triangulation theorem for actions of compact Lie groups.* Math. Ann. **262** (1983), 487–501.

[28] ———, *Existence and uniqueness of equivariant triangulations of smooth proper G-manifolds with some applications to equivariant Whitehead torsion.* J. Reine Angew. Math. **524** (2000), 129–183.

[29] K Iwasawa, *On some types of topological groups.* Ann. of Math. (2) **50**, (1949), 507–558.

[30] M Joachim, *Higher coherences for equivariant K-theory.* Structured ring spectra, 87–114, London Math. Soc. Lecture Note Ser., 315, Cambridge University Press, Cambridge, 2004.

[31] G M Kelly, *Basic concepts of enriched category theory.* Reprint of the 1982 original. Repr. Theory Appl. Categ. No. 10 (2005). vi+137 pp.

[32] H Krause, *A Brown representability theorem via coherent functors.* Topology **41** (2002), no. 4, 853–861.

[33] M Langer, W Lück, *On the group cohomology of the semi-direct product $\mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m$ and a conjecture of Adem-Ge-Pan-Petrosyan.* J. Pure Appl. Algebra **216** (2012), no. 6, 1318–1339.

[34] L G Lewis, Jr, *The stable category and generalized Thom spectra.* Ph.D. thesis, University of Chicago, 1978.

[35] L G Lewis, Jr, J P May, M Steinberger, *Equivariant stable homotopy theory.* Lecture Notes in Mathematics, Vol. 1213, Springer-Verlag, 1986. x+538 pp.

[36] W Lück, *Transformation groups and algebraic K-theory.* Lecture Notes in Mathematics, Vol. 1408. Mathematica Gottingensis. Springer-Verlag, 1989. xii+443 pp.

[37] ———, *Equivariant cohomological Chern characters.* Internat. J. Algebra Comput. 15 (2005), no. 5-6, 1025–1052.

[38] ———, *The Burnside ring and equivariant stable cohomotopy for infinite groups.* Pure Appl. Math. Q. **1** (2005), 479–541.

[39] ———, *Survey on classifying spaces for families of subgroups.* In: *Infinite groups: geometric, combinatorial and dynamical aspects*, 269–322. Progr. Math. **248**, Birkhäuser, Basel, 2005.

[40] ———, *Rational computations of the topological K-theory of classifying spaces of discrete groups.* J. Reine Angew. Math. **611** (2007), 163–187.

[41] ———, *The Segal conjecture for infinite groups.* Algebr. Geom. Topol. 20 (2020), no. 2, 965–986.

[42] W Lück, B Oliver, *The completion theorem in K-theory for proper actions of a discrete group.* Topology **40** (2001), 585–616.

[43] W Lück, H Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory.* Handbook of K-theory. Vol. 1, 2, 703–842, Springer, Berlin, 2005.

[44] W Lück, B Uribe, *Equivariant principal bundles and their classifying spaces.* Algebr. Geom. Topol. 14 (2014), no. 4, 1925–1995.

[45] A Malcev, *On the theory of the Lie groups in the large.* Rec. Math. [Mat. Sbornik] N. S. 16(58), (1945), 163–190.

[46] G Maltsiniotis, *Le théorème de Quillen, d'adjonction des foncteurs dérivés, revisité.* C. R. Math. Acad. Sci. Paris **344** (2007), no. 9, 549–552.

[47] M A Mandell, J P May, S Schwede, B Shipley, *Model categories of diagram spectra.* Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512.

[48] M A Mandell, J P May, *Equivariant orthogonal spectra and S-modules.* Mem. Amer. Math. Soc. **159** (2002), no. 755. x+108 pp.

[49] C Martinez-Pérez, B E A Nucinkis, *Cohomological dimension of Mackey functors for infinite groups.* J. London Math. Soc. (2) **74** (2006), 379–396.

[50] J McCleary, *A user's guide to spectral sequences.* Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001. xvi+561 pp.

[51] M C McCord, *Classifying spaces and infinite symmetric products.* Trans. Amer. Math. Soc. **146** (1969), 273–298.

[52] D Montgomery, L Zippin, *A theorem on Lie groups.* Bull. Amer. Math. Soc. 48, (1942), 448–452.
[53] A Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability.* J. Amer. Math. Soc. **9** (1996), 205–236.
[54] ———, *Triangulated categories.* Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001. viii+449 pp.
[55] T Nikolaus, P Scholze, *On topological cyclic homology.* Acta Math. **221** (2018), no. 2, 203–409.
[56] N C Phillips, *Equivariant K-theory for proper actions.* Pitman Research Notes in Mathematics Series, 178. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. viii+182 pp.
[57] D Quillen, *Homotopical algebra.* Lecture Notes in Mathematics, Vol. 43, Springer-Verlag, 1967. iv+156 pp.
[58] S Schwede, *The p-order of topological triangulated categories.* J. Topol. **6** (2013), no. 4, 868–914.
[59] ———, *Equivariant properties of symmetric products.* J. Amer. Math. Soc. **30** (2017), 673–711.
[60] ———, *Global homotopy theory.* New Mathematical Monographs **34**, Cambridge University Press, 2018. xviii+828 pp.
[61] ———, *Orbispaces, orthogonal spaces, and the universal compact Lie group.* Math. Z. **294** (2020), 71–107.
[62] S Schwede, B E Shipley, *Algebras and modules in monoidal model categories.* Proc. London Math. Soc. (3) **80** (2000), 491–511.
[63] ———, *A uniqueness theorem for stable homotopy theory.* Math. Z. **239** (2002), 803–828.
[64] ———, *Stable model categories are categories of modules.* Topology **42** (2003), 103–153.
[65] G Segal, *Equivariant K-theory.* Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 129–151.
[66] G. Segal, *Equivariant stable homotopy theory.* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 59–63, 1971.
[67] G. Segal, *Some results in equivariant homotopy theory.* Preprint, 1978.
[68] ———, *K-homology theory and algebraic K-theory.* K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 113–127. Lecture Notes in Mathematics, Vol. 575, Springer-Verlag, Berlin, 1977.
[69] K Shimakawa, *Infinite loop G-spaces associated to monoidal G-graded categories.* Publ. Res. Inst. Math. Sci. **25** (1989), 239–262.
[70] M Stolz, *Equivariant structure on smash powers of commutative ring spectra.* PhD thesis, University of Bergen, 2011.
[71] N P Strickland, *The category of CGWH spaces.* Preprint, available from the author's homepage.
[72] T tom Dieck, *Bordism of G-manifolds and integrality theorems.* Topology **9** (1970), 345–358.
[73] T tom Dieck, *Orbittypen und äquivariante Homologie. I.* Arch. Math. (Basel) **23** (1972), no. 1, 307–317.
[74] T tom Dieck, *Orbittypen und äquivariante Homologie. II.* Arch. Math. (Basel) **26** (1975), no. 6, 650–662.
[75] T tom Dieck, *Transformation Groups.* De Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co., Berlin, 1987. x+312 pp.
[76] T tom Dieck, *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. xii+567 pp.
[77] A Valette, *Introduction to the Baum-Connes conjecture.* From notes taken by Indira Chatterji. With an appendix by Guido Mislin. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. x+104 pp.
[78] K Wirthmüller, *Equivariant homology and duality.* Manuscripta Math. **11** (1974), 373–390.

Index