Global homotopy theory

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Preface

Equivariant stable homotopy theory has a long tradition, starting from geometrically motivated questions about symmetries of manifolds. The homotopy theoretic foundations of the subject were laid by tom Dieck, Segal and May and their students and collaborators in the 70's, and during the last decades equivariant stable homotopy theory has been very useful to solve computational and conceptual problems in algebraic topology, geometric topology and algebraic K-theory. Various important equivariant theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class. Prominent examples of this are equivariant stable homotopy, equivariant K-theory or equivariant bordism. *Global* equivariant homotopy theory studies such uniform phenomena, i.e., the adjective 'global' refers to simultaneous and compatible actions of all compact Lie groups.

This book introduces a new context for global homotopy theory. Various ways to provide a home for global stable homotopy types have previously been explored in [100, Ch. II], [68, Sec. 5], [18] and [19]. We use a different approach: we work with the well-known category of orthogonal spectra, but use a much finer notion of equivalence, the *global equivalences*, than what is traditionally considered. The basic underlying observation is that an orthogonal spectrum gives rise to an orthogonal G-spectrum for every compact Lie group G, and the fact that all these individual equivariant objects come from one orthogonal spectrum implicitly encodes strong compatibility conditions as the group G varies. An orthogonal spectrum thus has G-equivariant homotopy groups for every compact Lie group, and a global equivalence is a morphism of orthogonal spectra that induces isomorphisms for all equivariant homotopy groups for all compact Lie groups (based on 'complete G-universes', compare Definition 4.1.3).

The structure on the equivariant homotopy groups of an orthogonal spectrum gives an idea of the information encoded in a global homotopy type in our sense: the equivariant homotopy groups $\pi_k^G(X)$ are contravariantly func-

torial for continuous group homomorphisms ('restriction maps'), and they are covariantly functorial for inclusions of closed subgroups ('transfer maps'). The restriction and transfer maps enjoy various transitivity properties and interact via a double coset formula. This kind of algebraic structure has been studied before under different names, e.g., 'global Mackey functor', 'inflation functor', ... From a purely algebraic perspective, there are various parameters here than one can vary, namely the class of groups to which a value is assigned and the classes of homomorphisms to which restriction maps or transfer maps are assigned, and lots of variations have been explored. However, the decision to work with orthogonal spectra and equivariant homotopy groups on complete universes dictates a canonical choice: we prove in Theorem 4.2.6 that the algebra of natural operations between the equivariant homotopy groups of orthogonal spectra is freely generated by restriction maps along continuous group homomorphisms and transfer maps along closed subgroup inclusion, subject to explicitly understood relations.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. Every global equivalence is in particular a non-equivariant stable equivalence, so there is a 'forgetful' functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ on localizations, where $\mathcal{SH}$ denotes the traditional non-equivariant stable homotopy category. By Theorem 4.5.1 this forgetful functor has a left adjoint L and a right adjoint R, both fully faithful, that participate in a recollement of triangulated categories:

$$\begin{tikzcd} \mathcal{GH}^+ \arrow[r, "i_*"] & \mathcal{GH} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{SH} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

Here $\mathcal{GH}^+$ denotes the full subcategory of the global stable homotopy category spanned by the orthogonal spectra that are stably contractible in the traditional, non-equivariant sense.

The global sphere spectrum and suspension spectra are in the image of the left adjoint (Example 4.5.11). Global Borel cohomology theories are the image of the right adjoint (Example 4.5.19). The 'natural' global versions of Eilenberg-Mac Lane spectra (Construction 5.3.8), Thom spectra (Section 6.1), or topological K-theory spectra (Sections 6.3 and 6.4) are not in the image of either of the two adjoints. Periodic global K-theory, however, is right induced from finite cyclic groups, i.e., in the image of the analogous right adjoint from an intermediate global homotopy category $\mathcal{GH}_{cyc}$ based on finite cyclic groups (Example 6.4.27).

Looking at orthogonal spectra through the eyes of global equivalences is like using a prism: the latter breaks up white light into a spectrum of colors,

and global equivalences split a traditional, non-equivariant homotopy type into many different global homotopy types. The first example of this phenomenon that we will encounter refines the classifying space of a compact Lie group G. On the one hand, there is the constant orthogonal space with value a non-equivariant model for BG; and there is the *global classifying space* $B_{\mathrm{gl}}G$ (see Definition 1.1.27). The global classifying space is analogous to the geometric classifying space of a linear algebraic group in motivic homotopy theory [123, 4.2], and it is the counterpart to the stack of G-principal bundles in the world of stacks.

Another good example is the splitting up of the non-equivariant homotopy type of the classifying space of the infinite orthogonal group. Again there is the constant orthogonal space with value BO, the Grassmannian model $\mathbf{BO}$ (Example 2.4.1), a different Grassmannian model $\mathbf{bO}$ (Example 2.4.18), the bar construction model $\mathbf{B}^\circ\mathbf{O}$ (Example 2.4.14), and finally a certain 'cofree' orthogonal space $R(BO)$. The orthogonal space $\mathbf{bO}$ is also a homotopy colimit, as n goes to infinity, of the global classifying spaces $B_{\mathrm{gl}}O(n)$. We discuss these different global forms of BO in some detail in Section 2.4, and the associated Thom spectra in Section 6.1.

In the stable global world, every non-equivariant homotopy type has two extreme global refinements, the 'left induced' (the global analog of a constant orthogonal space, see Example 4.5.10) and the 'right induced' global homotopy type (representing Borel cohomology theories, see Example 4.5.19). Many important stable homotopy types have other natural global forms. The non-equivariant Eilenberg-Mac Lane spectrum of the integers has a 'free abelian group functor' model (Construction 5.3.8), and another incarnation as the Eilenberg-Mac Lane spectrum of the constant global functor with value $\mathbb{Z}$ (Remark 4.4.12). These two global refinements of the integral Eilenberg-Mac Lane spectrum agree on finite groups, but differ for compact Lie groups of positive dimensions.

As already indicated, there is a great variety of orthogonal Thom spectra, in real (or unoriented) flavors as **mO** and **MO**, as complex (or unitary) versions **mU** and **MU**, and there are periodic versions **mOP**, **MOP**, **mUP** and **MUP** of these; we discuss these spectra in Section 6.1. The theories represented by **mO** and **mU** have the closest ties to geometry; for example, the equivariant homotopy groups of **mO** receive Thom-Pontryagin maps from equivariant bordism rings, and these are isomorphisms for products of finite groups and tori (compare Theorem 6.2.33). The theories represented by **MO** are tom Dieck's homotopical equivariant bordism, isomorphic to 'stable equivariant bordism'.

Connective topological K-theory also has two fairly natural global refinements, in addition to the left and right induced ones. The 'orthogonal subspace' model **ku** (Construction 6.3.9) represents connective equivariant K-

theory on the class of finite groups; on the other hand, global connective K-theory $\mathbf{ku}^c$ (Construction 6.4.32) is the global synthesis of equivariant connective K-theory in the sense of Greenlees [66]. The periodic global K-theory spectrum **KU** is introduced in Construction 6.4.9; as the name suggests, **KU** is Bott periodic and represents equivariant K-theory.

The global equivalences are part of a closed model structure (see Theorem 4.3.18), so the methods of homotopical algebra can be used to study the stable global homotopy category. This works more generally relative to a class $\mathcal{F}$ of compact Lie groups, where we define $\mathcal{F}$-equivalences by requiring that $\pi_k^G(f)$ is an isomorphism for all integers and all groups in $\mathcal{F}$. We call a class $\mathcal{F}$ of compact Lie groups a *global family* if it is closed under isomorphisms, subgroups and quotients. For global families we refine the $\mathcal{F}$-equivalences to a stable model structure, the *$\mathcal{F}$-global model structure*, see Theorem 4.3.17. Besides all compact Lie groups, interesting global families are the classes of all finite groups, or all abelian compact Lie groups. The class of trivial groups is also admissible here, but then we just recover the 'traditional' stable category. If the family $\mathcal{F}$ is multiplicative, then the $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra and satisfies the monoid axiom (Proposition 4.3.28). Hence this model structure lifts to modules over an orthogonal ring spectrum and to algebras over an ultra-commutative ring spectrum (Corollary 4.3.29).

Ultra-commutativity. A recurring theme throughout this book is a phenomenon that I call *ultra-commutativity*. I use this term in the unstable and stable context for the homotopy theory of strictly commutative objects under global equivalences. An ultra-commutative multiplication is significantly more structure than just a coherently homotopy-commutative product (usually called an E_∞-multiplication). For example, the extra structure gives rise to power operations that can be turned into transfer maps (in additive notation) and norm maps (in multiplicative notation). Another difference is that an unstable global E_∞-structure would give rise to naive deloopings (i.e., by trivial representations). As I hope discuss elsewhere, a global ultra-commutative multiplication, in contrast, gives rise to 'genuine' deloopings (i.e., by non-trivial representations). As far as the objects are concerned, ultra-commutative monoids and ultra-commutative ring spectra are not at all new and have been much studied before; so one could dismiss the name 'ultra-commutativity' as a mere marketing maneuver. However, the homotopy theory of ultra-commutative monoids and ultra-commutative ring spectra with respect to global equivalences is new and, in the authors opinion, important. And important concepts deserve catchy names.

Global homotopy types as orbifold cohomology theories. I would like to briefly mention another reason for why one might be interested in global stable homotopy theory. In short, global stable homotopy types represent genuine cohomology theories on stacks, orbifolds, and orbispaces. Stacks and orbifolds are concepts from algebraic geometry and geometric topology that allow us to talk about objects that locally look like the quotient of a smooth object by a group action, in a way that remembers information about the isotropy groups of the action. Such 'stacky' objects can behave like smooth objects even if the underlying spaces have singularities. As for spaces, manifolds and schemes, cohomology theories are important invariants also for stacks and orbifolds, and examples such as ordinary cohomology or K-theory lend themselves to generalization. Special cases of orbifolds are 'global quotients', often denoted $M/\!\!/G$, for example for a smooth action of a compact Lie group G on a smooth manifold M. In such examples, the orbifold cohomology of $M/\!\!/G$ is supposed to be the G-equivariant cohomology of M. This suggests a way to *define* orbifold cohomology theories by means of equivariant stable homotopy theory, via suitable G-spectra E_G. However, since the group G is not intrinsic and can vary, one needs equivariant cohomology theories for all groups G, with some compatibility.

Part of the compatibility can be deduced from the fact that the same orbifold can be presented in different ways; for example, if G is a closed subgroup of K, then the global quotients $M/\!\!/G$ and $(M \times_G K)/\!\!/K$ describe the same orbifold. So if the orbifold cohomology theory is represented by equivariant spectra $\{E_G\}_G$ as indicated above, then necessarily $E_G \simeq \mathrm{res}^K_G(E_K)$, i.e., the equivariant homotopy types are consistent under restriction. This is the characteristic feature of *global* equivariant homotopy types, and it suggest that the latter ought to define orbifold cohomology theories.

The approach to global homotopy theory presented in this book in particular provides a way to turn the above outline into rigorous mathematics. There are different formal frameworks for stacks and orbifolds (algebro-geometric, smooth, topological), and these objects can be studied with respect to various notions of 'equivalence'. The approach that most easily feeds into our present context are the notions of *topological stacks* and *orbispaces* as developed by Gepner and Henriques in their paper [61]. Their homotopy theory of topological stacks is rigged up so that the derived mapping spaces out of the classifying stacks for principal G-bundles detect equivalences. In our setup, the global classifying spaces of compact Lie groups (see Definition 1.1.27) play exactly the same role, and this is another hint of a deeper connection. In fact, the global homotopy theory of orthogonal spaces as developed in Chapter 1 is a model for the homotopy theory of orbispaces in the sense of Gepner and Henriques. For a formal comparison of the two models we refer the reader to the author's paper

[145]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. Here the universal compact Lie group (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$, and in [145] we establish a global model structure on the category of $\mathcal{L}$-spaces.

If we now accept that one can pass between stacks, orbispaces and orthogonal spaces in homotopically meaningful way, a consequence is that every global stable homotopy type (i.e., every orthogonal spectrum) gives rise to a cohomology theory on stacks and orbifolds. Indeed, by taking the unreduced suspension spectrum, every unstable global homotopy type is transferred into the triangulated global stable homotopy category $\mathcal{GH}$. In particular, taking morphisms in $\mathcal{GH}$ into an orthogonal spectrum E defines $\mathbb{Z}$-graded E-cohomology groups. The counterpart of a global quotient $M/\!\!/G$ in the global homotopy theory of orthogonal spaces is the semifree orthogonal space $\mathbf{L}_{G,V}M$ introduced in Construction 1.1.22. By the adjunction relating the global and G-equivariant stable homotopy categories (see Theorem 4.5.24), the morphisms $[\![\Sigma^\infty_+\mathbf{L}_{G,V}M, E]\!]$ in the global stable homotopy category biject with the G-equivariant E-cohomology groups of M. In other words, when evaluated on a global quotient $M/\!\!/G$, our recipe for generating an orbifold cohomology theory from a global stable homotopy type precisely returns the G-equivariant cohomology of M, which was the original design criterion.

The procedure sketched so far actually applies to more general objects than our global stable homotopy types: indeed, all that was needed to produce the orbifold cohomology theory was a sufficiently exact functor from the homotopy theory of orbispaces to a triangulated category. If we aim for a stable homotopy theory (as opposed to its triangulated homotopy category), then there is a universal example, namely the stabilization of the homotopy theory of orbispaces, obtained by formally inverting suspension. Our global theory is, however, richer than this 'naive' stabilization. Indeed, there is a forgetful functor from the global stable homotopy category to the G-equivariant stable homotopy category, based on a complete G–universe; the equivariant cohomology theories represented by such objects are usually called 'genuine' (as opposed to 'naive'). Genuine equivariant cohomology theories have much more structure than naive ones; this structure manifests itself in different forms, for example as transfer maps, stability under 'twisted suspension' (i.e., smash product with linear representation spheres), an extension of the $\mathbb{Z}$-graded cohomology groups to an $RO(G)$-graded theory, and an equivariant refinement of additivity (the so called *Wirthmüller isomorphism*). Hence global stable homotopy types in the sense of this book represent *genuine* (as opposed to 'naive') orbifold cohomology theories.

Organization. In Chapter 1 we set up unstable global homotopy theory using orthogonal spaces, i.e., continuous functors from the category of finite-dimensional inner product spaces and linear isometric embeddings to spaces. We introduce global equivalences (Definition 1.1.2), discuss global classifying spaces of compact Lie groups (Definition 1.1.27), and set up the global model structures on the category of orthogonal spaces (Theorem 1.2.21). In Section 1.3 we investigate the box product of orthogonal spaces from a global equivariant perspective. Section 1.4 introduces a variant of unstable global homotopy theory based on a *global family*, i.e., a class $\mathcal{F}$ of compact Lie groups with certain closure properties. We discuss the $\mathcal{F}$-global model structure and record that for multiplicative global families, it lifts to category of modules and algebras (Corollary 1.4.15). In Section 1.5 we discuss the G-equivariant homotopy sets of orthogonal spaces and identify the natural structure between them (restriction maps along continuous group homomorphisms). The study of natural operations on $\pi_0^G(Y)$ is a recurring theme throughout this book; in the later chapters we return to it in the contexts of ultra-commutative monoids, orthogonal spectra and ultra-commutative ring spectra.

Chapter 2 is devoted to ultra-commutative monoids (a.k.a. commutative monoids with respect to the box product, or lax symmetric monoidal functors), which we want to advertise as a rigidified notion of 'global E_∞-space'. In Section 2.1 we establish a global model structure for ultra-commutative monoids (Theorem 2.1.15). Section 2.2 introduces and studies global power monoids, the algebraic structure that an ultra-commutative multiplication gives rise to on the homotopy group Rep-functor $\underline{\pi}_0(R)$. Section 2.3 contains a large collection of examples of ultra-commutative monoids and interesting morphisms between them. In Section 2.4 we discuss and compare different global refinements of the non-equivariant homotopy type BO, the classifying space for the infinite orthogonal group. Section 2.5 discusses 'units' and 'group completions' of ultra-commutative monoids. As an application of this technology we formulate and prove a global, highly structured version of Bott periodicity, see Theorem 2.5.41.

Chapter 3 is a largely self-contained exposition of many basics about equivariant stable homotopy theory for a fixed compact Lie group, modeled by orthogonal G-spectra. In Section 3.1 we recall orthogonal G-spectra and equivariant homotopy groups and prove their basic properties, such as the suspension isomorphism and long exact sequences of mapping cones and homotopy fibers, and the additivity of equivariant homotopy groups on sums and products. Section 3.2 discusses the Wirthmüller isomorphism and the closely related transfers. In Section 3.3 we introduce and study geometric fixed-point homotopy groups, an alternative invariant to characterize equivariant stable equivalences. Section 3.4 contains a proof of the double coset formula for the

composite of a transfer followed by the restriction to a closed subgroup. We review Mackey functors for finite groups and show that after inverting the group order, the category of G-Mackey functors splits as a product, indexed by conjugacy classes of subgroups, of module categories over the Weyl groups (Theorem 3.4.22). A topological consequence is that after inverting the group order, equivariant homotopy groups and geometric fixed-point homotopy groups determine each other in a completely algebraic fashion, compare Proposition 3.4.26 and Corollary 3.4.28. Section 3.5 is devoted to multiplicative aspects of equivariant stable homotopy theory.

Chapter 4 sets the stage for stable global homotopy theory, based on the notion of global equivalences for orthogonal spectra (Definition 4.1.3). We discuss semifree orthogonal spectra and identify certain morphisms between semifree orthogonal spectra as global equivalences (Theorem 4.1.29). In Section 4.2 we investigate *global functors*, the natural algebraic structure on the collection of equivariant homotopy groups of a global stable homotopy type. Among other things, we explicitly calculate the algebra of natural operations on equivariant homotopy groups (Theorem 4.2.6). In Section 4.3 we complement the global equivalences of orthogonal spectra by a stable model structure that we call the *global model structure*. Its fibrant objects are the 'global Ω-spectra' (Definition 4.3.8), the natural concept of a 'global infinite loop space' in our setting. Here we work more generally relative to a global family $\mathcal{F}$ and consider the $\mathcal{F}$-equivalences (i.e., equivariant stable equivalences for all compact Lie groups in the family $\mathcal{F}$). We follow the familiar outline: a certain $\mathcal{F}$-level model structure is Bousfield localized to an $\mathcal{F}$-global model structure (see Theorem 4.3.17). In Section 4.4 we develop some basic theory around the global stable homotopy category; since it comes from a stable model structure, this category is naturally triangulated and we show that the suspension spectra of global classifying spaces form a set of compact generators (Theorem 4.4.3). In Section 4.5 we vary the global family: we construct and study left and right adjoints to the forgetful functors associated with a change of global family (Theorem 4.5.1). As an application of Morita theory for stable model categories we show that rationally the global homotopy category for finite groups has an algebraic model, namely the derived category of rational global functors (Theorem 4.5.29).

Chapter 5 focuses on *ultra-commutative ring spectra*, i.e., commutative orthogonal ring spectra under multiplicative global equivalences. Section 5.1 introduces 'global power functors', the algebraic structure on the equivariant homotopy groups of ultra-commutative ring spectra. Roughly speaking, global power functors are global Green functors equipped with additional power operations, satisfying various properties reminiscent of those of the power maps $x \mapsto x^m$ in a commutative ring. The power operations give rise to norm maps

('multiplicative transfers') along finite index inclusions, and in our global context, the norm maps conversely determine the power operations, compare Remark 5.1.7. As we show in Theorem 5.1.11, the 0th equivariant homotopy groups of an ultra-commutative ring spectrum form a global power functor. In Section 5.2 we develop a description of the category of global power functors via the comonad of 'exponential sequences' (Theorem 5.2.13) and discuss localization of global power functors at a multiplicative subset of the underlying ring (Theorem 5.2.18). In Section 5.3 we give various examples of global power functors, such as the Burnside ring global power functor, the global functor represented by an abelian compact Lie group, free global power functors, constant global power functors, and the complex representation ring global functor. In Section 5.4 we establish the global model structure for ultra-commutative ring spectra (Theorem 5.4.3) and show that every global power functor is realized by an ultra-commutative ring spectrum (Theorem 5.4.14).

Chapter 6 is devoted to interesting examples of ultra-commutative ring spectra. Section 6.1 discusses two orthogonal Thom spectra **mO** and **MO**. The spectrum **mO** is globally connective and closely related to equivariant bordism. The global functor $\underline{\pi}_0(\mathbf{mO})$ admits a short and elegant algebraic presentation: it is obtained from the Burnside ring global functor by imposing the single relation $\mathrm{tr}_e^{C_2} = 0$, compare Theorem 6.1.44. The Thom spectrum **MO** was first considered by tom Dieck and it represents 'stable' equivariant bordism; it is periodic for orthogonal representations of compact Lie groups, and admits Thom isomorphisms for equivariant vector bundles. The equivariant homology theory represented by **MO** can be obtained from the one represented by **mO** in an algebraic fashion, by inverting the collection of 'inverse Thom classes', compare Corollary 6.1.35. Section 6.2 recalls the geometrically defined equivariant bordism theories. The Thom-Pontryagin construction maps the unoriented G-equivariant bordism ring $\mathcal{N}_*^G$ to the equivariant homotopy ring $\pi_*^G(\mathbf{mO})$, and that map is an isomorphism when G is a product of a finite group and a torus, see Theorem 6.2.33. We discuss global K-theory in Sections 6.3 and 6.4, which comes in three interesting flavors as connective global K-theory **ku**, global connective K-theory $\mathbf{ku}^c$ and periodic global K-theory **KU** (and in the real versions **ko**, $\mathbf{ko}^c$ and **KO**).

We include three appendices where we collect material that is mostly well-known, but that is either scattered through the literature or where we found the existing expositions too sketchy. Appendix A is a self-contained review of compactly generated spaces, our basic category to work in. Appendix B deals with fundamental properties of equivariant spaces, including the basic model structure in Proposition B.7. We also provide an exposition of the equivariant $\mathbf{\Gamma}$-space machinery, culminating in a version of the Segal-Shimakawa deloop-

ing machine. In Appendix C we review the basic definitions, properties and constructions involving categories of enriched functors.

While most of the material in the appendices is well-known, there are a few results I could not find in the literature. These results include the fact that compactly generated spaces are closed under geometric realization (Proposition A.35 (iii)), fixed-points commute with geometric realization and latching objects (Proposition B.1 (iv)), and compactly generated spaces are closed under prolongation of $\mathbf{\Gamma}$-spaces (Proposition B.26). Also apparently new are the results that prolongation of G-cofibrant $\mathbf{\Gamma}$-G-spaces to finite G-CW-complexes is homotopically meaningful (Proposition B.48), and that prolongation of G-cofibrant $\mathbf{\Gamma}$-G-spaces to spheres gives rise to G-Ω-spectra (for very special $\mathbf{\Gamma}$-G-spaces, see Theorem B.61) and to positive G-Ω-spectra (for special $\mathbf{\Gamma}$-G-spaces, see Theorem B.65). Here the key ideas all go back to Segal [155] and Shimakawa [157]; however, we formulate our results for the prolongation (i.e., categorical Kan extension), whereas Segal and Shimakawa work with a bar construction (also known as a homotopy coend or homotopy Kan extension) instead. We also give a partial extension of the machinery to compact Lie groups, whereas previous papers on the subject restrict attention to finite groups. As we explain in Remark B.66, there is no hope to obtain a G-Ω-spectrum by evaluation on spheres for compact Lie groups of positive dimension. However, we do prove in Theorem B.65 that evaluating a G-cofibrant special $\mathbf{\Gamma}$-G-space on spheres yields a 'G°-trivial positive G-Ω-spectrum', where G° is the identity component of G. Our Appendix B substantially overlaps with the paper [115] by May, Merling and Osorno that provides comparisons of prolongation, bar construction and the operadic approach to equivariant deloopings.

Relation to other work. The idea of global equivariant homotopy theory is not at all new and has previously been explored in different contexts. For example, in Chapter II of [100], Lewis and May define *coherent families* of equivariant spectra; these consist of collections of equivariant coordinate-free spectra in the sense Lewis, May and Steinberger, equipped with comparison maps involving change of groups and change of universe functors.

The approach closest to ours are the *global $\mathcal{I}_*$-functors* introduced by Greenlees and May in [68, Sec. 5]. These objects are 'global orthogonal spectra' in that they are indexed on pairs (G, V) consisting of a compact Lie group and a G-representation V. The corresponding objects with commutative multiplication are called *global $\mathcal{I}_*$-functors with smash products* in [68, Sec. 5] and it is for these that Greenlees and May define and study multiplicative norm maps. Clearly, an orthogonal spectrum gives rise to a global $\mathcal{I}_*$-functors in the sense of Greenlees and May. In the second chapter of her thesis [18], Bohmann com-

pares the approaches of Lewis-May and Greenlees-May; in the paper [19] she also relates these to orthogonal spectra.

Symmetric spectra in the sense of Hovey, Shipley and Smith [81] are another prominent model for the (non-equivariant) stable homotopy category. Much of what we do here with orthogonal spectra can also be done with symmetric spectra, if one is willing to restrict to finite groups (as opposed to general compact Lie groups). This restriction arises because only finite groups embed into symmetric groups, while every compact Lie group embeds into an orthogonal group. Hausmann [72, 73] has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric spectra with the global model structure. While some parts of the symmetric and orthogonal theories are similar, there are serious technical complications arising from the fact that for symmetric spectra the naively defined equivariant homotopy groups are not 'correct', a phenomenon that is already present non-equivariantly.

Prerequisites. This book assumes a solid background in algebraic topology and (non-equivariant) homotopy theory, including topics such as singular homology and cohomology, CW-complexes, homotopy groups, mapping spaces, loop spaces, fibrations and fiber bundles, Eilenberg-Mac Lane spaces, smooth manifolds, Grassmannian and Stiefel manifolds. Two modern references that contain all we need (and much more) are the textbooks by Hatcher [71] and tom Dieck [180]. Some knowledge of non-equivariant stable homotopy theory is helpful to appreciate the equivariant and global features of the structures and examples we discuss; from a strictly logical perspective, however, the non-equivariant theory is a degenerate special case of the global theory for the global family of trivial Lie groups. In particular, by simply ignoring all group actions, the examples presented in this book give models for many interesting and prominent non-equivariant stable homotopy types.

Since actions of compact Lie groups are central to this book, some familiarity with the structure and representation theory of compact Lie groups is obviously helpful, but we give references to the literature whenever we need any non-trivial facts. Many of our objects of study organize themselves into model categories in the sense of Quillen [134], so some background on model categories is necessary to understand the respective sections. The article [48] by Dwyer and Spalinski is a good introduction, and Hovey's book [80] is still the definitive reference. Some acquaintance with unstable equivariant homotopy theory is useful (but not logically necessary). In contrast, we do not assume any prior knowledge of equivariant *stable* homotopy theory, and Chapter 3 is a self-contained introduction based on equivariant orthogonal spectra. The

last two sections of Chapter 4 study the global stable homotopy category, and here we freely use the language of triangulated categories. The first chapter of Neeman's book [128] is a possible reference for the necessary background.

Throughout the book we work in the category of compactly generated spaces in the sense of McCord [118], i.e., a k-spaces (also called *Kelley spaces*) that satisfy the weak Hausdorff condition, see Definition A.1. Since the various useful properties of compactly generated spaces are scattered through the literature, we include a detailed discussion in Appendix A.

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1
Unstable global homotopy theory

In this chapter we develop a framework for unstable global homotopy theory via *orthogonal spaces*, i.e., continuous functors from the category of linear isometries **L** to spaces. In Section 1.1 we define global equivalences of orthogonal spaces and establish many basic properties of this class of morphisms. We also introduce global classifying spaces of compact Lie groups, the basic building blocks of global homotopy types. In Section 1.2 we complement the global equivalences by a global model structure on the category of orthogonal spaces. The construction follows a familiar pattern, by Bousfield localization of an auxiliary 'strong level model structure'. Section 1.2 also contains a discussion of cofree orthogonal spaces, i.e., global homotopy types that are 'right induced' from non-equivariant homotopy types. In Section 1.3 we recall the box product of orthogonal spaces, a Day convolution product based on the orthogonal direct sum of inner product spaces. The box product is a symmetric monoidal product, fully homotopical under global equivalences, and globally equivalent to the cartesian product. Section 1.4 introduces an important variation of our theme, where we discuss unstable global homotopy theory for a 'global family', i.e., a class of compact Lie groups with certain closure properties. In Section 1.5 we introduce the G-equivariant homotopy set $\pi_0^G(Y)$ of an orthogonal space and identify the natural structure on these sets (restriction maps along continuous group homomorphisms). The study of natural operations on the sets $\pi_0^G(Y)$ is a recurring theme throughout this book, and we will revisit and extend the results in the later chapters for ultra-commutative monoids, orthogonal spectra and ultra-commutative ring spectra.

Our main reason for working with orthogonal spaces is that they are the direct unstable analog of orthogonal spectra, and in this unstable model for global homotopy theory the passage to the stable theory in Chapter 4 is especially simple. However, there are other models for unstable global homotopy theory, most notably *topological stacks* and *orbispaces* as developed by Gepner and Henriques in their paper [61]. For a comparison of these two models

to our orthogonal space model we refer to the author's paper [145]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. Here the universal compact Lie group (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$, and in [145] we establish a global model structure on the category of $\mathcal{L}$-spaces.

1.1 Orthogonal spaces and global equivalences

In this section we introduce *orthogonal spaces*, along with the notion of *global equivalences*, our setup to rigorously formulate the idea of 'compatible equivariant homotopy types for all compact Lie groups'. We introduce various basic techniques to manipulate global equivalences of orthogonal spaces, such as recognition criteria by homotopy or strict colimits over representations (Propositions 1.1.7 and 1.1.17), and a list of standard constructions that preserve global equivalences (Proposition 1.1.9). Theorem 1.1.10 is a cofinality result for orthogonal spaces, showing that fairly general changes in the indexing category of linear isometries do not affect the global homotopy type. Definition 1.1.27 introduces global classifying spaces of compact Lie groups, the basic building blocks of global homotopy theory. Proposition 1.1.30 justifies the name by explaining the sense in which the global classifying space $B_{\mathrm{gl}}G$ 'globally classifies' principal G-bundles.

Before we start, let us fix some notation and conventions. By a 'space' we mean a *compactly generated space* in the sense of [118], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition, see Definition A.1. We denote the category of compactly generated spaces by $\mathbf{T}$ and review its basic properties in Appendix A.

An *inner product space* is a finite-dimensional real vector space equipped with a scalar product, i.e., a positive-definite symmetric bilinear form. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. The category $\mathbf{L}$ is a topological category in the sense that the morphism spaces come with a preferred topology: if $\varphi : V \longrightarrow W$ is a linear isometric embedding, then the action of the orthogonal group $O(W)$, by post-composition, induces a bijection

$$O(W)/O(\varphi^\perp) \cong \mathbf{L}(V, W) , \quad A \cdot O(\varphi^\perp) \longmapsto A \circ \varphi ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V, W)$ so that this bijection is a homeomorphism, and this topology is independent of φ. If $(v_1, \ldots, v_k)$ is an orthonormal basis of V, then for

every linear isometric embedding $\varphi : V \longrightarrow W$ the tuple $(\varphi(v_1), \dots, \varphi(v_k))$ is an orthonormal k-frame of W. This assignment is a homeomorphism from $\mathbf{L}(V, W)$ to the Stiefel manifold of k-frames in W.

An example of an inner product spaces is the vector space $\mathbb{R}^n$ with the standard scalar product

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n .$$

In fact, every inner product space V is isometrically isomorphic to the inner product space $\mathbb{R}^n$, for n the dimension of V. So the full topological subcategory with objects the $\mathbb{R}^n$ is a small skeleton of $\mathbf{L}$.

Definition 1.1.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathbf{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote the category of orthogonal spaces by *spc*.

The use of continuous functors from the category $\mathbf{L}$ to spaces has a long history in homotopy theory. The systematic use of inner product spaces (as opposed to numbers) to index objects in stable homotopy theory seems to go back to Boardman's thesis [15]. The category $\mathbf{L}$ (or its extension that also contains countably infinite-dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [16], and this notation is also used in [112]; other sources [102] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: on the one hand, our use of orthogonal spaces comes with a shift in emphasis, away from a focus on non-equivariant homotopy types, and towards viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Secondly, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global world with the latter the corresponding stable global world.

Now we define our main new concept, the notion of 'global equivalence' between orthogonal spaces. We let G be a compact Lie group. By a *G-representation* we mean a finite-dimensional orthogonal representation, i.e., a real inner product space equipped with a continuous G-action by linear isometries. In other words, a G-representation consists of an inner product space V and a continuous homomorphism $\rho : G \longrightarrow O(V)$. In this context, and throughout the book, we will often use without explicit mentioning that continuous homomorphisms between Lie groups are automatically smooth, see for example [28, Prop. I.3.12]. For every orthogonal space Y and every G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$, the induced map $Y(\varphi) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

We denote by

$$D^k \;=\; \{x \in \mathbb{R}^k \;:\; \langle x, x\rangle \le 1\} \quad \text{and} \quad \partial D^k \;=\; \{x \in \mathbb{R}^k \;:\; \langle x, x\rangle = 1\}$$

the unit disc in $\mathbb{R}^k$ and its boundary, a sphere of dimension $k-1$, respectively. In particular, $D^0 = \{0\}$ is a one-point space and $\partial D^0 = \emptyset$ is empty.

Definition 1.1.2. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \ge 0$ and all continuous maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

In other words, for every commutative square on the left

$$\begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(\varphi)^G"] & X(W)^G \arrow[d, "f(W)^G"] \\ D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

there exists the lift λ on the right-hand side that makes the upper left triangle commute on the nose, and the lower right triangle commute up to homotopy relative to ∂D^k. In such a situation we will often refer to the pair (α,β) as a 'lifting problem' and we will say that the pair (φ, λ) *solves the lifting problem*.

Example 1.1.3. If $X = \underline{A}$ and $Y = \underline{B}$ are the constant orthogonal spaces with values the spaces A and B, and $f = \underline{g}$ the constant morphism associated with a continuous map $g : A \longrightarrow B$, then $\underline{g}$ is a global equivalence if and only if for every commutative square

$$\begin{tikzcd} \partial D^k \arrow[r] \arrow[d, "\mathrm{incl}"'] & A \arrow[d, "g"] \\ D^k \arrow[r] \arrow[ur, dashed, "\lambda"] & B \end{tikzcd}$$

there exists a lift λ that makes the upper left triangle commute, and the lower right triangle commute up to homotopy relative to ∂D^k. But this is one of the equivalent ways of characterizing weak equivalences of spaces, compare [114, Sec. 9.6, Lemma]. So $\underline{g}$ is a global equivalence if and only if g is a weak equivalence.

Remark 1.1.4. The notion of global equivalence is meant to capture the idea

that for every compact Lie group G, some induced morphism

$$\mathrm{hocolim}_V\, f(V) \ : \ \mathrm{hocolim}_V\, X(V) \ \longrightarrow \ \mathrm{hocolim}_V\, Y(V)$$

is a G-weak equivalence, where 'hocolim$_V$' is a suitable homotopy colimit over all G-representations V along all equivariant linear isometric embeddings. This is a useful way to think about global equivalences, and it could be made precise by letting V run over the poset of finite-dimensional subrepresentations of a complete G-universe and using the Bousfield-Kan construction of a homotopy colimit over this poset. Since the 'poset of all G-representations' has a cofinal subsequence, called an *exhaustive sequence* in Definition 1.1.6, we can also model the 'homotopy colimit over all G-representations' as the mapping telescope over an exhaustive sequence. However, the actual definition we work with has the advantage that it does not refer to universes and we do not have to define or manipulate homotopy colimits.

In many examples of interest, all the structure maps of an orthogonal space Y are closed embeddings. When this is the case, the actual colimit (over the subrepresentations of a complete universe) of the G-spaces $Y(V)$ serves the purpose of a 'homotopy colimit over all representations', and it can be used to detect global equivalences, compare Proposition 1.1.17 below.

We will now establish some useful criteria for detecting global equivalences. We call a continuous map $f : A \longrightarrow B$ an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \longrightarrow X$ and a homotopy $H : A \times [0,1] \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : B \times [0,1] \longrightarrow X$ starting with φ such that $\bar{H} \circ (f \times [0,1]) = H$. Below we will write $H_t = H(-,t) : A \longrightarrow X$. All h-cofibrations in the category of compactly generated spaces are closed embeddings, compare Proposition A.31. The following somewhat technical lemma should be well known, but I was unable to find a reference.

Lemma 1.1.5. *Let A be a subspace of a space B such that the inclusion $A \longrightarrow B$ is an h-cofibration. Let $f : X \longrightarrow Y$ be a continuous map and*

$$H \ : \ A \times [0,1] \ \longrightarrow \ X \qquad \textit{and} \qquad K \ : \ B \times [0,1] \ \longrightarrow \ Y$$

homotopies such that $K|_{A\times[0,1]} = fH$. Then the lifting problem (H_0, K_0) has a solution if and only if the lifting problem (H_1, K_1) has a solution.

Proof The problem is symmetric, so we only show one direction. We suppose that the lifting problem (H_0, K_0) has a solution consisting of a continuous map $\lambda : B \longrightarrow X$ such that $\lambda|_A = H_0$ and a homotopy $G : B \times [0,1] \longrightarrow Y$ such that

$$G_0 \ = \ f \circ \lambda\,, \quad G_1 \ = \ K_0 \qquad \text{and} \qquad (G_t)|_A \ = \ f \circ H_0$$

for all $t \in [0,1]$. The homotopy extension property provides a homotopy H' :

$B \times [0,1] \longrightarrow X$ such that

$$H'_0 = \lambda \qquad \text{and} \qquad H'|_{A\times[0,1]} = H \ .$$

Then the map $\lambda' = H'_1 : B \longrightarrow X$ satisfies

$$\lambda'|_A = (H'_1)|_A = H_1 \ .$$

We define a continuous map $J : B \times [0,3] \longrightarrow Y$ by

$$J_t = \begin{cases} f \circ H'_{1-t} & \text{for } 0 \le t \le 1, \\ G_{t-1} & \text{for } 1 \le t \le 2, \text{ and} \\ K_{t-2} & \text{for } 2 \le t \le 3. \end{cases}$$

In particular,

$$J_0 = f \circ \lambda' \qquad \text{and} \qquad J_3 = K_1 \ ;$$

so J almost witnesses the fact that λ' solves the lifting problem (H_1, K_1), except that J is *not* a relative homotopy.

We improve J to a relative homotopy from $f \circ \lambda'$ to K_1. We define a continuous map $L : A \times [0,3] \times [0,1] \longrightarrow Y$ by

$$L(-,t,s) = \begin{cases} f \circ H_{1-t} & \text{for } 0 \le t \le s, \\ f \circ H_{1-s} & \text{for } s \le t \le 3-s, \text{ and} \\ f \circ H_{t-2} & \text{for } 3-s \le t \le 3. \end{cases}$$

Then $L(-,-,0)$ is the constant homotopy at the map $f \circ H_1$, and

$$L(-,-,1) = J|_{A\times[0,3]} \ : \ A \times [0,3] \longrightarrow Y \ .$$

Since the inclusion of A into B is an h-cofibration, the inclusion of $B\times\{0\}\cup_{A\times\{0\}} A \times [0,1]$ into $B \times [0,1]$ has a continuous retraction; hence the inclusion

$$B \times \{0\} \times [0,1] \cup_{A\times\{0\}\times[0,1]} A \times [0,1] \times [0,1] \longrightarrow B \times [0,1] \times [0,1]$$

also has a continuous retraction. We abbreviate $D = [0,3] \times \{1\} \cup \{0,3\} \times [0,1]$; the pair of spaces $([0,3] \times [0,1], D)$ is pair-homeomorphic to $([0,1] \times [0,1], \{0\} \times [0,1])$. So the inclusion

$$B \times D \cup_{A\times D} A \times [0,3] \times [0,1] \longrightarrow B \times [0,3] \times [0,1]$$

has a continuous retraction. The map L and the map

$$J \cup \mathrm{const}_{f\lambda} \cup \mathrm{const}_{K_1} \ : \ B \times D = B \times ([0,3] \times \{1\} \cup \{0,3\} \times [0,1]) \longrightarrow Y$$

agree on $A \times D$, so there is a continuous map $\bar{L} : B \times [0,3] \times [0,1] \longrightarrow Y$ such that

$$\bar{L}(-,-,1) = J \ , \qquad \bar{L}|_{A\times[0,3]\times[0,1]} = L \ ,$$

and

$$\bar{L}(-,0,s) \;=\; f\circ\lambda \qquad\text{and}\qquad \bar{L}(-,1,s) \;=\; K_1$$

for all $s \in [0,1]$. The map $\bar{J} = \bar{L}(-,-,0) : B\times[0,3] \longrightarrow Y$ then satisfies

$$\bar{J}|_{A\times[0,3]} \;=\; \bar{L}(-,-,0)|_{A\times[0,3]} \;=\; L(-,-,0)\,,$$

which is the constant homotopy at the map $f\circ H_1$; so $\bar{J}$ is a homotopy (parametrized by $[0,3]$ instead of $[0,1]$) *relative* to A. Because

$$\bar{J}_0 \;=\; \bar{L}(-,0,0) \;=\; f\circ\lambda \qquad\text{and}\qquad \bar{J}_3 \;=\; \bar{L}(-,3,0) \;=\; K_1\,,$$

the homotopy $\bar{J}$ witnesses that λ' solves the lifting problem (H_1, K_1). □

Definition 1.1.6. Let G be a compact Lie group. An *exhaustive sequence* is a nested sequence

$$V_1 \;\subset\; V_2 \;\subset\; \ldots \;\subset\; V_n \;\subset\; \ldots$$

of finite-dimensional G-representations such that every finite-dimensional G-representation admits a linear isometric G-embedding into some V_n.

Given an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations and an orthogonal space Y, the values at the representations and their inclusions form a sequence of G-spaces and G-equivariant continuous maps

$$Y(V_1) \;\longrightarrow\; Y(V_2) \;\longrightarrow\; \cdots \;\longrightarrow\; Y(V_i) \;\longrightarrow\; \cdots\,.$$

We denote by

$$\mathrm{tel}_i\; Y(V_i)$$

the mapping telescope of this sequence of G-spaces; this telescope inherits a natural G-action.

We recall that a G-equivariant continuous map $f : A \longrightarrow B$ between G-spaces is a *G-weak equivalence* if for every closed subgroup H of G the map $f^H : A^H \longrightarrow B^H$ of H-fixed-points is a weak homotopy equivalence (in the non-equivariant sense).

Proposition 1.1.7. *For every morphism of orthogonal spaces $f : X \longrightarrow Y$, the following three conditions are equivalent.*

- (i) *The morphism f is a global equivalence.*
- (ii) *For every compact Lie group G, every G-representation V, every finite G-CW-pair (B,A) and all continuous G-maps $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ such that $\beta|_A = f(V)\circ\alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi)\circ\alpha$ and such that $f(W)\circ\lambda$ is G-homotopic, relative to A, to $Y(\varphi)\circ\beta$.*

(iii) *For every compact Lie group G and every exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations, the induced map*

$$\mathrm{tel}_i\, f(V_i) \ :\ \mathrm{tel}_i\, X(V_i) \ \longrightarrow\ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence.

Proof At various places in the proof we use without explicitly mentioning it that taking G-fixed-points commutes with formation of the mapping telescopes; this follows from the fact that taking G-fixed-points commutes with pushouts along closed embeddings and with sequential colimits along closed embeddings, compare Proposition B.1.

(i)$\Longrightarrow$(ii) We argue by induction over the number of the relative G-cells in (B, A). If $B = A$, then $\lambda = \alpha$ solves the lifting problem, and there is nothing to show. Now we suppose that A is a proper subcomplex of B. We choose a G-CW-subcomplex B' that contains A and such that (B, B') has exactly one equivariant cell. Then (B', A) has strictly fewer cells, and the restricted equivariant lifting problem $(\alpha : A \longrightarrow X(V), \beta' = \beta|_{B'} : B' \longrightarrow Y(V))$ has a solution $(\varphi : V \longrightarrow U, \lambda' : B' \longrightarrow X(U))$ by the inductive hypothesis.

We choose a characteristic map for the last cell, i.e., a pushout square of G-spaces

$$\begin{tikzcd} G/H \times \partial D^k \arrow[r, "\chi"] \arrow[d, "\mathrm{incl}"'] & B' \arrow[d, "\mathrm{incl}"] \\ G/H \times D^k \arrow[r, "\chi"'] & B \end{tikzcd}$$

in which H is a closed subgroup of G. We arrive at the non-equivariant lifting problem on the left:

$$\begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar{\chi}"] \arrow[d, "\mathrm{incl}"'] & X(U)^H \arrow[d, "f(U)^H"] \\ D^k \arrow[r, "Y(\varphi)^H \circ \beta^H \circ \bar{\chi}"'] & Y(U)^H \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar{\chi}"] \arrow[d, "\mathrm{incl}"'] & X(U)^H \arrow[r, "X(\psi)^H"] & X(W)^H \arrow[d, "f(W)^H"] \\ D^k \arrow[r, "Y(\varphi)^H \circ \beta^H \circ \bar{\chi}"'] \arrow[urr, dashed, "\lambda"] & Y(U)^H \arrow[r, "Y(\psi)^H"'] & Y(W)^H \end{tikzcd}$$

Here $\bar{\chi} = \chi(eH, -) : D^k \longrightarrow B^H$. Since f is a global equivalence, there is an H-equivariant linear isometric embedding $\psi : U \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^H$ such that $\lambda|_{\partial D^k} = X(\psi)^H \circ (\lambda')^H \circ \bar{\chi}$ and $f(W)^H \circ \lambda$ is homotopic, relative ∂D^k, to $Y(\psi)^H \circ Y(\varphi)^H \circ \beta^H \circ \bar{\chi}$, as illustrated by the diagram on the right above. By enlarging W, if necessary, we can assume without loss of generality that W is underlying a G-representation and even that ψ is G-equivariant.

The G-equivariant extension of λ

$$G/H \times D^k \longrightarrow X(W) , \quad (gH, x) \longmapsto g \cdot \lambda(x)$$

and the map $X(\psi) \circ \lambda' : B' \longrightarrow X(W)$ then agree on $G/H \times \partial D^k$, so they glue to a G-map $\tilde{\lambda} : B \longrightarrow X(W)$. The pair $(\psi\varphi : V \longrightarrow W, \tilde{\lambda} : B \longrightarrow X(W))$ then solves the original lifting problem (α, β).

(ii)$\Longrightarrow$(iii) We suppose that f satisfies (ii), and we let G be any compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-representations. We consider an equivariant lifting problem, i.e., a finite G-CW-pair (B, A) and a commutative square:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & \mathrm{tel}_i\, X(V_i) \arrow[d, "\mathrm{tel}_i\, f(V_i)"] \\ B \arrow[r, "\beta"'] & \mathrm{tel}_i\, Y(V_i) \end{tikzcd}$$

We show that every such lifting problem has an equivariant solution. Since B and A are compact, there is an $n \geq 0$ such that α has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, X(V_i)$ and β has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, Y(V_i)$ (see Proposition A.15 (i)). There is a natural equivariant homotopy from the identity of the truncated telescope $\mathrm{tel}_{[0,n]}\, X(V_i)$ to the composite

$$\mathrm{tel}_{[0,n]}\, X(V_i) \xrightarrow{\mathrm{proj}} X(V_n) \xrightarrow{\mathrm{incl}} \mathrm{tel}_{[0,n]}\, X(V_i) .$$

Naturality means that this homotopy is compatible with the same homotopy for the telescope of the G-spaces $Y(V_i)$. Lemma 1.1.5 (or rather its G-equivariant generalization) applies to these homotopies, so instead of the original lifting problem we may solve the homotopic lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "\mathrm{incl}"'] & X(V_n) \arrow[r, "i_n"] \arrow[d, "f(V_n)"'] & \mathrm{tel}_i\, X(V_i) \arrow[d, "\mathrm{tel}_i\, f(V_i)"] \\ B \arrow[r, "\beta'"'] & Y(V_n) \arrow[r, "i_n"'] & \mathrm{tel}_i\, Y(V_i) \end{tikzcd}$$

where α' is the composite of the projection $\mathrm{tel}_{[0,n]}\, X(V_i) \longrightarrow X(V_n)$ with α, viewed as a map into the truncated telescope, and similarly for β'.

Since f satisfies (ii), the lifting problem $(\alpha' : A \longrightarrow X(V_n), \beta' : B \longrightarrow Y(V_n))$ has a solution after enlarging V_n along some linear isometric G-embedding. Since the sequence $\{V_i\}_{i\geq 1}$ is exhaustive, we can take this embedding as the inclusion $i : V_n \longrightarrow V_m$ for some $m \geq n$, i.e., there is a continuous G-map $\lambda : B \longrightarrow X(V_m)$ such that $\lambda|_A = X(i)^G \circ \alpha'$ and such that $f(V_m)^G \circ \lambda$ is

G-homotopic, relative A, to $Y(i)^G \circ \beta'$, compare the diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "\mathrm{incl}"'] & X(V_n) \arrow[r, "X(i)"] & X(V_m) \arrow[d, "f(V_m)"] \\ B \arrow[r, "\beta'"'] \arrow[urr, dashed, "\lambda"] & Y(V_n) \arrow[r, "Y(i)"'] & Y(V_m) \end{tikzcd}$$

The composite

$$X(V_n) \xrightarrow{X(i)} X(V_m) \xrightarrow{i_m} \mathrm{tel}_i\, X(V_i)$$

does *not* agree with $i_n : X(V_n) \longrightarrow \mathrm{tel}_i\, X(V_i)$, so the composite $i_n \circ \lambda : B \longrightarrow \mathrm{tel}_i\, X(V_i)$ does not quite solve the (modified) lifting problem $(i_n \circ \alpha', i_n \circ \beta')$. But there is a G-equivariant homotopy $H : X(V_n) \times [0,1] \longrightarrow \mathrm{tel}_i\, X(V_i)$ between $i_m \circ X(i)$ and i_n, and a similar homotopy $K : Y(V_n) \times [0,1] \longrightarrow \mathrm{tel}_i\, Y(V_i)$ for Y instead of X. These homotopies satisfy

$$K \circ (f(V_n) \times [0,1]) \ = \ (\mathrm{tel}_i\, f(V_i)) \circ H\ ,$$

so Lemma 1.1.5 implies that the modified lifting problem, and hence the original lifting problem, has an equivariant solution.

(iii)$\Longrightarrow$(i) We let G be a compact Lie group, V a G-representation, $k \geq 0$ and $(\alpha : \partial D^k \longrightarrow X(V)^G, \beta : D^k \longrightarrow Y(V)^G)$ a lifting problem, i.e., such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$. We choose an exhaustive sequence $\{V_i\}$ of G-representations; then we can embed V into some V_n by a linear isometric G-map and thereby assume without loss of generality that $V = V_n$.

We let $i_n : X(V_n) \longrightarrow \mathrm{tel}_i\, X(V_i)$ and $i_n : Y(V_n) \longrightarrow \mathrm{tel}_i\, Y(V_i)$ be the canonical maps. Since $\mathrm{tel}_i\, f(V_i) : \mathrm{tel}_i\, X(V_i) \longrightarrow \mathrm{tel}_i\, Y(V_i)$ is a G-weak equivalence, there is a continuous map $\lambda : D^k \longrightarrow (\mathrm{tel}_i\, X(V_i))^G$ such that $\lambda|_{\partial D^k} = i_n^G \circ \alpha$ and $(\mathrm{tel}_i\, f(V_i))^G \circ \lambda$ is homotopic, relative ∂D^k, to $i_n^G \circ \beta$. Since fixed-points commute with mapping telescopes and since D^k is compact, there is an $m \geq n$ such that λ and the relative homotopy that witnesses the relation $(\mathrm{tel}_i\, f(V_i))^G \circ \lambda \simeq i_n^G \circ \beta$ both have image in $\mathrm{tel}_{[0,m]}\, X(V_i)^G$, the truncated telescope of the G-fixed-points. The following diagram commutes

$$\begin{tikzcd} X(V_n)^G \arrow[r, "\mathrm{can}"'] \arrow[rrr, bend left, "X(\mathrm{incl})^G"] \arrow[d, "f(V_n)^G"'] & \mathrm{tel}_{[0,n]}\, X(V_i)^G \arrow[r, "\mathrm{incl}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"'] & \mathrm{tel}_{[0,m]}\, X(V_i)^G \arrow[r, "\mathrm{proj}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"] & X(V_m)^G \arrow[d, "f(V_m)^G"] \\ Y(V_n)^G \arrow[r, "\mathrm{can}"] \arrow[rrr, bend right, "Y(\mathrm{incl})^G"'] & \mathrm{tel}_{[0,n]}\, Y(V_i)^G \arrow[r, "\mathrm{incl}"] & \mathrm{tel}_{[0,m]}\, Y(V_i)^G \arrow[r, "\mathrm{proj}"] & Y(V_m)^G \end{tikzcd}$$

where the right horizontal maps are the projections of the truncated telescope to

the last term. So projecting from $\mathrm{tel}_{[0,m]} X(V_i)^G$ to $X(V_m)^G$ and from $\mathrm{tel}_{[0,m]} Y(V_i)^G$ to $Y(V_m)^G$ produces the desired solution to the lifting problem. □

We establish some more basic facts about the class of global equivalences. A *homotopy* between two morphisms of orthogonal spaces $f, f' : X \longrightarrow Y$ is a morphism

$$H : X \times [0,1] \longrightarrow Y$$

such that $H(-,0) = f$ and $H(-,1) = f'$.

Definition 1.1.8. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *homotopy equivalence* if there is a morphism $g : Y \longrightarrow X$ such that gf and fg are homotopic to the respective identity morphisms. The morphism f is a *strong level equivalence* if for every compact Lie group G and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence. The morphism f is a *strong level fibration* if for every compact Lie group G and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration.

If $f, f' : X \longrightarrow Y$ are homotopic morphisms of orthogonal spaces, then the maps $f(V)^G, f'(V)^G : X(V)^G \longrightarrow Y(V)^G$ are homotopic for every compact Lie group G and every G-representation V. So if f is a homotopy equivalence of orthogonal spaces, then the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a non-equivariant homotopy equivalence for every G-representation V. So every homotopy equivalence is in particular a strong level equivalence. By the following proposition, strong level equivalences are global equivalences.

A continuous map $\varphi : A \longrightarrow B$ is a *closed embedding* if it is injective and a closed map. Such a map is then a homeomorphism of A onto the closed subspace $\varphi(A)$ of B. If a compact Lie group G acts on two spaces A and B and $\varphi : A \longrightarrow B$ is a G-equivariant closed embedding, then the restriction $\varphi^G : A^G \longrightarrow B^G$ to G-fixed-points is also a closed embedding.

We call a morphism $f : A \longrightarrow B$ of orthogonal spaces an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism of orthogonal spaces $\varphi : B \longrightarrow X$ and a homotopy $H : A \times [0,1] \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : B \times [0,1] \longrightarrow X$ starting with φ such that $\bar{H} \circ (f \times [0,1]) = H$.

Proposition 1.1.9. (i) *Every strong level equivalence is a global equivalence.*
(ii) *The composite of two global equivalences is a global equivalence.*
(iii) *If f, g and h are composable morphisms of orthogonal spaces such that hg and gf are global equivalences, then f, g, h and hgf are also global equivalences.*
(iv) *Every retract of a global equivalence is a global equivalence.*
(v) *A coproduct of any set of global equivalences is a global equivalence.*

(vi) *A finite product of global equivalences is a global equivalence.*

(vii) *For every space K the functor $- \times K$ preserves global equivalences of orthogonal spaces.*

(viii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spaces that are objectwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be global equivalences of orthogonal spaces that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a global equivalence.*

(ix) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be a global equivalence of orthogonal spaces that is objectwise a closed embedding, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to the colimit of the sequence $\{f_n\}_{n \geq 0}$ is a global equivalence.*

(x) *Let*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram of orthogonal spaces such that f and f' are h-cofibrations. If the morphisms α, β and γ are global equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ C' \cup_{A'} B' \ .$$

(xi) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is a global equivalence. If, in addition, f or g is an h-cofibration, then the morphism k is a global equivalence.

(xii) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is a global equiva-

lence. If, in addition, f or h is a strong level fibration, then the morphism g is also a global equivalence.

Proof (i) We let $f : X \longrightarrow Y$ be a strong level equivalence, G a compact Lie group, V a G-representation and $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ continuous maps such that $f(V)^G \circ \alpha = \beta|_{\partial D^k}$. Since f is a strong level equivalence, the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence, so there is a continuous map $\lambda : D^k \longrightarrow X(V)^G$ such that $\lambda|_{\partial D^k} = \alpha$ and $f(V)^G \circ \lambda$ is homotopic to β relative to ∂D^k. So the pair (Id_V, λ) solves the lifting problem, and hence f is a global equivalence.

(ii) We let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be global equivalences, G a compact Lie group, (B, A) a finite G-CW-pair, V a G-representation and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Z(V)$ continuous G-maps such that $(gf)(V)\circ\alpha = \beta|_A$. Since g is a global equivalence, the equivariant lifting problem $(f(V) \circ \alpha, \beta)$ has a solution $(\varphi : V \longrightarrow W,\ \lambda : B \longrightarrow Y(W))$ such that

$$\lambda|_A \;=\; Y(\varphi) \circ f(V) \circ \alpha \;=\; f(W) \circ X(\varphi) \circ \alpha \;,$$

and $g(W) \circ \lambda$ is homotopic to $Z(\varphi) \circ \beta$ relative to A. Since f is a global equivalence, the equivariant lifting problem $(X(\varphi) \circ \alpha, \lambda)$ has a solution $(\psi : W \longrightarrow U,\ \lambda' : B \longrightarrow X(U))$ such that

$$\lambda'|_A \;=\; X(\psi) \circ X(\varphi) \circ \alpha$$

and such that $f(U)\circ\lambda'$ is G-homotopic to $Y(\psi)\circ\lambda$ relative to A. Then $(gf)(U)\circ\lambda'$ is G-homotopic, relative to A, to

$$g(U) \circ Y(\psi) \circ \lambda \;=\; Z(\psi) \circ g(W) \circ \lambda$$

which in turn is G-homotopic to $Z(\psi\varphi)\circ\beta$, also relative to A. So the pair $(\psi\varphi, \lambda')$ solves the original lifting problem for the morphism $gf : X \longrightarrow Z$.

(iii) We let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Q$ be the three composable morphisms such that $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Q$ are global equivalences. We let G be a compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-representations. Evaluating everything in sight on the representations and forming mapping telescopes yields three composable continuous G-maps

$$\mathrm{tel}_i\, X(V_i) \xrightarrow{\mathrm{tel}_i\, f(V_i)} \mathrm{tel}_i\, Y(V_i) \xrightarrow{\mathrm{tel}_i\, g(V_i)} \mathrm{tel}_i\, Z(V_i) \xrightarrow{\mathrm{tel}_i\, h(V_i)} \mathrm{tel}_i\, Q(V_i) \;.$$

Proposition 1.1.7 shows that the G-maps

$$\begin{aligned}(\mathrm{tel}_i\, g(V_i)) \circ (\mathrm{tel}_i\, f(V_i)) \;&=\; \mathrm{tel}_i(gf)(V_i) \quad \text{and} \\ (\mathrm{tel}_i\, h(V_i)) \circ (\mathrm{tel}_i\, g(V_i)) \;&=\; \mathrm{tel}_i(hg)(V_i)\end{aligned}$$

are G-weak equivalences. Since G-weak equivalences satisfy the 2-out-of-6-property, we conclude that the G-maps $\mathrm{tel}_i\, f(V_i)$, $\mathrm{tel}_i\, g(V_i)$, $\mathrm{tel}_i\, h(V_i)$ and

$$(\mathrm{tel}_i\, h(V_i)) \circ (\mathrm{tel}_i\, g(V_i)) \circ (\mathrm{tel}_i\, f(V_i)) \;=\; \mathrm{tel}_i(hgf)(V_i)$$

are G-weak equivalences. Another application of Proposition 1.1.7 then shows that f, g, h and hgf are global equivalences.

(iv) Let g be a global equivalence and f a retract of g. So there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & \bar{X} \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"'] & \bar{Y} \arrow[r, "s"'] & Y \end{tikzcd}$$

such that $ri = \mathrm{Id}_X$ and $sj = \mathrm{Id}_Y$. We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ continuous G-maps such that $f(V) \circ \alpha = \beta|_A$. Since g is a global equivalence and

$$g(V) \circ i(V) \circ \alpha \;=\; j(V) \circ f(V) \circ \alpha \;=\; (j(V) \circ \beta)|_A \ ,$$

there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow \bar{X}(W)$ such that $\lambda|_A = \bar{X}(\varphi) \circ i(V) \circ \alpha$ and $g(W) \circ \lambda$ is G-homotopic to $\bar{Y}(\varphi) \circ j(V) \circ \beta$ relative to A. Then

$$(r(W) \circ \lambda)|_A \;=\; r(W) \circ \bar{X}(\varphi) \circ i(V) \circ \alpha \;=\; X(\varphi) \circ r(V) \circ i(V) \circ \alpha \;=\; X(\varphi) \circ \alpha$$

and

$$f(W) \circ r(W) \circ \lambda \;=\; s(W) \circ g(W) \circ \lambda$$

is G-homotopic to

$$s(W) \circ \bar{Y}(\varphi) \circ j(V) \circ \beta \;=\; Y(\varphi) \circ s(V) \circ j(V) \circ \beta \;=\; Y(\varphi) \circ \beta$$

relative to A. So the pair $(\varphi, r(W) \circ \lambda)$ solves the original lifting problem for the morphism $f : X \longrightarrow Y$; thus f is a global equivalence.

Part (v) holds because the disc D^k is connected, so any lifting problem for a coproduct of orthogonal spaces is located in one of the summands.

For part (vi) it suffices to consider a product of two global equivalences $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$. Because global equivalences are closed under composition (part (ii)) and $f \times f' = (f \times Y') \circ (X \times f')$, it suffices to show that for every global equivalence $f : X \longrightarrow Y$ and every orthogonal space Z the morphism $f \times Z : X \times Z \longrightarrow Y \times Z$ is a global equivalence. But this is straightforward: we let G be a compact Lie group, V a G-representation, (B, A)

a finite G-CW-pair and $\alpha : A \longrightarrow (X \times Z)(V)$ and $\beta : B \longrightarrow (Y \times Z)(V)$ continuous G-maps such that $(f \times Z)(V) \circ \alpha = \beta|_A$. Because

$$(X \times Z)(V) \;=\; X(V) \times Z(V)$$

and similarly for $(Y \times Z)(V)$, we have $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ for continuous G-maps $\alpha_1 : A \longrightarrow X(V)$, $\alpha_2 : A \longrightarrow Z(V)$, $\beta_1 : B \longrightarrow Y(V)$ and $\beta_2 : B \longrightarrow Z(V)$. The relation $(f \times Z)(V) \circ (\alpha_1, \alpha_2) = (\beta_1, \beta_2)|_A$ shows that $\alpha_2 = (\beta_2)|_A$. Since f is a global equivalence, the equivariant lifting problem (α_1, β_1) for $f(V)$ has a solution $(\varphi : V \longrightarrow W, \lambda : B \longrightarrow X(W))$ such that $\lambda|_A = X(\varphi) \circ \alpha_1$ and $f(W) \circ \lambda$ is G-homotopic to $Y(\varphi) \circ \beta_1$ relative to A. Then the pair $(\varphi, (\lambda, Z(\varphi) \circ \beta_2))$ solves the original lifting problem, so $f \times Z$ is a global equivalence.

(vii) If X is an orthogonal space and K a space, then $X \times K$ is the product of X with the constant orthogonal space with values K. So part (vii) is a special case of (vi).

(viii) We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X_\infty(V)$ and $\beta : B \longrightarrow Y_\infty(V)$ continuous G-maps such that $\psi_\infty(V) \circ \alpha = \beta|_A : A \longrightarrow Y_\infty(V)$. Since A and B are compact and $X_\infty(V)$ and $Y_\infty(V)$ are colimits of sequences of closed embeddings, the maps α and β factor through maps

$$\bar{\alpha} : A \longrightarrow X_n(V) \qquad \text{and} \qquad \bar{\beta} : B \longrightarrow Y_n(V)$$

for some $n \geq 0$, see Proposition A.15 (i). Since the canonical maps $X_n(V) \longrightarrow X_\infty(V)$ and $Y_n(V) \longrightarrow Y_\infty(V)$ are injective, $\bar{\alpha}$ and $\bar{\beta}$ are again G-equivariant. Moreover, the relation $\psi_n(V) \circ \bar{\alpha} = \bar{\beta}|_A : A \longrightarrow Y_n(V)$ holds because it holds after composition with the injective map $Y_n(V) \longrightarrow Y_\infty(V)$.

Since ψ_n is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X_n(W)$ such that $\lambda|_A = X_n(\varphi) \circ \bar{\alpha}$ and $\psi_n(W) \circ \lambda$ is G-homotopic to $Y_n(\varphi) \circ \bar{\beta}$ relative to A. We let $\lambda' : B \longrightarrow X_\infty(W)$ be the composite of λ and the canonical map $X_n(W) \longrightarrow X_\infty(W)$. Then the pair (φ, λ') is a solution for the original lifting problem, and hence $\psi_\infty : X_\infty \longrightarrow Y_\infty$ is a global equivalence.

(ix) This is a special case of part (viii) where we set $X_n = Y_0$, $e_n = \mathrm{Id}_{Y_0}$ and $\psi_n = f_{n-1} \circ \cdots \circ f_0 : Y_0 \longrightarrow Y_n$. The morphism ψ_n is then a global equivalence by part (ii), and Y_0 is a colimit of the constant first sequence. Since the morphism ψ_∞ induced on the colimits of the two sequences is the canonical map $Y_0 \longrightarrow Y_\infty$, part (viii) proves the claim.

(x) Let G be a compact Lie group. We consider an exhaustive sequence $\{V_i\}_{i \geq 1}$ of finite-dimensional G-representations. Since α, β and γ are global equivalences, the three vertical maps in the following commutative diagram

of G-spaces are G-weak equivalences, by Proposition 1.1.7:

$$\begin{tikzcd}
\mathrm{tel}_i\, C(V_i) \arrow[d, "\mathrm{tel}\,\gamma(V_i)"'] & \mathrm{tel}_i\, A(V_i) \arrow[l, "\mathrm{tel}\,g(V_i)"'] \arrow[r, "\mathrm{tel}\,f(V_i)"] \arrow[d, "\mathrm{tel}\,\alpha(V_i)"] & \mathrm{tel}_i\, B(V_i) \arrow[d, "\mathrm{tel}\,\beta(V_i)"] \\
\mathrm{tel}_i\, C'(V_i) & \mathrm{tel}_i\, A'(V_i) \arrow[l, "\mathrm{tel}\,g'(V_i)"] \arrow[r, "\mathrm{tel}\,f'(V_i)"'] & \mathrm{tel}_i\, B'(V_i)
\end{tikzcd}$$

Since mapping telescopes commute with product with $[0,1]$ and retracts, the maps $\mathrm{tel}_i\, f(V_i)$ and $\mathrm{tel}_i\, f'(V_i)$ are h-cofibrations of G-spaces. The induced map of the horizontal pushouts is thus a G-weak equivalence by Proposition B.6. Since formation of mapping telescopes commutes with pushouts, the map

$$\mathrm{tel}_i(\gamma \cup \beta)(V_i) \;:\; \mathrm{tel}_i(C \cup_A B)(V_i) \;\longrightarrow\; \mathrm{tel}_i(C' \cup_{A'} B')(V_i)$$

is a G-weak equivalence. The claim thus follows by another application of the telescope criterion for global equivalences, Proposition 1.1.7.

(xi) In the commutative diagram

$$\begin{tikzcd}
C \arrow[d, equal] & A \arrow[l, "g"'] \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "f"] \\
C & A \arrow[l, "g"] \arrow[r, "f"'] & B
\end{tikzcd}$$

all vertical morphisms are global equivalences. If f is an h-cofibration, we apply part (x) to this diagram to get the desired conclusion. If g is an h-cofibration, we apply part (x) after interchanging the roles of left and right horizontal morphisms.

(xii) We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow P(V)$ and $\beta : B \longrightarrow Z(V)$ continuous G-maps such that $g(V) \circ \alpha = \beta|_A$. Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ k(V) \circ \alpha$ and such that $f(W) \circ \lambda$ is G-homotopic, relative to A, to $Y(\varphi) \circ h(V) \circ \beta$. We let $H : B \times [0,1] \longrightarrow Y(W)$ be a relative G-homotopy from $Y(\varphi) \circ h(V) \circ \beta = h(W) \circ Z(\varphi) \circ \beta$ to $f(W) \circ \lambda$. Now we distinguish two cases.

Case 1: The morphism h is a strong level fibration. We can choose a lift $\bar{H}$ in the square

$$\begin{tikzcd}
B \times 0 \cup_{A \times 0} A \times [0,1] \arrow[r, "(Z(\varphi)\circ\beta)\cup K"] \arrow[d, "\sim"'] & Z(W) \arrow[d, "h(W)"] \\
B \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"] & Y(W)
\end{tikzcd}$$

where $K : A\times[0,1] \longrightarrow Z(W)$ is the constant homotopy from $g(W)\circ P(\varphi)\circ\alpha$ to itself. Since the square is a pullback and $h(W)\circ\bar{H}(-,1) = H(-,1) = f(W)\circ\lambda$, there is a unique continuous G-map $\bar{\lambda} : B \longrightarrow P(W)$ that satisfies

$$g(W)\circ\bar{\lambda} \;=\; \bar{H}(-,1) \qquad \text{and} \qquad k(W)\circ\bar{\lambda} \;=\; \lambda\ .$$

The restriction of $\bar{\lambda}$ to A satisfies

$$\begin{aligned} g(W)\circ\bar{\lambda}|_A \;&=\; \bar{H}(-,1)|_A \;=\; g(W)\circ P(\varphi)\circ\alpha \qquad \text{and}\\ k(W)\circ\bar{\lambda}|_A \;&=\; \lambda|_A \;=\; X(\varphi)\circ k(V)\circ\alpha \;=\; k(W)\circ P(\varphi)\circ\alpha\ ; \end{aligned}$$

the pullback property thus implies that $\bar{\lambda}|_A = P(\varphi)\circ\alpha$.

Finally, the composite $g(W)\circ\bar{\lambda}$ is homotopic, relative A and via $\bar{H}$, to $\bar{H}(-,0) = Z(\varphi)\circ\beta$. This is the required lifting data, and we have thus verified the defining property of a global equivalence for the morphism g.

Case 2: The morphism f is a strong level fibration. The argument is similar as in the first case. Now we can choose a lift H' in the square

$$\begin{tikzcd} B\times 1 \cup_{A\times 1} A\times[0,1] \arrow[r, "\lambda\cup K'"] \arrow[d, "\sim"'] & X(W) \arrow[d, "f(W)"] \\ B\times[0,1] \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y(W) \end{tikzcd}$$

where $K' : A\times[0,1] \longrightarrow X(W)$ is the constant homotopy from $X(\varphi)\circ k(V)\circ\alpha$ to itself. Since the square is a pullback and $f(W)\circ H'(-,0) = H(-,0) = h(W)\circ Z(\varphi)\circ\beta$, there is a unique continuous G-map $\bar{\lambda} : B \longrightarrow P(W)$ that satisfies

$$g(W)\circ\bar{\lambda} \;=\; Z(\varphi)\circ\beta \qquad \text{and} \qquad k(W)\circ\bar{\lambda} \;=\; H'(-,0)\ .$$

The restriction of $\bar{\lambda}$ to A satisfies

$$\begin{aligned} g(W)\circ\bar{\lambda}|_A \;&=\; Z(\varphi)\circ g(V)\circ\alpha \;=\; g(W)\circ P(\varphi)\circ\alpha \qquad \text{and}\\ k(W)\circ\bar{\lambda}|_A \;&=\; H'(-,0)|_A \;=\; X(\varphi)\circ k(V)\circ\alpha \;=\; k(W)\circ P(\varphi)\circ\alpha\ . \end{aligned}$$

The pullback property thus implies that $\bar{\lambda}|_A = P(\varphi)\circ\alpha$. Since $g(W)\circ\bar{\lambda} = Z(\varphi)\circ\beta$, this is the required lifting data, and we have verified the global equivalence criterion of Proposition 1.1.7 (ii) for the morphism g. □

The restriction to *finite* products is essential in part (vi) of the previous Proposition 1.1.9; i.e., an infinite product of global equivalences need not be a global equivalence. The following simple example illustrates this. We let Y_n denote the orthogonal space with

$$Y_n(V) \;=\; \begin{cases} \emptyset & \text{if } \dim(V) < n, \text{ and}\\ \{*\} & \text{if } \dim(V) \geq n. \end{cases}$$

The unique morphism $Y_n \longrightarrow Y_0$ is a global equivalence for every $n \geq 0$. However, the product $\prod_{n\geq 0} Y_n$ is the empty orthogonal space, whereas the product $\prod_{n\geq 0} Y_0$ is a terminal orthogonal space. The unique morphism from the initial (i.e., empty) to a terminal orthogonal space is *not* a global equivalence.

The following proposition provides a lot of flexibility for changing an orthogonal space into a globally equivalent one by modifying the input variable. We will use it multiple times in this book.

Theorem 1.1.10. *Let $F : \mathbf{L} \longrightarrow \mathbf{L}$ be a continuous endofunctor of the category of linear isometries and $i : \mathrm{Id} \longrightarrow F$ a natural transformation. Then for every orthogonal space Y the morphism*

$$Y \circ i \; : \; Y \longrightarrow Y \circ F$$

is a global equivalence of orthogonal spaces.

Proof In the first step we show an auxiliary statement. We let V be an inner product space and $z \in F(V)$ an element that is orthogonal to the subspace $i_V(V)$, the image of the linear isometric embedding $i_V : V \longrightarrow F(V)$. We claim that for every linear isometric embedding $\varphi : V \longrightarrow W$ the element $F(\varphi)(z)$ of $F(W)$ is orthogonal to the subspace $i_W(W)$. To prove the claim we write any given element of W as $\varphi(v) + y$ for some $v \in V$ and $y \in W$ orthogonal to $\varphi(V)$. Then

$$\langle F(\varphi)(z), i_W(\varphi(v))\rangle \; = \; \langle F(\varphi)(z), F(\varphi)(i_V(v))\rangle \; = \; \langle z, i_V(v)\rangle \; = \; 0$$

by the hypotheses on z. Now we define $A \in O(W)$ as the linear isometry that is the identity on $\varphi(V)$ and the negative of the identity on the orthogonal complement of $\varphi(V)$. Then $A \circ \varphi = \varphi$ and

$$\begin{aligned} \langle F(\varphi)(z), i_W(y)\rangle \; &= \; \langle F(A)(F(\varphi)(z)), F(A)(i_W(y))\rangle \\ &= \; \langle F(A\varphi)(z), i_W(A(y))\rangle \; = \; -\langle F(\varphi)(z), i_W(y)\rangle \, , \end{aligned}$$

and hence $\langle F(\varphi)(z), i_W(y)\rangle = 0$. Altogether this shows that $\langle F(\varphi)(z), i_W(\varphi(v) + y)\rangle = 0$, which establishes the claim.

Now we consider a compact Lie group G, a G-representation V, a finite G-CW-pair (B, A) and a lifting problem $\alpha : A \longrightarrow Y(V)$ and $\beta : B \longrightarrow Y(F(V))$ for $(Y \circ i)(V)$. Then $\beta|_A = Y(i_V) \circ \alpha$ by hypothesis, and we claim that $Y(i_{F(V)}) \circ \beta$ is G-homotopic to $Y(F(i_V)) \circ \beta = (Y \circ F)(i_V) \circ \beta$, relative A; granting this for the moment, we conclude that the pair $(i_V : V \longrightarrow F(V), \beta)$ solves the lifting problem.

It remains to construct the relative homotopy. The two embeddings

$$F(i_V) \, , \; i_{F(V)} \; : \; F(V) \; \longrightarrow \; F(F(V))$$

are homotopic, relative to $i_V : V \longrightarrow F(V)$, through G-equivariant isometric embeddings, via

$$\begin{aligned} H \ : \ F(V)\times[0,1] \ &\longrightarrow \qquad F(F(V)) \\ (v+w,t) \qquad &\longmapsto F(i_V)(v) \ + \ t\cdot i_{F(V)}(w) \ + \ \sqrt{1-t^2}\cdot F(i_V)(w)\ , \end{aligned}$$

where $v \in i_V(V)$ and w is orthogonal to $i_V(V)$. The verification that $H(-,t) : F(V) \longrightarrow F(F(V))$ is indeed a linear isometric embedding for every $t \in [0,1]$ uses that $i_{F(V)} = F(i_V)$ on the subspace $i_V(V)$ of $F(V)$, and that $i_{F(V)}(w)$ is orthogonal to $F(i_V)(w)$, by the claim proved above. The continuous functor Y takes this homotopy of equivariant linear isometric embeddings to a G-equivariant homotopy $Y(H(-,t))$ from $Y(F(i_V))$ to $Y(i_{F(V)})$, relative to $Y(i_V)$. Composing with β gives the required relative G-homotopy from $Y(F(i_V))\circ\beta$ to $Y(i_{F(V)})\circ\beta$. □

Example 1.1.11 (Additive and multiplicative shift)**.** Here are some typical examples to which the previous theorem applies. Every inner product space W defines an 'additive shift functor' and a 'multiplicative shift functor' on the category of orthogonal spaces, defined by pre-composition with the continuous endofunctors

$$-\oplus W \ : \ \mathbf{L} \longrightarrow \mathbf{L} \qquad \text{and} \qquad -\otimes W \ : \ \mathbf{L} \longrightarrow \mathbf{L}\ .$$

In other words, the additive and multiplicative W-shift of an orthogonal space Y have values

$$(\mathrm{sh}^W_\oplus Y)(V) \ = \ Y(V\oplus W) \qquad \text{and} \qquad (\mathrm{sh}^W_\otimes Y)(V) \ = \ Y(V\otimes W)\ .$$

Here, and in the rest of the book, we endow the tensor product $V\otimes W$ of two inner product spaces V and W with the inner product characterized by

$$\langle v\otimes w, \bar v\otimes \bar w\rangle \ = \ \langle v,\bar v\rangle\cdot\langle w,\bar w\rangle$$

for all $v,\bar v \in V$ and $w,\bar w \in W$. Another way to say this is that for every orthonormal basis $\{b_i\}_{i\in I}$ of V and every orthonormal basis $\{d_j\}_{j\in J}$ of W the family $\{b_i\otimes d_j\}_{(i,j)\in I\times J}$ forms an orthonormal basis of $V\otimes W$. Theorem 1.1.10 then shows that the morphism $Y \longrightarrow \mathrm{sh}^W_\oplus Y$ given by applying Y to the first summand embedding $V \longrightarrow V\oplus W$ is a global equivalence. To get a similar statement for the multiplicative shift we have to assume that $W \neq 0$; then for every vector $w \in W$ of length 1 the map

$$V \ \longrightarrow \ V\otimes W\ , \quad v \ \longmapsto \ v\otimes w$$

is a natural linear isometric embedding. So Theorem 1.1.10 shows that the morphism $Y(-\otimes w) : Y \longrightarrow \mathrm{sh}^W_\otimes Y$ is a global equivalence.

For the following discussion of universes we recall that a finite group has finitely many isomorphism classes of irreducible orthogonal representations, and a compact Lie group of positive dimension has countably infinitely many such isomorphism classes. I have not yet found an explicit reference with a proof of this well-known fact, but one can argue as follows.

We first consider *unitary* representations of G. If G is finite, then the characters of irreducible unitary representations form a basis of the $\mathbb{C}$-vector space of conjugation invariant $\mathbb{C}$-valued functions on G. So the number of isomorphism classes of irreducible unitary representations agrees with the number of conjugacy classes of elements of G, and is thus finite. If G is of positive dimension, then the characters of irreducible unitary representations form an orthonormal basis of the complex Hilbert space of square integrable (with respect to the Haar measure), conjugation invariant functions on G, see for example [91, §11, Thm. 2]. Since G is compact and of positive dimension, this Hilbert space is infinite-dimensional and separable, so there are countably infinitely many isomorphism classes of irreducible unitary representations.

To treat the case of orthogonal representations of G, we recall from [28, II Prop. 6.9] that complexification can be used to construct a map from the set of isomorphism classes of irreducible orthogonal representations to the set of isomorphism classes of irreducible unitary representations of G. There is a caveat, however: the complexification of an irreducible orthogonal representation need not be irreducible. More precisely, the reducibility behavior under complexification depends on the 'type' of the irreducible orthogonal representation λ. By Schur's lemma, the endomorphism algebra $\mathrm{Hom}^G_{\mathbb{R}}(\lambda, \lambda)$ is a finite-dimensional skew-field extension of $\mathbb{R}$, hence isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

- If $\mathrm{Hom}^G_{\mathbb{R}}(\lambda, \lambda)$ is isomorphic to $\mathbb{R}$, then λ is of *real type*. In this case the complexification $\lambda_{\mathbb{C}}$ is irreducible as a unitary G-representation.
- If $\mathrm{Hom}^G_{\mathbb{R}}(\lambda, \lambda)$ is isomorphic to $\mathbb{C}$ or $\mathbb{H}$, then λ is of *complex type* or of *quaternionic type*, respectively. In this case there is an irreducible unitary G-representation ρ such that $\lambda_{\mathbb{C}}$ is isomorphic to the direct sum of ρ and its conjugate $\bar{\rho}$. If λ is of complex type, then ρ is not isomorphic to its conjugate; if λ is of quaternionic type, then ρ is self-conjugate, i.e., isomorphic to its conjugate.

Since the underlying orthogonal representation of $\lambda_{\mathbb{C}}$ is isomorphic to the direct sum of two copies of λ, two non-isomorphic irreducible orthogonal representations cannot become isomorphic after complexification. So the above construction gives an injective map from the set of irreducible orthogonal representations to the set of irreducible unitary representations. Altogether this shows that there at most countably many isomorphism classes of irreducible orthogonal representations of a compact Lie group.

Definition 1.1.12. Let G be a compact Lie group. A *G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following two properties:

- the representation $\mathcal{U}$ has non-zero G-fixed-points,
- if a finite-dimensional G-representation V embeds into $\mathcal{U}$, then a countable infinite direct sum of copies of V also embeds into $\mathcal{U}$.

A G-universe is *complete* if every finite-dimensional G-representation embeds into it.

A G-universe is characterized, up to equivariant linear isometry, by the set of irreducible G-representations that embed into it. We let $\Lambda = \{\lambda\}$ be a complete set of pairwise non-isomorphic irreducible G-representations that embed into $\mathcal{U}$. The first condition says that Λ contains a trivial 1-dimensional representation, and the second condition is equivalent to the requirement that

$$\mathcal{U} \cong \bigoplus_{\lambda\in\Lambda} \bigoplus_{\mathbb{N}} \lambda \, .$$

Moreover, $\mathcal{U}$ is complete if and only if Λ contains (representatives of) *all* irreducible G-representations. Since there are only countably many isomorphism classes of irreducible orthogonal G-representations, a complete G-universe exists.

Remark 1.1.13. We let H be a closed subgroup of a compact Lie group G. We will frequently use the fact that the underlying H-representation of a complete G-universe $\mathcal{U}$ is a complete H-universe. Indeed, if U is an H-representation, then there is a G-representation V and an H-equivariant linear isometric embedding $U \longrightarrow V$, see for example [131, Prop. 1.4.2] or [28, III Thm. 4.5]. Since V embeds G-equivariantly into $\mathcal{U}$, the original representation U embeds H-equivariantly into $\mathcal{U}$.

In the following, for every compact Lie group G we fix a complete G-universe $\mathcal{U}_G$. We let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite-dimensional G-subrepresentations of $\mathcal{U}_G$.

Definition 1.1.14. For an orthogonal space Y and a compact Lie group G we define the *underlying G-space* as

$$Y(\mathcal{U}_G) \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)} Y(V) \, ,$$

the colimit of the G-spaces $Y(V)$.

Remark 1.1.15. The underlying G-space $Y(\mathcal{U}_G)$ can always be written as a

sequential colimit of values of Y. Indeed, we can choose a nested sequence of finite-dimensional G-subrepresentations

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

whose union is all of $\mathcal{U}_G$. This is then in particular an exhaustive sequence in the sense of Definition 1.1.6. Since the subposet $\{V_n\}_{n\geq 0}$ is cofinal in $s(\mathcal{U}_G)$, the colimit of the functor $V \mapsto Y(V)$ over $s(\mathcal{U}_G)$ is also a colimit over the subsequence $Y(V_n)$.

If the group G is finite, then we can define a complete universe as

$$\mathcal{U}_G = \bigoplus_{\mathbb{N}} \rho_G ,$$

a countably infinite sum of copies of the regular representation $\rho_G = \mathbb{R}[G]$, with G as orthonormal basis. Then $\mathcal{U}_G$ is filtered by the finite sums $n \cdot \rho_G$, and we get

$$Y(\mathcal{U}_G) = \operatorname{colim}_n Y(n \cdot \rho_G) ,$$

where the colimit is taken along the inclusions $n \cdot \rho_G \longrightarrow (n+1) \cdot \rho_G$ that miss the final summand.

Definition 1.1.16. An orthogonal space Y is *closed* if it takes every linear isometric embedding $\varphi : V \longrightarrow W$ of inner product spaces to a closed embedding $Y(\varphi) : Y(V) \longrightarrow Y(W)$.

In particular, for every closed orthogonal space Y and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the induced map on G-fixed-points $Y(\varphi)^G : Y(V)^G \longrightarrow Y(W)^G$ is also a closed embedding.

Proposition 1.1.17. *Let $f : X \longrightarrow Y$ be a morphism between closed orthogonal spaces. Then f is a global equivalence if and only if for every compact Lie group G the map*

$$f(\mathcal{U}_G)^G : X(\mathcal{U}_G)^G \longrightarrow Y(\mathcal{U}_G)^G$$

is a weak equivalence.

Proof The poset $s(\mathcal{U}_G)$ has a cofinal subsequence, so all colimits over $s(\mathcal{U}_G)$ can be realized as sequential colimits. The claim is then a straightforward consequence of the fact that fixed-points commute with sequential colimits along closed embeddings (see Proposition B.1 (ii)) and continuous maps from compact spaces such as D^k and ∂D^k to sequential colimits along closed embeddings factor through a finite stage (see Proposition A.15 (i)). □

Now we turn to semifree orthogonal spaces. The basic building blocks of global homotopy theory, the global classifying spaces of compact Lie groups

(Definition 1.1.27), are special cases of this construction. Free and semifree orthogonal spaces are made from spaces of linear isometric embeddings, so we start by recalling various properties of certain spaces of linear isometric embeddings. We consider two compact Lie groups G and K, a finite-dimensional G-representation V, and a K-representation $\mathcal{U}$, possibly of countably infinite dimension. If $\mathcal{U}$ is infinite-dimensional, we topologize the space $\mathbf{L}(V, \mathcal{U})$ of linear isometric embeddings as the filtered colimit of the spaces $\mathbf{L}(V, U)$, taken over the poset of finite-dimensional subspaces U of $\mathcal{U}$. The space $\mathbf{L}(V, \mathcal{U})$ inherits a continuous left K-action and a compatible continuous right G-action from the actions on the target and source, respectively. We turn these two actions into a single left action of the group $K \times G$ by defining

$$((k, g) \cdot \varphi)(v) \;=\; k \cdot \varphi(g^{-1} \cdot v) \tag{1.1.18}$$

for $\varphi \in \mathbf{L}(V, \mathcal{U})$ and $(k, g) \in K \times G$. We recall that a continuous $(K \times G)$-equivariant map is a $(K \times G)$-*cofibration* if it has the left lifting property with respect to all morphisms of $(K \times G)$-spaces $f : X \longrightarrow Y$ such that the map $f^\Gamma : X^\Gamma \longrightarrow Y^\Gamma$ is a weak equivalence and a Serre fibration for every closed subgroup Γ of $K \times G$.

Proposition 1.1.19. *Let G and K be compact Lie groups, V a finite-dimensional G-representation, and $\mathcal{U}$ a K-representation of finite or countably infinite dimension.*

(i) *For every finite-dimensional K-subrepresentation U of $\mathcal{U}$, the inclusion induces a $(K \times G)$-cofibration*

$$\mathbf{L}(V, U) \;\longrightarrow\; \mathbf{L}(V, \mathcal{U})$$

and a K-cofibration of orbit spaces

$$\mathbf{L}(V, U)/G \;\longrightarrow\; \mathbf{L}(V, \mathcal{U})/G\,.$$

(ii) *The $(K \times G)$-space $\mathbf{L}(V, \mathcal{U})$ is $(K \times G)$-cofibrant. The K-space $\mathbf{L}(V, \mathcal{U})/G$ is K-cofibrant.*

Proof (i) We consider two natural numbers $m, n \geq 0$. The space $\mathbf{L}(V, \mathbb{R}^{m+n})$ is homeomorphic to the Stiefel manifold of $\dim(V)$-frames in $\mathbb{R}^{m+n}$, and is hence a compact smooth manifold, and the action of $O(m) \times O(n) \times G$ is smooth. Illman's theorem [84, Cor. 7.2] thus provides an $(O(m) \times O(n) \times G)$-CW-structure on $\mathbf{L}(V, \mathbb{R}^{m+n})$. In particular, $\mathbf{L}(V, \mathbb{R}^{m+n})$ is cofibrant as an $(O(m) \times O(n) \times G)$-space. The group $N = e \times O(n) \times e$ is a closed normal subgroup of $O(m) \times O(n) \times G$, so the inclusion of the N-fixed-points into $\mathbf{L}(V, \mathbb{R}^{m+n})$ is an $(O(m) \times O(n) \times G)$-cofibration (compare Proposition B.12). The map

$$\mathbf{L}(V, \mathbb{R}^m) \;\longrightarrow\; \mathbf{L}(V, \mathbb{R}^{m+n})\,, \tag{1.1.20}$$

induced by the embedding $\mathbb{R}^m \longrightarrow \mathbb{R}^{m+n}$ as the first m coordinates, is a homeomorphism from $\mathbf{L}(V, \mathbb{R}^m)$ onto the N-fixed-points of $\mathbf{L}(V, \mathbb{R}^{m+n})$; so the map (1.1.20) is an $(O(m) \times O(n) \times G)$-cofibration.

Now we can prove the proposition when $\mathcal{U}$ is finite-dimensional. We can assume that $\mathcal{U}$ is $\mathbb{R}^{m+n}$ with the standard scalar product, and that U is the subspace in which the last n coordinates vanish. The K-action on $\mathcal{U}$ is given by a continuous homomorphism $\psi : K \longrightarrow O(m{+}n)$. Since U is a K-subrepresentation, the image of ψ must be contained in the subgroup $O(m) \times O(n)$. The $(K \times G)$-action on the map (1.1.20) is then obtained by restriction of the $(O(m) \times O(n) \times G)$-action along the homomorphism

$$\psi \times \mathrm{Id} \ : \ K \times G \ \longrightarrow \ O(m) \times O(n) \times G \ .$$

Restriction along any continuous homomorphism between compact Lie groups preserves cofibrations by Proposition B.14 (i), so the map (1.1.20) is a $(K \times G)$-cofibration by the first part.

Now we treat the case when the dimension of $\mathcal{U}$ is infinite. We choose an exhausting nested sequence of K-subrepresentations

$$U = U_0 \ \subset \ U_1 \ \subset \ U_2 \ \subset \ \ldots \ .$$

Then all the morphisms $\mathbf{L}(V, U_{n-1}) \longrightarrow \mathbf{L}(V, U_n)$ are $(K \times G)$-cofibrations by the above. Since cofibrations are closed under sequential composites, the morphism

$$\mathbf{L}(V, U_0) \ \longrightarrow \ \mathrm{colim}_n \ \mathbf{L}(V, U_n) \ = \ \mathbf{L}(V, \mathcal{U})$$

is also a $(K \times G)$-cofibration.

Applying Proposition B.14 (iii) to the normal subgroup $e \times G$ of $K \times G$ shows that the functor

$$(e \times G)\backslash - \ : \ (K \times G)\mathbf{T} \ \longrightarrow \ K\mathbf{T}$$

takes $(K \times G)$-cofibrations to K-cofibrations. This proves the second claim.

(ii) This is the special case $U = \{0\}$. The space $\mathbf{L}(V, \{0\})$ is either empty or consists of a single point; in either case $\mathbf{L}(V, \{0\})$ is $(K \times G)$-cofibrant. Part (i) then implies that $\mathbf{L}(V, \mathcal{U})$ is $(K \times G)$-cofibrant and $\mathbf{L}(V, \mathcal{U})/G$ is K-cofibrant. $\square$

The following fundamental contractibility property goes back, at least, to Boardman and Vogt [16]. The equivalent version that we need can be found in [100, Lemma II 1.5].

Proposition 1.1.21. *Let G be a compact Lie group, V a G-representation and $\mathcal{U}$ a G-universe such that V embeds into $\mathcal{U}$. Then the space $\mathbf{L}(V, \mathcal{U})$, equipped with the conjugation action by G, is G-equivariantly contractible.*

Proof We start by showing that the space $\mathbf{L}^G(V,\mathcal{U})$ of G-equivariant linear isometric embeddings is weakly contractible. We let $\mathcal{U}$ be a G-representation of finite or countably infinite dimension. Then the map

$$H \;:\; [0,1]\times\mathbf{L}^G(V,\mathcal{U}) \;\longrightarrow\; \mathbf{L}^G(V,\mathcal{U}\oplus V)$$

defined by

$$H(t,\varphi)(v) \;=\; (t\cdot\varphi(v),\; \sqrt{1-t^2}\cdot v)$$

is a homotopy from the constant map with value $i_2 : V \longrightarrow \mathcal{U}\oplus V$ to the map $i_1\circ -$ (post-composition with $i_1 : \mathcal{U}\longrightarrow\mathcal{U}\oplus V$).

Since V embeds into $\mathcal{U}$ and $\mathcal{U}$ is a G-universe, it contains infinitely many orthogonal copies of V. In other words, we can assume that

$$\mathcal{U} \;=\; \mathcal{U}'\oplus V^\infty$$

for some G-representation $\mathcal{U}'$. Then

$$\mathbf{L}^G(V,\mathcal{U}) \;=\; \mathbf{L}^G(V,\mathcal{U}'\oplus V^\infty) \;=\; \operatorname{colim}_{n\geq 0}\mathbf{L}^G(V,\mathcal{U}'\oplus V^n)\;;$$

the colimit is formed along the post-composition maps with the direct sum embedding $\mathcal{U}'\oplus V^n \longrightarrow \mathcal{U}'\oplus V^{n+1}$. Every map in the colimit system is a closed embedding and homotopic to a constant map, by the previous paragraph. So the colimit is weakly contractible.

Applying the previous paragraph to a closed subgroup H of G shows that the fixed-point space $\mathbf{L}^H(V,\mathcal{U})$ is weakly contractible; in other words, $\mathbf{L}(V,\mathcal{U})$ is G-weakly contractible. The space $\mathbf{L}(V,\mathcal{U})$ comes with a $(G\times G)$-action as in (1.1.18), and it is $(G\times G)$-cofibrant by Proposition 1.1.19 (ii). Then $\mathbf{L}(V,\mathcal{U})$ is also cofibrant as a G-space for the diagonal action, by Proposition B.14 (i). Since $\mathbf{L}(V,\mathcal{U})$ is G-cofibrant and weakly G-contractible, it is actually equivariantly contractible. □

Now we turn to semifree orthogonal spaces.

Construction 1.1.22. Given a compact Lie group G and a G-representation V, the functor

$$\mathrm{ev}_{G,V} \;:\; spc \;\longrightarrow\; G\mathbf{T}$$

that sends an orthogonal space Y to the G-space $Y(V)$ has a left adjoint

$$\mathbf{L}_{G,V} \;:\; G\mathbf{T} \;\longrightarrow\; spc\,. \tag{1.1.23}$$

To construct the left adjoint we note that G acts from the right on $\mathbf{L}(V,W)$ by

$$(\varphi\cdot g)(v) \;=\; \varphi(gv)$$

for $\varphi \in \mathbf{L}(V, W)$, $g \in G$ and $v \in V$. Given a G-space A, the value of $\mathbf{L}_{G,V}A$ at an inner product space W is

$$(\mathbf{L}_{G,V}A)(W) \ = \ \mathbf{L}(V, W) \times_G A \ = \ (\mathbf{L}(V, W) \times A) / (\varphi g, a) \sim (\varphi, ga) \ .$$

We refer to $\mathbf{L}_{G,V}A$ as the *semifree orthogonal space* generated by A at (G, V). We also denote by $\mathbf{L}_{G,V}$ the orthogonal space with

$$\mathbf{L}_{G,V}(W) \ = \ \mathbf{L}(V, W)/G \ .$$

So $\mathbf{L}_{G,V}$ is isomorphic to the semifree orthogonal space generated at (G, V) by a one-point G-space.

The 'freeness' property of $\mathbf{L}_{G,V}A$ is a consequence of the enriched Yoneda lemma, see Remark C.2 or [90, Sec. 1.9]; it means explicitly that for every orthogonal space Y and every continuous G-map $f : A \longrightarrow Y(V)$ there is a unique morphism $f^\flat : \mathbf{L}_{G,V}A \longrightarrow Y$ of orthogonal spaces such that the composite

$$A \xrightarrow{[\mathrm{Id},-]} \mathbf{L}(V, V) \times_G A = (\mathbf{L}_{G,V}A)(V) \xrightarrow{f^\flat(V)} Y(V)$$

is f. Indeed, the map $f^\flat(W)$ is the composite

$$\mathbf{L}(V, W) \times_G A \xrightarrow{\mathrm{Id} \times_G f} \mathbf{L}(V, W) \times_G Y(V) \xrightarrow{[\varphi, y] \mapsto Y(\varphi)(y)} Y(W) \ .$$

Example 1.1.24. For every compact Lie group G, every G-representation V and every G-space A the semifree orthogonal space $\mathbf{L}_{G,V}A$ is closed. To see this we let $\varphi : U \longrightarrow W$ be a linear isometric embedding; since $\mathbf{L}(V, U)$ is compact, the continuous injection

$$\mathbf{L}(V, \varphi) \ : \ \mathbf{L}(V, U) \ \longrightarrow \ \mathbf{L}(V, W)$$

is a closed embedding. So the map $\mathbf{L}(V, \varphi) \times A$ is a closed embedding as well. The orbit map

$$\mathbf{L}(V, \varphi) \times_G A \ : \ \mathbf{L}(V, U) \times_G A \ \longrightarrow \ \mathbf{L}(V, W) \times_G A$$

is then a closed embedding by Proposition B.13 (iii).

The next proposition identifies the fixed-point spaces of a semifree orthogonal space $\mathbf{L}_{G,V}$. A certain family $\mathcal{F}(K; G)$ of subgroups (which we call 'graph subgroups') of $K \times G$ arises naturally.

Definition 1.1.25. Let K and G be compact Lie groups. The family $\mathcal{F}(K; G)$ of *graph subgroups* consists of those closed subgroups Γ of $K \times G$ that intersect $1 \times G$ only in the neutral element $(1, 1)$.

The name ‘graph subgroup’ stems from the fact that $\mathcal{F}(K;G)$ consists precisely of the graphs of all ‘subhomomorphisms’, i.e., continuous homomorphisms $\alpha : L \longrightarrow G$ from a closed subgroup L of K. Clearly, the graph $\Gamma(\alpha) = \{(l, \alpha(l)) \mid l \in L\}$ of every such homomorphism belongs to $\mathcal{F}(K;G)$. Conversely, for $\Gamma \in \mathcal{F}(K;G)$ we let $L \leq K$ be the image of Γ under the projection $K \times G \longrightarrow K$. Since $\Gamma \cap (1 \times G) = \{(1,1)\}$, every element $l \in L$ then has a unique preimage $(l, \alpha(l))$ under the projection, and the assignment $l \mapsto \alpha(l)$ is a continuous homomorphism from L to G whose graph is Γ.

We recall that a *universal G-space* for a family $\mathcal{F}$ of closed subgroups is a cofibrant G-space E such that

- all isotropy groups of E belong to the family $\mathcal{F}$, and
- for every $H \in \mathcal{F}$ the fixed-point space E^H is weakly contractible.

If V and W are G-representations, then restriction of a linear isometry from $V \oplus W$ to V defines a G-equivariant morphism of orthogonal spaces

$$\rho_{V,W} \; : \; \mathbf{L}_{V \oplus W} \; \longrightarrow \; \mathbf{L}_V \; .$$

If U is a K-representation, then we combine the left K-action and the right G-action on $\mathbf{L}(V, U)$ into a left action of $K \times G$ as in (1.1.18).

Proposition 1.1.26. *Let G and K be compact Lie groups and V a faithful G-representation.*

(i) *The $(K \times G)$-space $\mathbf{L}_V(\mathcal{U}_K) = \mathbf{L}(V, \mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups.*

(ii) *If W is another G-representation, then the restriction map*

$$\rho_{V,W}(\mathcal{U}_K) \; : \; \mathbf{L}(V \oplus W, \mathcal{U}_K) \; \longrightarrow \; \mathbf{L}(V, \mathcal{U}_K)$$

is a $(K \times G)$-homotopy equivalence. For every G-space A, the map

$$(\rho_{V,W} \times_G A)(\mathcal{U}_K) \; : \; (\mathbf{L}_{G,V \oplus W} A)(\mathcal{U}_K) \; \longrightarrow \; (\mathbf{L}_{G,V} A)(\mathcal{U}_K)$$

is a K-homotopy equivalence and the morphism of orthogonal spaces

$$\rho_{V,W} \times_G A \; : \; \mathbf{L}_{G,V \oplus W} A \; \longrightarrow \; \mathbf{L}_{G,V} A$$

is a global equivalence.

Proof (i) We let Γ be any closed subgroup of $K \times G$. Since the G-action on V is faithful, the induced right G-action on $\mathbf{L}(V, \mathcal{U}_K)$ is free. So if Γ intersects $1 \times G$ non-trivially, then $\mathbf{L}(V, \mathcal{U}_K)^\Gamma$ is empty. On the other hand, if $\Gamma \cap (1 \times G) = \{(1,1)\}$, then Γ is the graph of a unique continuous homomorphism $\alpha : L \longrightarrow G$, where L is the projection of Γ to K. Then

$$\mathbf{L}(V, \mathcal{U}_K)^\Gamma \; = \; \mathbf{L}^L(\alpha^* V, \mathcal{U}_K)$$

is the space of L-equivariant linear isometric embeddings from the L-representation α^*V to the underlying L-universe of $\mathcal{U}_K$. Since $\mathcal{U}_K$ is a complete K-universe, the underlying L-universe is also complete (Remark 1.1.13), thus the space $\mathbf{L}^L(\alpha^*V, \mathcal{U}_K)$ is contractible by Proposition 1.1.21. The space $\mathbf{L}(V, \mathcal{U}_K)$ is cofibrant as a $(K \times G)$-space by Proposition 1.1.19 (ii).

(ii) Since G acts faithfully on V, and hence also on $V \oplus W$, the $(K \times G)$-spaces $\mathbf{L}(V \oplus W, \mathcal{U}_K)$ and $\mathbf{L}(V, \mathcal{U}_K)$ are universal spaces for the same family $\mathcal{F}(K; G)$, by part (i). So the map $\rho_{V,W}(\mathcal{U}_K) : \mathbf{L}(V \oplus W, \mathcal{U}_K) \longrightarrow \mathbf{L}(V, \mathcal{U}_K)$ is a $(K \times G)$-equivariant homotopy equivalence, see Proposition B.11 (ii). The functor $- \times_G A$ preserves homotopies, so the restriction map $(\rho_{V,W} \times_G A)(\mathcal{U}_K)$ is a K-homotopy equivalence.

The orthogonal spaces $\mathbf{L}_{G,V\oplus W}A$ and $\mathbf{L}_{G,V}A$ are closed by Example 1.1.24, so Proposition 1.1.17 applies and shows that $\rho_{V,W} \times_G A$ is a global equivalence. □

Definition 1.1.27. The *global classifying space* $B_{\mathrm{gl}}G$ of a compact Lie group G is the semifree orthogonal space

$$B_{\mathrm{gl}}G \;=\; \mathbf{L}_{G,V} \;=\; \mathbf{L}(V, -)/G\ ,$$

where V is any faithful G-representation.

The global classifying space $B_{\mathrm{gl}}G$ is well-defined up to preferred zigzag of global equivalences of orthogonal spaces. Indeed, if V and $\bar{V}$ are two faithful G-representations, then $V \oplus \bar{V}$ is yet another one, and the two restriction morphisms

$$\mathbf{L}_{G,V} \;\longleftarrow\; \mathbf{L}_{G,V\oplus\bar{V}} \;\longrightarrow\; \mathbf{L}_{G,\bar{V}}$$

are global equivalences by Proposition 1.1.26 (ii).

Example 1.1.28. We make the global classifying space more explicit for the smallest non-trivial example: the cyclic group C_2 of order 2. The sign representation σ of C_2 is faithful, so we can take $B_{\mathrm{gl}}C_2$ to be the semifree orthogonal space generated by (C_2, σ); its value at an inner product space W is

$$(B_{\mathrm{gl}}C_2)(W) \;=\; \mathbf{L}_{C_2,\sigma}(W) \;=\; \mathbf{L}(\sigma, W)/C_2\ .$$

Evaluation at any of the two unit vectors in σ is a homeomorphism from the space $\mathbf{L}(\sigma, W)$ to $S(W)$, the unit sphere of W. Moreover, the C_2-action on the left becomes the antipodal action on $S(W)$. So the map descends to a homeomorphism between $\mathbf{L}(\sigma, W)/C_2$ and $P(W)$, the projective space of W, and hence

$$(B_{\mathrm{gl}}C_2)(W) \;\cong\; P(W)\ .$$

So for a compact Lie group K, the underlying K-space of $B_{\mathrm{gl}}C_2$ is $P(\mathcal{U}_K)$, the

projective space of a complete K-universe. In particular, the underlying non-equivariant space is homeomorphic to $\mathbb{R}P^\infty$.

Remark 1.1.29 ($B_{\mathrm{gl}}G$ globally classifies principal G-bundles)**.** The term 'global classifying space' is justified by the fact that $B_{\mathrm{gl}}G$ 'globally classifies principal G-bundles'. We recall that a (K, G)*-bundle*, also called a K*-equivariant* G*-principal bundle*, is a principal G-bundle in the category of K-spaces, i.e., a G-principal bundle $p : E \longrightarrow B$ that is also a morphism of K-spaces and such that the actions of G and K on the total space E commute (see for example [179, Ch. I (8.7)]). For every compact Lie group K, the quotient map

$$q \; : \; \mathbf{L}(V, \mathcal{U}_K) \; \longrightarrow \; \mathbf{L}(V, \mathcal{U}_K)/G \; = \; \mathbf{L}_{G,V}(\mathcal{U}_K) \; = \; (B_{\mathrm{gl}}G)(\mathcal{U}_K)$$

is a principal (K, G)-bundle. Indeed, the total space $\mathbf{L}(V, \mathcal{U}_K)$ is homeomorphic to the Stiefel manifold of $\dim(V)$-frames in $\mathbb{R}^\infty$, and hence it admits a CW-structure. Every CW-complex is a normal Hausdorff space (see for example [71, Prop. A.3] or [57, Prop. 1.2.1]), hence is completely regular. So $\mathbf{L}(V, \mathcal{U}_K)$ is completely regular. Since the G-action on $\mathbf{L}(V, \mathcal{U}_K)$ is free, the quotient map q is a G-principal bundle by [131, Prop. 1.7.35] or [26, II Thm. 5.8]. Moreover, this bundle is universal in the sense of the following proposition. Every G-space that admits a G-CW-structure is paracompact, see [125, Thm. 3.2] (this reference is rather sketchy, but one can follow the non-equivariant argument spelled out in more detail in [57, Thm. 1.3.5]). So the next proposition applies in particular to all G-CW-complexes.

Proposition 1.1.30. *Let V be a faithful representation of a compact Lie group G. Then for every paracompact K-space A the map*

$$[A, \mathbf{L}(V, \mathcal{U}_K)/G]^K \;\longrightarrow\; \mathrm{Prin}_{(K,G)}(A) \, , \quad [f] \;\longmapsto\; [f^*(q)]$$

from the set of equivariant homotopy classes of K-maps to the set of isomorphism classes of (K, G)-bundles is bijective.

Proof I do not know a reference for the result in precisely this form, so I sketch how to deduce it from various results in the literature about equivariant fiber bundles. A principal (K, G)-bundle $p : E \longrightarrow B$ is *equivariantly trivializable* if there is a closed subgroup L of K, a continuous homomorphism $\alpha : L \longrightarrow G$, an L-space X and an isomorphism of (K, G)-bundles between p and the projection

$$(K \times G) \times_L X \;\longrightarrow\; K \times_L X \, ;$$

here the source is the quotient space of $K \times G \times X$ by the equivalence relation $(k, g, lx) \sim (kl, g\alpha(l), x)$ for all $(k, g, l, x) \in K \times G \times L \times X$. A principal (K, G)-bundle $p : E \longrightarrow B$ is *numerable* if B has a trivializing (in the above

sense) open cover by K-invariant subsets such that the cover moreover admits a subordinate partition of unity by K-invariant functions.

A *universal* (K,G)*-principal bundle* is a numerable (K,G)-bundle $p^u : E^u \longrightarrow B^u$ such that for every K-space A the map

$$[A, B^u]^K \longrightarrow \mathrm{Prin}^{\mathrm{num}}_{(K,G)}(A)\ , \quad [f] \longmapsto [f^*(p^u)]$$

is bijective, where now the target is the set of isomorphism classes of *numerable* principal (K,G)-bundles. Every principal (K,G)-bundle is numerable over a paracompact base, by [94, Cor. 1.5]. So we are done if we can show that $q : \mathbf{L}(V, \mathcal{U}_K) \longrightarrow \mathbf{L}(V, \mathcal{U}_K)/G$ is a universal (K,G)-principal bundle in the above sense.

Universal (K,G)-principal bundles can be built in different ways; the most common construction is a version of Milnor's infinite join [120], see for example [174, 3.1 Satz] or [179, I Theorem (8.12)]. Another method is via bar construction, compare [183]. I do not know of a reference that explicitly identifies the projection $q : \mathbf{L}(V, \mathcal{U}_K) \longrightarrow \mathbf{L}(V, \mathcal{U}_K)/G$ as a universal bundle in the present sense, so we appeal to Lashof's criterion [94, Thm. 2.14]. The base space $\mathbf{L}(V, \mathcal{U}_K)/G$ is the union, along h-cofibrations, of the compact spaces $\mathbf{L}(V, W_i)/G$, where $\{W_i\}_{i\geq 1}$ is any exhausting sequence of subrepresentations of $\mathcal{U}_K$. Since compact spaces are paracompact, the union $\mathbf{L}(V, \mathcal{U}_K)/G$ is paracompact, see for example [57, Prop. A.5.1 (v)]. Since $\mathbf{L}(V, \mathcal{U}_K)/G$ is also normal, hence completely regular, the bundle q is numerable by Corollaries 1.5 and 1.13 of [94]. Moreover, the fixed-points of $\mathbf{L}(V, \mathcal{U}_K)$ under any graph subgroup of $(K \times G)$ are contractible by Proposition 1.1.26 (i), so Theorem 2.14 of [94] applies and shows that $q : \mathbf{L}(V, \mathcal{U}_K) \longrightarrow \mathbf{L}(V, \mathcal{U}_K)/G$ is strongly universal, and hence a universal principal (K,G)-bundle. □

As another example we look at the case $G = O(n)$, the nth orthogonal group. The category of principal $O(n)$-bundles is equivalent to the category of euclidean vector bundles of rank n, via the associated frame bundle. By the same construction, principal $(K, O(n))$-bundles can be identified with K-equivariant euclidean vector bundles of rank n over K-spaces. The space $\mathbf{L}(\mathbb{R}^n, \mathcal{U}_K)/O(n)$ is homeomorphic to $Gr_n(\mathcal{U}_K)$, the Grassmannian of n-planes in $\mathcal{U}_K$. In the case when K is a trivial group, the fact that $Gr_n(\mathbb{R}^\infty)$ is a classifying space for rank n vector bundles over paracompact spaces is proved in various textbooks. Since $O(1)$ is a cyclic group of order 2, this gives another perspective on Example 1.1.28.

1.2 Global model structure for orthogonal spaces

In this section we establish the global model structure on the category of orthogonal spaces, see Theorem 1.2.21. Towards this aim we first discuss a 'strong level model structure' which we then localize. In Proposition 1.2.27 we use the global model structure to relate unstable global homotopy theory to the homotopy theory of K-spaces for a fixed compact Lie group K. We also show that global classifying spaces of compact Lie groups with abelian identity component are 'cofree', i.e., right induced from non-equivariant classifying spaces, see Theorem 1.2.32. At the end of this section we briefly discuss the realization of simplicial orthogonal spaces in Construction 1.2.34; we show that the realization takes level-wise global equivalences to global equivalences, under a certain 'Reedy flatness' condition (Proposition 1.2.37).

There is a functorial way to write an orthogonal space as a sequential colimit of orthogonal spaces which are made from the information below a fixed dimension. We refer to this as the *skeleton filtration* of an orthogonal space. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal space need not be injective.

The skeleton filtration is in fact a special case of a more general skeleton filtration on certain enriched functor categories that we discuss in Appendix C. Indeed, if we specialize the base category to $\mathcal{V} = \mathbf{T}$, the category of spaces under cartesian product, and the index category to $\mathcal{D} = \mathbf{L}$, then the functor category $\mathcal{D}^*$ becomes the category spc of orthogonal spaces. The dimension function needed in the construction and analysis of skeleta is the vector space dimension.

We denote by $\mathbf{L}^{\leq m}$ the full topological subcategory of the linear isometries category $\mathbf{L}$ whose objects are the inner product spaces of dimension at most m. We denote by $spc^{\leq m}$ the category of continuous functors from $\mathbf{L}^{\leq m}$ to $\mathbf{T}$. The restriction functor

$$spc \longrightarrow spc^{\leq m}, \quad Y \longmapsto Y^{\leq m} = Y|_{\mathbf{L}^{\leq m}}$$

has a left adjoint

$$l_m : spc^{\leq m} \longrightarrow spc$$

given by an enriched Kan extension as follows. The extension $l_m(Z)$ of a continuous functor $Z : \mathbf{L}^{\leq m} \longrightarrow \mathbf{T}$ is a coequalizer of the two morphisms of orthogonal spaces

$$\coprod_{0\leq j\leq k\leq m} \mathbf{L}(\mathbb{R}^k, -) \times \mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \times Z(\mathbb{R}^j) \rightrightarrows \coprod_{0\leq i\leq m} \mathbf{L}(\mathbb{R}^i, -) \times Z(\mathbb{R}^i) \tag{1.2.1}$$

One morphism arises from the composition morphisms

$$\mathbf{L}(\mathbb{R}^k, -) \times \mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \longrightarrow \mathbf{L}(\mathbb{R}^j, -)$$

and the identity on $Z(\mathbb{R}^j)$; the other morphism arises from the action maps

$$\mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \times Z(\mathbb{R}^j) \longrightarrow Z(\mathbb{R}^k)$$

and the identity on the free orthogonal space $\mathbf{L}(\mathbb{R}^k, -)$. Colimits in the category of orthogonal spaces are created objectwise, so the value $l_m(Z)(V)$ at an inner product space can be calculated by plugging V into the variable slot in the coequalizer diagram (1.2.1).

It is a general property of Kan extensions along a fully faithful functor (such as the inclusion $\mathbf{L}^{\leq m} \longrightarrow \mathbf{L}$) that the values do not change on the given subcategory, see for example [90, Prop. 4.23]. More precisely, the adjunction unit

$$Z \longrightarrow (l_m(Z))^{\leq m}$$

is an isomorphism for every continuous functor $Z : \mathbf{L}^{\leq m} \longrightarrow \mathbf{T}$.

Definition 1.2.2. The *m-skeleton*, for $m \geq 0$, of an orthogonal space Y is the orthogonal space

$$\mathrm{sk}^m Y = l_m(Y^{\leq m}),$$

the extension of the restriction of Y to $\mathbf{L}^{\leq m}$. It comes with a natural morphism $i_m : \mathrm{sk}^m Y \longrightarrow Y$, the counit of the adjunction $(l_m, (-)^{\leq m})$. The *mth latching space* of Y is the $O(m)$-space

$$L_m Y = (\mathrm{sk}^{m-1} Y)(\mathbb{R}^m);$$

it comes with a natural $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) : L_m Y \longrightarrow Y(\mathbb{R}^m),$$

the *mth latching map*.

We agree to set $\mathrm{sk}^{-1} Y = \emptyset$, the empty orthogonal space, and $L_0 Y = \emptyset$, the empty space. The value

$$i_m(V) : (\mathrm{sk}^m Y)(V) \longrightarrow Y(V)$$

of the morphism i_m is an isomorphism for all inner product spaces V of dimension at most m.

The two morphisms $i_{m-1} : \mathrm{sk}^{m-1} Y \longrightarrow Y$ and $i_m : \mathrm{sk}^m Y \longrightarrow Y$ both restrict to isomorphisms on $\mathbf{L}^{\leq m-1}$, so there is a unique morphism $j_m : \mathrm{sk}^{m-1} Y \longrightarrow \mathrm{sk}^m Y$ such that $i_m \circ j_m = i_{m-1}$. The sequence of skeleta stabilizes to Y in a very strong sense. For every inner product space V, the maps $j_m(V)$ and $i_m(V)$ are homeomorphisms as soon as $m > \dim(V)$. In particular, $Y(V)$ is a colimit of the

sequence of maps $j_m(V)$ with respect to the morphisms $i_m(V)$. Since colimits in the category of orthogonal spaces are created objectwise, we deduce that the orthogonal space Y is a colimit of the sequence of morphisms j_m with respect to the morphisms i_m.

We denote the left adjoint to the functor $Y \mapsto Y(\mathbb{R}^m)$ by

$$G_m \; : \; O(m)\mathbf{T} \; \longrightarrow \; spc \; .$$

So G_m is a shorthand notation for $\mathbf{L}_{O(m),\mathbb{R}^m}$, the semifree functor (1.1.23) indexed by the tautological $O(m)$-representation. Proposition C.17 specializes to:

Proposition 1.2.3. *For every orthogonal space Y and every $m \geq 0$ the commutative square*

$$\begin{tikzcd} G_m L_m Y \arrow[r, "G_m \nu_m"] \arrow[d] & G_m Y(\mathbb{R}^m) \arrow[d] \\ \mathrm{sk}^{m-1} Y \arrow[r, "j_m"'] & \mathrm{sk}^m Y \end{tikzcd} \tag{1.2.4}$$

is a pushout of orthogonal spaces. The left and right vertical morphisms are adjoint to the identity of $L_m Y$ and of $Y(\mathbb{R}^m)$, respectively.

Example 1.2.5. As an illustration of the definition, we describe the skeleta and latching objects for small values of m. We have

$$\mathrm{sk}^0 Y \; = \; \mathrm{const}(Y(0)) \; ,$$

the constant orthogonal space with value $Y(0)$; the latching map

$$\nu_1 \; : \; L_1 Y \; = \; (\mathrm{sk}^0 Y)(\mathbb{R}) \; = \; Y(0) \; \xrightarrow{Y(u)} \; Y(\mathbb{R})$$

is the map induced by the unique linear isometric embedding $u : 0 \longrightarrow \mathbb{R}$. Now we evaluate the pushout square (1.2.4) for $m = 1$ at an inner product space V; the result is a pushout square of $O(1)$-spaces

$$\begin{tikzcd} P(V) \times Y(0) \arrow[r] \arrow[d, "\mathrm{proj}"'] & \mathbf{L}(\mathbb{R}, V) \times_{O(1)} Y(\mathbb{R}) \arrow[d] \\ Y(0) \arrow[r] & (\mathrm{sk}^1 Y)(V) \end{tikzcd}$$

where $P(V)$ is the projective space of V. Here we exploit the fact that $O(1)$ acts trivially on $L_1 Y = Y(0)$ and we can thus identify

$$(G_1 L_1 Y)(V) \; = \; \mathbf{L}(\mathbb{R}, V) \times_{O(1)} Y(0) \; \cong \; P(V) \times Y(0) \; , \quad [\varphi, y] \; \longmapsto \; (\varphi(\mathbb{R}), y) \; .$$

The upper horizontal map sends $(\varphi(\mathbb{R}), y)$ to $[\varphi, Y(u)(y)]$.

Example 1.2.6 (Latching objects of free orthogonal spaces)**.** We let V be an n-dimensional representation of a compact Lie group G, and A a G-space. Then the semifree orthogonal space (1.1.23) generated by A in level V is 'purely n-dimensional' in the following sense. The evaluation functor

$$\mathrm{ev}_{G,V} \; : \; spc \longrightarrow G\mathbf{T}$$

factors through the category $\mathbf{L}^{\leq n}$ as the composite

$$spc \longrightarrow spc^{\leq n} \xrightarrow{\mathrm{ev}_{G,V}} G\mathbf{T} \, .$$

So the left adjoint semifree functor $\mathbf{L}_{G,V}$ can be chosen to be the composite of the two individual left adjoints

$$\mathbf{L}_{G,V} \; = \; l_n \circ l_{G,V} \, .$$

Here $l_{G,V} : G\mathbf{T} \longrightarrow spc^{\leq n}$ is given at a G-space A and an inner product space W of dimension at most n by

$$(l_{G,V}A)(W) \; = \; \begin{cases} \mathbf{L}(V, W) \times_G A & \text{if } \dim(W) = n, \\ \quad \emptyset & \text{if } \dim(W) < n. \end{cases}$$

The space $(\mathbf{L}_{G,V}A)_m$ is trivial for $m < n$, hence the latching space $L_m(\mathbf{L}_{G,V}A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(\mathbf{L}_{G,V}A) \longrightarrow (\mathbf{L}_{G,V}A)_m$ is an isomorphism. So for $m < n$ the skeleton $\mathrm{sk}^m(\mathbf{L}_{G,V}A)$ is trivial, and for $m \geq n$ the skeleton $\mathrm{sk}^m(\mathbf{L}_{G,V}A) = \mathbf{L}_{G,V}A$ is the entire orthogonal space.

Now we work our way towards the strong level model structure of orthogonal spaces. Proposition C.23 is a fairly general recipe for constructing level model structures on a category such as orthogonal spaces. We specialize the general construction to the situation at hand. We recall from Definition 1.1.8 that a morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *strong level equivalence* (or *strong level fibration*) if for every compact Lie group G and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence (or Serre fibration).

Lemma 1.2.7. *For every morphism $f : X \longrightarrow Y$ of orthogonal spaces, the following are equivalent.*

(i) *The morphism f is a strong level equivalence.*
(ii) *For every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence.*
(iii) *The map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-weak equivalence for every $m \geq 0$.*

Proof Clearly, condition (i) implies condition (ii), and that implies condition (iii) (because the tautological action of $O(m)$ on $\mathbb{R}^m$ is faithful). So we suppose that $f(\mathbb{R}^m)$ is an $O(m)$-weak equivalence for every $m \geq 0$, and we show that f is a strong level equivalence. Given a G-representation V of dimension m, we choose a linear isometry $\varphi : V \cong \mathbb{R}^m$; conjugation by φ turns the G-action on V into a homomorphism $\rho : G \longrightarrow O(m)$, i.e.,

$$\rho(g) \;=\; \varphi \circ (g \cdot -) \circ \varphi^{-1} \;.$$

The homeomorphism $X(\varphi) : X(V) \longrightarrow X(\mathbb{R}^m)$ then restricts to a homeomorphism

$$X(V)^G \;\cong\; X(\mathbb{R}^m)^{\rho(G)} \;.$$

This homeomorphism is natural for morphisms of orthogonal spaces, so the hypothesis that $f(\mathbb{R}^m)^{\rho(G)} : X(\mathbb{R}^m)^{\rho(G)} \longrightarrow Y(\mathbb{R}^m)^{\rho(G)}$ is a weak equivalence implies that also the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence. □

The same kind of reasoning as in Lemma 1.2.7 shows:

Lemma 1.2.8. *The following are equivalent for every morphism $f : X \longrightarrow Y$ of orthogonal spaces.*

(i) *The morphism f is a strong level fibration.*
(ii) *For every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is a fibration in the projective model structure of G-spaces.*
(iii) *The map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-fibration for every $m \geq 0$.*

Definition 1.2.9. A morphism of orthogonal spaces $i : A \longrightarrow B$ is a *flat cofibration* if the latching morphism

$$\nu_m i \;=\; i(\mathbb{R}^m) \cup \nu_m^B \;:\; A(\mathbb{R}^m) \cup_{L_m A} L_m B \;\longrightarrow\; B(\mathbb{R}^m)$$

is an $O(m)$-cofibration for all $m \geq 0$. An orthogonal space B is *flat* if the unique morphism from the empty orthogonal space to B is a flat cofibration. Equivalently, for every $m \geq 0$ the latching map $\nu_m : L_m B \longrightarrow B(\mathbb{R}^m)$ is an $O(m)$-cofibration.

We are ready to establish the strong level model structure.

Proposition 1.2.10. *The strong level equivalences, strong level fibrations and flat cofibrations form a topological cofibrantly generated model structure, the* strong level model structure*, on the category of orthogonal spaces.*

Proof We apply Proposition C.23 as follows. We let $C(m)$ be the projective model structure on the category of $O(m)$-spaces (with respect to the set of

all closed subgroups of $O(m)$), compare Proposition B.7. The classes of level equivalences, level fibrations and cofibrations in the sense of Proposition C.23 then become precisely the strong level equivalences, strong level fibrations and flat cofibrations.

In this situation the consistency condition (see Definition C.22) is a consequence of a stronger property, namely that the functor

$$\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times_{O(m)} - \;:\; O(m)\mathbf{T} \;\longrightarrow\; O(m+n)\mathbf{T}$$

takes acyclic cofibrations to acyclic cofibrations (in the two relevant projective model structures). Since the functor is a left adjoint, it suffices to prove the claim for the generating acyclic cofibrations, i.e., the maps

$$O(m)/H \times j_k$$

for all $k \geq 0$ and all closed subgroups H of $O(m)$, where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0, 1]$ is the inclusion. The functor under consideration takes this generator to the map $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/H \times j_k$, which is an acyclic $O(m+n)$-cofibration because $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/H$ is cofibrant as an $O(m+n)$-space, by Proposition 1.1.19 (iii).

We describe explicit sets of generating cofibrations and generating acyclic cofibrations. We let I^{str} be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (B.8). Then the set I^{str} detects the acyclic fibrations in the strong level model structure, by Proposition C.23 (iii). Similarly, we let J^{str} be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (B.9). Again by Proposition C.23 (iii), J^{str} detects the fibrations in the strong level model structure.

The model structure is topological by Proposition B.5, where we take $\mathcal{G}$ as the set of orthogonal spaces $L_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all closed subgroups H of $O(m)$, and we take $\mathcal{Z} = \emptyset$. □

For easier reference we make the generating (acyclic) cofibrations of the strong level model structure even more explicit. Using the isomorphism

$$G_m(O(m)/H) \;=\; \mathbf{L}(\mathbb{R}^m, -) \times_{O(m)} (O(m)/H) \;\cong\; \mathbf{L}(\mathbb{R}^m, -)/H \;=\mathbf{L}_{H,\mathbb{R}^m}\;,$$

we can identify I^{str} with the set of all morphisms

$$\mathbf{L}_{H,\mathbb{R}^m} \times i_k \;:\; \mathbf{L}_{H,\mathbb{R}^m} \times \partial D^k \;\longrightarrow\; \mathbf{L}_{H,\mathbb{R}^m} \times D^k$$

for all $k, m \geq 0$ and all closed subgroups H of $O(m)$. The tautological action of H on $\mathbb{R}^m$ is faithful; conversely every pair (G, V) consisting of a compact Lie group and a faithful representation is isomorphic to a pair $(H, \mathbb{R}^m)$ for some

closed subgroup H of $\mathbb{R}^m$. We conclude that I^{str} is a set of representatives of the isomorphism classes of morphisms

$$\mathbf{L}_{G,V} \times i_k \ : \ \mathbf{L}_{G,V} \times \partial D^k \ \longrightarrow \ \mathbf{L}_{G,V} \times D^k$$

for G a compact Lie group, V a faithful G-representation and $k \geq 0$. Similarly, J^{str} is a set of representatives of the isomorphism classes of morphisms

$$\mathbf{L}_{G,V} \times j_k \ : \ \mathbf{L}_{G,V} \times D^k \times \{0\} \ \longrightarrow \ \mathbf{L}_{G,V} \times D^k \times [0,1]$$

for G a compact Lie group, V a faithful G-representation and $k \geq 0$.

Proposition 1.2.11. *Let K be a compact Lie group and $\varphi : W \longrightarrow U$ a linear isometric embedding of K-representations, where W is finite-dimensional, and U is finite-dimensional or countably infinite-dimensional.*

(i) *For every flat cofibration of orthogonal spaces $i : A \longrightarrow B$ the maps*

$$\begin{aligned} i(U) \ &: \ A(U) \ \longrightarrow \ B(U) \quad \textit{and} \\ i(U) \cup B(\varphi) \ &: \ A(U) \cup_{A(W)} B(W) \ \longrightarrow \ B(U) \end{aligned}$$

are K-cofibrations of K-spaces.

(ii) *For every flat orthogonal space B the map $B(\varphi) : B(W) \longrightarrow B(U)$ is a K-cofibration of K-spaces and the K-space $B(U)$ is K-cofibrant.*

(iii) *Every flat orthogonal space is closed.*

Proof (i) The class of those morphisms of orthogonal spaces i such that the map $i(U) \cup B(\varphi)$ is a K-cofibration of K-spaces is closed under coproducts, cobase change, composition and retracts. Similarly, the class of those morphisms of orthogonal spaces i such that the map $i(U)$ is a K-cofibration of K-spaces is closed under coproducts, cobase change, composition and retracts. So it suffices to show each of the two claims for a set of generating cofibrations. We do this for the morphisms $\mathbf{L}_{G,V} \times i_k$ for all $k \geq 0$, all compact Lie groups G and all G-representations V, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. In this case the first map specializes to $\mathbf{L}(V,U)/G \times i_k$. The map i_k is a cofibration and $\mathbf{L}(V,U)/G$ is cofibrant as a K-space by Proposition 1.1.19 (ii). So $\mathbf{L}(V,U)/G \times i_k$ is a K-cofibration of K-spaces.

The second map in question becomes the pushout product of the sphere inclusion i_k with the map

$$\mathbf{L}(V,\varphi)/G \ : \ \mathbf{L}(V,W)/G \ \longrightarrow \ \mathbf{L}(V,U)/G \ .$$

The map i_k is a cofibration and $\mathbf{L}(V,\varphi)/G$ is a K-cofibration by Proposition 1.1.19 (i). So their pushout product is again a K-cofibration.

Part (ii) is the special case of part (i) where $A = \emptyset$ is the empty orthogonal space. Part (iii) is the special case of (ii) where K is a trivial group, using

that cofibrations of spaces are in particular h-cofibrations (Corollary A.30) and hence closed embeddings (Proposition A.31). □

Now we proceed towards the *global model structure* on the category of orthogonal spaces, see Theorem 1.2.21. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. The fibrations in the global model structure are defined as follows.

Definition 1.2.12. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every faithful G-representation V and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the map

$$(f(V)^G, X(\varphi)^G) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(W)^G} X(W)^G$$

is a weak equivalence.

An orthogonal space X is *static* if for every compact Lie group G, every faithful G-representation V, and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the structure map

$$X(\varphi) \ : \ X(V) \ \longrightarrow \ X(W)$$

is a G-weak equivalence.

Equivalently, a morphism f is a global fibration if and only if f is a strong level fibration and for every compact Lie group G, every faithful G-representation V and equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the square of G-fixed-point spaces

$$\begin{tikzcd} X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\ Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd} \tag{1.2.13}$$

is homotopy cartesian.

Clearly, an orthogonal space X is static if and only if the unique morphism to a terminal orthogonal space is a global fibration; the static orthogonal spaces will thus turn out to be the fibrant objects in the global model structure. The static orthogonal spaces are those that, roughly speaking, don't change the equivariant homotopy type once a faithful representation has been reached.

Proposition 1.2.14. (i) *Every global equivalence that is also a global fibration is a strong level equivalence.*

(ii) *Every global equivalence between static orthogonal spaces is a strong level equivalence.*

Proof (i) We let $f : X \longrightarrow Y$ be a morphism of orthogonal spaces that is both a global fibration and a global equivalence. We consider a compact Lie group G, a faithful G-representation V, a finite G-CW-pair (B, A) and a commutative square:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V) \arrow[d, "f(V)"] \\ B \arrow[r, "\beta"'] & Y(V) \end{tikzcd}$$

We will exhibit a continuous G-map $\mu : B \longrightarrow X(V)$ such that $\mu|_A = \alpha$ and such that $f(V) \circ \mu$ is homotopic, relative A, to β. This shows that the map $f(V)$ is a G-weak equivalence, so f is a strong level equivalence.

Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha : A \longrightarrow X(W)$ and such that $f(W) \circ \lambda : B \longrightarrow Y(W)$ is G-homotopic, relative to A, to $Y(\varphi) \circ \beta$. Since f is a strong level fibration, we can improve λ into a continuous G-map $\lambda' : B \longrightarrow X(W)$ such that $\lambda'|_A = \lambda|_A = X(\varphi) \circ \alpha$ and such that $f(W) \circ \lambda'$ is equal to $Y(\varphi) \circ \beta$.

Since f is a global fibration the G-map

$$(f(V), X(\varphi)) \ : \ X(V) \ \longrightarrow \ Y(V) \times_{Y(W)} X(W)$$

is a G-weak equivalence. So we can find a continuous G-map $\mu : B \longrightarrow X(V)$ such that $\mu|_A = \alpha$ and $(f(V), X(\varphi)) \circ \mu$ is G-homotopic, relative A, to $(\beta, \lambda') : B \longrightarrow Y(V) \times_{Y(W)} X(W)$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V) \arrow[d, "{(f(V), X(\varphi))}"] \\ B \arrow[r, "{(\beta, \lambda')}"'] \arrow[ur, dashed, "\mu"] & Y(V) \times_{Y(W)} X(W) \end{tikzcd}$$

This is the desired map.

(ii) We let $f : X \longrightarrow Y$ be a global equivalence between static orthogonal spaces. We let G be a compact Lie group, V a faithful G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ continuous G-maps such that $f(V) \circ \alpha = \beta|_A$. Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha$ and $f(W) \circ \lambda$ is G-homotopic to $Y(\varphi) \circ \beta$ relative A. Since X is static, the map $X(\varphi) : X(V) \longrightarrow X(W)$ is a G-weak equivalence, so there is a continuous G-map $\bar{\lambda} : B \longrightarrow X(V)$ such that $\bar{\lambda}|_A = \alpha$ and $X(\varphi) \circ \bar{\lambda}$ is G-homotopic to λ relative A. The two G-maps $f(V) \circ \bar{\lambda}$ and $\beta : B \longrightarrow Y(V)$ then agree on A and become G-homotopic, relative A, after

composition with $Y(\varphi) : Y(V) \longrightarrow Y(W)$. Since Y is static, the map $Y(\varphi)$ is a G-weak equivalence, so $f(V) \circ \bar{\lambda}$ and $\beta : B \longrightarrow Y(V)$ are already G-homotopic relative A. This shows that $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence, and hence f is a strong level equivalence. □

Construction 1.2.15. We let $j : A \longrightarrow B$ be a morphism in a topological model category. We factor j through the mapping cylinder as the composite

$$A \xrightarrow{c(j)} Z(j) = (A \times [0,1]) \cup_j B \xrightarrow{r(j)} B ,$$

where $c(j)$ is the 'front' mapping cylinder inclusion and $r(j)$ is the projection, which is a homotopy equivalence. In our applications we will assume that both A and B are cofibrant; then the morphism $c(j)$ is a cofibration by the pushout product property. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$c(j)\square i_k \ : \ A \times D^k \cup_{A \times \partial D^k} Z(j) \times \partial D^k \ \longrightarrow \ Z(j) \times D^k$$

for $k \geq 0$, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion.

Proposition 1.2.16. *Let C be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ a fibration. Then the following two conditions are equivalent:*

(i) *The square of spaces*

$$\begin{tikzcd} \mathrm{map}(B,X) \arrow[r, "\mathrm{map}(j,X)"] \arrow[d, "\mathrm{map}(B,f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(B,Y) \arrow[r, "\mathrm{map}(j,Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{1.2.17}$$

is homotopy cartesian.

(ii) *The morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$.*

Proof The square (1.2.17) maps to the square

$$\begin{tikzcd} \mathrm{map}(Z(j),X) \arrow[r, "\mathrm{map}(c(j),X)"] \arrow[d, "\mathrm{map}(Z(j),f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(Z(j),Y) \arrow[r, "\mathrm{map}(c(j),Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{1.2.18}$$

via the map induced by $r(j) : Z(j) \longrightarrow B$ on the left part and the identity on the right part. Since $r(j)$ is a homotopy equivalence, the map of squares is a weak equivalence at all four corners. So the square (1.2.17) is homotopy cartesian if and only if the square (1.2.18) is homotopy cartesian.

Since A is cofibrant and f a fibration, $\mathrm{map}(A, f)$ is a Serre fibration. So the square (1.2.18) is homotopy cartesian if and only if the map

$$(\mathrm{map}(Z(j), f), \mathrm{map}(c(j), X)) \ : \qquad (1.2.19)$$
$$\mathrm{map}(Z(j), X) \longrightarrow \mathrm{map}(Z(j), Y) \times_{\mathrm{map}(A,Y)} \mathrm{map}(A, X)$$

is a weak equivalence. Since $c(j)$ is a cofibration and f is a fibration, the map (1.2.19) is always a Serre fibration. So (1.2.19) is a weak equivalence if and only if it is an acyclic fibration, which is equivalent to the right lifting property for the inclusions $i_k : \partial D^k \longrightarrow D^k$ for all $k \geq 0$. By adjointness, the map (1.2.19) has the right lifting property with respect to the maps i_k if and only if the morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$. □

The set J^{str} was defined in the proof of Proposition 1.2.10 as the set of morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (B.9). The set J^{str} detects the fibrations in the strong level model structure. We add another set of morphisms K that detects when the squares (1.2.13) are homotopy cartesian. Given any compact Lie group G and G-representations V and W, the restriction morphism

$$\rho_{G,V,W} = \rho_{V,W}/G \ : \ \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$$

restricts (the G-orbit of) a linear isometric embedding from $V \oplus W$ to V. If the representation V is faithful, then this morphism is a global equivalence by Proposition 1.1.26 (ii). We set

$$K = \bigcup_{G,V,W} \mathcal{Z}(\rho_{G,V,W}) ,$$

the set of all pushout products of boundary inclusions $\partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G, a faithful G-representation V and an arbitrary G-representation W. The morphism $\rho_{G,V,W}$ represents the map of G-fixed-point spaces $X(i_{V,W})^G : X(V)^G \longrightarrow X(V \oplus W)^G$; every G-equivariant linear isometric embedding is isomorphic to a direct summand inclusion $i_{V,W}$; the right lifting property with respect to the union $J^{\mathrm{str}} \cup K$ characterizes the global fibrations, by Proposition 1.2.16. We have shown:

Proposition 1.2.20. *A morphism of orthogonal spaces is a global fibration if and only if it has the right lifting property with respect to the set* $J^{\mathrm{str}} \cup K$.

Now we are ready for the main result of this section.

Theorem 1.2.21 (Global model structure)**.** *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spaces. The fibrant objects in the global model structure are the static orthogonal spaces. The global model structure is proper, topological and cofibrantly generated.*

Proof We number the model category axioms as in [48, 3.3]. The category of orthogonal spaces is complete and cocomplete, so axiom MC1 holds. Global equivalences satisfy the 2-out-of-6 property by Proposition 1.1.9 (iii), so they also satisfy the 2-out-of-3 property MC2. Global equivalences are closed under retracts by Proposition 1.1.9 (iv); it is straightforward that cofibrations and global fibrations are closed under retracts, so axiom MC3 holds.

The strong level model structure shows that every morphism of orthogonal spaces can be factored as $f \circ i$ for a flat cofibration i followed by a strong level equivalence f that is also a strong level fibration. For every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ between faithful G-representations, both vertical maps in the commutative square of fixed-point spaces (1.2.13) are then weak equivalences, so the square is homotopy cartesian. The morphism f is thus a global fibration and a global equivalence, which provides one of the factorizations as required by MC5. For the other half of the factorization axiom MC5 we apply the small object argument (see for example [48, 7.12] or [80, Thm. 2.1.14]) to the set $J^{\mathrm{str}} \cup K$. All morphisms in J^{str} are flat cofibrations and strong level equivalences. Since $\mathbf{L}_{G,V\oplus W}$ and $\mathbf{L}_{G,V}$ are flat, the morphisms in K are also flat cofibrations, and they are global equivalences because the morphisms $\rho_{G,V,W}$ are (Proposition 1.1.26 (ii)). The small object argument provides a functorial factorization of every morphism $X \longrightarrow Y$ of orthogonal spaces as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in $J^{\mathrm{str}} \cup K$, and q has the right lifting property with respect to $J^{\mathrm{str}} \cup K$. Since all morphisms in $J^{\mathrm{str}} \cup K$ are flat cofibrations and global equivalences, the morphism i is a flat cofibration and a global equivalence by the closure properties of Proposition 1.1.9. Moreover, q is a global fibration by Proposition 1.2.20.

Now we show the lifting properties MC4. By Proposition 1.2.14 (i) a morphism that is both a global equivalence and a global fibration is a strong level equivalence, and hence an acyclic fibration in the strong level model structure. So every morphism that is simultaneously a global equivalence and a global fibration has the right lifting property with respect to flat cofibrations. Now we let $j : A \longrightarrow B$ be a flat cofibration that is also a global equivalence and

we show that it has the left lifting property with respect to all global fibrations. By the small object argument we factor $j = q \circ i$, where $i : A \longrightarrow W$ is a $(J^{\mathrm{str}} \cup K)$-cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and is hence an acyclic fibration in the strong level model structure, again by Proposition 1.2.14 (i). Since j is a flat cofibration, a lifting exists in:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

Thus j is a retract of the morphism i that has the left lifting property with respect to global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. In doing so we have also specified sets of generating flat cofibrations I^{str} and generating acyclic cofibrations $J^{\mathrm{str}} \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations, so the global model structure is cofibrantly generated.

Left properness of the global model structure follows from Proposition 1.1.9 (xi) and the fact that flat cofibrations are h-cofibrations (Corollary A.30 (iii)). Right properness follows from Proposition 1.1.9 (xii) because global fibrations are in particular strong level fibrations.

The global model structure is topological by Proposition B.5, with $\mathcal{G}$ the set of semifree orthogonal spaces $\mathbf{L}_{G,V}$ indexed by a set of representatives (G, V) of the isomorphism classes of pairs consisting of a compact Lie group G and a faithful G-representation V, and with $\mathcal{Z}$ the set of mapping cylinder inclusions $c(\rho_{G,V,W})$ of the morphisms $\rho_{G,V,W}$. □

The global model structure of orthogonal spaces is also monoidal, in fact with respect to two different monoidal structures. Indeed, the categorical product of orthogonal spaces has the pushout product property for flat cofibrations, by Proposition 1.3.9 below. Moreover, Proposition 1.4.12 (iii) (for the global family of all compact Lie groups) shows that global model structure of orthogonal spaces satisfies the pushout product property with respect to the box product of orthogonal spaces.

We also introduce a 'positive' version of the global model structure for orthogonal spaces; our main use of this variation is for the global model structure of ultra-commutative monoids in Section 2.1. As is well known from similar contexts (for example, the stable model structure for commutative orthogonal ring spectra), model structures cannot usually be lifted naively to multiplicative objects with strictly commutative products. The solution is to lift a 'positive'

version of the global model structure in which the values at the trivial inner product space are homotopically meaningless and where the fibrant objects are the 'positively static' orthogonal spaces.

Definition 1.2.22. A morphism of orthogonal spaces $f : A \longrightarrow B$ is a *positive cofibration* if it is a flat cofibration and the map $f(0) : A(0) \longrightarrow B(0)$ is a homeomorphism. An orthogonal space Y is *positively static* if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$, the structure map

$$Y(\varphi) \ : \ Y(V) \ \longrightarrow \ Y(W)$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positively static orthogonal space is static (in the absolute sense) if the structure map $Y(0) \longrightarrow Y(\mathbb{R})$ is a non-equivariant weak equivalence.

Proposition 1.2.23 (Positive global model structure)**.** *The global equivalences and positive cofibrations are part of a cofibrantly generated, proper, topological model structure, the* positive global model structure *on the category of orthogonal spaces. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a fibration in the positive global model structure if and only if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$, the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration and the square of G-fixed-point spaces*

$$\begin{tikzcd}
X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\
Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G
\end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positively static orthogonal spaces.

Proof We start by establishing a *positive strong level model structure*. We call a morphism $f : X \longrightarrow Y$ of orthogonal spaces a *positive strong level equivalence* (or *positive strong level fibration*) if for every inner product space V with $V \neq 0$ the map $f(V) : X(V) \longrightarrow Y(V)$ is an $O(V)$-weak equivalence (or $O(V)$-fibration). Then we claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a topological model structure on the category of orthogonal spaces.

The proof is another application of the general construction method for level

model structures in Proposition C.23. Indeed, we let $C(0)$ be the degenerate model structure on the category $\mathbf{T}$ of spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m \geq 1$ we let $C(m)$ be the projective model structure (for the set of all closed subgroups) on the category of $O(m)$-spaces, compare Proposition B.7. With respect to these choices of model structures, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition C.23 become the positive strong level equivalences, positive strong level fibrations and positive cofibrations, respectively. The consistency condition (Definition C.22) holds, as it is now strictly weaker than for the strong level model structure.

We obtain the positive global model structure for orthogonal spaces by 'mixing' the positive strong level model structure with the global model structure of Theorem 1.2.21. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [38, Thm. 2.1], which is our first claim. By [38, Cor. 3.7] (or rather its dual formulation), an orthogonal space is fibrant in the positive global model structure if and only if it is weakly equivalent to a static orthogonal space in the positive strong level model structure; this is equivalent to being positively static.

Cofibrant generation, properness and topologicalness of the positive global model structure are proved in much the same way as for the absolute global model structure in Theorem 1.2.21. □

Remark 1.2.24. We can relate the unstable global homotopy theory of orthogonal spaces to the homotopy theory of G-spaces for a fixed compact Lie group G. Evaluation at a faithful G-representation V and the semifree functor at (G, V) are a pair of adjoint functors

$$\mathbf{L}_{G,V} \ : \ G\mathbf{T} \rightleftarrows spc \ : \ \mathrm{ev}_{G,V}$$

between the categories of G-spaces and orthogonal spaces. This adjoint pair is a Quillen pair with respect to the global model structure of orthogonal spaces and the 'genuine' model structure of G-spaces (i.e., the projective model structure with respect to the family of all subgroups, compare Proposition B.7). The adjoint total derived functors

$$L(\mathbf{L}_{G,V}) \ : \ \mathrm{Ho}(G\mathbf{T}) \rightleftarrows \mathrm{Ho}(spc) \ : \ R(\mathrm{ev}_{G,V})$$

are independent of the faithful representation V up to preferred natural isomorphism, by Proposition 1.1.26 (ii).

Every G-space is G-weakly equivalent to a G-CW-complex, and these are built from the orbits G/H. So the derived left adjoint $L(\mathbf{L}_{G,V}) : G\mathbf{T} \longrightarrow spc$ is

essentially determined by its values on the coset spaces G/H. Since $\mathbf{L}_{G,V}(G/H)$ is isomorphic to $\mathbf{L}_{H,V} = B_{\mathrm{gl}}H$, the derived left adjoint takes the homogeneous space G/H to a global classifying space of H.

The derived right adjoint also has a more explicit description, at least for closed orthogonal spaces Y, as the underlying G-space $Y(\mathcal{U}_G)$. Indeed, we can choose a fibrant replacement of Y in the global model structure, i.e., a flat cofibration $j : Y \longrightarrow Z$ that is also a global equivalence, and such that Z is globally fibrant (i.e., static). Then Z is also closed, and so the induced map $j(\mathcal{U}_G) : Y(\mathcal{U}_G) \longrightarrow Z(\mathcal{U}_G)$ is a G-weak equivalence by Proposition 1.1.17. We may assume that V is a subrepresentation of $\mathcal{U}_G$; we choose a nested sequence

$$V = V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$$

of finite-dimensional G-subrepresentations that exhaust $\mathcal{U}_G$. Since V is faithful and Z is closed and static, the induced maps

$$Z(V) = Z(V_1) \longrightarrow Z(V_2) \longrightarrow \ldots \longrightarrow Z(V_n) \longrightarrow \ldots$$

are all closed embeddings and G-weak equivalences. So the canonical map

$$Z(V) \longrightarrow \operatorname{colim}_{n\geq 1} Z(V_n) = Z(\mathcal{U}_G)$$

is also a G-weak equivalence. Since Z is a globally fibrant replacement of Y, the G-space $Z(V)$ calculates the right derived functor of $\mathrm{ev}_{G,V}$ at Y. This exhibits a chain of two G-weak equivalences

$$R(\mathrm{ev}_{G,V})(Y) = Z(V) \xrightarrow{\simeq} Z(\mathcal{U}_G) \xleftarrow{\simeq} Y(\mathcal{U}_G) .$$

Construction 1.2.25 (Cofree orthogonal spaces)**.** We will now define for every compact Lie group K a right adjoint to the functor that takes an orthogonal space Y to the underlying K-space $Y(\mathcal{U}_K)$. We refer to the right adjoint R_K as the *cofree functor*. We consider the continuous functor

$$\mathbf{L}(-,\mathcal{U}_K) : \mathbf{L} \longrightarrow (K\mathbf{T})^{\mathrm{op}} , \quad V \longmapsto \mathbf{L}(V,\mathcal{U}_K) ,$$

with functoriality given by pre-composition with linear isometric embeddings. The group K acts on the values of this functor through the action on the complete universe $\mathcal{U}_K$. The cofree orthogonal space $R_K(A)$ associated to a K-space A is then the composite

$$\mathbf{L} \xrightarrow{\mathbf{L}(-,\mathcal{U}_K)} K\mathbf{T}^{\mathrm{op}} \xrightarrow{\mathrm{map}^K(-,A)} \mathbf{T} .$$

The unit of the adjunction is the morphism

$$\eta_Y : Y \longrightarrow R_K(Y(\mathcal{U}_K)) \qquad (1.2.26)$$

whose value at an inner product space V is the adjoint of the action map

$$\mathbf{L}(V,\mathcal{U}_K)\times Y(V) \longrightarrow Y(\mathcal{U}_K)\ , \quad (\varphi,y) \longmapsto Y(\varphi)(y)\ .$$

The counit of the adjunction is the continuous K-map

$$\epsilon_A \ : \ R_K(A)(\mathcal{U}_K) \longrightarrow A$$

assembled from the compatible K-maps

$$R_K(A)(V) \ = \ \mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K),A) \longrightarrow A\ , \quad f \longmapsto f(i_V)\ ,$$

for $V \in s(\mathcal{U}_K)$, where $i_V : V \longrightarrow \mathcal{U}_K$ is the inclusion. This data makes the functors

$$(-)(\mathcal{U}_K) \ : \ spc \rightleftarrows K\mathbf{T} \ : \ R_K$$

an adjoint pair.

Proposition 1.2.27. *Let K be a compact Lie group.*

(i) *The adjoint functor pair $((-)(\mathcal{U}_K), R_K)$ is a Quillen pair for the global model structure of orthogonal spaces and the projective model structure of K-spaces.*
(ii) *For every K-space A the orthogonal space $R_K(A)$ is static.*
(iii) *For every closed orthogonal space Y the map*

$$(\eta_Y)^* \circ R_K \ : \ \mathrm{Ho}(K\mathbf{T})(Y(\mathcal{U}_K),A) \longrightarrow \mathrm{Ho}(spc)(Y,R_K(A))$$

is bijective.

Proof (i) We let $f : X \longrightarrow Y$ be a fibration of K-spaces. We let G be another compact Lie group and V a faithful G-representation. Then the K-space $\mathbf{L}(V,\mathcal{U}_K)/G$ is K-cofibrant by Proposition 1.1.19 (ii). The projective model structure on K-spaces is topological, so $\mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K)/G,-)$ takes fibrations of K-spaces to fibrations of spaces. Because

$$\mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K)/G,X) \ = \ (R_K(X)(V))^G\ ,$$

this means that R_K takes fibrations of K-spaces to strong level fibrations of orthogonal spaces. By the same argument, R_K takes acyclic fibrations of K-spaces to acyclic fibrations in the strong level model structure, which coincide with the acyclic fibrations in the global model structure of orthogonal spaces.

Now we let $\varphi : V \longrightarrow W$ be a G-equivariant linear isometric embedding. Then the map

$$\rho_{V,W}(\mathcal{U}_K)/G \ : \ \mathbf{L}(V\oplus W,\mathcal{U}_K)/G \longrightarrow \mathbf{L}(V,\mathcal{U}_K)/G$$

is a K-homotopy equivalence by Proposition 1.1.26 (ii). So the induced map

$$(R_K(X)(\varphi))^G \ : \ (R_K(X)(V))^G \ \longrightarrow \ (R_K(X)(V \oplus W))^G$$

is a homotopy equivalence. So in the commutative square

$$\begin{tikzcd}
(R_K(X)(V))^G \arrow[r, "R_K(X)(\varphi)^G"] \arrow[d, "(R_K(f)(V))^G"'] & (R_K(X)(V \oplus W))^G \arrow[d, "(R_K(f)(V\oplus W))^G"] \\
(R_K(Y)(V))^G \arrow[r, "R_K(Y)(\varphi)^G"'] & (R_K(Y)(V \oplus W))^G
\end{tikzcd}$$

both vertical maps are Serre fibrations and both horizontal maps are weak equivalences. The square is then homotopy cartesian, and so the morphism $R_K(f) : R_K(X) \longrightarrow R_K(Y)$ is a global fibration of orthogonal spaces. Altogether this shows that the right adjoint R_K preserves fibrations and acyclic fibrations, so $((-)(\mathcal{U}_K), R_K)$ is a Quillen pair.

(ii) Every K-space A is fibrant in the projective model structure. So $R_K(A)$ is fibrant in the global model structure of orthogonal spaces; by Theorem 1.2.21 these fibrant objects are precisely the static orthogonal spaces.

(iii) We choose a global equivalence $f : Y^c \longrightarrow Y$ with flat source. Then Y^c and Y are both closed, the former by Proposition 1.2.11 (iii). So the map $f(\mathcal{U}_K) : Y^c(\mathcal{U}_K) \longrightarrow Y(\mathcal{U}_K)$ is a K-weak equivalence by Proposition 1.1.17. So the morphism f induces bijections on both sides of the map in question, and it suffices to prove the claim for Y^c instead of Y. But Y^c is cofibrant and A is fibrant, so in this case the claim is just the derived adjunction isomorphism. □

For $K = e$, the trivial group, we drop the subscript and abbreviate the cofree functor R_e to R.

Definition 1.2.28. An orthogonal space Y is *cofree* if it is globally equivalent to an orthogonal space of the form RA for some space A.

We will now develop criteria for detecting cofree orthogonal spaces, and then recall some non-tautological examples. One criterion involves the unit of the adjunction, the special case

$$\eta_Y \ : \ Y \ \longrightarrow \ R(Y(\mathbb{R}^\infty))$$

of (1.2.26) for $\mathcal{U}_K = \mathbb{R}^\infty$. The next proposition shows that the morphism η_Y is always a non-equivariant weak equivalence, provided Y is closed.

Proposition 1.2.29. *For every closed orthogonal space Y the morphism η_Y : $Y \longrightarrow R(Y(\mathbb{R}^\infty))$ induces a weak equivalence*

$$\eta_Y(\mathbb{R}^\infty) \ : Y(\mathbb{R}^\infty) \ \longrightarrow \ R(Y(\mathbb{R}^\infty))(\mathbb{R}^\infty)$$

of underlying non-equivariant spaces.

Proof We start with a general observation about cofree orthogonal spaces. Since RA is static (Proposition 1.2.27 (ii)) and closed, the map $(RA)(\varphi) : (RA)(V) \longrightarrow (RA)(W)$ induced by any linear isometric embedding $\varphi : V \longrightarrow W$ is a weak equivalence and a closed embedding. So the canonical map

$$\begin{aligned} A \;\cong\; \mathrm{map}(\mathbf{L}(0,\mathbb{R}^\infty), A) \;&= (RA)(0) \\ &\longrightarrow \mathrm{colim}_{W\in s(\mathbb{R}^\infty)} (RA)(W) \;=\; (RA)(\mathbb{R}^\infty) \end{aligned} \tag{1.2.30}$$

is a weak equivalence as well. The adjunction counit $\epsilon_A : (RA)(\mathbb{R}^\infty) \longrightarrow A$ is a retraction to the map (1.2.30), so ϵ_A is also a weak equivalence.

Now we turn to the proof of the proposition. Even though the map $\eta_Y(\mathbb{R}^\infty)$ under consideration is *not* the same as the canonical map (1.2.30) for $A = Y(\mathbb{R}^\infty)$, the counit $\epsilon_{Y(\mathbb{R}^\infty)} : R(Y(\mathbb{R}^\infty))(\mathbb{R}^\infty) \longrightarrow Y(\mathbb{R}^\infty)$ is also a retraction to $\eta_Y(\mathbb{R}^\infty)$. Since $\epsilon_{Y(\mathbb{R}^\infty)}$ is a weak equivalence, so is $\eta_Y(\mathbb{R}^\infty)$. □

While the morphism $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ tends to be a non-equivariant equivalence, it is typically *not* a global equivalence. We will now see that for a closed orthogonal space Y the morphism η_Y is a global equivalence if and only if Y is cofree.

We recall that for a compact Lie group K a *universal free K-space* is a K-cofibrant free K-space whose underlying space in non-equivariantly contractible. Any two universal free K-spaces are K-homotopy equivalent, see Proposition B.11. We call a K-space A *cofree* if the map

$$\mathrm{const} \;:\; A \;\longrightarrow\; \mathrm{map}(EK, A)$$

that sends a point to the corresponding constant map is a K-weak equivalence for some (hence any) universal free K-space EK.

Proposition 1.2.31. *For a closed orthogonal space Y the following three conditions are equivalent.*

(i) *The orthogonal space Y is cofree.*
(ii) *For every compact Lie group K the K-space $Y(\mathcal{U}_K)$ is cofree.*
(iii) *The adjunction unit $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ is a global equivalence.*

Proof In a first step we show that for every space A and every compact Lie group K, the K-space $(RA)(\mathcal{U}_K)$ is cofree. We choose a faithful K-representation W. Then $\mathbf{L}(W,\mathbb{R}^\infty)$ is a universal free K-space by Proposition 1.1.26 (i). So the projection from $EK \times \mathbf{L}(W,\mathbb{R}^\infty)$ to the second factor is a K-weak equivalence between cofibrant K-spaces, hence a K-homotopy equivalence. So the

induced map

$$\begin{aligned} \mathrm{const} \; : \; (RA)(W) \; &= \; \mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), A) \\ &\longrightarrow \; \mathrm{map}(EK \times \mathbf{L}(W,\mathbb{R}^\infty), A) \; \cong \; \mathrm{map}(EK, (RA)(W)) \end{aligned}$$

is a K-homotopy equivalence. Hence the K-space $(RA)(W)$ is cofree as soon as K acts faithfully on W. Since RA is static (by Proposition 1.2.27 (ii)) and closed, the canonical map

$$(RA)(W) \;\longrightarrow\; (RA)(\mathcal{U}_K)$$

is a K-weak equivalence. So $(RA)(\mathcal{U}_K)$ is K-cofree. Now we prove the equivalence of conditions (i), (ii) and (iii).

(i)$\Longrightarrow$(ii) The global equivalences are part of the global model structure on the category of orthogonal spaces, compare Theorem 1.2.21. Moreover, the orthogonal space RA is static, hence fibrant in the global model structure. So if Y is globally equivalent to RA, then for some (hence any) global equivalence $p : Y^c \longrightarrow Y$ with cofibrant (i.e., flat) source, there is a global equivalence $f : Y^c \longrightarrow RA$.

Now we let K be any compact Lie group. The orthogonal space Y^c is closed by Proposition 1.2.11 (iii). Since Y and RA are also closed, the global equivalences induce K-weak equivalences

$$Y(\mathcal{U}_K) \xleftarrow[\sim]{p(\mathcal{U}_K)} Y^c(\mathcal{U}_K) \xrightarrow[\sim]{f(\mathcal{U}_K)} (RA)(\mathcal{U}_K)$$

by Proposition 1.1.17. Since $(RA)(\mathcal{U}_K)$ is K-cofree by the introductory remark, so is $Y(\mathcal{U}_K)$.

(ii)$\Longrightarrow$(iii) We start with a preliminary observation. We let Y and Z be two closed orthogonal spaces such that the K-spaces $Y(\mathcal{U}_K)$ and $Z(\mathcal{U}_K)$ are cofree for all compact Lie groups K. We claim that every morphism $f : Y \longrightarrow Z$ of orthogonal spaces such that $f(\mathbb{R}^\infty) : Y(\mathbb{R}^\infty) \longrightarrow Z(\mathbb{R}^\infty)$ is a non-equivariant weak equivalence is already a global equivalence. Indeed, for every compact Lie group K the two vertical maps in the commutative square of K-spaces

$$\begin{tikzcd} Y(\mathcal{U}_K) \arrow[r, "f(\mathcal{U}_K)"] \arrow[d, "\mathrm{const}"'] & Z(\mathcal{U}_K) \arrow[d, "\mathrm{const}"] \\ \mathrm{map}(EK, Y(\mathcal{U}_K)) \arrow[r, "\simeq", "\mathrm{map}(EK, f(\mathcal{U}_K))"'] & \mathrm{map}(EK, Z(\mathcal{U}_K)) \end{tikzcd}$$

are K-weak equivalences by hypothesis. Since $\mathcal{U}_K$ is non-equivariantly isometrically isomorphic to $\mathbb{R}^\infty$, the K-map $f(\mathcal{U}_K) : Y(\mathcal{U}_K) \longrightarrow Z(\mathcal{U}_K)$ is a non-equivariant weak equivalence by hypothesis. So the lower horizontal map

is a K-weak equivalence. We conclude that the upper horizontal map is a K-weak equivalence. Since Y and Z are closed, the criterion of Proposition 1.1.17 shows that f is a global equivalence.

Now we apply the criterion to the morphism $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$. The map $\eta_Y(\mathbb{R}^\infty)$ is a weak equivalence by Proposition 1.2.29. Moreover, for every compact Lie group K, the space $Y(\mathcal{U}_K)$ is K-cofree by hypothesis (ii), and $R(Y(\mathbb{R}^\infty))(\mathcal{U}_K)$ is K-cofree by the introductory remark. The criterion of the previous paragraph thus applies and shows that the morphism $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ is a global equivalence.

Condition (i) is a special case of (iii). □

We recall now that the global classifying spaces of certain compact Lie groups are cofree, namely of those with abelian identity path component. Said differently, the group must be an extension of a finite group by a torus. The following theorem is a reinterpretation of the main result of Rezk's paper [138], who calls these groups '1-truncated' because the homotopy groups of the underlying spaces vanish in dimensions larger than 1. The special case of *abelian* compact Lie groups was proved earlier by Lashof, May and Segal [95]. The case of finite groups seems to be folklore, going all the way back to Hurewicz [82] who proved that for finite groups G, homotopy classes of continuous maps $BK \longrightarrow BG$ are in bijection with conjugacy classes of continuous group homomorphisms from K to G.

Theorem 1.2.32. *Let G be a compact Lie group whose identity path component is abelian. Then the global classifying space $B_{\mathrm{gl}}G$ is cofree.*

Proof We let V be any faithful G-representation, so that $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$. Then $\mathbf{L}_{G,V}(\mathcal{U}_K) = \mathbf{L}(V, \mathcal{U}_K)/G$ is a classifying space for principal (K,G)-bundles, by Proposition 1.1.26. Since G has abelian identity component, it is 1-truncated in the sense of Rezk, and so the K-space $\mathbf{L}_{G,V}(\mathcal{U}_K)$ is cofree by [138, Thm. 1.4]. So criterion (ii) of Proposition 1.2.31 is satisfied; since the orthogonal space $\mathbf{L}_{G,V}$ is closed, we have thus shown that it is cofree. □

Remark 1.2.33. The global classifying space $B_{\mathrm{gl}}G$ is *not* cofree in general, e.g., when the identity component of G is not abelian. Indeed, for another compact Lie group K, the homotopy set $\pi_0^K(B_{\mathrm{gl}}G)$ (to be introduced in Definition 1.5.5 below) is in bijection with conjugacy classes of continuous homomorphisms from K to G, by Proposition 1.5.12 (ii). On the other hand, the set $\pi_0^K(R(BG))$ is in bijection with homotopy classes of continuous maps from BK to BG. However, there are continuous maps $BK \longrightarrow BG$ that are not homotopic to $B\alpha$ for any continuous homomorphism $\alpha : K \longrightarrow G$. Whenever this happens, the adjunction unit $\eta_{B_{\mathrm{gl}}G} : B_{\mathrm{gl}}G \longrightarrow R(BG)$ is not surjective on π_0^K,

and hence not a global equivalence (by Corollary 1.5.7 below). The first examples of such 'exotic' maps between classifying spaces of compact Lie groups were constructed by Sullivan and appeared in his widely circulated and highly influential MIT lecture notes; an edited version of Sullivan's notes was eventually published in [168]. Indeed, Corollary 5.11 of [168] constructs 'unstable Adams operations' $\psi^p : BU(n) \longrightarrow BU(n)$ for a prime p and all $n < p$; for $n > 1$ these maps are not induced by any continuous homomorphism.

Construction 1.2.34 (Realization of simplicial objects)**.** We will occasionally want to realize simplicial objects, so we quickly recall the necessary background. We let C be a cocomplete category tensored over the category $\mathbf{T}$ of spaces. We let $\mathbf{\Delta}$ denote the simplicial indexing category, with objects the finite totally ordered sets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for $n \geq 0$. Morphisms in $\mathbf{\Delta}$ are all weakly monotone maps. We let

$$\Delta^n \;=\; \{(t_1, \ldots, t_n) \in [0,1]^n \mid t_1 \leq t_2 \leq \cdots \leq t_n\}$$

be the topological n-simplex. As n varies, these topological simplices assemble into a covariant functor

$$\mathbf{\Delta} \longrightarrow \mathbf{T}\,, \quad [n] \longmapsto \Delta^n\,;$$

the coface maps are given by

$$(d_i)_*(t_1, \ldots, t_n) \;=\; \begin{cases} (0, t_1, \ldots, t_n) & \text{for } i = 0, \\ (t_1, \ldots, t_i, t_i, \ldots, t_n) & \text{for } 0 < i < n, \\ (t_1, \ldots, t_n, 1) & \text{for } i = n. \end{cases}$$

For $0 \leq i \leq n-1$, the codegeneracy map $(s_i)_* : \Delta^n \longrightarrow \Delta^{n-1}$ drops the entry t_{i+1}.

A *simplicial object* in C is functor $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow C$, i.e., a contravariant functor from $\mathbf{\Delta}$. We use the customary notation $X_n = X([n])$ for the value of a simplicial object at $[n]$. The *realization* of X is the coend

$$|X| \;=\; \int^{[n]\in\Delta} X_n \times \Delta^n$$

of the functor

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \;\longrightarrow\; C\,, \quad ([m],[n]) \;\longmapsto\; X_m \times \Delta^n\,.$$

We also need to recall the *latching objects* of a simplicial object. We let $\mathbf{\Delta}(n)$ denote category whose objects are the weakly monotone surjections $\sigma : [n] \longrightarrow [k]$; a morphism from $\sigma : [n] \longrightarrow [k]$ to $\sigma' : [n] \longrightarrow [k']$ is a morphism $\alpha : [k] \longrightarrow [k']$ in $\mathbf{\Delta}$ (necessarily surjective as well) with $\alpha \circ \sigma = \sigma'$. We let $\mathbf{\Delta}(n)_\circ$ denote the full subcategory of $\mathbf{\Delta}(n)$ consisting of all objects *except*

the identity of $[n]$. A simplicial object X can be restricted along the forgetful functor

$$\mathbf{\Delta}(n)^{\mathrm{op}}_{\circ} \xrightarrow{u} \mathbf{\Delta}^{\mathrm{op}} , \quad (\sigma : [n] \longrightarrow [k]) \longmapsto [k] .$$

The nth *latching object* of X is the colimit over $\mathbf{\Delta}(n)^{\mathrm{op}}_{\circ}$ of the restricted functor:

$$L^{\mathbf{\Delta}}_n(X) = \mathrm{colim}_{\mathbf{\Delta}(n)^{\mathrm{op}}_{\circ}} (X \circ u) . \tag{1.2.35}$$

The morphisms

$$\sigma^* : X_k = (X \circ u)(\sigma : [n] \longrightarrow [k]) \longrightarrow X_n$$

assemble into a *latching morphism*

$$l_n : L^{\mathbf{\Delta}}_n(X) \longrightarrow X_n$$

from the nth latching object of X to the value at $[n]$. For example, the category $\mathbf{\Delta}(0)_{\circ}$ is empty, so $L^{\mathbf{\Delta}}_0(X)$ is an initial object of $\mathcal{C}$. The category $\mathbf{\Delta}(1)_{\circ}$ has a unique object $s_0 : [1] \longrightarrow [0]$, so $L^{\mathbf{\Delta}}_1(X) = X_0$ and the latching morphism is given by $s_0^* : X_0 \longrightarrow X_1$. The category $\mathbf{\Delta}(2)_{\circ}$ has three objects and two non-identity morphisms, and $L^{\mathbf{\Delta}}_2(X)$ is a pushout of the diagram

$$X_1 \xleftarrow{s_0^*} X_0 \xrightarrow{s_0^*} X_1 .$$

We specialize the above to realizations of simplicial orthogonal spaces, i.e., simplicial objects in the category of orthogonal spaces. For orthogonal spaces, coends and product with Δ^n are objectwise, and hence

$$|X|(V) = |X(V)| ,$$

i.e., the value of $|X|$ at an inner product space V is the realization of the simplicial space $[n] \mapsto X_n(V)$, as discussed in Construction A.32. By Proposition A.35, the realization can be formed in the ambient category of all topological spaces, and the result is automatically compactly generated.

Definition 1.2.36. A simplicial orthogonal space X is *Reedy flat* if the latching morphism $l_n : L^{\mathbf{\Delta}}_n(X) \longrightarrow X_n$ is a flat cofibration of orthogonal spaces for every $n \geq 0$.

The terminology ‘Reedy flat’ stems from Reedy’s theorem [136] that the simplicial objects in any model category admit a certain model structure, nowadays called the ‘Reedy model structure’, in which the equivalences are the level equivalences of simplicial objects. Reedy’s paper – though highly influential – remains unpublished, but an account of the Reedy model structure can for example be found in [63, VII Prop. 2.11]. If we form the Reedy model structure starting with the global model structure of orthogonal spaces, then the cofibrant objects are precisely the Reedy flat simplicial orthogonal spaces.

The most important consequence of Reedy flatness for our purposes is that realization is homotopical for these simplicial orthogonal spaces. Indeed, the global model structure of orthogonal spaces is a topological model structure by Theorem 1.2.21. We can turn this into a simplicial model structure by defining the tensor of an orthogonal space X with a simplicial set A as

$$X \otimes A \;=\; X \times |A| \;,$$

the objectwise product with the geometric realization of A. The following proposition is then a special case of the fact that realization is a left Quillen functor for the Reedy model structure on simplicial orthogonal spaces, see [63, VII Prop. 3.6].

Proposition 1.2.37. (i) *The realization of every Reedy flat simplicial orthogonal space is flat.*

(ii) *Let $f : X \longrightarrow Y$ be a morphism of Reedy flat simplicial orthogonal spaces. If $f_n : X_n \longrightarrow Y_n$ is a global equivalence for every $n \geq 0$, then the morphism of realizations $|f| : |X| \longrightarrow |Y|$ is a global equivalence.*

1.3 Monoidal structures

This section is devoted to monoidal products in the category of orthogonal spaces, with emphasis on global homotopical features. Our main focus is the box product of orthogonal spaces, a special case of a Day type convolution product, and the 'good' monoidal structure for orthogonal spaces. We prove in Theorem 1.3.2 that the box product is fully homotopical with respect to global equivalences. While the box product is the most useful monoidal structure for orthogonal spaces, the cartesian product is also relevant for our purposes. We show in Proposition 1.3.9 that the categorical product, too, satisfies the pushout product property for flat cofibrations. In particular, the product of two flat orthogonal spaces is again flat.

The last part of this section introduces complex analogues $\mathbf{L}^{\mathbb{C}}_{G,W}$ of the semifree orthogonal spaces, indexed by unitary G-representations W, see Construction 1.3.10. While these complex versions are not (semi)free in any categorical sense, they are similar to the semifree orthogonal spaces in many ways; for example, the orthogonal spaces $\mathbf{L}^{\mathbb{C}}_{G,W}$ are flat (Proposition 1.3.11 (ii)) and behave well multiplicatively under box and cartesian product (Proposition 1.3.12).

We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spaces (X, Y) to another orthogonal space Z as a collection of continuous maps

$$b_{V,W} \;:\; X(V) \times Y(W) \;\longrightarrow\; Z(V \oplus W),$$

for all inner product spaces V and W, such that for all linear isometric embeddings $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ the following square commutes:

$$\begin{tikzcd}
X(V) \times Y(W) \arrow[r, "b_{V,W}"] \arrow[d, "X(\varphi)\times Y(\psi)"'] & Z(V \oplus W) \arrow[d, "Z(\varphi\oplus\psi)"] \\
X(V') \times Y(W') \arrow[r, "b_{V',W'}"'] & Z(V' \oplus W')
\end{tikzcd}$$

We define a box product of X and Y as a universal example of an orthogonal space with a bimorphism from X and Y. More precisely, a box product is a pair $(X \boxtimes Y, i)$ consisting of an orthogonal space $X \boxtimes Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \boxtimes Y$, i.e., such that for every orthogonal space Z the map

$$spc(X \boxtimes Y, Z) \longrightarrow \mathrm{Bimor}((X, Y), Z) , \quad f \longmapsto fi = \{f(V \oplus W) \circ i_{V,W}\}_{V,W}$$

is bijective. We will often refer to this bijection as the *universal property* of the box product of orthogonal spaces. Very often only the object $X \boxtimes Y$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism.

The existence of a universal bimorphism out of any pair of orthogonal spaces X and Y, and thus of a box product $X \boxtimes Y$, is a special case of the existence of Day type convolution products on certain functor categories; the construction is an enriched Kan extension of the 'pointwise' cartesian product of X and Y along the direct sum functor $\oplus : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}$ (see Proposition C.5), or more explicitly an enriched coend (see Remark C.6).

Also by the general theory of convolution products, the box product $X \boxtimes Y$ is a functor in both variables (Construction C.8) and it supports a preferred symmetric monoidal structure (see Theorem C.10); so there are specific natural associativity and symmetry isomorphisms

$$(X \boxtimes Y) \boxtimes Z \longrightarrow X \boxtimes (Y \boxtimes Z) \qquad \text{and} \qquad X \boxtimes Y \longrightarrow Y \boxtimes X$$

and a strict unit, the terminal orthogonal space $\mathbf{1}$, i.e., such that $\mathbf{1} \boxtimes X = X = X \boxtimes \mathbf{1}$. The upshot is that the associativity and symmetry isomorphisms make the box product of orthogonal spaces a symmetric monoidal product with the terminal orthogonal space as unit object. The box product of orthogonal spaces is *closed* symmetric monoidal in the sense that the box product is adjoint to an internal Hom orthogonal space. We won't use the internal function object, so we do not elaborate on it.

The next result proves a key feature, namely that the box product of orthogonal spaces coincides with the categorical product, up to global equivalence.

Given two orthogonal spaces X and Y, the maps

$$X(V) \times Y(W) \xrightarrow{X(i_1)\times Y(i_2)} X(V \oplus W) \times Y(V \oplus W) \;=\; (X \times Y)(V \oplus W)$$

form a bimorphism $(X, Y) \longrightarrow X \times Y$ as V and W vary over all inner product spaces; here $i_1 : V \longrightarrow V \oplus W$ and $i_2 : W \longrightarrow V \oplus W$ are the two direct summand embeddings. This bimorphism is represented by a morphism

$$\rho_{X,Y} \; : \; X \boxtimes Y \longrightarrow X \times Y \tag{1.3.1}$$

of orthogonal spaces that is natural in both variables.

Theorem 1.3.2. *Let X and Y be orthogonal spaces.*

(i) *The morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence.*
(ii) *The functor $X \boxtimes -$ preserves global equivalences.*

Proof (i) For an orthogonal space Z we denote by $\mathrm{sh}\, Z$ the orthogonal space defined by

$$(\mathrm{sh}\, Z)(V) \;=\; Z(V \oplus V) \qquad \text{and} \qquad (\mathrm{sh}\, Z)(\varphi) \;=\; Z(\varphi \oplus \varphi)\;;$$

thus $\mathrm{sh}\, Z$ is isomorphic to $\mathrm{sh}^{\mathbb{R}^2}_{\otimes}(Z)$, the multiplicative shift of Z by $\mathbb{R}^2$ as defined in Example 1.1.11. We define a morphism of orthogonal spaces

$$\lambda \; : \; X \times Y \longrightarrow \mathrm{sh}(X \boxtimes Y)$$

at an inner product space V as the composite

$$X(V) \times Y(V) \xrightarrow{i_{V,V}} (X \boxtimes Y)(V \oplus V) \;=\; (\mathrm{sh}(X \boxtimes Y))(V)\,.$$

Now we consider the two composites $\lambda \circ \rho_{X,Y}$ and $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$:

$$X \boxtimes Y \xrightarrow{\rho_{X,Y}} X \times Y \xrightarrow{\lambda} \mathrm{sh}(X \boxtimes Y) \xrightarrow{\mathrm{sh}(\rho_{X,Y})} \mathrm{sh}(X \times Y)$$

We claim that the composite $\lambda \circ \rho_{X,Y} : X \boxtimes Y \longrightarrow \mathrm{sh}(X \boxtimes Y)$ is homotopic to the morphism $(X \boxtimes Y) \circ i_1$, where i_1 is the natural linear isometric embedding $V \longrightarrow V \oplus V$ as the first summand. Indeed, for every $t \in [0, 1]$ we define a natural linear isometric embedding

$$j_t \; : \; V \oplus W \longrightarrow V \oplus W \oplus V \oplus W \quad \text{by} \quad j_t(v, w) \;=\; (v,\; t \cdot w,\; 0,\; \sqrt{1 - t^2} \cdot w)\,.$$

Then the maps

$$\begin{aligned} X(V) \times Y(W) &\xrightarrow{i_{V,W}} (X \boxtimes Y)(V \oplus W) \\ &\xrightarrow{(X\boxtimes Y)(j_t)} (X \boxtimes Y)(V \oplus W \oplus V \oplus W) \;=\; (\mathrm{sh}(X \boxtimes Y))(V \oplus W) \end{aligned}$$

form a bimorphism as V and W vary; the universal property of the box product turns this into a morphism of orthogonal spaces

$$f_t \;:\; X \boxtimes Y \longrightarrow \mathrm{sh}(X \boxtimes Y)\ .$$

The linear isometric embeddings j_t vary continuously with t, hence the morphisms f_t do as well. Moreover, $f_0 = \lambda \circ \rho_{X,Y}$ and $f_1 = (X \boxtimes Y) \circ i_1$, so this is the desired homotopy. The morphism $(X \boxtimes Y) \circ i_1$ is a global equivalence by Theorem 1.1.10, hence so is the morphism $\lambda \circ \rho_{X,Y}$.

The shift functor preserves products, and under the canonical isomorphism $\mathrm{sh}(X \times Y) \cong (\mathrm{sh}\, X) \times (\mathrm{sh}\, Y)$ the morphism $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ becomes the product of the two morphisms

$$X \circ i_1 \;:\; X \longrightarrow \mathrm{sh}\, X \qquad \text{and} \qquad Y \circ i_2 \;:\; Y \longrightarrow \mathrm{sh}\, Y\ .$$

The morphisms $X \circ i_1$ and $Y \circ i_2$ are global equivalences by Theorem 1.1.10, hence so is their product (by Proposition 1.1.9 (vi)). The global equivalences satisfy the 2-out-of-6 property by Proposition 1.1.9 (iii); since $\lambda \circ \rho_{X,Y}$ and $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ are global equivalences, so is the morphism $\rho_{X,Y}$.

(ii) The cartesian product $X \times -$ preserves global equivalences by Proposition 1.1.9 (vi). Together with part (i) this implies part (ii). □

Example 1.3.3 (Box product of semifree orthogonal spaces)**.** We show that the box product of two semifree orthogonal spaces is another semifree orthogonal space. This can be deduced from the general fact that a convolution product of two representable functors is again representable (see Remark C.11); however, the argument is simple enough that we make it explicit for orthogonal spaces.

We consider two compact Lie groups G and K, a G-representation V, a K-representation W, a G-space A and a K-space B. Then $V \oplus W$ is a $(G \times K)$-representation via

$$(g, k) \cdot (v, w) \;=\; (gv,\, kw)\ ,$$

and $A \times B$ is a $(G \times K)$-space in much the same way. The map

$$\begin{aligned} A \times B &\xrightarrow{(\mathrm{Id}_V \cdot G, -) \times (\mathrm{Id}_W \cdot K, -)} (\mathbf{L}_{G,V} A)(V) \times (\mathbf{L}_{K,W} B)(W) \\ &\xrightarrow{\quad i_{V,W} \quad} ((\mathbf{L}_{G,V} A) \boxtimes (\mathbf{L}_{K,W} B))(V \oplus W) \end{aligned}$$

is $(G \times K)$-equivariant, so it extends freely to a morphism of orthogonal spaces

$$\mathbf{L}_{G \times K, V \oplus W}(A \times B) \longrightarrow (\mathbf{L}_{G,V} A) \boxtimes (\mathbf{L}_{K,W} B)\ . \tag{1.3.4}$$

The maps

$$\begin{aligned} (\mathbf{L}(V, U) \times_G A) \times (\mathbf{L}(W, U') \times_K B) &\longrightarrow \mathbf{L}(V \oplus W, U \oplus U') \times_{G \times K} (A \times B) \\ ([\varphi, a],\, [\psi, b]) &\longmapsto [\varphi \oplus \psi, (a, b)] \end{aligned}$$

form a bimorphism from $(\mathbf{L}_{G,V}A, \mathbf{L}_{K,W}B)$ to $\mathbf{L}_{G\times K,V\oplus W}(A\times B)$ as the inner product spaces U and U' vary. The universal property of the box product translates this into a morphism

$$(\mathbf{L}_{G,V}A) \boxtimes (\mathbf{L}_{K,W}B) \longrightarrow \mathbf{L}_{G\times K,V\oplus W}(A \times B) .$$

These two morphisms are mutually inverse isomorphisms, i.e., the box product $(\mathbf{L}_{G,V}A) \boxtimes (\mathbf{L}_{K,W}B)$ is isomorphic to $\mathbf{L}_{G\times K,V\times W}(A \times B)$. A special case of this shows that the box product of two global classifying spaces is another global classifying space. Indeed, if G acts faithfully on V, and K acts faithfully on W, then the $(G \times K)$-action on $V \oplus W$ is also faithful, hence

$$(B_{\mathrm{gl}}G) \boxtimes (B_{\mathrm{gl}}K) = \mathbf{L}_{G,V} \boxtimes \mathbf{L}_{K,W} \cong \mathbf{L}_{G\times K,V\oplus W} = B_{\mathrm{gl}}(G \times K) . \qquad (1.3.5)$$

If we compose the inverse of the isomorphism (1.3.5) with the global equivalence $\rho_{\mathbf{L}_{G,V},\mathbf{L}_{K,W}}$ from (1.3.1), we obtain a global equivalence of orthogonal spaces

$$B_{\mathrm{gl}}(G \times K) \xrightarrow{\sim} (B_{\mathrm{gl}}G) \times (B_{\mathrm{gl}}K) .$$

Now we show that the categorical product of orthogonal spaces has the pushout product property for flat cofibrations, see Proposition 1.3.9 below. In particular, the product of two flat orthogonal spaces is again flat, which is not completely obvious from the outset. To this end we establish a useful sufficient condition for flatness; the criterion is inspired by a flatness criterion for **I**-spaces proved by Sagave and Schlichtkrull in [141, Prop. 3.11]. The conditions may seem technical at first sight, but we give two examples where they are easily verified, see Propositions 1.3.8 and 1.3.11. The category of linear isometries **L** does not have very many limits; however, for property (b) of the next proposition we note that it does have pullbacks.

Proposition 1.3.6. *Let Y be an orthogonal space.*

(i) *Suppose that Y satisfies the following conditions.*

   (a) *For every inner product space V, the space $Y(V)$ is compact.*
   (b) *As a functor from the category* **L** *to sets, Y preserves pullbacks.*

   Then for all $m \geq 0$ the canonical morphism $i_m : \mathrm{sk}^m Y \longrightarrow Y$ from the m-skeleton is objectwise a closed embedding.

(ii) *Let G be a compact Lie group acting continuously on Y through automorphisms of orthogonal spaces. Suppose that in addition to the conditions* (a) *and* (b)*, the following also hold for all inner product spaces V:*

   (c) *The $(G \times O(V))$-space $Y(V)$ admits a $(G \times O(V))$-CW-structure.*
   (d) *If an element of $Y(V)$ is fixed by a reflection in $O(V)$, then it is in the image of $Y(U)$ for some proper subspace U of V.*

Then the orthogonal orbit space Y/G is flat.

Proof We let V be any inner product space and consider the composite

$$\coprod_{0\le i\le m}\mathbf{L}(\mathbb{R}^i,V)\times Y(\mathbb{R}^i) \longrightarrow (\mathrm{sk}^m Y)(V) \xrightarrow{i_m(V)} Y(V)\ , \tag{1.3.7}$$

where the first map is as in the definition of $\mathrm{sk}^m Y$ as a coequalizer (1.2.1). The first map in (1.3.7) is a quotient map, and its source is compact by hypothesis (a) and because the spaces $\mathbf{L}(\mathbb{R}^i,V)$ are all compact. So the space $(\mathrm{sk}^m Y)(V)$ is quasi-compact. Since $Y(V)$ is Hausdorff, the map $i_m(V)$ is a closed map. So all that remains is to show that $i_m(V)$ is injective.

We consider two pairs

$$(\varphi,x)\ \in\ \mathbf{L}(\mathbb{R}^i,V)\times Y(\mathbb{R}^i) \qquad \text{and} \qquad (\varphi',x')\ \in\ \mathbf{L}(\mathbb{R}^j,V)\times Y(\mathbb{R}^j)$$

with $i,j\le m$, such that

$$Y(\varphi)(x)\ =\ Y(\varphi')(x')\ \in\ Y(V)\ .$$

We show that the two pairs represent the same element in the coequalizer $(\mathrm{sk}^m Y)(V)$. We choose a pullback square in the category $\mathbf{L}$:

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "h'"] \arrow[d, "h"'] & \mathbb{R}^j \arrow[d, "\varphi'"] \\
\mathbb{R}^i \arrow[r, "\varphi"'] & V
\end{tikzcd}$$

Condition (b) provides an element $z\in Y(\mathbb{R}^k)$ such that $Y(h)(z)=x$ and $Y(h')(z)=x'$. Then

$$(\varphi,x)=(\varphi,Y(h)(z))\sim(\varphi h,z)=(\varphi' h',z)\sim(\varphi',Y(h')(z))=(\varphi',x')\ .$$

So the pairs (φ,x) and (φ',x') describe the same element in $(\mathrm{sk}^m Y)(V)$.

Now we let G act on Y and also assume conditions (c) and (d). By the first part, the latching morphism

$$\nu_m^Y\ =\ i_{m-1}(\mathbb{R}^m)\ :\ L_m(Y)\ =\ (\mathrm{sk}^{m-1} Y)(\mathbb{R}^m)\ \longrightarrow\ Y(\mathbb{R}^m)$$

is a closed embedding for every $m\ge 0$, and the space $Y(\mathbb{R}^m)$ admits a $(G\times O(m))$-CW-structure by hypothesis (c). We claim that ν_m^Y is a $(G\times O(m))$-cofibration; to this end we characterize its image by a stabilizer condition. An element of $y\in Y(\mathbb{R}^m)$ is in the image of the latching map if and only if it is in the image of the map $Y(U)\longrightarrow Y(\mathbb{R}^m)$ for some $(m-1)$-dimensional subspace U of $\mathbb{R}^m$. Then the orthogonal reflection in the hyperplane U fixes y. Conversely, if y is fixed by a reflection, then it is in the image of $Y(U)$ for some proper subspace U, by hypothesis (d). So the image of the latching morphism

coincides with the subspace of all elements of $Y(\mathbb{R}^m)$ whose stabilizer group contains the reflection in some hyperplane of $\mathbb{R}^m$.

If K is any compact Lie group and X a K-CW-complex, then the set of elements of X whose stabilizer group contains a conjugate of some fixed subgroup is automatically a K-CW-subcomplex. In the situation at hand this means that the image of the latching map ν_m^Y is a $(G\times O(m))$-subcomplex in any $(G\times O(m))$-CW-structure on $Y(\mathbb{R}^m)$. So the latching map is a $(G \times O(m))$-cofibration.

The latching space construction commutes with colimits, so the canonical map

$$(L_m Y)/G \longrightarrow L_m(Y/G)$$

is a homeomorphism, and the latching map for Y/G is obtained from the latching map for Y by passage to G-orbits. Since the latching map for Y is a $(G \times O(m))$-cofibration, the induced map on G-orbits is then an $O(m)$-cofibration by Proposition B.14 (iii). This shows that the orthogonal space Y/G is flat. □

We apply the flatness criterion to show that the product of two semifree orthogonal spaces is flat.

Proposition 1.3.8. *Let G and K be compact Lie groups and let V and W be faithful representations of G and K, respectively. Then the orthogonal space $\mathbf{L}_{G,V} \times \mathbf{L}_{K,W}$ is flat.*

Proof We verify the conditions (a)-(d) of Proposition 1.3.6 for the orthogonal space $\mathbf{L}_V \times \mathbf{L}_W$ with the action of $G \times K$ by pre-composition on linear isometric embeddings. The space $\mathbf{L}(V, U)$ is compact and a Stiefel manifold, so it comes with a 'standard' smooth structure; moreover, the actions of G by pre-composition and of $O(U)$ by post-composition are smooth. By the same argument, $\mathbf{L}(W, U)$ admits the structure of a smooth closed $(K\times O(U))$-manifold. Hence the product $\mathbf{L}(V, U)\times\mathbf{L}(W, U)$ underlies a closed smooth $(G\times K\times O(U))$-manifold. So Illman's theorem [84, Cor. 7.2] provides a finite $(G\times K\times O(U))$-CW-structure. This verifies conditions (a) and (c).

Conditions (b) is straightforward. Finally, if a pair

$$(\varphi, \psi) \in \mathbf{L}(V, U) \times \mathbf{L}(W, U)$$

is fixed by the reflection in some hyperplane U' of U, then the images of φ and ψ are contained in U', and hence (φ, ψ) lies in the image of $\mathbf{L}(V, U')\times\mathbf{L}(W, U')$. This verifies condition (d). Proposition 1.3.6 thus applies, and shows that the orthogonal space $(\mathbf{L}_V\times\mathbf{L}_W)/(G\times K)$ is flat. So the isomorphic orthogonal space $\mathbf{L}_{G,V} \times \mathbf{L}_{K,W}$ is flat as well. □

Since the semifree orthogonal spaces $\mathbf{L}_{G,V}$ 'generate' the flat cofibrations,

the previous Proposition 1.3.8 is the key input for the pushout product property for the categorical product of orthogonal spaces.

Proposition 1.3.9. *Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be flat cofibrations of orthogonal spaces. Then the pushout product morphism*

$$f\Box g \;=\; (f\times Y)\cup(B\times g)\;:\; A\times Y\cup_{A\times X} B\times X \;\longrightarrow\; B\times Y$$

is a flat cofibration. In particular, the product of two flat orthogonal spaces is again flat.

Proof Since the cartesian product preserves colimits in both variables, it suffices to show the claim for two generating flat cofibrations

$$\mathbf{L}_{G,V}\times i_k \;:\; \mathbf{L}_{G,V}\times\partial D^k \;\longrightarrow\; \mathbf{L}_{G,V}\times D^k$$

and $\mathbf{L}_{K,W}\times i_m$, where G and K are compact Lie groups, V and W are faithful representations of G and K, respectively, and $k, m \geq 0$. The pushout product $(\mathbf{L}_{G,V}\times i_k)\Box(\mathbf{L}_{K,W}\times i_m)$ of two such generators is isomorphic to the morphism

$$\mathbf{L}_{G,V}\times\mathbf{L}_{K,W}\times i_{k+m}\ .$$

This morphism is a flat cofibration since the strong level model structure of orthogonal spaces is topological and $\mathbf{L}_{G,V}\times\mathbf{L}_{K,W}$ is flat (Proposition 1.3.8). □

In Definition 1.1.27 we introduced global classifying spaces as the semifree orthogonal spaces defined from faithful orthogonal representations of compact Lie groups. As we shall explain in Proposition 1.3.11 below, we can also use faithful *unitary* representations instead. This extra flexibility will come in handy when we study global objects with an intrinsic complex flavor, such as global classifying spaces of unitary groups, complex Grassmannians, or complex Bott periodicity. The next construction introduces the orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ for a unitary G-representation W. In contrast to their orthogonal cousins $\mathbf{L}_{G,V}$, the unitary analogs are not representable nor semifree in any sense. However, the unitary versions also enjoy various useful properties, for example that they are flat (see Proposition 1.3.11 (ii)) and behave well under box product (see Proposition 1.3.12).

Construction 1.3.10. To define the orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ for a unitary G-representation W, we introduce notation for going back and forth between euclidean inner product spaces over $\mathbb{R}$ and hermitian inner product spaces over $\mathbb{C}$. Throughout, we shall denote euclidean inner products on $\mathbb{R}$-vector spaces by pointy brackets $\langle -, -\rangle$, and hermitian inner products on $\mathbb{C}$-vector spaces by round parentheses $(-, -)$. For a euclidean inner product space V we let

$$V_{\mathbb{C}} \;=\; \mathbb{C}\otimes_{\mathbb{R}} V$$

be the complexification; the euclidean inner product $\langle -, - \rangle$ on V induces a hermitian inner product $(-, -)$ on the complexification $V_{\mathbb{C}}$, defined as the unique sesquilinear form that satisfies

$$(1 \otimes v, 1 \otimes v') = \langle v, v' \rangle$$

for all $v, v' \in V$. For a hermitian inner product space W, we let uW denote the underlying $\mathbb{R}$-vector space, equipped with the euclidean inner product

$$\langle w, w' \rangle = \mathrm{Re}(w, w') ,$$

the real part of the given hermitian inner product. Every $\mathbb{C}$-linear isometric embedding is in particular an $\mathbb{R}$-linear isometric embedding of underlying euclidean vector spaces, so $U(W) \subseteq O(uW)$, i.e., the unitary group of W is a subgroup of the orthogonal group of uW. We thus view a unitary representation on W as an orthogonal representation on uW. If V and W are two finite-dimensional $\mathbb{C}$-vector spaces equipped with hermitian inner products, we denote by $\mathbf{L}^{\mathbb{C}}(V, W)$ the space of $\mathbb{C}$-linear isometric embeddings. We topologize this as a complex Stiefel manifold, i.e., so it is homeomorphic to the space of complex $\dim_{\mathbb{C}}(V)$-frames in W.

Now we can define complex analogs of semifree orthogonal spaces. We let G be a compact Lie group and W a finite-dimensional unitary G-representation. We define the orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ by

$$\mathbf{L}^{\mathbb{C}}_{G,W}(V) = \mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})/G .$$

We define a morphism of orthogonal spaces

$$f_{G,W} : \mathbf{L}_{G,uW} \longrightarrow \mathbf{L}^{\mathbb{C}}_{G,W}$$

as follows. The map

$$j_W : W \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} W = (uW)_{\mathbb{C}} , \quad j_W(w) = 1/\sqrt{2} \cdot (1 \otimes w - i \otimes (iw))$$

is a G-equivariant $\mathbb{C}$-linear isometric embedding. At a real inner product space V, we can thus define

$$f_{G,W}(V) : \mathbf{L}(uW, V)/G \longrightarrow \mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})/G \quad \text{by} \quad f_{G,W}(V)(\varphi G) = (\varphi_{\mathbb{C}} \circ j_W)G .$$

Proposition 1.3.11. *Let G be a compact Lie group and W a faithful unitary G-representation.*

(i) *The morphism $f_{G,W} : \mathbf{L}_{G,uW} \longrightarrow \mathbf{L}^{\mathbb{C}}_{G,W}$ is a global equivalence.*
(ii) *The orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ is flat.*

Proof (i) Both source and target of f are closed orthogonal spaces; so by Proposition 1.1.17 we may show that for every compact Lie group K the map

$$f_{G,W}(\mathcal{U}_K) : \mathbf{L}(uW, \mathcal{U}_K)/G \longrightarrow \mathbf{L}^{\mathbb{C}}(W, \mathbb{C} \otimes_{\mathbb{R}} \mathcal{U}_K)/G$$

is a K-weak equivalence. We consider the $(K\times G)$-equivariant continuous map

$$\tilde{f} \;:\; \mathbf{L}(uW, \mathcal{U}_K) \longrightarrow \mathbf{L}^{\mathbb{C}}(W, \mathbb{C}\otimes_{\mathbb{R}} \mathcal{U}_K)\,, \quad \varphi \longmapsto \varphi_{\mathbb{C}} \circ j_W$$

that 'covers' $f_{G,W}(\mathcal{U}_K)$. The source of $\tilde{f}$ is a universal $(K \times G)$-space for the family $\mathcal{F}(K;G)$ of graph subgroups, by Proposition 1.1.26 (i). Since $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{U}_K$ is a complete complex G-universe, the complex analog of Proposition 1.1.26 (i), proved in much the same way, shows that the target of $\tilde{f}$ is also such a universal space for the same family of subgroups of $K\times G$. So $\tilde{f}$ is a $(K\times G)$-equivariant homotopy equivalence, compare Proposition B.11. The map $f_{G,W}(\mathcal{U}_K) = \tilde{f}/G$ induced on G-orbit spaces is thus a K-equivariant homotopy equivalence.

(ii) We verify conditions (a)-(d) of Proposition 1.3.6 for the orthogonal G-space $\mathbf{L}^{\mathbb{C}}_W$. The space $\mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})$ is compact and a complex Stiefel manifold, so it comes with a 'standard' smooth structure; moreover, the action of $G \times O(V)$ by pre- and post-composition is smooth. So Illman's theorem [84, Cor. 7.2] provides a $(G \times O(V))$-CW-structure on $\mathbf{L}^{\mathbb{C}}_W(V)$. This verifies conditions (a) and (c).

For condition (b) we observe that complexification takes pullback squares in $\mathbf{L}$ to pullback squares of hermitian inner product spaces and complex linear isometric embeddings. So the functor $\mathbf{L}^{\mathbb{C}}(W, (-)_{\mathbb{C}})$ preserves pullbacks. Finally, if $\varphi \in \mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})$ is fixed by the reflection in some hyperplane U of V, then the image of φ is contained in $U_{\mathbb{C}}$, and hence φ lies in the image of $\mathbf{L}^{\mathbb{C}}(W, U_{\mathbb{C}})$. This verifies condition (d). Proposition 1.3.6 thus applies, and shows that the orthogonal space $\mathbf{L}^{\mathbb{C}}_W/G = \mathbf{L}^{\mathbb{C}}_{G,W}$ is flat. □

Now we explain in which way the orthogonal spaces $\mathbf{L}^{\mathbb{C}}_{G,W}$ are multiplicative in the pair (G, W). We let K be another compact Lie group and U a unitary K-representation. The maps

$$\begin{array}{ccc} \mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})/G \times \mathbf{L}^{\mathbb{C}}(U, V'_{\mathbb{C}})/K & \longrightarrow & \mathbf{L}^{\mathbb{C}}(W\oplus U, (V\oplus V')_{\mathbb{C}})/(G\times K) \\ (\varphi\cdot G,\, \psi\cdot K) & \longmapsto & (\varphi\oplus\psi)\cdot(G\times K) \end{array}$$

form a bimorphism from $(\mathbf{L}^{\mathbb{C}}_{G,W}, \mathbf{L}^{\mathbb{C}}_{K,U})$ to $\mathbf{L}^{\mathbb{C}}_{G\times K, W\oplus U}$ as the inner product spaces V and V' vary. The universal property of the box product translates this into a morphism of orthogonal spaces

$$\zeta_{G,K;W,U} \;:\; \mathbf{L}^{\mathbb{C}}_{G,W} \boxtimes \mathbf{L}^{\mathbb{C}}_{K,U} \longrightarrow \mathbf{L}^{\mathbb{C}}_{G\times K, W\oplus U}\,.$$

The analogous morphism for semifree orthogonal spaces (i.e., for orthogonal representations and without the superscript $^{\mathbb{C}}$) is an isomorphism, see Example 1.3.3. For the complex analogs, an isomorphism would be too much to hope for, but the next best thing is true:

Proposition 1.3.12. *Let G and K be compact Lie groups, W a unitary G-representation and U a unitary K-representation. Then the morphism $\zeta_{G,K;W,U}$ is a global equivalence.*

Proof The global equivalences discussed in Proposition 1.3.11 (i) make the following square commute:

$$\begin{tikzcd}
\mathbf{L}_{G,uW} \boxtimes \mathbf{L}_{K,uU} \arrow[r, "\cong", "(1.3.5)"'] \arrow[d, "f_{G,W} \boxtimes f_{K,U}"', "\simeq"] & \mathbf{L}_{G\times K,u(W\oplus U)} \arrow[d, "f_{G\times K,W\oplus U}", "\simeq"'] \\
\mathbf{L}^{\mathbb{C}}_{G,W} \boxtimes \mathbf{L}^{\mathbb{C}}_{K,U} \arrow[r, "\zeta_{G,K;W,U}"'] & \mathbf{L}^{\mathbb{C}}_{G\times K,W\oplus U}
\end{tikzcd}$$

The left vertical morphism is a global equivalence because these are stable under box product (Theorem 1.3.2 (ii)). Since the vertical morphisms are global equivalences and the upper horizontal morphism is an isomorphism, the lower horizontal morphism $\zeta_{G,K;W,U}$ is also a global equivalence. □

1.4 Global families

In this section we explain a variant of unstable global homotopy theory based on a *global family*, i.e., a class $\mathcal{F}$ of compact Lie groups with certain closure properties. We introduce $\mathcal{F}$-equivalences, a relative version of global equivalences, and establish an $\mathcal{F}$-relative version of the global model structure in Theorem 1.4.8. We also discuss the compatibility of the $\mathcal{F}$-global model structure with the box product of orthogonal spaces, see Proposition 1.4.12. Finally, we record that, for multiplicative global families, the $\mathcal{F}$-global model structure lifts to categories of modules and algebras, see Corollary 1.4.15.

Definition 1.4.1. A *global family* is a non-empty class of compact Lie groups that is closed under isomorphisms, closed subgroups and quotient groups.

Some relevant examples of global families are: all compact Lie groups; all finite groups; all abelian compact Lie groups; all finite abelian groups; all finite cyclic groups; all finite p-groups. Another example is the global family $\langle G \rangle$ generated by a compact Lie group G, i.e., the class of all compact Lie groups isomorphic to a quotient of a closed subgroup of G. A degenerate case is the global family $\langle e \rangle$ of all trivial groups. In this case our theory specializes to the non-equivariant homotopy theory of orthogonal spaces.

For a global family $\mathcal{F}$ and a compact Lie group G we write $\mathcal{F} \cap G$ for the family of those closed subgroups of G that belong to $\mathcal{F}$. We also write $\mathcal{F}(m)$ for $\mathcal{F} \cap O(m)$, the family of closed subgroups of $O(m)$ that belong to $\mathcal{F}$. We recall that an equivariant continuous map of $O(m)$-spaces is an $\mathcal{F}(m)$-*cofibration* if

it has the left lifting property with respect to all morphisms $q : A \longrightarrow B$ of $O(m)$-spaces such that the map $q^H : A^H \longrightarrow B^H$ is a weak equivalence and Serre fibration for all $H \in \mathcal{F}(m)$.

The following definitions of $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations are direct relativizations of the corresponding concepts in the strong level model structure of orthogonal spaces.

Definition 1.4.2. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is

- an *$\mathcal{F}$-level equivalence* if for every compact Lie group G in $\mathcal{F}$ and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence;
- an *$\mathcal{F}$-level fibration* if for every compact Lie group G in $\mathcal{F}$ and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration; and
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-cofibration for all $m \geq 0$.

Every inner product space V is isometrically isomorphic to $\mathbb{R}^m$ with the standard scalar product, where m is the dimension of V. So a morphism $f : X \longrightarrow Y$ of orthogonal spaces is an $\mathcal{F}$-level equivalence (or $\mathcal{F}$-level fibration) precisely if for every $m \geq 0$ the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-equivalence (or $\mathcal{F}(m)$-projective fibration). The formal argument is analogous to Lemmas 1.2.7 and 1.2.8 which treat the case $\mathcal{F} = \mathcal{A}ll$. Clearly, the classes of $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations are closed under composition and retracts.

Now we discuss the $\mathcal{F}$-level model structures on orthogonal spaces. When $\mathcal{F} = \mathcal{A}ll$ is the global family of all compact Lie groups, then $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$. For this maximal global family, an $\mathcal{A}ll$-level equivalence is just a strong level equivalence in the sense of Definition 1.1.8. Moreover, the $\mathcal{A}ll$-level fibrations coincide with the strong level fibrations. The $\mathcal{A}ll$-cofibrations coincide with the flat cofibrations. So for the global family of all compact Lie groups the $\mathcal{A}ll$-level model structure on orthogonal spaces is the strong level model structure of Proposition 1.2.10.

Proposition 1.4.3. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a topological and cofibrantly generated model structure, the* $\mathcal{F}$-level model structure*, on the category of orthogonal spaces.*

Proof We specialize Proposition C.23 by letting $C(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces, compare Proposition B.7. With respect to these choices of model structures $C(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition C.23

become precisely the $\mathcal{F}$-level equivalences, $\mathcal{F}$-fibrations and $\mathcal{F}$-cofibrations. Every acyclic cofibration in the $\mathcal{F}(m)$-projective model structure of $O(m)$-spaces is also an acyclic cofibration in the $\mathcal{A}ll$-projective model structure of $O(m)$-spaces. So the consistency condition (see Definition C.22) in the present situation is a special case of the consistency condition for the strong level model structure that we established in the proof of Proposition 1.2.10.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-level model structure. We let $I_{\mathcal{F}}$ be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.8). Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure by Proposition C.23 (iii). Similarly, we let $J_{\mathcal{F}}$ be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.9). Again by Proposition C.23 (iii), $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure. The $\mathcal{F}$-level model structure is topological by Proposition B.5, where we take $\mathcal{G}$ as the set of orthogonal spaces $L_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all $H \in \mathcal{F}(m)$. □

Now we proceed towards the construction of the *$\mathcal{F}$-global model structure*, see Theorem 1.4.8 below. The weak equivalences in this model structures are the $\mathcal{F}$-equivalences of the following definition, the direct generalization of global equivalences in the presence of a global family.

Definition 1.4.4. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an *$\mathcal{F}$-equivalence* if the following condition holds: for every compact Lie group G in $\mathcal{F}$, every G-representation V, every $k \geq 0$ and all maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $f(V)^G \circ \alpha = \beta|_{\partial D^k}$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then $\mathcal{A}ll$-equivalences are precisely the global equivalences. The following proposition generalizes Proposition 1.1.7, and it is proved in much the same way.

Proposition 1.4.5. *Let $\mathcal{F}$ be a global family. For every morphism of orthogonal spaces $f : X \longrightarrow Y$, the following three conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *Let G be a compact Lie group, V a G-representation and (B, A) a finite G-CW-pair all of whose isotropy groups belong to $\mathcal{F}$. Then for all continuous G-maps $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ such that $\beta|_A = f(V) \circ \alpha$,*

there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha$ and such that $f(W) \circ \lambda$ is G-homotopic, relative to A, to $Y(\varphi) \circ \beta$.

(iii) *For every compact Lie group G in the family $\mathcal{F}$ and every exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations the induced map*

$$\mathrm{tel}_i\, f(V_i) \ : \ \mathrm{tel}_i\, X(V_i) \ \longrightarrow \ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence.

Definition 1.4.6. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an $\mathcal{F}$-*global fibration* if it is an $\mathcal{F}$-level fibration and for every compact Lie group G in the family $\mathcal{F}$, every faithful G-representation V and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the map

$$(f(V)^G, X(\varphi)^G) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(W)^G} X(W)^G$$

is a weak equivalence.

The next proposition contains various properties of $\mathcal{F}$-equivalences that generalize Proposition 1.1.9 and Proposition 1.2.14 (i).

Proposition 1.4.7. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-level equivalence is an $\mathcal{F}$-equivalence.*
(ii) *The composite of two $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(iii) *If f, g and h are composable morphisms of orthogonal spaces such that gf and hg are $\mathcal{F}$-equivalences, then f, g, h and hgf are also $\mathcal{F}$-equivalences.*
(iv) *Every retract of an $\mathcal{F}$-equivalence is an $\mathcal{F}$-equivalence.*
(v) *A coproduct of any set of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(vi) *A finite product of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(vii) *For every space K the functor $-\times K$ preserves $\mathcal{F}$-equivalences of orthogonal spaces.*
(viii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spaces that are objectwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be $\mathcal{F}$-equivalences of orthogonal spaces that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is an $\mathcal{F}$-equivalence.*
(ix) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be an $\mathcal{F}$-equivalence and a closed embedding of orthogonal spaces, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is an $\mathcal{F}$-equivalence.*

(x) *Let*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram of orthogonal spaces such that f and f' are h-cofibrations. If the morphisms α,β and γ are $\mathcal{F}$-equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ C' \cup_{A'} B' \ .$$

(xi) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is an $\mathcal{F}$-equivalence. If in addition f or g is an h-cofibration, then the morphism k is an $\mathcal{F}$-equivalence.

(xii) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is an $\mathcal{F}$-equivalence. If in addition one of the morphisms f or h is an $\mathcal{F}$-level fibration, then the morphism g is also an $\mathcal{F}$-equivalence.

(xiii) *Every $\mathcal{F}$-equivalence that is also an $\mathcal{F}$-global fibration is an $\mathcal{F}$-level equivalence.*

(xiv) *The box product of two $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*

Proof The proofs of (i) through (xii) are almost verbatim the same as the corresponding parts of Proposition 1.1.9, and we omit them. Part (xiii) is proved in the same way as Proposition 1.2.14 (i).

(xiv) The product of orthogonal spaces preserves $\mathcal{F}$-equivalences in both variables by part (vi). The morphism $\rho_{X,Y} \ : \ X \boxtimes Y \ \longrightarrow \ X \times Y$ is a global equivalence, hence an $\mathcal{F}$-equivalence, for all orthogonal spaces X and Y, by Theorem 1.3.2 (i); this implies the claim. □

Now we establish the $\mathcal{F}$-global model structures on the category of orthogonal spaces. We spell out sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-global model structures. In Proposition 1.4.3 we introduced $I_{\mathcal{F}}$ as the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.8). The set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure, which coincide with the acyclic fibrations in the $\mathcal{F}$-global model structure.

Also in Proposition 1.4.3 we defined $J_{\mathcal{F}}$ as the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.9). The set $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure. We add another set of morphisms $K_{\mathcal{F}}$ that detects when the squares (1.2.13) are homotopy cartesian for $G \in \mathcal{F}$. We set

$$K_{\mathcal{F}} \ = \bigcup_{G,V,W \,:\, G\in\mathcal{F}} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $i_k : \partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$, compare Construction 1.2.15; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G in $\mathcal{F}$, a faithful G-representation V and an arbitrary G-representation W. By Proposition 1.2.16, the right lifting property with respect to the union $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ characterizes the $\mathcal{F}$-global fibrations.

The proof of the following theorem proceeds by mimicking the proof in the special case $\mathcal{F} = \mathcal{A}ll$, and all arguments in the proof of Theorem 1.2.21 go through almost verbatim. Whenever the small object argument is used, it now has to be taken with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ (as opposed to the set $J^{\mathrm{str}} \cup K$).

Theorem 1.4.8 ($\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $\mathcal{F}$-cofibrations form a model structure, the $\mathcal{F}$-*global model structure*, on the category of orthogonal spaces.*

(ii) *The fibrant objects in the $\mathcal{F}$-global model structure are the $\mathcal{F}$-*static orthogonal spaces*, i.e., those orthogonal spaces X such that for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the map of G-fixed-point spaces $X(\varphi)^G : X(V)^G \longrightarrow X(W)^G$ is a weak equivalence.*

(iii) *A morphism of orthogonal spaces is:*

- *an acyclic fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$; and*

- *a fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

(iv) *The $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Example 1.4.9. In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-global homotopy theory of orthogonal spaces just another model for the (non-equivariant) homotopy theory of spaces. Indeed, the evaluation functor $\mathrm{ev}_0 : spc \longrightarrow \mathbf{T}$ is a right Quillen equivalence with respect to the $\langle e \rangle$-global model structure. So the total derived functor

$$\mathrm{Ho}(\mathrm{ev}_0) \ : \ \mathrm{Ho}^{\langle e \rangle}(spc) \ \longrightarrow \mathrm{Ho}(\mathbf{T})$$

is an equivalence of homotopy categories.

In fact, for the global family $\mathcal{F} = \langle e \rangle$, most of what we do here has already been studied before: The $\langle e \rangle$-global model structure and the fact that it is Quillen equivalent to the model category of spaces were established by Lind [102, Thm. 1.1]; in [102], orthogonal spaces are called '$\mathcal{I}$-spaces' and $\langle e \rangle$-global equivalences are called 'weak homotopy equivalences' and are defined as those morphisms that induce weak equivalences on homotopy colimits.

Corollary 1.4.10. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spaces and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *The morphism can be written as $f = w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*
(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the $\mathcal{F}$-level model structure and every $\mathcal{F}$-static orthogonal space Y the induced map*

$$[f^c, Y] \ : \ [B^c, Y] \ \longrightarrow \ [A^c, Y]$$

on homotopy classes of morphisms is bijective.

Proof (i)$\Longleftrightarrow$(ii) The class of $\mathcal{F}$-equivalences contains the global equivalences by definition, and the $\mathcal{F}$-level equivalences by 1.4.7 (i), and is closed under composition 1.4.7 (ii), so all composites $w_2 \circ w_1$ as in the claim are $\mathcal{F}$-equivalences. On the other hand, every $\mathcal{F}$-equivalence f can be factored in the global model structure of Theorem 1.2.21 as $f = qj$ where j is a global equivalence and q is a global fibration. Since f and j are $\mathcal{F}$-equivalences, so is q by Proposition 1.4.7 (iii). So q is an $\mathcal{F}$-equivalence and a global fibration, hence an $\mathcal{F}$-level equivalence by Proposition 1.4.7 (xiii).

(i)$\Longleftrightarrow$(iii) The morphism f is an $\mathcal{F}$-equivalence if and only if the $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ is an $\mathcal{F}$-equivalence. Since A^c and B^c are

$\mathcal{F}$-cofibrant, they are cofibrant in the $\mathcal{F}$-global model structure. So by general model model category theory (see for example [78, Cor. 7.7.4]), f^c is an $\mathcal{F}$-equivalence if and only if the induced map $[f^c, X]$ is bijective for every fibrant object in the $\mathcal{F}$-global model structure. By Theorem 1.4.8 (ii) these fibrant objects are precisely the $\mathcal{F}$-static orthogonal spaces. □

Remark 1.4.11 (Mixed global model structures)**.** Cole's 'mixing theorem' for model structures [38, Thm. 2.1] allows us to construct many more 'mixed' $\mathcal{F}$-global model structures on the category of orthogonal spaces. We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the $\mathcal{E}$-global model structure is a fibration in the $\mathcal{F}$-global model structure. By Cole's theorem [38, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spaces. By [38, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts of all composites $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal space is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal space [38, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [38]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structures on the category of orthogonal spaces that are all Quillen equivalent to the model category of (non-equivariant) spaces, with respect to weak equivalences.

The next topic is the compatibility of the $\mathcal{F}$-global model structure with the box product of orthogonal spaces. Given two morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ of orthogonal spaces we denote the *pushout product* morphism by

$$f \Box g = (f \boxtimes Y) \cup (B \boxtimes g) \ : \ A \boxtimes Y \cup_{A \boxtimes X} B \boxtimes X \ \longrightarrow \ B \boxtimes Y \, .$$

We recall that a model structure on a symmetric monoidal category satisfies the *pushout product property* if the following two conditions hold:

- for every pair of cofibrations f and g the pushout product morphism $f \Box g$ is also a cofibration; and
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f \Box g$.

We let $\mathcal{E}$ and $\mathcal{F}$ be two global families. We denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$. So a compact Lie group H belongs to $\mathcal{E} \times \mathcal{F}$ if and only if H is isomorphic to a

closed subgroup of a group of the form $(G \times K)/N$ for some groups $G \in \mathcal{E}$ and $K \in \mathcal{F}$, and some closed normal subgroup N of $G \times K$.

Proposition 1.4.12. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of a flat cofibration that is also an $\mathcal{F}$-equivalence with any morphism of orthogonal spaces is an $\mathcal{F}$-equivalence.*
(iii) *Let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Then the $\mathcal{F}$-global model structure satisfies the pushout product property with respect to the box product of orthogonal spaces.*
(iv) *The positive global model structure satisfies the pushout product property with respect to the box product of orthogonal spaces.*

Proof (i) It suffices to show the claim for sets of generating cofibrations. The $\mathcal{E}$-cofibrations are generated by the morphisms

$$\mathbf{L}_{G,V} \times i_k \ : \ \mathbf{L}_{G,V} \times \partial D^k \ \longrightarrow \ \mathbf{L}_{G,V} \times D^k$$

for $G \in \mathcal{E}$, V a G-representation and $k \geq 0$. Similarly, the $\mathcal{F}$-cofibrations are generated by the morphisms $\mathbf{L}_{K,W} \times i_m$ for $K \in \mathcal{F}$, W a K-representation and $m \geq 0$. The pushout product of two such generators is isomorphic to the morphism

$$\mathbf{L}_{G \times K, V \oplus W} \times i_{k+m} \ : \ \mathbf{L}_{G \times K, V \oplus W} \times \partial D^{k+m} \ \longrightarrow \ \mathbf{L}_{G \times K, V \oplus W} \times D^{k+m} \ ,$$

compare Example 1.3.3. Since $G \times K$ belongs to the family $\mathcal{E} \times \mathcal{F}$, this pushout product morphism is an $(\mathcal{E} \times \mathcal{F})$-cofibration.

(ii) We let $i : A \longrightarrow B$ and $j : K \longrightarrow L$ be morphisms of orthogonal spaces such that i is a flat cofibration and an $\mathcal{F}$-equivalence. Then $i \boxtimes K$ and $i \boxtimes L$ are $\mathcal{F}$-equivalences by Proposition 1.4.7 (xiv). Moreover, i is an h-cofibration by Corollary A.30 (iii), hence so is $i \boxtimes K : A \boxtimes K \longrightarrow B \boxtimes K$. Thus its cobase change, the canonical morphism

$$A \boxtimes L \ \longrightarrow \ A \boxtimes L \cup_{A \boxtimes K} B \boxtimes K$$

is an $\mathcal{F}$-equivalence by Proposition 1.4.7 (xi). Since $i \boxtimes L : A \boxtimes L \longrightarrow B \boxtimes L$ is also an $\mathcal{F}$-equivalence, so is the pushout product map, by 2-out-of-6 (Proposition 1.4.7 (iii)).

(iii) The part of the pushout product property that refers only to cofibrations is true by part (i) with $\mathcal{E} = \mathcal{F}$ and the hypothesis that $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Every cofibration in the $\mathcal{F}$-global model structure is in particular a flat cofibration, so the part of the pushout product property that also refers to acyclic cofibrations in the $\mathcal{F}$-global model structure is a special case of (ii).

Part (iv) is proved in the essentially the same way as (iii), for the global family of all compact Lie groups. □

Finally, we will discuss another important relationship between the $\mathcal{F}$-global model structures and the box product, namely the *monoid axiom* [146, Def. 3.3]. We only discuss a slightly weaker form of the monoid axiom in the sense that we only cover sequential (as opposed to more general transfinite) compositions.

Proposition 1.4.13 (Monoid axiom)**.** *We let $\mathcal{F}$ be a global family. For every flat cofibration $i : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal space Y the morphism*

$$i \boxtimes Y \ : \ A \boxtimes Y \longrightarrow B \boxtimes Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions.

Proof Every flat cofibration is an h-cofibration (Corollary A.30 (iii) applied to the strong level model structure), and h-cofibrations are closed under box product with any orthogonal space (Corollary A.30 (ii)), so $i \boxtimes Y$ is an h-cofibration. Since i is an $\mathcal{F}$-equivalence, so is $i \boxtimes Y$ by Proposition 1.4.7 (xiv).

Proposition 1.4.7 shows that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions. □

We let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. The constant one-point orthogonal space $\mathbf{1}$ is the unit object for the box product of orthogonal spaces, and it is 'free', i.e., $\langle e \rangle$-cofibrant. So $\mathbf{1}$ is cofibrant in the $\mathcal{F}$-global model structure. So with respect to the box product, the $\mathcal{F}$-global model structure is a symmetric monoidal model category in the sense of [80, Def. 4.2.6]. A corollary is that the unstable $\mathcal{F}$-global homotopy category, i.e., the localization of the category of orthogonal spaces at the class of $\mathcal{F}$-equivalences, inherits a closed symmetric monoidal structure, compare [80, Thm. 4.3.3]. This 'derived box product' is nothing new, though: since the morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence for all orthogonal spaces X and Y, the derived box product is just a categorical product in $\mathrm{Ho}^{\mathcal{F}}(spc)$.

Definition 1.4.14. An *orthogonal monoid space* is an orthogonal space R equipped with unit morphism $\eta : \mathbf{1} \longrightarrow R$ and a multiplication morphism

$\mu : R \boxtimes R \longrightarrow R$ that is unital and associative in the sense that the diagram

$$\begin{tikzcd} (R \boxtimes R) \boxtimes R \arrow[r, "\text{associativity}", "\cong"'] \arrow[d, "\mu \boxtimes R"'] & R \boxtimes (R \boxtimes R) \arrow[r, "R\boxtimes\mu"] & R \boxtimes R \arrow[d, "\mu"] \\ R \boxtimes R \arrow[rr, "\mu"'] & & R \end{tikzcd}$$

commutes. An orthogonal monoid space R is *commutative* if moreover $\mu \circ \tau_{R,R} = \mu$, where $\tau_{R,R} : R \boxtimes R \longrightarrow R \boxtimes R$ is the symmetry isomorphism of the box product. A *morphism* of orthogonal monoid spaces is a morphism of orthogonal spaces $f : R \longrightarrow S$ such that $f \circ \mu^R = \mu^S \circ (f \boxtimes f)$ and $f \circ \eta_R = \eta_S$.

One can expand the data of an orthogonal monoid space into an 'external' form as follows. The unit morphism $\eta : \mathbf{1} \longrightarrow R$ is determined by a unit element $0 \in R(0)$, the image of the map $\eta(0) : \mathbf{1}(0) \longrightarrow R(0)$. The multiplication map corresponds to continuous maps $\mu_{V,W} : R(V) \times R(W) \longrightarrow R(V \oplus W)$ for all inner product spaces V and W, which form a bimorphism as (V, W) varies and satisfy

$$\mu_{V,0}(x, 0) = x \qquad \text{and} \qquad \mu_{0,W}(0, y) = y .$$

Put another way, the data of an orthogonal monoid space in external form is that of a lax monoidal functor. The commutativity condition can be expressed in terms of the external multiplication as the commutativity of the diagrams

$$\begin{tikzcd} R(V) \times R(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\text{twist}"'] & R(V \oplus W) \arrow[d, "R(\tau_{V,W})"] \\ R(W) \times R(V) \arrow[r, "\mu_{W,V}"'] & R(W \oplus V) \end{tikzcd}$$

where $\tau_{V,W} : V \oplus W \longrightarrow W \oplus V$ interchanges the summands. So commutative orthogonal monoid spaces in external form are lax symmetric monoidal functors. We will later refer to commutative orthogonal monoid spaces as *ultra-commutative monoids*.

Every $\mathcal{F}$-cofibration is in particular a flat cofibration, so the monoid axiom in the $\mathcal{F}$-global model structure holds. If the global family $\mathcal{F}$ is closed under products, Theorem 4.1 of [146] applies to the $\mathcal{F}$-global model structure of Theorem 1.4.8 and shows:

Corollary 1.4.15. *Let R be an orthogonal monoid space and $\mathcal{F}$ a multiplicative global family.*

(i) *The category of R-modules admits the $\mathcal{F}$-*global model structure *in which a morphism is an equivalence (or fibration) if the underlying morphism*

of orthogonal spaces is an $\mathcal{F}$-equivalence (or $\mathcal{F}$-global fibration). This model structure is cofibrantly generated. Every cofibration in this $\mathcal{F}$-global model structure is an h-cofibration of underlying orthogonal spaces. If the underlying orthogonal space of R is $\mathcal{F}$-cofibrant, then every cofibration of R-modules is an $\mathcal{F}$-cofibration of underlying orthogonal spaces.

(ii) *If R is commutative, then with respect to $\boxtimes_R$ the $\mathcal{F}$-global model structure of R-modules is a monoidal model category that satisfies the monoid axiom.*

(iii) *If R is commutative, then the category of R-algebras admits the $\mathcal{F}$-*global model structure *in which a morphism is an equivalence (or fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (or $\mathcal{F}$-global fibration). Every cofibrant R-algebra is also cofibrant as an R-module.*

Proof Almost of the statements are in Theorem 4.1 of [146]. The only additional claims that require justification are the two statements in part (i) that concern the behavior of the forgetful functor on the cofibrations in the $\mathcal{F}$-global model structure.

Since the forgetful functor from R-modules to orthogonal spaces preserves all colimits and since the classes of h-cofibrations and of $\mathcal{F}$-cofibrations of orthogonal spaces are both closed under coproducts, cobase change, sequential colimits and retracts, it suffices to show each claim for the generating cofibrations in the $\mathcal{F}$-global model structure on the category of R-modules. These are of the form

$$(R \boxtimes \mathbf{L}_{H,\mathbb{R}^m}) \times i_k$$

for some $k, m \geq 0$ and H a closed subgroup of $O(m)$ that belongs to the global family $\mathcal{F}$; as usual $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. Since i_k is an h-cofibration of spaces, the morphisms $(R \boxtimes \mathbf{L}_{H,\mathbb{R}^m}) \times i_k$ are h-cofibrations of orthogonal spaces. This concludes the proof that every cofibration of R-modules is an h-cofibration of underlying orthogonal spaces.

Now we suppose that the underlying orthogonal space of R is $\mathcal{F}$-cofibrant. Because H belongs to $\mathcal{F}$, the orthogonal space $\mathbf{L}_{H,\mathbb{R}^m}$ is $\mathcal{F}$-cofibrant. Hence the orthogonal space $R \boxtimes \mathbf{L}_{H,\mathbb{R}^m}$ is $\mathcal{F}$-cofibrant by Proposition 1.4.12 (iii). So $(R \boxtimes \mathbf{L}_{H,\mathbb{R}^m}) \times i_k$ is an $\mathcal{F}$-cofibration of orthogonal spaces. This concludes the proof that every cofibration of R-modules is an $\mathcal{F}$-cofibration of underlying orthogonal spaces. □

Strictly speaking, Theorem 4.1 of [146] does not apply verbatim to the $\mathcal{F}$-global model structure because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied, and our version of the monoid axiom in Proposition 1.4.13 is weaker than Theorem 3.3 of [146] in that we do

not close under *transfinite* compositions. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to sequential compositions of flat cofibrations, and this suffices to run the countable small object argument (compare also Remark 2.4 of [146]).

Proposition 1.4.16. *Let R be an orthogonal monoid space and N a right R-module that is cofibrant in the $\mathcal{A}ll$-global model structure of Corollary* 1.4.15 (i). *Then for every global family $\mathcal{F}$, the functor $N \boxtimes_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spaces.*

Proof For the course of this proof we call an R-module N *homotopical* if the functor $N \boxtimes_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spaces. Since the $\mathcal{A}ll$-global model structure on the category of right R-modules is obtained by lifting the global model structure of orthogonal spaces along the free and forgetful adjoint functor pair, every cofibrant right R-module is a retract of an R-module that arises as the colimit of a sequence

$$\emptyset = M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_k \longrightarrow \ldots \tag{1.4.17}$$

in which each M_k is obtained from M_{k-1} as a pushout

$$\begin{tikzcd} A_k \boxtimes R \arrow[r, "f_k \boxtimes R"] \arrow[d] & B_k \boxtimes R \arrow[d] \\ M_{k-1} \arrow[r] & M_k \end{tikzcd}$$

for some flat cofibration $f_k : A_k \longrightarrow B_k$ of orthogonal spaces. For example, f_k can be chosen as a disjoint union of morphisms in the set I^{str} of generating flat cofibrations. We show by induction on k that each module M_k is homotopical. The induction starts with the empty R-module, where there is nothing to show. Now we suppose that M_{k-1} is homotopical and claim that then M_k is homotopical as well. To see this we consider an $\mathcal{F}$-equivalence of left R-modules $\varphi : X \longrightarrow Y$. Then $M_k \boxtimes_R \varphi$ is obtained by passing to horizontal pushouts in the following commutative diagram of orthogonal spaces:

$$\begin{tikzcd} M_{k-1} \boxtimes_R X \arrow[d, "M_{k-1}\boxtimes_R\varphi"'] & A_k \boxtimes X \arrow[l] \arrow[r, "f_k \boxtimes X"] \arrow[d, "A_k\boxtimes\varphi"] & B_k \boxtimes X \arrow[d, "B_k\boxtimes\varphi"] \\ M_{k-1} \boxtimes_R Y & A_k \boxtimes Y \arrow[l] \arrow[r, "f_k \boxtimes Y"'] & B_k \boxtimes Y \end{tikzcd}$$

Here we exploit the fact that $(A_k \boxtimes R) \boxtimes_R X$ is naturally isomorphic to $A_k \boxtimes X$. The left vertical morphism in the diagram is an $\mathcal{F}$-equivalence by hypothesis. The middle and right vertical morphisms are $\mathcal{F}$-equivalences because the box product is homotopical for $\mathcal{F}$-equivalences (Proposition 1.4.7 (xiv)). Moreover,

the morphism f_k is a flat cofibration since it is an h-cofibration (by Corollary A.30 (iii)), and so the morphisms $f_k \boxtimes X$ and $f_k \boxtimes Y$ are h-cofibrations. Proposition 1.4.7 (x) then shows that the induced morphism on horizontal pushouts $M_k \boxtimes_R \varphi$ is again an $\mathcal{F}$-equivalence.

Now we let M be a colimit of the sequence (1.4.17). Then $M \boxtimes_R X$ is a colimit of the sequence $M_k \boxtimes_R X$. Moreover, since $f_k : A_k \longrightarrow B_k$ is an h-cofibration, so is the morphism $f_k \boxtimes R$, and hence also its cobase change $M_{k-1} \longrightarrow M_k$. So the sequence whose colimit is $M \boxtimes_R X$ consists of h-cofibrations, which are objectwise closed embeddings by Proposition A.31. The same is true for $M \boxtimes_R Y$. Since each M_k is homotopical and colimits of orthogonal spaces along closed embeddings are homotopical (by Proposition 1.4.7 (viii)), we conclude that the morphism $M \boxtimes_R \varphi : M \boxtimes_R X \longrightarrow M \boxtimes_R Y$ is an $\mathcal{F}$-equivalence, so that M is homotopical. Thus the class of homotopical R-modules is closed under retracts, since $\mathcal{F}$-equivalences are closed under retracts, and so every cofibrant right R-module is homotopical. □

1.5 Equivariant homotopy sets

In this section we define the equivariant homotopy sets $\pi_0^G(Y)$ of orthogonal spaces and relate them by restriction maps defined from continuous homomorphisms between compact Lie groups. The resulting structure, as the Lie groups vary, is a 'Rep-functor' $\underline{\pi}_0(Y)$, i.e., a contravariant functor from the category of compact Lie groups and conjugacy classes of continuous homomorphisms. The Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}G)$ associated with a global classifying space is the Rep-functor represented by G, by Proposition 1.5.12. We identify the category of all natural operations with the category Rep of conjugacy classes of continuous homomorphisms, compare Corollary 1.5.14. Construction 1.5.15 introduces a pairing of equivariant homotopy sets

$$\times \ : \ \pi_0^G(X) \times \pi_0^G(Y) \ \longrightarrow \ \pi_0^G(X \boxtimes Y)$$

for any pair of orthogonal spaces, and Proposition 1.5.17 summarizes its main properties.

We recall that $\mathcal{U}_G$ is a chosen complete G-universe and $s(\mathcal{U}_G)$ denotes the poset, under inclusion, of finite-dimensional G-subrepresentations of $\mathcal{U}_G$.

Definition 1.5.1. Let Y be an orthogonal space, G be a compact Lie group and A a G-space. We define

$$[A, Y]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \, [A, Y(V)]^G \ ,$$

the colimit over the poset $s(\mathcal{U}_G)$ of the sets of G-homotopy classes of G-maps from A to $Y(V)$.

The canonical G-maps $Y(V) \longrightarrow Y(\mathcal{U}_G)$ induce maps from $[A, Y(V)]^G$ to $[A, Y(\mathcal{U}_G)]^G$ and hence a canonical map

$$[A, Y]^G \longrightarrow [A, Y(\mathcal{U}_G)]^G . \tag{1.5.2}$$

In general there is no reason for this map to be injective or surjective. If Y is closed and A is compact, the situation improves:

Proposition 1.5.3. *Let G be a compact Lie group.*

(i) *The canonical map* (1.5.2) *is bijective for every closed orthogonal space Y and every compact G-space A.*
(ii) *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ an $\mathcal{F}$-equivalence of orthogonal spaces. Then for every finite G-CW-complex A all of whose isotropy groups belong to $\mathcal{F}$, the induced map*

$$[A, f]^G \ : \ [A, X]^G \longrightarrow [A, Y]^G$$

is bijective.
(iii) *For every pair of orthogonal spaces X and Y and every G-space A, the canonical map*

$$([A, p_X]^G, [A, p_Y]^G) \ : \ [A, X \times Y]^G \longrightarrow [A, X]^G \times [A, Y]^G$$

is bijective (where p_X and p_Y are the projections).

Proof (i) Since the poset $s(\mathcal{U}_G)$ contains a cofinal subsequence, $Y(\mathcal{U}_G)$ is a sequential colimit of values of Y along closed embeddings. By Proposition A.15 (i), every continuous G-map $A \longrightarrow Y(\mathcal{U}_G)$ thus factors through $Y(V)$ for some finite-dimensional $V \in s(\mathcal{U}_G)$, which shows surjectivity. Injectivity follows by the same argument applied to the compact G-space $A \times [0, 1]$.

(ii) We let $\beta : A \longrightarrow Y(V)$ be a continuous G-map, for some $V \in s(\mathcal{U}_G)$, that represents an element of $[A, Y]^G$. Together with the unique map from the empty space this specifies an equivariant lifting problem on the left:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X(V) \arrow[d, "f(V)"] \\ A \arrow[r, "\beta"'] & Y(V) \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X(V) \arrow[r, "X(\varphi)"] & X(W) \arrow[d, "f(W)"] \\ A \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V) \arrow[r, "Y(\varphi)"'] & Y(W) \end{tikzcd}$$

Since $(A, \emptyset)$ is a finite G-CW-pair with isotropy in $\mathcal{F}$ and f an $\mathcal{F}$-equivalence, Proposition 1.4.5 (ii) provides a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map λ on the right-hand side such that $f(W) \circ \lambda$

is G-homotopic to $Y(\varphi) \circ \beta$. We choose a G-equivariant linear isometric embedding $j : W \longrightarrow \mathcal{U}_G$ extending the inclusion of V. Then the class in $[A, X]^G$ represented by the G-map

$$X(j) \circ \lambda \;:\; A \;\longrightarrow\; X(j(W))$$

is taken to $[\beta]$ by the map $[A, f]^G$. This shows that $[A, f]^G$ is surjective.

For injectivity we consider two G-maps $g, g' : A \longrightarrow X(V)$, for some $V \in s(\mathcal{U}_G)$, such that $[A, f]^G[g] = [A, f]^G[g']$. By enlarging V, if necessary, we can assume that the two composites $f(V) \circ g$ and $f(V) \circ g'$ are G-homotopic. A choice of such a homotopy specifies an equivariant lifting problem on the left:

$$\begin{tikzcd} A \times \{0,1\} \arrow[r, "{g,g'}"] \arrow[d] & X(V) \arrow[d, "f(V)"] \\ A \times [0,1] \arrow[r, "\beta"'] & Y(V) \end{tikzcd} \qquad \begin{tikzcd} A \times \{0,1\} \arrow[r, "{g,g'}"] \arrow[d] & X(V) \arrow[r, "X(\varphi)"] & X(W) \arrow[d, "f(W)"] \\ A \times [0,1] \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V) \arrow[r, "Y(\varphi)"'] & Y(W) \end{tikzcd}$$

Proposition 1.4.5 (ii) provides a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a lift λ on the right-hand side such that $\lambda(-, 0) = g$, $\lambda(-, 1) = g'$ and $f(W) \circ \lambda$ is G-homotopic, relative $A \times \{0, 1\}$, to $Y(\varphi) \circ \beta$. As in the first part, we use a G-equivariant linear isometric embedding $j : W \longrightarrow \mathcal{U}_G$, extending the inclusion of V, to transform λ into the G-homotopy

$$X(j) \circ \lambda \;:\; A \times [0, 1] \;\longrightarrow\; X(j(W))$$

that connects the images of g and g' in $X(j(W))$. This shows that $[g] = [g']$ in $[A, X]^G$, so $[A, f]^G$ is also injective.

(iii) Products of orthogonal spaces are formed objectwise, so the canonical map

$$[A, (X \times Y)(V)]^G \;\longrightarrow\; [A, X(V)]^G \times [A, Y(V)]^G$$

is bijective for every G-representation V. Filtered colimits commute with finite products, so the claim follows by passage to colimits over the poset $s(\mathcal{U}_G)$. □

Example 1.5.4. We specialize to the case where $Y = B_{\mathrm{gl}}G$ is the global classifying space of a compact Lie group G. Proposition 1.1.30 above already gave an explanation for the name 'global classifying space' by exhibiting $(B_{\mathrm{gl}}G)(\mathcal{U}_K)$ as a classifying space for principal (K, G)-bundles over paracompact K-spaces. We now reinterpret this result as follows.

We choose a faithful G-representation V, so that $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$. We let A be a compact K-space and consider the composite

$$[A, B_{\mathrm{gl}}G]^K \xrightarrow{\cong} [A, (B_{\mathrm{gl}}G)(\mathcal{U}_K)]^K \xrightarrow{\cong} \mathrm{Prin}_{(K,G)}(A)\,,$$

where the first map is the bijection of Proposition 1.5.3 (i), exploiting that the

orthogonal space $\mathbf{L}_{G,V}$ is closed. The second map is the bijection provided by Proposition 1.1.30. The composite bijection

$$[A, B_{\mathrm{gl}}G]^K \cong \mathrm{Prin}_{(K,G)}(A)$$

sends the class represented by a continuous K-equivariant map $f : A \longrightarrow \mathbf{L}(V, W)/G$ to the class of the pullback principal (K, G)-bundle f^*q over A, where $q : \mathbf{L}(V, W) \longrightarrow \mathbf{L}(V, W)/G$ is the projection.

Now we specialize the equivariant homotopy sets $[A, Y]^G$ to the case $A = \{*\}$ of a one-point G-space, and then give it a new name.

Definition 1.5.5. Let G be a compact Lie group. The *G-equivariant homotopy set* of an orthogonal space Y is the set

$$\pi_0^G(Y) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} \pi_0(Y(V)^G) . \tag{1.5.6}$$

Specializing Proposition 1.5.3 to a one-point G-space yields:

Corollary 1.5.7. *Let G be a compact Lie group.*

(i) *For every closed orthogonal space Y the canonical map*

$$\pi_0^G(Y) \longrightarrow \pi_0(Y(\mathcal{U}_G)^G)$$

is bijective.

(ii) *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ an $\mathcal{F}$-equivalence of orthogonal spaces. Then for every compact Lie group G in $\mathcal{F}$ the induced map*

$$\pi_0^G(f) : \pi_0^G(X) \longrightarrow \pi_0^G(Y)$$

of equivariant homotopy sets is bijective.

The homotopy sets $\pi_0^G(Y)$ have contravariant functoriality in G: every continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups induces a restriction map $\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y)$, as we shall now explain. We denote by α^* the restriction functor from G-spaces to K-spaces (or from G-representations to K-representations) along α, i.e., α^*Z (or α^*V) is the same topological space as Z (or the same inner product space as V) endowed with a K-action via

$$k \cdot z = \alpha(k) \cdot z .$$

Given an orthogonal space Y, we note that the K-spaces $\alpha^*(Y(V))$ and $Y(\alpha^*V)$ are equal (not just isomorphic) for every G-representation V.

The restriction $\alpha^*(\mathcal{U}_G)$ is a K-universe, but if α has a non-trivial kernel, then this K-universe is not complete. When α is injective, $\alpha^*(\mathcal{U}_G)$ is a complete K-universe (by Remark 1.1.13), but typically different from the chosen complete K-universe $\mathcal{U}_K$. To deal with this we explain how a G-fixed-point $y \in Y(U)^G$,

for an arbitrary G-representation U, gives rise to an unambiguously defined element $\langle y \rangle$ in $\pi_0^G(Y)$. The point here is that U need not be a subrepresentation of the chosen universe $\mathcal{U}_G$ and the resulting class does not depend on any additional choices.

To construct $\langle y \rangle$ we choose a G-equivariant linear isometry $j : U \longrightarrow V$ onto a G-subrepresentation V of $\mathcal{U}_G$. Then $Y(j)(y)$ is a G-fixed-point of $Y(V)$, so we obtain an element

$$\langle y \rangle \ = \ [Y(j)(y)] \ \in \ \pi_0^G(Y) \ .$$

It is crucial, but not completely obvious, that $\langle f \rangle$ does not depend on the choice of isometry j.

Proposition 1.5.8. *Let Y be an orthogonal space, G a compact Lie group, U a G-representation and $y \in Y(U)^G$ a G-fixed-point.*

(i) *The class $\langle y \rangle$ in $\pi_0^G(Y)$ is independent of the choice of linear isometry from U to a subrepresentation of $\mathcal{U}_G$.*
(ii) *For every G-equivariant linear isometric embedding $\varphi : U \longrightarrow W$ the relation*

$$\langle Y(\varphi)(y) \rangle \ = \ \langle y \rangle \qquad \textit{holds in} \quad \pi_0^G(Y).$$

Proof (i) We let $j : U \longrightarrow V$ and $j' : U \longrightarrow V'$ be two G-equivariant linear isometries, with $V, V' \in s(\mathcal{U}_G)$. We choose a third G-equivariant linear isometry $j'' : U \longrightarrow V''$ such that $V'' \in s(\mathcal{U}_G)$ and V'' is orthogonal to both V and V'. We let W be the span of V, V' and V'' inside $\mathcal{U}_G$. We can then view j, j' and j'' as equivariant linear isometric embeddings from U to W.

Since the images of j and j'' are orthogonal, they are homotopic through G-equivariant linear isometric embeddings into W, via the homotopy $H : U \times [0,1] \longrightarrow W$ given by

$$H(u,t) \ = \ \sqrt{1-t^2} \ \cdot j(u) \ + \ t \cdot j''(u) \ .$$

By the same argument, j' and j'' are homotopic through G-equivariant linear isometric embeddings. In particular, j and j' are homotopic to each other. If $H(-,t) : U \longrightarrow W$ is a continuous 1-parameter family of G-equivariant linear isometric embeddings from j to j', then

$$t \ \longmapsto \ Y(H(-,t))(y)$$

is a path in $Y(W)^G$ from $Y(j)(y)$ to $Y(j')(y)$, so $[Y(j)(y)] = [Y(j')(y)]$ in $\pi_0^G(Y)$.

(ii) If $j : W \longrightarrow V$ is an equivariant linear isometry with $V \in s(\mathcal{U}_G)$, we define $\bar{V} = j(\varphi(U))$ and we let $k : U \longrightarrow \bar{V}$ be the equivariant linear isometry

that is defined by $k(u) = j(\varphi(u))$ (i.e., k is essentially $j \circ \varphi$, but with range $\bar{V}$ instead of V). Then

$$\langle Y(\varphi)(y)\rangle = [Y(j)(Y(\varphi)(y))] = [Y(j\varphi)(y)] = [Y(k)(y)] = \langle y\rangle . \qquad \square$$

We can now define the *restriction map* associated to a continuous group homomorphism $\alpha : K \longrightarrow G$ by

$$\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y), \quad [y] \longmapsto \langle y\rangle . \tag{1.5.9}$$

This makes sense because every G-fixed-point of $Y(V)$ is also a K-fixed-point of $\alpha^*(Y(V)) = Y(\alpha^* V)$. For a second continuous group homomorphism $\beta : L \longrightarrow K$ we have

$$\beta^* \circ \alpha^* = (\alpha\beta)^* : \pi_0^G(Y) \longrightarrow \pi_0^L(Y) .$$

Clearly, restriction along the identity homomorphism is the identity, so we have made the collection of equivariant homotopy sets $\pi_0^G(Y)$ a contravariant functor in the group variable.

An important special case of the restriction homomorphisms are conjugation maps. Here we consider a closed subgroup H of G, an element $g \in G$ and denote the conjugation homomorphism by

$$c_g : H \longrightarrow H^g , \quad c_g(h) = g^{-1}hg ,$$

where $H^g = \{g^{-1}hg \mid h \in H\}$ is the conjugate subgroup. As for any group homomorphism, c_g induces a restriction map of equivariant homotopy sets

$$g_\star = (c_g)^* : \pi_0^{H^g}(Y) \longrightarrow \pi_0^H(Y) .$$

For $g, \bar{g} \in G$ we have $c_{g\bar{g}} = c_{\bar{g}} \circ c_g : H \longrightarrow H^{g\bar{g}}$ and thus

$$(g\bar{g})_\star = (c_{g\bar{g}})^* = (c_{\bar{g}} \circ c_g)^* = (c_g)^* \circ (c_{\bar{g}})^* = g_\star \circ \bar{g}_\star .$$

A key fact is that inner automorphisms act trivially, i.e., the restriction map $g_\star$ is the identity on $\pi_0^G(Y)$. So the action by the restriction maps of the automorphism group of G on $\pi_0^G(Y)$ factors through the outer automorphism group.

Proposition 1.5.10. *For every orthogonal space Y, every compact Lie group G, and every $g \in G$, the conjugation map $g_\star : \pi_0^G(Y) \longrightarrow \pi_0^G(Y)$ is the identity.*

Proof We consider a finite-dimensional G-subrepresentation V of $\mathcal{U}_G$ and a G-fixed-point $y \in Y(V)^G$ that represents an element in $\pi_0^G(Y)$. Then the map $l_g : c_g^*(V) \longrightarrow V$ given by left multiplication by g is a G-equivariant linear isometry. So

$$g_\star[y] = (c_g)^*[y] = [Y(l_g)(y)] = [g \cdot y] = [y] ,$$

by the very definition of the restriction map. The third equation is the definition

of the G-action on $Y(V)$ through the G-action on V. The fourth equation is the hypothesis that y is G-fixed. □

We note that because inner automorphisms act as the identity, the restriction map α^* only depends on the homotopy class of α. More precisely, suppose that $\alpha, \alpha' : K \longrightarrow G$ are homotopic through continuous group homomorphisms. Then α and α' belong to the same path component of the space $\hom(K, G)$ of continuous homomorphisms, so they are conjugate by an element of G, compare Proposition A.25.

We denote by Rep the category whose objects are the compact Lie groups and whose morphisms are conjugacy classes of continuous group homomorphisms. We can summarize the discussion thus far by saying that for every orthogonal space Y the restriction maps make the equivariant homotopy sets $\{\pi_0^G(Y)\}$ a functor

$$\underline{\pi}_0(Y) \ : \ \mathrm{Rep}^{\mathrm{op}} \longrightarrow (\mathrm{sets})\ .$$

We will refer to such a functor as a *Rep-functor*.

Our next aim is to show that the homotopy Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}G)$ of a global classifying space is represented by G. For every G-representation V we define the *tautological class*

$$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V}) \tag{1.5.11}$$

to be the path component of the G-fixed-point

$$\mathrm{Id}_V \cdot G \ \in \ (\mathbf{L}(V, V)/G)^G = (\mathbf{L}_{G,V}(V))^G\ ,$$

the G-orbit of the identity of V.

Proposition 1.5.12. *Let G and K be compact Lie groups and V a faithful G-representation.*

(i) *The K-fixed-point space $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α.*

(ii) *The map*

$$\mathrm{Rep}(K, G) \ \longrightarrow \ \pi_0^K(\mathbf{L}_{G,V})\ , \quad [\alpha : K \longrightarrow G] \ \longmapsto \ \alpha^*(u_{G,V})$$

is bijective.

Proof Part (i) works for any universal $(K\times G)$-space E for the family $\mathcal{F}(K; G)$ of graph subgroups, for example for $E = \mathbf{L}(V, \mathcal{U}_K)$. The argument can be found in Theorem 2.17 of [94] or Proposition 5 of [99]. We repeat the proof for the

convenience of the reader. For a continuous homomorphism $\alpha : K \longrightarrow G$, we let $C(\alpha)$ denote the centralizer, in G, of the image of α, and we set

$$E^\alpha \;=\; \{x \in E \mid (k, \alpha(k)) \cdot x = x \text{ for all } k \in K\} ,$$

the space of fixed-points of the graph of α. Since the G-action on the universal space E is free, Proposition B.17 provides a homeomorphism

$$\coprod \alpha^\flat \;:\; \coprod\nolimits_{\langle\alpha\rangle} E^\alpha/C(\alpha) \;\longrightarrow\; (E/G)^K ,$$

where the disjoint union is indexed by conjugacy classes of continuous homomorphisms. The graph of α belongs to the family $\mathcal{F}(K;G)$, so E^α is a contractible space. The action of $C(\alpha)$ on E^α is a restriction of the G-action on E, and is hence free. Since E is $(K \times G)$-cofibrant, the fixed-point space E^α is cofibrant for the action of the normalizer (inside $K \times G$) of the graph of α, by Proposition B.12. Hence E^α is also cofibrant as a $C(\alpha)$-space, by Proposition B.14 (i). So for every homomorphism α the space $E^\alpha/C(\alpha)$ is a classifying space for the group $C(\alpha)$. This shows part (i).

(ii) Since the classifying space of a topological group is connected, part (i) identifies the path components of $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ with the conjugacy classes of continuous homomorphisms $\alpha : K \longrightarrow G$. The bijection sends the class of α to $\alpha^*(u_{G,V})$. The claim then follows by applying Corollary 1.5.7 (i). □

Now we show that the restriction maps along continuous group homomorphisms give all natural operations between equivariant homotopy sets of orthogonal spaces. We will perform similar calculations several other times in this book, so we state the argument in the more general situation of a category C related to the category of orthogonal spaces by an adjoint functor pair. Our present context is the degenerate case $C = spc$ and the identity functors. Later we will also consider the cases of the categories of ultra-commutative monoids, of orthogonal spectra and of ultra-commutative ring spectra.

We recall that the restriction morphism

$$\rho_{G,V,W} \;=\; \rho_{V,W}/G \;:\; \mathbf{L}_{G,V\oplus W} \;=\; \mathbf{L}(V \oplus W, -)/G \;\longrightarrow\; \mathbf{L}(W,-)/G \;=\; \mathbf{L}_{G,W}$$

restricts the orbit of a linear isometric embedding from $V \oplus W$ to the second summand W. This morphism is a global equivalence of orthogonal spaces by Proposition 1.1.26 (ii), as long as G acts faithfully on W.

Proposition 1.5.13. *Let C be a category and*

$$\Lambda \;:\; spc \rightleftarrows C \;:\; U$$

an adjoint functor pair such that the composite functor $U\Lambda : spc \longrightarrow spc$

is continuous. Suppose in addition that for every compact Lie group G, all G-representations V and all non-zero faithful G-representations W, the morphism of Rep-*functors*

$$\underline{\pi}_0(U\Lambda(\rho_{G,V,W})) \; : \; \underline{\pi}_0(U\Lambda(\mathbf{L}_{G,V\oplus W})) \longrightarrow \underline{\pi}_0(U\Lambda(\mathbf{L}_{G,W}))$$

is an isomorphism. Let W be a non-zero faithful G-representation, and set

$$u^C_{G,W} \; = \; \eta_*(u_{G,W}) \; \in \; \pi^G_0(U\Lambda(\mathbf{L}_{G,W})) \; ,$$

where $\eta : \mathbf{L}_{G,W} \longrightarrow U\Lambda(\mathbf{L}_{G,W})$ *is the adjunction unit. Then for every compact Lie group K, evaluation at the class* $u^C_{G,W}$ *is a bijection*

$$\mathrm{Nat}_{C\to(\mathrm{sets})}(\pi^G_0 \circ U, \pi^K_0 \circ U) \longrightarrow \pi^K_0(U\Lambda(\mathbf{L}_{G,W})) \; , \quad \tau \longmapsto \tau(u^C_{G,W})$$

between the set of natural transformations of functors from $\pi^G_0 \circ U$ *to* $\pi^K_0 \circ U$*, and the set* $\pi^K_0(U\Lambda(\mathbf{L}_{G,W}))$*.*

Proof To show that the evaluation map is injective we show that every natural transformation $\tau : \pi^G_0 \circ U \longrightarrow \pi^K_0 \circ U$ is determined by the element $\tau(u^C_{G,W})$. We let X be any object of C and $[x] \in \pi^G_0(UX)$ a G-equivariant homotopy class, represented by a G-fixed-point

$$x \; \in \; (UX)(V)^G \; ,$$

for some G-representation V. We can stabilize with the representation W and obtain another representative

$$(UX)(i)(x) \; \in \; (UX)(V \oplus W)^G \; ,$$

where $i : V \longrightarrow V \oplus W$ is the embedding of the first summand. This G-fixed-point is adjoint to a morphism of orthogonal spaces

$$\hat{x} \; : \; \mathbf{L}_{G,V\oplus W} \longrightarrow UX$$

and hence adjoint to a C-morphism

$$x^\flat \; : \; \Lambda(\mathbf{L}_{G,V\oplus W}) \longrightarrow X$$

that satisfies

$$\pi^G_0(Ux^\flat)(u^C_{G,V\oplus W}) \; = \; [x] \quad \text{in } \pi^G_0(UX) \; .$$

The restriction morphism of orthogonal spaces $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,W}$ sends $u_{G,V\oplus W}$ to $u_{G,W}$. The morphism of orthogonal spaces

$$U\Lambda(\rho_{G,V,W}) \; : \; U\Lambda(\mathbf{L}_{G,V\oplus W}) \longrightarrow U\Lambda(\mathbf{L}_{G,W})$$

then sends $u^C_{G,V\oplus W}$ to $u^C_{G,W}$. The diagram

$$\begin{tikzcd}
\pi_0^G(U\Lambda(\mathbf{L}_{G,W})) \arrow[d, "\tau"] & \pi_0^G(U\Lambda(\mathbf{L}_{G,V\oplus W})) \arrow[l, "\pi_0^G(U\Lambda(\rho_{G,V,W}))", "\cong"'] \arrow[r, "\pi_0^G(Ux^\flat)"] \arrow[d, "\tau"] & \pi_0^G(UX) \arrow[d, "\tau"] \\
\pi_0^K(U\Lambda(\mathbf{L}_{G,W})) & \pi_0^K(U\Lambda(\mathbf{L}_{G,V\oplus W})) \arrow[l, "\cong", "\pi_0^K(U\Lambda(\rho_{G,V,W}))"'] \arrow[r, "\pi_0^K(Ux^\flat)"'] & \pi_0^K(UX)
\end{tikzcd}$$

commutes and the two left horizontal maps are bijective by hypothesis. Since

$$[x] \;=\; \pi_0^G(Ux^\flat)(\pi_0^G(U\Lambda(\rho_{G,V,W}))^{-1}(u^C_{G,W}))\ ,$$

naturality yields that

$$\begin{aligned}
\tau[x] \;&=\; \tau(\pi_0^G(Ux^\flat)(\pi_0^G(U\Lambda(\rho_{G,V,W}))^{-1}(u^C_{G,W}))) \\
&=\; \pi_0^K(Ux^\flat)(\pi_0^K(U\Lambda(\rho_{G,V,W}))^{-1}(\tau(u^C_{G,W})))\ .
\end{aligned}$$

This shows that the transformation τ is determined by the value $\tau(u^C_{G,W})$.

It remains to construct, for every element $z \in \pi_0^K(U\Lambda(\mathbf{L}_{G,W}))$, a natural transformation $\tau : \pi_0^G \circ U \longrightarrow \pi_0^K \circ U$ with $\tau(u^C_{G,W}) = z$. The previous paragraph dictates what to do: we represent a given class in $\pi_0^G(UX)$ by a G-fixed-point $x \in (UX)(V \oplus W)^G$ and set

$$\tau[x] \;=\; \pi_0^K(Ux^\flat)(\pi_0^K(U\Lambda(\rho_{G,V,W}))^{-1}(z))\ .$$

We must verify that the element $\tau[x]$ is independent of the representative. If $y \in (UX)(V \oplus W)^G$ is in the same path component as x, then any path adjoins to a homotopy of morphisms of orthogonal spaces

$$H \;:\; \mathbf{L}_{G,V\oplus W} \times [0,1] \;\longrightarrow UX$$

from $\hat{x}$ to $\hat{y}$. Since the functor $U\Lambda$ is continuous, the composite morphisms

$$U\Lambda(\mathbf{L}_{G,V\oplus W}) \;\xrightarrow{U\Lambda(H(-,t))}\; U\Lambda(UX) \;\xrightarrow{U(\epsilon_X)}\; UX$$

form a continuous 1-parameter family of morphisms for $t \in [0,1]$, where $\epsilon_X : \Lambda(UX) \longrightarrow X$ is the counit of the adjunction. This witnesses that the morphism $Ux^\flat$ is homotopic to the morphism $Uy^\flat$. So $\pi_0^K(Ux^\flat) = \pi_0^K(Uy^\flat)$, and the class $\tau[x]$ is independent of the representative in the given path component of $(UX)(V \oplus W)^G$.

Now we let V' be another G-representation and $\varphi : V \longrightarrow V'$ a G-equivariant linear isometric embedding. Then

$$y \;=\; (UX)(\varphi \oplus W)(x) \qquad \text{in} \quad (UX)(V' \oplus W)^G$$

is another representative of the class $[x]$. The restriction morphism

$$\varphi^\sharp \;=\; \mathbf{L}(\varphi \oplus W, -)/G \;:\; \mathbf{L}_{G,V'\oplus W} \;\longrightarrow\; \mathbf{L}_{G,V\oplus W}$$

makes the following diagram of orthogonal spaces commute:

$$\begin{tikzcd} \mathbf{L}_{G,W} & \mathbf{L}_{G,V'\oplus W} \arrow[l, "\rho_{G,V',W}"'] \arrow[r, "\hat{y}"] \arrow[d, "\varphi^\sharp"'] & UX \\ & \mathbf{L}_{G,V\oplus W} \arrow[ul, "\rho_{G,V,W}"] \arrow[ur, "\hat{x}"'] & \end{tikzcd}$$

Passing to adjoints and applying U yields another commutative diagram:

$$\begin{tikzcd} U\Lambda(\mathbf{L}_{G,W}) & U\Lambda(\mathbf{L}_{G,V'\oplus W}) \arrow[l, "U\Lambda(\rho_{G,V',W})"'] \arrow[r, "Uy^\flat"] \arrow[d, "U\Lambda(\varphi^\sharp)"'] & UX \\ & U\Lambda(\mathbf{L}_{G,V\oplus W}) \arrow[ul, "U\Lambda(\rho_{G,V,W})"] \arrow[ur, "Ux^\flat"'] & \end{tikzcd}$$

So

$$\begin{aligned} \pi_0^K(Ux^\flat)\circ\pi_0^K(U\Lambda(\rho_{G,V,W}))^{-1} &= \pi_0^K(Ux^\flat)\circ\pi_0^K(U\Lambda(\varphi^\sharp))\circ\pi_0^K(U\Lambda(\rho_{G,V',W}))^{-1} \\ &= \pi_0^K(Uy^\flat)\circ\pi_0^K(U\Lambda(\rho_{G,V',W}))^{-1}\ , \end{aligned}$$

and hence the class $\tau[x]$ remains unchanged upon stabilization of x along φ. Altogether this shows that $\tau[x]$ is well-defined.

Naturality of τ is straightforward: if $\psi : X \longrightarrow Y$ is a C-morphism and $x \in (UX)(V\oplus W)^G$ is as above, then $(U\psi)(V\oplus W)(x) \in (UY)(V\oplus W)^G$ represents the class $\pi_0^G(U\psi)[x]$. Moreover, the adjoint of $(U\psi)(V\oplus W)(x)$ coincides with the composite

$$\Lambda(\mathbf{L}_{G,V\oplus W}) \xrightarrow{x^\flat} X \xrightarrow{\psi} Y\ .$$

So naturality follows:

$$\begin{aligned} \tau(\pi_0^G(U\psi)[x]) &= \pi_0^K(U\psi\circ Ux^\flat)(\pi_0^K(U\Lambda(\rho_{G,V,W}))^{-1}(z)) \\ &= \pi_0^K(U\psi)\left(\pi_0^K(Ux^\flat)(\pi_0^K(U\Lambda(\rho_{G,V,W}))^{-1}(z))\right) = \pi_0^K(U\psi)(\tau[x])\ . \end{aligned}$$

Finally, the class $u^C_{G,W}$ is represented by the G-fixed-point

$$\eta_{\mathbf{L}_{G,W}}(\mathrm{Id}_W\cdot G) \in (U\Lambda(\mathbf{L}_{G,W}))(W)^G\ ,$$

which is adjoint to the identity of $\Lambda(\mathbf{L}_{G,W})$. Hence $\tau(u^C_{G,W}) = \pi_0^K(\mathrm{Id})(z) = z$. □

Corollary 1.5.14. *Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set-valued functors on the category of orthogonal spaces is of the form α^* for a continuous group homomorphism $\alpha : K \longrightarrow G$, unique up to conjugacy.*

Proof We let W be any non-zero faithful G-representation. The composite

$$\mathrm{Rep}(K,G) \xrightarrow{[\alpha]\mapsto\alpha^*} \mathrm{Nat}(\pi_0^G,\pi_0^K) \xrightarrow{\mathrm{ev}} \pi_0^K(\mathbf{L}_{G,W})$$

is bijective by Proposition 1.5.12 (ii), where the second map is evaluation at the tautological class $u_{G,W}$. The evaluation map is bijective by Proposition 1.5.13, applied to $C = spc$ and the identity adjoint functor pair. So the first map is bijective as well. □

Construction 1.5.15. Given two orthogonal spaces X and Y, we endow the equivariant homotopy sets with a pairing

$$\times \;:\; \pi_0^G(X) \times \pi_0^G(Y) \longrightarrow \pi_0^G(X \boxtimes Y)\,, \tag{1.5.16}$$

where G is any compact Lie group. We suppose that V and W are G-representations and $x \in X(V)^G$ and $y \in Y(W)^G$ are fixed-points that represent classes in $\pi_0^G(X)$ and $\pi_0^G(Y)$, respectively. We denote by $x \times y$ the image of the G-fixed-point (x, y) under the G-map

$$i_{V,W} \;:\; X(V) \times Y(W) \longrightarrow (X \boxtimes Y)(V \oplus W)$$

that is part of the universal bimorphism. If $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ are equivariant linear isometric embeddings, then

$$\begin{aligned} X(\varphi)(x) \times Y(\psi)(y) \;&=\; i_{V',W'}(X(\varphi)(x), Y(\psi)(y)) \\ &=\; (X \boxtimes Y)(\varphi \oplus \psi)(i_{V,W}(x,y)) \;=\; (X \boxtimes Y)(\varphi \oplus \psi)(x \times y)\,. \end{aligned}$$

So by Proposition 1.5.8 (ii) the classes $\langle x \times y\rangle$ and $\langle X(\varphi)(x) \times Y(\psi)(y)\rangle$ coincide in $\pi_0^G(X \boxtimes Y)$. The upshot is that the assignment

$$[x] \times [y] \;=\; \langle x \times y\rangle \;\in\; \pi_0^G(X \boxtimes Y)$$

is well-defined.

The pairings of equivariant homotopy sets have several expected properties which we summarize in the next proposition.

Proposition 1.5.17. *Let G be a compact Lie group and X, Y and Z orthogonal spaces.*

(i) (Unitality) *Let 1 be the unique element of the set $\pi_0^G(\mathbf{1})$. Then $1 \times x = x = x \times 1$ for all $x \in \pi_0^G(X)$.*
(ii) (Associativity) *For all classes $x \in \pi_0^G(X)$, $y \in \pi_0^G(Y)$ and $z \in \pi_0^G(Z)$ the relation*

$$a_*((x \times y) \times z) \;=\; x \times (y \times z)$$

holds in $\pi_0^G(X \boxtimes (Y \boxtimes Z))$, where $a : (X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z)$ is the associativity isomorphism.

(iii) (Commutativity) *For all classes* $x \in \pi_0^G(X)$ *and* $y \in \pi_0^G(Y)$ *the relation*

$$\tau_*(x \times y) \ = \ y \times x$$

holds in $\pi_0^G(Y \boxtimes X)$, *where* $\tau : X \boxtimes Y \longrightarrow Y \boxtimes X$ *is the symmetry isomorphism.*

(iv) (Restriction) *For all classes* $x \in \pi_0^G(X)$ *and* $y \in \pi_0^G(Y)$ *and all continuous homomorphisms* $\alpha : K \longrightarrow G$, *the relation*

$$\alpha^*(x) \times \alpha^*(y) \ = \ \alpha^*(x \times y)$$

holds in $\pi_0^K(X \boxtimes Y)$.

Proof The unitality property (i), the associativity property (ii) and compatibility with restriction (iv) are straightforward from the definitions. Part (iii) exploits that the square

$$\begin{tikzcd} X(V) \times Y(W) \arrow[r, "i_{V,W}"] \arrow[d, "\text{twist}"'] & (X \boxtimes Y)(V \oplus W) \arrow[d, "\tau(\chi_{V,W})"] \\ Y(W) \times X(V) \arrow[r, "i_{W,V}"'] & (Y \boxtimes X)(W \oplus V) \end{tikzcd}$$

commutes. The image of (x, y) under the upper right composite represents $\tau_*(x \times y)$, whereas the image of (y, x) under the lower left composite represents $y \times x$, so $\tau_*(x \times y) = y \times x$. □

Remark 1.5.18 (External versus internal products)**.** If G and K are two compact Lie groups, we can define an 'external' form of the pairing (1.5.16) as the composite

$$\pi_0^G(X) \times \pi_0^K(Y) \xrightarrow{p_1^* \times p_2^*} \pi_0^{G\times K}(X) \times \pi_0^{G\times K}(Y) \xrightarrow{\times} \pi_0^{G\times K}(X \boxtimes Y) \,, \quad (1.5.19)$$

where $p_1 : G \times K \longrightarrow G$ and $p_2 : G \times K \longrightarrow K$ are the two projections. These external pairings also satisfy various naturality, unitality, associativity and commutativity properties which we do not spell out. On the other hand, the internal pairing (1.5.16) can be recovered from the external products (1.5.19) by taking $G = K$ and restricting along the diagonal embedding $\Delta_G : G \longrightarrow G \times G$. Indeed, the $p_1 \circ \Delta_G = p_2 \circ \Delta_G = \mathrm{Id}_G$, and hence

$$\Delta_G^*(p_1^*(x) \times p_2^*(y)) \ = \ \Delta_G^*(p_1^*(x)) \times \Delta_G^*(p_2^*(y)) \ = \ x \times y \,.$$

Theorem 1.3.2 (i) and the fact that the functor π_0^G commutes with finite products (Proposition 1.5.3 (iii) for $A = *$) imply:

Corollary 1.5.20. *For every compact Lie group G and all orthogonal spaces X and Y, the three maps*

$$\pi_0^G(X)\times\pi_0^G(Y) \xrightarrow{\times} \pi_0^G(X\boxtimes Y) \xrightarrow{(\rho_{X,Y})_*} \pi_0^G(X\times Y) \xrightarrow{((p_X)_*,(p_Y)_*)} \pi_0^G(X)\times\pi_0^G(Y)$$

are bijective, where $p_X : X\times Y \longrightarrow X$ and $p_Y : X\times Y \longrightarrow Y$ are the projections. Moreover, the composite is the identity.

Construction 1.5.21 (Infinite box products)**.** We end this section with a generalization of the previous corollary to 'infinite box products' of based orthogonal spaces, but we first have to clarify what we mean by that. We let I be an indexing set and $\{X_i\}_{i\in I}$ a family of based orthogonal spaces, i.e., each equipped with a distinguished point $x_i \in X_i(0)$. If $K \subset J$ are two nested, *finite* subsets of I, then the basepoints of X_k for $k \in J - K$ provide a morphism

$$\boxtimes_{k\in K} X_k \longrightarrow \boxtimes_{j\in J} X_j \ . \tag{1.5.22}$$

In terms of the universal property of the box product, this morphism arises from the maps

$$\prod\nolimits_{k\in K} X_k(V_k) \longrightarrow \prod\nolimits_{k\in K} X_k(V_k)\times\prod\nolimits_{j\in J-K} X_j(0) \longrightarrow (\boxtimes_{j\in J} X_j)(\oplus_{k\in K} V_k) \ ,$$

where the second map is part of the universal multi-morphism. We can thus define the *infinite box product* as the colimit of the finite box products over the filtered poset of finite subsets of I:

$$\boxtimes'_{i\in I} X_i \ = \ \mathrm{colim}_{J\subset I, |J|<\infty} \ \boxtimes_{j\in J} X_j \ .$$

If I happens to be finite, then this recovers the iterated box product.

The distinguished basepoint of X_i represents a distinguished basepoint in the equivariant homotopy set $\pi_0^G(X_i)$ for every compact Lie group G. In fact, these points all arise from the basepoint in $\pi_0^e(X_i)$ by restriction along the unique homomorphism $G \longrightarrow e$. The *weak product* $\prod'_{i\in I}\pi_0^G(X_i)$ is the subset of the product consisting of all tuples $(y_i)_{i\in I}$ with the property that almost all y_i are the distinguished basepoint. Equivalently, the weak product is the filtered colimit of the finite products over the poset of finite subsets of I.

If we iterate the pairing (1.5.16), it provides a multi-pairing

$$\prod\nolimits_{j\in J} \pi_0^G(X_j) \longrightarrow \pi_0^G(\boxtimes_{j\in J} X_j)$$

for every finite set J. Passing to colimits over finite subsets of I on both sides yields a map

$$\prod\nolimits'_{i\in I} \pi_0^G(X_i) \longrightarrow \pi_0^G(\boxtimes'_{i\in I} X_i) \ . \tag{1.5.23}$$

Proposition 1.5.24. *Let I be a set and $\{X_i\}_{i\in I}$ a family of based orthogonal spaces. Then for every compact Lie group G the map* (1.5.23) *is bijective.*

Proof For every $k \in I$ we define a 'projection'

$$\Pi_k \; : \; \boxtimes'_{i\in I} X_i \longrightarrow X_k$$

as follows. Since the infinite box product is defined as a colimit, we must specify the 'restriction' of Π_k to $\boxtimes_{j\in J} X_j$ for every finite subset J of I, compatibly as J increases. For $k \notin J$ we define this restriction as the constant morphism factoring through the basepoint of X_k. For $k \in J$ we define the restriction

$$\boxtimes_{j\in J} X_j \longrightarrow X_k$$

as the morphism corresponding, under the universal property of the box product, to the multi-morphism with components

$$\prod_{j\in J} X_j(V_j) \xrightarrow{\mathrm{proj}_k} X_k(V_k) \xrightarrow{X_k(\mathrm{incl})} X_k(\oplus_{j\in J} V_j) \; .$$

Then the composite

$$\prod'_{i\in I} \pi_0^G(X_i) \xrightarrow{(1.5.23)} \pi_0^G(\boxtimes'_{i\in I} X_i) \xrightarrow{\pi_0^G(\Pi_k)} \pi_0^G(X_k)$$

is the projection onto the kth factor. So if two tuples in the weak product have the same image under the map (1.5.23), they coincide. This shows injectivity.

Now we show surjectivity. Every element of $\pi_0^G(\boxtimes'_{i\in I} X_i)$ is represented by a G-fixed-point of $(\boxtimes'_{i\in I} X_i)(V)$ for some G-representation V. Colimits of orthogonal spaces are formed objectwise, so $(\boxtimes'_{i\in I} X_i)(V)$ is a colimit of the spaces $(\boxtimes_{j\in J} X_j)(V)$, formed over the filtered poset of finite subsets J of I. For every nested pair of finite subsets $K \subset J$ of I the morphism (1.5.22) has a retraction, by 'projection'. So at every inner product space V, the map

$$(\boxtimes_{k\in K} X_k)(V) \longrightarrow (\boxtimes_{j\in J} X_j)(V)$$

is a closed embedding by Proposition A.12. For fixed V, the colimit $(\boxtimes'_{i\in I} X_i)(V)$ in the category $\mathbf{T}$ of compactly generated spaces can thus be calculated in the ambient category of all topological spaces, by Proposition A.14 (ii). In particular, every G-fixed-point of $(\boxtimes'_{i\in I} X_i)(V)$ arises from a G-fixed-point of $(\boxtimes_{j\in J} X_j)(V)$ for some finite subset J of I. In other words, the canonical map

$$\mathrm{colim}_{J\subset I, |J|<\infty}\, \pi_0^G(\boxtimes_{j\in J} X_j) \longrightarrow \pi_0^G(\boxtimes'_{i\in I} X_i)$$

is surjective. For finite sets J the map $\prod_{j\in J} \pi_0^G(X_j) \longrightarrow \pi_0^G(\boxtimes_{j\in J} X_j)$ is bijective by Corollary 1.5.20, so this shows surjectivity. □

2
Ultra-commutative monoids

Orthogonal monoid spaces are the lax monoidal continuous functors from the linear isometries category $\mathbf{L}$ to the category of spaces. Orthogonal monoid spaces with strictly commutative multiplication (i.e., the lax *symmetric* monoidal functors) play a special role, and we honor this by special terminology, referring to them as *ultra-commutative monoids*. This chapter is devoted to the study of ultra-commutative monoids, including a global model structure, an algebraic study of their homotopy operations, and many examples.

I want to motivate the adjective 'ultra-commutative'. In various contexts of homotopy theory, highly structured multiplications that are only associative or commutative up to higher coherence homotopies can in fact be rigidified to multiplications that are strictly associative or possibly strictly commutative. One example is the fact that E_∞-spaces can be rigidified to strictly commutative $\mathcal{I}$-space monoids [141, Thm. 1.3]; another example is the fact that E_∞-ring objects internal to symmetric spectra can be rigidified to strictly commutative symmetric ring spectra, see for example [51, Thm. 1.4] and the paragraph immediately following it. More to the point of our present discussion, in [102, Thm. 1.3] Lind establishes a Quillen equivalence between the non-equivariant homotopy theory of E_∞-spaces (i.e., spaces with an action of the linear isometries operad) and the non-equivariant homotopy theory of commutative orthogonal monoid spaces (there called 'commutative $\mathcal{I}$-FCPs').

Our use of the word 'ultra-commutative' is intended as a reminder that the slogan 'E_∞=commutative' is no longer true in equivariant or global contexts. More specifically, one can consider orthogonal monoid spaces with an action of an E_∞-operad; up to non-equivariant equivalence, these objects model E_∞-spaces, and they can be replaced by equivalent strictly commutative orthogonal monoid spaces. The analogous statement for global equivalences is false, i.e., E_∞-orthogonal spaces cannot in general be replaced by globally equivalent ultra-commutative monoids. In fact, the definition of power operations and

transfer maps requires strict commutativity, and Remark 2.4.25 illustrates how the lack of transfers obstructs ultra-commutativity.

The study of orthogonal monoid spaces goes back to Boardman and Vogt [16], who introduce them as a delooping machine in a non-equivariant context. More precisely, they show that for every ultra-commutative monoid R the space $R(\mathbb{R}^\infty)$ has the structure of an 'E-space' (nowadays called an E_∞-space), and if in addition $\pi_0(R(\mathbb{R}^\infty))$ is a group, then $R(\mathbb{R}^\infty)$ is an infinite loop space. Ultra-commutative monoids also appear, with an extra point-set topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [112, IV Def. 2.1]; in [102], they are studied under the name 'commutative $\mathcal{I}$-FCPs'.

In Section 2.1 we formally define ultra-commutative monoids and establish the global model structure. Section 2.2 is devoted to the algebraic structure on the homotopy Rep-functor $\underline{\pi}_0(R)$ of an ultra-commutative monoid. We refer to this structure as a 'global power monoid'; it consist of an abelian monoid structure on the set $\pi_0^G(R)$ for every compact Lie group G, natural for restriction along continuous homomorphisms, and an additional structure that can equivalently be encoded as power operations (see Definition 2.2.8) or as transfer maps (see Construction 2.2.29). In this section we also show that these operations are the entire natural structure (see Theorem 2.2.24). Section 2.3 collects various examples of ultra-commutative monoids: among these are ones made from the infinite families of classical Lie groups (orthogonal, special orthogonal, unitary, special unitary, symplectic, spin and spin^c); examples consisting of Grassmannians under direct sum of subspaces (in real, oriented, complex or quaternionic flavors); examples made from Grassmannians under tensor product of subspaces (in a real or complex version); and ultra-commutative multiplicative models for global classifying spaces of abelian compact Lie groups.

Section 2.4 is a case study of how non-equivariant homotopy types can 'fold up' into many different global homotopy types. We define, discuss and compare different ultra-commutative and E_∞-orthogonal monoid spaces whose underlying non-equivariant homotopy type is BO, a classifying space for the infinite orthogonal group; in all examples we also identify the associated global power monoids and fixed-point spaces. Section 2.5 discusses 'units' and 'group completion' of ultra-commutative monoids. The two constructions are dual to each other, and they are homotopically right adjoint and left adjoint to the inclusion of group-like ultra-commutative monoids. On the algebraic level of global power monoids, the topological constructions pick out the invertible elements and perform the algebraic group completion. A naturally occurring example of a global group completion is the morphism from the additive Grassmannians to the periodic global version of BO. As an application we end the section with a global, highly structured version of Bott periodicity: Theorem

2.5.41 shows that **BUP** is globally equivalent, as an ultra-commutative monoid, to $\Omega\mathbf{U}$.

2.1 Global model structure

In this section we formally define ultra-commutative monoids and establish various formal properties of the category *umon* of ultra-commutative monoids. We introduce free ultra-commutative monoids in Example 2.1.5. The main result is the model structure with global equivalences as the weak equivalences, see Theorem 2.1.15.

Definition 2.1.1. An *ultra-commutative monoid* is a commutative orthogonal monoid space. We write *umon* for the category of ultra-commutative monoids.

As we explained after Definition 1.4.14, the data of an ultra-commutative monoid is the same as that of a lax symmetric monoidal continuous functor from the linear isometries category $\mathbf{L}$ (under orthogonal direct sum) to the category $\mathbf{T}$ of spaces (under cartesian product).

Remark 2.1.2. One can think of an ultra-commutative monoid as encoding a collection of E_∞-G-spaces, one for every compact Lie group G, compatible under restriction. If R is a closed orthogonal space and G a compact Lie group, then the G-equivariant homotopy type encoded in R can be accessed as the underlying G-space

$$R(\mathcal{U}_G) \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)} R(V)\ .$$

The additional structure of an ultra-commutative monoid gives rise to an action of a specific E_∞-G-operad on this G-space, namely the linear isometries operad of the complete G-universe $\mathcal{U}_G$. The nth space of this operad is the space $\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G)$ of linear isometric embeddings (not necessarily equivariant) of $\mathcal{U}_G^n$ into $\mathcal{U}_G$. The group G acts on $\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G)$ by conjugation, and the operad structure is by direct sum and composition of linear isometric embeddings. The symmetric group Σ_n permutes the summands in the source. The space $\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G)$ is G-equivariantly contractible by [100, II Lemma 1.5], and the Σ_n-action is free; in fact, $\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G)$ even has the $(G\times\Sigma_n)$-equivariant homotopy type of a universal space for (G,Σ_n)-bundles.

By simultaneous passage to colimit over $s(\mathcal{U}_G)$ in all n variables, the iterated multiplication maps

$$R(V_1)\times\cdots\times R(V_n) \;\longrightarrow\; R(V_1\oplus\cdots\oplus V_n)$$

give rise to a map $\mu_{(n)} : R(\mathcal{U}_G)^n \longrightarrow R(\mathcal{U}_G^n)$. A linear isometric embedding

$\psi : \mathcal{U} \longrightarrow \mathcal{U}'$ between countably infinite-dimensional inner product spaces induces a map $R(\psi) : R(\mathcal{U}) \longrightarrow R(\mathcal{U}')$; the resulting 'action map'

$$\mathbf{L}(\mathcal{U}, \mathcal{U}') \times R(\mathcal{U}) \longrightarrow R(\mathcal{U}') , \quad (\psi, y) \longmapsto R(\psi)(y)$$

is continuous. The operadic action map is then simply the composite

$$\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G) \times R(\mathcal{U}_G)^n \xrightarrow{\mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G) \times \mu_{(n)}} \mathbf{L}(\mathcal{U}_G^n, \mathcal{U}_G) \times R(\mathcal{U}_G^n) \xrightarrow{\text{act}} R(\mathcal{U}_G) .$$

Now we work towards the main result of this section, the global model structure for ultra-commutative monoids. Before we start with the homotopical considerations, we get some of the necessary category theory out of the way. For a moment we consider more generally any symmetric monoidal category C with monoidal product $\boxtimes$ and unit object I. We can then study operads in C and algebras over a fixed operad. The following (co-)completeness and preservation results can be found in [137, Prop. 2.3.5] or [55, Prop. 3.3.1].

Proposition 2.1.3. *Let $(C, \boxtimes, I)$ be a complete and cocomplete symmetric monoidal category such that the monoidal product preserves colimits in each variable. Let $\mathcal{P}$ be an operad in C. Then the category of $\mathcal{P}$-algebras is complete and cocomplete. Moreover, the forgetful functor from the category of $\mathcal{P}$-algebras to the underlying category C creates all limits, all filtered colimits and those coequalizers that are reflexive in the underlying category C.*

We let *Com* denote the incarnation of the commutative operad internal to the category of orthogonal spaces, under box product. So for every $n \geq 0$ the orthogonal space $Com(n)$ of n-ary operations is constant with value a one-point space. Equivalently, *Com* is a terminal operad in orthogonal spaces. Endowing an orthogonal space with an ultra-commutative multiplication is the same as giving it an algebra structure over the commutative operad *Com*. More formally, the category of ultra-commutative monoids is isomorphic to the category of *Com*-algebras. So Proposition 2.1.3 has the following special case:

Corollary 2.1.4. *The category of ultra-commutative monoids is complete and cocomplete. The forgetful functor from the category of ultra-commutative monoids to the category of orthogonal spaces creates all limits, all filtered colimits and those coequalizers that are reflexive in the category of orthogonal spaces.*

Example 2.1.5 (Free ultra-commutative monoids)**.** We quickly recall that ultra-commutative monoids are monadic over the category of orthogonal spaces; this is not particular to our context, and the analogous fact holds for commutative monoids in any cocomplete symmetric monoidal category. For every orthogonal space Y and $m \geq 0$ we denote by

$$\mathbb{P}^m(Y) = Y^{\boxtimes m}/\Sigma_m$$

the m-symmetric power, with respect to the box product, of Y. In particular, $\mathbb{P}^0(Y)$ is the terminal, constant one-point orthogonal space, and $\mathbb{P}^1(Y) = Y$. Then the orthogonal space

$$\mathbb{P}(Y) \;=\; \coprod\nolimits_{m\geq 0} \mathbb{P}^m(Y) \;=\; \coprod\nolimits_{m\geq 0} Y^{\boxtimes m}/\Sigma_m$$

is an ultra-commutative monoid under the concatenation product, and it is in fact the free ultra-commutative monoid generated by Y. More precisely, the functor

$$\mathbb{P} \;:\; spc \;\longrightarrow\; umon$$

becomes a left adjoint to the forgetful functor with respect to the morphism $\eta_Y : Y = \mathbb{P}^1(Y) \longrightarrow \mathbb{P}Y$, the inclusion of the 'linear' summand, as the adjunction unit. In other words, the following composite is bijective for every ultra-commutative monoid R:

$$umon(\mathbb{P}Y, R) \xrightarrow{\text{forget}} spc(\mathbb{P}Y, R) \xrightarrow{\eta_Y^*} spc(Y, R) \,.$$

Moreover, this adjunction is monadic, i.e., the category of ultra-commutative monoids is isomorphic to the category of algebras over the monad $\mathbb{P}$.

Construction 2.1.6. We will also exploit the fact that the category of ultra-commutative monoids is tensored and cotensored over the category $\mathbf{T}$ of spaces, so let us spend a few words explaining this enrichment. In fact, the constructions work more generally for algebras over continuous monads on any category enriched in spaces, see for example [117, Lemma 2.8]. We only spell out the case of ultra-commutative monoids, which are the algebras over the free ultra-commutative monoid monad $\mathbb{P} : spc \longrightarrow spc$.

The mapping space of morphisms between two orthogonal spaces X and Y is defined as follows. Since every inner product space is isometrically isomorphic to $\mathbb{R}^n$ for some n, the map

$$spc(X, Y) \;\longrightarrow\; \prod\nolimits_{n\geq 0} \operatorname{map}(X(\mathbb{R}^n), Y(\mathbb{R}^n)) \,, \quad f \;\longmapsto\; \{f(\mathbb{R}^n)\}_{n\geq 0}$$

is injective with closed image. So we endow $spc(X, Y)$ with the subspace topology of the product (which is taken internal to the category $\mathbf{T}$ of compactly generated spaces, i.e., it is the Kelleyfied product topology).

If R and S are ultra-commutative monoids, then the set $umon(R, S)$ of morphisms of ultra-commutative monoids is a closed subset of the space $spc(R, S)$, and we give it the subspace topology. We omit the verification that composition is continuous in this topology, so we have indeed defined an enrichment of the category of ultra-commutative monoids in spaces.

The cotensors of ultra-commutative monoids are defined 'pointwise'. In

more detail, we consider an ultra-commutative monoid R and a space A. Then the orthogonal space $\mathrm{map}(A, R)$ inherits an ultra-commutative multiplication

$$\mathrm{map}(A, R) \boxtimes \mathrm{map}(A, R) \longrightarrow \mathrm{map}(A, R)$$

from the bimorphism with (V, W)-component

$$\mathrm{map}(A, R(V)) \times \mathrm{map}(A, R(W)) \longrightarrow \mathrm{map}(A, R(V \oplus W)) , \ (f, g) \longmapsto \mu_{V,W} \circ (f, g) .$$

The multiplicative unit is the constant map with value the unit of R.

The tensor of an ultra-commutative monoid R with a space A is, however, *not* pointwise. To avoid confusion with the objectwise product we denote this tensor by $R \otimes A$; its defining property is that it represents the functor

$$\mathrm{map}(A, umon(R, -)) \ : \ umon \longrightarrow (\mathrm{sets}) .$$

So $R \otimes A$ comes equipped with a continuous map $i : A \longrightarrow umon(R, R \otimes A)$ such that the map

$$umon(R \otimes A, S) \longrightarrow \mathrm{map}(A, umon(R, S)) , \quad f \longmapsto umon(R, f) \circ i$$

is bijective. One construction of a tensor $R \otimes A$ is as a coequalizer, in the category of ultra-commutative monoids, of the two morphisms:

$$\mathbb{P}((\mathbb{P}R) \times A) \ \underset{\nu}{\overset{\mathbb{P}(\alpha \times A)}{\rightrightarrows}} \ \mathbb{P}(R \times A)$$

Here $\alpha : \mathbb{P}R \longrightarrow R$ is the structure morphism (i.e., the counit of the free-forgetful adjunction) and ν is adjoint to the morphism of orthogonal spaces

$$(\mathbb{P}R) \times A \longrightarrow \mathbb{P}(R \times A)$$

that in turn is adjoint to the composite

$$A \xrightarrow{a \mapsto (-,a)} \mathrm{map}(R, R \times A) \xrightarrow{\mathbb{P}} \mathrm{map}(\mathbb{P}R, \mathbb{P}(R \times A)) .$$

The above coequalizer defining $R \otimes A$ is reflexive in the underlying category of orthogonal spaces, so it can be calculated in the underlying category, by Proposition 2.1.3.

In our discussion of global group completions in Section 2.5 we will want to realize simplicial ultra-commutative monoids. We refer to Construction 1.2.34 for generalities about the realization of simplicial objects. For a simplicial ultra-commutative monoid $B : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow umon$, the term ‘geometric realization’ actually has two potentially different interpretations, and we spend some time in clarifying this issue. On the one hand we can form the geometric realization

$|B|_{\text{un}}$ in the underlying category of orthogonal spaces; this is, by definition, a coend, in the category of orthogonal spaces, of the functor

$$\mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta} \longrightarrow spc , \quad ([m], [n]) \longmapsto B_m \times \Delta^n .$$

We call this the *underlying realization* of B. Coends of orthogonal spaces are calculated objectwise, so $|B|_{\text{un}}(V)$ is a realization of the simplicial space $[m] \mapsto B_m(V)$. It is not a priori obvious, however, whether this realization inherits any multiplication.

On the other hand, we explained in Construction 2.1.6 that the category of ultra-commutative monoids is tensored over the category $\mathbf{T}$ of spaces. We continue to write $R \otimes A$ for the tensor of an ultra-commutative monoid R with a space A, in order to distinguish it from the (objectwise) product of the underlying orthogonal space of R with A. We can also consider the realization $|B|_{\text{in}}$ *internal to ultra-commutative monoids*, i.e., a coend, in the category of ultra-commutative monoids, of the functor

$$\mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta} \longrightarrow umon , \quad ([m], [n]) \longmapsto B_m \otimes \Delta^n .$$

We call this the *internal realization*. The internal realization is, by definition, an ultra-commutative monoid, but it is not immediately clear how it relates to the underlying realization of $|B|_{\text{un}}$ as an orthogonal space. As we shall now show, the forgetful functor from a category of ultra-commutative monoids to orthogonal spaces commutes with realization of simplicial objects. We do not claim any originality here, and many results of this kind can be found in the literature, see for example [110, Thm. 12.2], [50, VII Prop. 3.3], [117, Prop. 4.5], [106, Prop. 12.4] or [62, Thm. 2.2].

Proposition 2.1.7. *Let B be a simplicial object in the category of ultra-commutative monoids. Then the canonical morphism $|B|_{\text{un}} \longrightarrow |B|_{\text{in}}$ from the underlying realization to the internal realization is an isomorphism of orthogonal spaces.*

Proof We adapt an argument given by Mandell in an unpublished preprint [106, Prop. 12.4]. We start by considering two simplicial orthogonal spaces $X, Y : \mathbf{\Delta}^{\text{op}} \longrightarrow spc$. We denote by $X \boxtimes Y$ the diagonal of the external box product, i.e., the composite simplicial orthogonal space

$$\mathbf{\Delta}^{\text{op}} \xrightarrow{\text{diagonal}} \mathbf{\Delta}^{\text{op}} \times \mathbf{\Delta}^{\text{op}} \xrightarrow{X \times Y} spc \times spc \xrightarrow{\boxtimes} spc .$$

For every $n \geq 0$ we consider the composite

$$\begin{aligned}(X_n \boxtimes Y_n) \times \Delta^n &\xrightarrow{\text{Id} \times \text{diag}} (X_n \boxtimes Y_n) \times (\Delta^n \times \Delta^n) \\ &\xrightarrow{\text{shuffle}} (X_n \times \Delta^n) \boxtimes (Y_n \times \Delta^n) \longrightarrow |X| \boxtimes |Y| ,\end{aligned}$$

where the last morphism is the box product of the two canonical morphisms $X_n \times \Delta^n \longrightarrow |X|$ and $Y_n \times \Delta^n \longrightarrow |Y|$. These composites are compatible with the coend relations, so together they form a morphism of orthogonal spaces

$$|X \boxtimes Y| \longrightarrow |X| \boxtimes |Y| .$$

We claim that this morphism is an isomorphism. Indeed, since $\boxtimes$ preserves colimits in each variable, the right-hand side is a coend of the functor

$$(\mathbf{\Delta}^2)^{\mathrm{op}} \times \mathbf{\Delta}^2 \longrightarrow spc , \quad ([k],[l],[m],[n]) \longmapsto (X_k \boxtimes Y_l) \times \Delta^m \times \Delta^n .$$

Coends of orthogonal spaces are calculated objectwise, and for bisimplicial spaces the bi-realization is homeomorphic to the realization of the diagonal (see [135, p. 94, Lemma] or Proposition A.37 (iii)).

By iterating, we obtain a Σ_m-equivariant isomorphism of orthogonal spaces

$$|X^{\boxtimes m}| \cong |X|^{\boxtimes m}$$

for every $m \geq 0$. Since coends commute with colimits, we can pass to Σ_m-orbits and take the coproduct over $m \geq 0$, resulting in an isomorphism

$$|\mathbb{P}(X)|_{\mathrm{un}} = |\coprod_{m\geq 0} (X^{\boxtimes m})/\Sigma_m| \cong \coprod_{m\geq 0} |X|^{\boxtimes m}/\Sigma_m = \mathbb{P}|X| .$$

Moreover, the ultra-commutative monoid $\mathbb{P}|X|$ has the universal property of the internal realization of the simplicial ultra-commutative monoid $\mathbb{P} \circ X$. This shows the claim in the special case where B is freely generated by a simplicial orthogonal space.

Now we treat the general case. The diagram

$$\mathbb{P}(\mathbb{P}B) \underset{\mu}{\overset{\mathbb{P}\alpha}{\rightrightarrows}} \mathbb{P}B \xrightarrow{\alpha} B$$

is a coequalizer diagram of simplicial ultra-commutative monoids. Here $\alpha : \mathbb{P}R \longrightarrow R$ is the adjunction counit and μ is the monad structure of the free functor. Moreover, the coequalizer is reflexive in the underlying category of orthogonal spaces, by the morphisms

$$\mathbb{P}(\mathbb{P}B) \xleftarrow{\eta_{\mathbb{P}B}} \mathbb{P}B \xleftarrow{\eta_B} B$$

where $\eta : R \longrightarrow \mathbb{P}R$ is the unit of the free-forget adjunction, i.e., the inclusion as the 'linear' summand $\mathbb{P}^1(R)$.

Applying the two functors under consideration gives a commutative diagram of orthogonal spaces

$$\begin{tikzcd}
|\mathbb{P}(\mathbb{P}B)|_{\mathrm{un}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & |\mathbb{P}B|_{\mathrm{un}} \arrow[r] \arrow[d] & |B|_{\mathrm{un}} \arrow[d] \\
|\mathbb{P}(\mathbb{P}B)|_{\mathrm{in}} \arrow[r, shift left] \arrow[r, shift right] & |\mathbb{P}B|_{\mathrm{in}} \arrow[r] & |B|_{\mathrm{in}}
\end{tikzcd} \tag{2.1.8}$$

We claim that both rows are coequalizer diagrams of orthogonal spaces. For the upper row we argue as follows. For every $n \geq 0$ the diagram

$$\mathbb{P}(\mathbb{P}B_n) \rightrightarrows \mathbb{P}B_n \longrightarrow B_n$$

is a coequalizer in the category of ultra-commutative monoids. Since the diagram splits in the underlying category of orthogonal spaces, it is also a coequalizers diagram there, by Proposition 2.1.3 or [105, IV.6, Lemma]. So the diagram

$$\mathbb{P}(\mathbb{P}B) \rightrightarrows \mathbb{P}B \longrightarrow B$$

is also a coequalizer diagram of simplicial orthogonal spaces. Since both product with Δ^n and coends commute with colimits in the category of orthogonal spaces, the diagram stays a coequalizer after (underlying) geometric realization. Since coends commute with all colimits, the bottom row of (2.1.8) is a coequalizer diagram of ultra-commutative monoids. Again the diagram splits in the underlying category of orthogonal spaces, so the lower diagram is also a coequalizer diagram of orthogonal spaces. Since the two left vertical morphisms in (2.1.8) are isomorphisms of orthogonal spaces by the special case above, this proves that the morphism $|B|_{\mathrm{un}} \longrightarrow |B|_{\mathrm{in}}$ is an isomorphism as well. □

Ultra-commutative monoids form a pointed category: the constant one-point orthogonal monoid space is a zero object. The enrichment, tensors and cotensors over spaces extend to enrichment, tensors and cotensors over the category of *based* topological spaces. We shall write $R \rhd A$ for the tensor of an ultra-commutative monoid R with a based space (A, a_0), in order to distinguish it from the (objectwise) smash product of the underlying based orthogonal space of R with A. Thus $R \rhd A$ is a pushout, in the category of ultra-commutative monoids, of the diagram

$$* \longleftarrow R \otimes \{a_0\} \xrightarrow{R \otimes \mathrm{incl}} R \otimes A\ . \tag{2.1.9}$$

As may be familiar from similar contexts, the bar construction $B(R)$ of an ultra-commutative monoid R can be interpreted as $R \rhd S^1$, the based tensor of R with the based space S^1, see (2.5.30) below. Another way to say this is that the bar construction is the internal suspension in the category of ultra-commutative monoids. We show a more general statement and consider a based simplicial set A. We define a simplicial object of ultra-commutative monoids by

$$B_m(R, A) \ = \ R \rhd A_m\ ,$$

with simplicial structure induced by that of A. Since A_m is a based set, $R \rhd A_m$

is in fact a categorical coproduct (i.e., box product) of copies of R, indexed by the non-basepoint elements of A_m.

The next proposition constructs an isomorphism of ultra-commutative monoids between $R \triangleright |A|$ and $|B_\bullet(R,A)|_{\mathrm{in}}$, the internal geometric realization. By Proposition 2.1.7, we can (and will) confuse the internal realization with the underlying realization of $B_\bullet(R,A)$ in the category of orthogonal spaces. Variations of the following proposition appear in various places in the literature, and they go back, at least, to the interpretation, by McClure, Schwänzl and Vogt [117], of the topological Hochschild homology of a commutative ring spectrum as the tensor with S^1.

Proposition 2.1.10. *Let R be an ultra-commutative monoid and A a based simplicial set. Then $R \triangleright |A|$ is an internal realization of the simplicial ultra-commutative monoid $B_\bullet(R,A)$.*

Proof The geometric realization $|A|$ is a coend of the functor

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \longrightarrow \mathbf{T}_*\ , \quad ([m],[n]) \longmapsto A_m \wedge \Delta^n_+\ .$$

Since the functor $R \triangleright -$ preserves colimits, $R \triangleright |A|$ is a coend, in the category of ultra-commutative monoids, of the functor

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \longrightarrow umon\ , \quad ([m],[n]) \longmapsto R \triangleright (A_m \wedge \Delta^n_+)\ .$$

The isomorphisms

$$R \triangleright (A_m \wedge \Delta^n_+) \cong (R \triangleright A_m) \triangleright \Delta^n_+ \cong (R \triangleright A_m) \otimes \Delta^n = B_m(A,R) \otimes \Delta^n$$

are natural in $([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}$, and they show that $R \triangleright |A|$ is an internal realization of the simplicial ultra-commutative monoid $B_\bullet(R,A)$. □

Now we approach the global model structure on the category of ultra-commutative monoids. We will establish this model structure as a special case of a lifting theorem for model structures to categories of commutative monoids that was formulated by White [188, Thm. 3.2]. Like its predecessor for associative monoids [146, Thm. 4.1 (3)], the input is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom. However, lifting a model structure to *commutative* monoids is more subtle and needs extra hypotheses; the essence of the additional condition is that, loosely speaking, symmetric powers must be 'sufficiently homotopy invariant'. Earlier, Gorchinskiy and Guletskii [64] had also studied symmetric power constructions in a symmetric monoidal model category, and there is a substantial overlap in the arguments of [64] and [188].

We let C be a symmetric monoidal category with monoidal product $\boxtimes$. To

simplify the exposition we follow the usual abuse and suppress the associativity and unit isomorphisms from the notation, i.e., we pretend that the underlying monoidal structure is strict (i.e., a *permutative* structure). We let $i : A \longrightarrow B$ be a C-morphism and arrange the n-fold $\boxtimes$-power of i into an n-dimensional cube $K^n(i)$ in C, i.e., a functor

$$K^n(i) \ : \ \mathcal{P}(n) \ \longrightarrow \ C$$

from the poset category of subsets of $\{1, 2, \dots, n\}$ and inclusions to C. More explicitly, if $S \subseteq \{1, 2, \dots, n\}$ is a subset, then the vertex of the cube at S is

$$K^n(i)(S) \ = C_1 \boxtimes C_2 \boxtimes \cdots \boxtimes C_n \qquad \text{with} \qquad C_j \ = \begin{cases} A & \text{if } j \notin S \\ B & \text{if } j \in S. \end{cases}$$

All morphisms in the cube $K^n(i)$ are $\boxtimes$-products of identities and copies of the morphism $i : A \longrightarrow B$. The initial vertex of the cube is $K^n(i)(\emptyset) = A^{\boxtimes n}$ and the terminal vertex is $K^n(i)(\{1, \dots, n\}) = B^{\boxtimes n}$.

We denote by $Q^n(i)$ the colimit of the punctured cube, i.e., the cube $K^n(i)$ with the terminal vertex removed, and by $i^{\Box n} \ : \ Q^n(i) \ \longrightarrow \ K^n(i)(\{1, \dots, n\}) = B^{\boxtimes n}$ the canonical morphism, an iterated pushout product morphism. Indeed, for $n = 2$ the cube $K^2(i)$ is a square and looks like

$$\begin{tikzcd}
A \boxtimes A \arrow[r, "A\boxtimes i"] \arrow[d, "i\boxtimes A"'] & A \boxtimes B \arrow[d, "i\boxtimes B"] \\
B \boxtimes A \arrow[r, "B\boxtimes i"'] & B \boxtimes B
\end{tikzcd}$$

Hence

$$i^{\Box 2} \ = \ i \Box i \ = \ (B \boxtimes i) \cup (i \boxtimes B) \ : \ B \boxtimes A \cup_{A\boxtimes A} A \boxtimes B \ \longrightarrow \ B \boxtimes B \, .$$

Similarly, $i^{\Box 3}$ is the morphism from the colimit of the punctured cube to the terminal vertex of the following cuboid:

$$\begin{tikzcd}
A \boxtimes A \boxtimes A \arrow[rr, "A\boxtimes A\boxtimes i"] \arrow[dr, "A\boxtimes i\boxtimes A"'] \arrow[dd, "i\boxtimes A\boxtimes A"'] & & A \boxtimes A \boxtimes B \arrow[dr, "A\boxtimes i\boxtimes B"] \arrow[dd, "i\boxtimes A\boxtimes B" near end] & \\
& A \boxtimes B \boxtimes A \arrow[rr, "A\boxtimes B\boxtimes i"' near start] \arrow[dd, "i\boxtimes B\boxtimes A" near end] & & A \boxtimes B \boxtimes B \arrow[dd, "i\boxtimes B\boxtimes B"] \\
B \boxtimes A \boxtimes A \arrow[rr, "B\boxtimes A\boxtimes i" near start] \arrow[dr, "B\boxtimes i\boxtimes A"'] & & B \boxtimes A \boxtimes B \arrow[dr, "B\boxtimes i\boxtimes B"] & \\
& B \boxtimes B \boxtimes A \arrow[rr, "B\boxtimes B\boxtimes i"'] & & B \boxtimes B \boxtimes B
\end{tikzcd}$$

We observe that the symmetric group Σ_n acts on $Q^n(i)$ and $B^{\boxtimes n}$ by permuting

the factors, and the iterated pushout product morphism $i^{\Box n} : Q^n(i) \longrightarrow B^{\boxtimes n}$ is Σ_n-equivariant. We recall from [64] the notions of *symmetrizable cofibration* and *symmetrizable acyclic cofibration*.

Definition 2.1.11. [64, Def. 3] Let C be a symmetric monoidal model category. A morphism $i : A \longrightarrow B$ is a *symmetrizable cofibration* (or a *symmetrizable acyclic cofibration*) if the morphism

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (or an acyclic cofibration) for every $n \geq 1$.

Since the morphism $i^{\Box 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration and every symmetrizable acyclic cofibration is in particular an acyclic cofibration. We will now proceed to prove that in the category of orthogonal spaces, all cofibrations and acyclic cofibrations in the positive global model structure are symmetrizable with respect to the box product. The next proposition will be used to verify this for the generating acyclic cofibrations. We recall from Construction 1.2.15 that given a morphism $j : A \longrightarrow B$, the set $\mathcal{Z}(j)$ consists of all pushout product maps

$$c(j)\Box i_k \ : \ A \times D^k \cup_{A\times\partial D^k} Z(j) \times \partial D^k \ \longrightarrow \ Z(j) \times D^k$$

of the mapping cylinder inclusion $c(j) : A \longrightarrow Z(j)$ with the sphere inclusions for $k \geq 0$.

Proposition 2.1.12. *Let C be a symmetric monoidal topological model category.*

(i) *For every $n \geq 1$ the functor $\mathbb{P}^n$ preserves the homotopy relation on morphisms and it preserves homotopy equivalences.*

(ii) *Let $j : A \longrightarrow B$ be a symmetrizable acyclic cofibration between cofibrant objects. Then for every $k \geq 0$, the pushout product map*

$$j\Box i_k \ : \ A \times D^k \cup_{A\times\partial D^k} B \times \partial D^k \ \longrightarrow \ B \times D^k$$

is a symmetrizable acyclic cofibration.

(iii) *Let $j : A \longrightarrow B$ be a morphism between cofibrant objects such that the morphism $\mathbb{P}^n(j) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(B)$ is a weak equivalence for every $n \geq 1$. Then every morphism in the set $\mathcal{Z}(j)$ is a symmetrizable acyclic cofibration.*

Proof (i) This is the topological version of [64, Lemma 1]. For every object A of C and every space K the morphism

$$A^{\boxtimes n} \times K \xrightarrow{A^{\boxtimes n}\times\Delta} A^{\boxtimes n} \times K^n \ \cong \ (A \times K)^{\boxtimes n}$$

is Σ_n-equivariant (with respect to the trivial Σ_n-action on K in the source) and factors over a natural morphism

$$\tilde{\Delta} \ : \ \mathbb{P}^n(A) \times K \ = \ (A^{\boxtimes n} \times K)/\Sigma_n \ \longrightarrow \ (A \times K)^{\boxtimes n}/\Sigma_n \ = \ \mathbb{P}^n(A \times K) \ .$$

If $H : A \times [0,1] \longrightarrow B$ is a homotopy from a morphism $f = H(-,0)$ to another morphism $g = H(-,1)$, then the composite

$$\mathbb{P}^n(A) \times [0,1] \ \xrightarrow{\tilde{\Delta}} \ \mathbb{P}^n(A \times [0,1]) \ \xrightarrow{\mathbb{P}^n(H)} \ \mathbb{P}^n(B)$$

is a homotopy from the morphism $\mathbb{P}^n(f)$ to $\mathbb{P}^n(g)$. So $\mathbb{P}^n$ preserves the homotopy relation, and hence also homotopy equivalences.

(ii) We argue by induction on k. For $k = 0$ the pushout product map $j \Box i_0$ is isomorphic to j, hence a symmetrizable acyclic cofibration by hypothesis. Now we assume the claim for some k, and deduce it for $k+1$. Since j is a symmetrizable acyclic cofibration between cofibrant objects, the morphism $\mathbb{P}^n(j)$ is a weak equivalence for every $n \geq 1$ by [64, Cor. 23]. Since the functors $\mathbb{P}^n$ preserve the homotopy relation and the projections $A \times D^k \longrightarrow A$ and $B \times D^k \longrightarrow B$ are homotopy equivalences, the morphism $\mathbb{P}^n(j \times D^k)$ is a weak equivalence for every $n \geq 1$. So $j \times D^k : A \times D^k \longrightarrow B \times D^k$ is a symmetrizable acyclic cofibration, again by [64, Cor. 23]. We write $\partial D^{k+1} = D^k_+ \cup_{\partial D^k} D^k_-$ as the union of the upper and lower hemisphere along the equator. The upper morphism in the pushout square

$$\begin{tikzcd}
D^k_+ \times A \arrow[r, "D^k_+ \times j"] \arrow[d] & D^k_+ \times B \arrow[d] \\
\partial D^{k+1} \times A \arrow[r] & \partial D^{k+1} \times A \cup_{D^k_+ \times A} D^k_+ \times B
\end{tikzcd}$$

is a symmetrizable acyclic cofibration by the previous paragraph. The class of symmetrizable acyclic cofibrations is closed under cobase change by [64, Thm. 7 (A)]; the lower morphism is thus a symmetrizable acyclic cofibration.

The square

$$\begin{tikzcd}
A \times D^k_- \cup_{A \times \partial D^k} B \times \partial D^k \arrow[r, "j \Box i_k"] \arrow[d] & B \times D^k_- \arrow[d] \\
A \times \partial D^{k+1} \cup_{A \times D^k_+} B \times D^k_+ \arrow[r, "j \Box \mathrm{incl}"'] & B \times \partial D^{k+1}
\end{tikzcd}$$

is a pushout. The upper morphism is a symmetrizable acyclic cofibration by the inductive hypothesis, hence so is the lower morphism, again by stability under cobase change. The morphism $j \times \partial D^{k+1} : A \times \partial D^{k+1} \longrightarrow B \times \partial D^{k+1}$

is thus the composite of two symmetrizable acyclic cofibrations, hence a symmetrizable acyclic cofibration itself, by [64, Thm. 7 (C)]. As a cobase change, the morphism

$$A \times D^{k+1} \longrightarrow A \times D^{k+1} \cup_{A \times \partial D^{k+1}} B \times \partial D^{k+1}$$

is then a symmetrizable acyclic cofibration by [64, Thm. 7 (A)]. The induced morphism

$$\mathbb{P}^n(A \times D^{k+1}) \longrightarrow \mathbb{P}^n(A \times D^{k+1} \cup_{A \times \partial D^{k+1}} B \times \partial D^{k+1})$$

is then a weak equivalence by [64, Cor. 23]. Since $\mathbb{P}^n(j \times D^{k+1}) : \mathbb{P}^n(A \times D^{k+1}) \longrightarrow \mathbb{P}^n(B \times D^{k+1})$ is a weak equivalence, so is the morphism

$$\mathbb{P}^n(j \Box i_{k+1}) \ : \ \mathbb{P}^n(A \times D^{k+1} \cup_{A \times \partial D^{k+1}} B \times \partial D^{k+1}) \longrightarrow \mathbb{P}^n(B \times D^{k+1}) \ .$$

One more time by [64, Cor. 23], this shows that $j \Box i_{k+1}$ is a symmetrizable acyclic cofibration. This completes the induction step.

(iii) Since A and B are cofibrant, the mapping cylinder inclusion

$$c(j) \ : \ A \longrightarrow (A \times [0,1]) \cup_j B = Z(j)$$

is a cofibration. Moreover, the projection $Z(j) \longrightarrow B$ is a homotopy equivalence, hence so is $\mathbb{P}^n(Z(j)) \longrightarrow \mathbb{P}^n(B)$ for every $n \geq 1$. Since $\mathbb{P}^n(j)$ is a weak equivalence by hypothesis, the morphism $\mathbb{P}^n(c(j)) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(Z(j))$ is a weak equivalence for every $n \geq 1$. So $c(j)$ is a symmetrizable acyclic cofibration by [64, Cor. 23]. Applying (ii) to the morphism $c(j)$ yields the claim. □

Now we can verify the symmetrizability of cofibrations and acyclic cofibrations for the positive global model structure of orthogonal spaces. The cofibration part (i) is in fact slightly stronger in that it does not need any positivity hypothesis.

Theorem 2.1.13. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spaces. Then for every $n \geq 1$ the morphism*

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \longrightarrow B^{\boxtimes n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spaces are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spaces that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \longrightarrow B^{\boxtimes n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spaces are symmetrizable.

Proof (i) We recall from the proof of Proposition 1.2.10 the set

$$I^{\mathrm{str}} = \{ G_m(O(m)/H \times i_k) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spaces, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. The set I^{str} detects the acyclic fibrations in the strong level model structure of orthogonal spaces. In particular, every flat cofibration is a retract of an I^{str}-cell complex. By [64, Cor. 9] or [188, Lemma A.1], it suffices to show that the generating flat cofibrations in I^{str} are symmetrizable.

The orthogonal space $G_m(O(m)/H \times K)$ is isomorphic to $\mathbf{L}_{H,\mathbb{R}^m} \times K$, so we show more generally that every morphism of the form

$$\mathbf{L}_{G,V} \times i_k \ : \ \mathbf{L}_{G,V} \times \partial D^k \ \longrightarrow \ \mathbf{L}_{G,V} \times D^k$$

is a symmetrizable cofibration, where V is any representation of a compact Lie group G. The symmetrized iterated pushout product

$$(\mathbf{L}_{G,V} \times i_k)^{\square n}/\Sigma_n \ : \ Q^n(\mathbf{L}_{G,V} \times i_k)/\Sigma_n \ \longrightarrow \ (\mathbf{L}_{G,V} \times i_k)^{\boxtimes n}/\Sigma_n \qquad (2.1.14)$$

is isomorphic to

$$\mathbf{L}_{\Sigma_n \wr G, V^n}(i_k^{\square n}) \ : \ \mathbf{L}_{\Sigma_n \wr G, V^n}(Q^n(i_k)) \ \longrightarrow \ \mathbf{L}_{\Sigma_n \wr G, V^n}((D^k)^n) \ ,$$

where

$$i_k^{\square n} \ : \ Q^n(i_k) \ \longrightarrow \ (D^k)^n$$

is the n-fold pushout product of the inclusion $i_k : \partial D^k \longrightarrow D^k$, with respect to the cartesian product of spaces. Here the wreath product $\Sigma_n \wr G$ acts on V^n by

$$(\sigma; g_1, \ldots, g_n) \cdot (v_1, \ldots, v_n) \ = \ (g_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)} v_{\sigma^{-1}(n)}) \ .$$

The map $i_k^{\square n}$ is Σ_n-equivariant, and we claim that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. One way to see this is to exploit the fact that i_k is homeomorphic to the geometric realization of the inclusion $\iota_k : \partial\Delta[k] \longrightarrow \Delta[k]$ of the boundary of the simplicial k-simplex. So $i_k^{\square n}$ is Σ_n-homeomorphic to the geometric realization of the inclusion $\iota_k^{\square n} : Q^n(\iota_k) \longrightarrow \Delta[k]^n$ of Σ_n-simplicial sets. The geometric realization of an equivariant embedding of simplicial sets is always an equivariant cofibration of spaces, so altogether this shows that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. Proposition B.14 (i) then shows that $i_k^{\square n}$ is also a cofibration of $(\Sigma_n \wr G)$-spaces via restriction along the projection $\Sigma_n \wr G \longrightarrow \Sigma_n$. So the morphism (2.1.14) is a flat cofibration.

(ii) Theorem 1.2.21 describes a set $J^{\mathrm{str}} \cup K$ of generating acyclic cofibrations for the global model structure on the category of orthogonal spaces. From this we obtain a set $J^+ \cup K^+$ of generating acyclic cofibrations for the *positive* global model structure of Proposition 1.2.23 by restricting to those morphisms

in $J^{\mathrm{str}} \cup K$ that are positive cofibrations, i.e., homeomorphisms in level 0. So explicitly, we set

$$J^+ \;=\; \{\, G_m(O(m)/H \times j_k) \mid m \geq 1, k \geq 0, H \leq O(m)\} \;,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+ \;=\; \bigcup_{G,V,W \,:\, V\neq 0} \mathcal{Z}(\rho_{G,V,W}) \;,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$. Here (G, V, W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a non-zero faithful G-representation V and an arbitrary G-representation W. By [64, Cor. 9] or [188, Lemma A.1] it suffices to show that all morphisms in $J^+ \cup K^+$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m(j_k \times O(m)/H)$ in J^+. For every $n \geq 1$, the morphism

$$(G_m(O(m)/H \times j_k))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the map j_k is a homotopy equivalence of spaces, so $G_m(O(m)/H\times j_k)$ is a homotopy equivalence of orthogonal spaces; the morphism $\mathbb{P}^n(G_m(O(m)/H\times j_k))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition 2.1.12 (i). Then [64, Cor. 23] shows that $G_m(O(m)/H \times j_k)$ is a symmetrizable acyclic cofibration. This takes care of the set J^+.

Now we consider the morphisms in the set K^+. Since G acts faithfully on the non-zero inner product space V, the action of the wreath product $\Sigma_n \wr G$ on V^n is again faithful. So the morphism

$$\rho_{\Sigma_n\wr G,V^n,W^n} \;:\; \mathbf{L}_{\Sigma_n\wr G,V^n\oplus W^n} \;\longrightarrow\; \mathbf{L}_{\Sigma_n\wr G,V^n}$$

is a global equivalence by Proposition 1.1.26 (ii). By the isomorphism

$$\mathbb{P}^n(\mathbf{L}_{G,V}) \;=\; \mathbf{L}_{G,V}^{\boxtimes n}/\Sigma_n \;\cong\; \mathbf{L}_{\Sigma_n\wr G,V^n} \;,$$

the morphism $\rho_{\Sigma_n\wr G,V^n,W^n}$ is isomorphic to $\mathbb{P}^n(\rho_{G,V,W}) : \mathbb{P}^n(\mathbf{L}_{G,V\oplus W}) \longrightarrow \mathbb{P}^n(\mathbf{L}_{G,V})$, which is thus a global equivalence. Proposition 2.1.12 (iii) then shows that all morphisms in $\mathcal{Z}(\rho_{G,V,W})$ are symmetrizable acyclic cofibrations. □

The hypothesis in Theorem 2.1.13 (ii) that i is a *positive* flat cofibration is really necessary. Indeed, the unique morphism $\rho : \mathbf{L}_{\mathbb{R}} \longrightarrow *$ to the terminal orthogonal space is a global equivalence, and the source and target of ρ are flat, but only the source is positively flat. Then the mapping cylinder inclusion $c(\rho) : \mathbf{L}_{\mathbb{R}} \longrightarrow C(\mathbf{L}_{\mathbb{R}})$ is a global equivalence between flat orthogonal spaces, but it is *not* a homeomorphism at 0. And indeed, for no $n \geq 2$ is the

morphism $\mathbb{P}^n(\mathbf{L}_{\mathbb{R}}) \longrightarrow \mathbb{P}^n(C(\mathbf{L}_{\mathbb{R}}))$ a global equivalence, because the source is isomorphic to $\mathbf{L}_{\Sigma_n,\mathbb{R}^n} = B_{\mathrm{gl}}\Sigma_n$, whereas the target is homotopy equivalent to the terminal orthogonal space.

Now we put all the pieces together and prove the global model structure for ultra-commutative monoids. We call a morphism of ultra-commutative monoids a *global equivalence* (or *positive global fibration*) if the underlying morphism of orthogonal spaces is a global equivalence (or fibration in the positive global model structure).

Theorem 2.1.15 (Global model structure for ultra-commutative monoids)**.**

(i) *The global equivalences and positive global fibrations are part of a cofibrantly generated, proper, topological model structure on the category of ultra-commutative monoids, the* global model structure*.*
(ii) *Let $j : R \longrightarrow S$ be a cofibration in the global model structure of ultra-commutative monoids.*
   (a) *The morphism of R-modules underlying j is a cofibration in the global model structure of R-modules of Corollary* 1.4.15 (i)*.*
   (b) *The morphism of orthogonal spaces underlying j is an h-cofibration, and hence a closed embedding.*
   (c) *If the underlying orthogonal space of R is flat, then j is a flat cofibration of orthogonal spaces.*

Proof (i) The positive global model structure of orthogonal spaces established in Proposition 1.2.23 is cofibrantly generated and monoidal (by Proposition 1.4.12 (iv)). The 'unit axiom' also holds: we let $f : I \longrightarrow *$ be any positive flat replacement of the monoidal unit, the constant one-point orthogonal space. Then for every orthogonal space Y the induced morphism $f \boxtimes X : I \boxtimes Y \longrightarrow * \boxtimes Y$ is a global equivalence by Theorem 1.3.2 (ii). The monoid axiom holds by Proposition 1.4.13. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 2.1.13, so the model structure satisfies the 'commutative monoid axiom' in the sense of [188, Def. 3.1]. The symmetric algebra functor $\mathbb{P}$ commutes with filtered colimits by Corollary 2.1.4. Theorem 3.2 of [188] thus shows that the positive global model structure of orthogonal spaces lifts to the category of ultra-commutative monoids.

The global model structure is topological by Proposition B.5, where we take $\mathcal{G}$ as the set of free ultra-commutative monoids $\mathbb{P}(L_{H,\mathbb{R}^m})$ for all $m \geq 1$ and all closed subgroups H of $O(m)$, and we take $\mathcal{Z}$ as the set of acyclic cofibrations $\mathbb{P}(c(\rho_{G.V.W}))$ for the mapping cone inclusions $c(\rho_{G,V,W})$ of the global equivalences $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$, indexed by representatives as in the definition of the set K^+. Since weak equivalences and fibrations of ultra-commutative

monoids are defined on underlying orthogonal spaces, and since pullbacks of ultra-commutative monoids are created on underlying orthogonal spaces, right properness is inherited from the positive global model structure of orthogonal spaces (Proposition 1.2.23). We defer the proof of left properness until after the proof of part (ii).

(ii) For (a) we recall that the global model structure on the category of R-modules is lifted, via the free and forgetful adjoint functor pair, from the absolute global model structure of Theorem 1.2.21. By Corollary 1.4.15 (i) and (ii) this model structure of R-modules is a cofibrantly generated monoidal model category that satisfies the monoid axiom. Moreover, the unit object R is cofibrant; for this it is relevant that we have lifted the *absolute* model structure (as opposed to the *positive model structure*). We claim that all cofibrations in this model structure are symmetrizable with respect to the box product of R-modules. By [64, Cor. 9] or [188, Lemma A.1] it suffices to show this for a set of generating cofibrations, which can be taken to be of the form $R \boxtimes i$ for i in a set of flat cofibrations of orthogonal spaces (for example the set I^{str} defined in the proof of the strong level model structure, Proposition 1.2.10). A box product, over R, of free R-modules induced from orthogonal spaces is isomorphic to the free R-module generated by the box product of underlying orthogonal spaces:

$$(R \boxtimes X) \boxtimes_R (R \boxtimes Y) \;\cong\; R \boxtimes (X \boxtimes Y)$$

Since $R \boxtimes -$ is a left adjoint, it commutes with pushouts and orbits by Σ_n-actions. Hence the analogous statement carries over to symmetrized iterated box products. In other words, for every morphism $i : A \longrightarrow B$ of orthogonal spaces there is a natural isomorphism in the arrow category of R-modules between

$$(R \boxtimes i)^{\Box_R n}/\Sigma_n \;:\; Q^n_R(R \boxtimes i)/\Sigma_n \;\longrightarrow\; \mathbb{P}^n_R(R \boxtimes B)$$

and

$$R \boxtimes (i^{\Box n}/\Sigma_n) \;:\; R \boxtimes (Q^n(i)/\Sigma_n) \;\longrightarrow\; R \boxtimes \mathbb{P}^n(B)\,.$$

If i is a flat cofibration of orthogonal spaces, then so is the morphism $i^{\Box n}/\Sigma_n$, by Theorem 2.1.13 (i). So the morphism $R \boxtimes (i^{\Box n}/\Sigma_n)$ is a cofibration of R-modules, hence so is the morphism $(R \boxtimes i)^{\Box_R n}/\Sigma_n$. This completes the proof that all cofibrations in the global model structure for R-modules of Corollary 1.4.15 (i) are symmetrizable with respect to $\boxtimes_R$.

Now we apply Corollary 3.6 of [188]; there is a slight caveat here, because the hypotheses ask for the validity of the 'strong commutative monoid axiom' (Definition 3.4 of [188]), which requires the symmetrizability of both the cofibrations and the acyclic cofibrations. Since the model structure on R-modules

was lifted from an absolute model structure, it is *not* the case that all acyclic cofibrations are symmetrizable. However, [188, Cor. 3.6] and its proof are only about cofibrations, and don't involve the weak equivalences at all. So the proof of [188, Cor. 3.6] only needs the symmetrizability of the cofibrations, which we just established for the global model category of R-modules of Corollary 1.4.15 (i). Since R is cofibrant as an R-module, [188, Cor. 3.6] shows that for every cofibrant commutative R-algebra S, the structure morphism $i : R \longrightarrow S$ is a cofibration of R-modules. Part (i) now follows because commutative R-algebras are morphisms of ultra-commutative monoids with source R. More precisely, the category of commutative R-algebras is isomorphic to the category of ultra-commutative monoids under R. Moreover, a commutative R-algebra S is cofibrant if and only if the structure morphism $i : R \longrightarrow S$ is a cofibration of ultra-commutative monoids.

(b) This is a combination of part (a) and the fact, proved in Corollary 1.4.15 (i), that all cofibrations of R-modules are h-cofibrations of orthogonal spaces.

(c) This is a combination of part (a) and the fact, also proved in Corollary 1.4.15 (i), that if R itself is flat, then all cofibrations of R-modules are flat cofibrations of orthogonal spaces.

If remains to prove left properness of the model structure. Pushouts in a category of commutative algebras are given by the relative monoidal product. For ultra-commutative monoids this means that a pushout square has the form

$$\begin{tikzcd} R \arrow[r, "f", "\simeq"'] \arrow[d, "j"'] & T \arrow[d, "j\boxtimes_R T"] \\ S \arrow[r, "S\boxtimes_R f"'] & S \boxtimes_R T \end{tikzcd}$$

where S and T are considered as R-modules by restriction along j and f, respectively. For left properness we now suppose that j is a cofibration and f is a global equivalence. By part (a) of (ii), the morphism j is then a cofibration of R-modules in the global model structure of Corollary 1.4.15 (i). Since R is cofibrant in that model structure, also S is cofibrant as an R-module. Proposition 1.4.16 then shows that the functor $S \boxtimes_R -$ preserves global equivalences. So the cobase change $S \boxtimes_R f$ of f is a global equivalence. This shows that the global model structure of ultra-commutative monoids is left proper. □

2.2 Global power monoids

In this section we investigate the algebraic structure that an ultra-commutative multiplication produces on the Rep-functor $\underline{\pi}_0(R)$. Besides an abelian monoid

structure on $\pi_0^G(R)$ for every compact Lie group G, this structure includes power operations and transfer maps. We formalize this algebraic structure under the name 'global power monoid', see Definition 2.2.8. Theorem 2.2.24 then says that global power monoids are precisely the natural algebraic structure, i.e., they parametrize all natural operations on $\underline{\pi}_0(R)$ for ultra-commutative monoids. In Construction 2.2.29 we introduce the transfer maps, which are an equivalent way of packaging the power operations in a global power monoid; the main properties of the transfers are summarized in Proposition 2.2.30.

Given an orthogonal monoid space R (not necessarily commutative) with multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ and a compact Lie group G, we define a binary operation

$$+ \ : \ \pi_0^G(R) \times \pi_0^G(R) \longrightarrow \pi_0^G(R) \tag{2.2.1}$$

on the G-equivariant homotopy set of R as the composite

$$\pi_0^G(R) \times \pi_0^G(R) \xrightarrow{\times} \pi_0^G(R \boxtimes R) \xrightarrow{\mu_*} \pi_0^G(R) .$$

The pairing $\times$ was defined in Construction 1.5.15. If we expand the definition, it boils down to the following explicit recipe: if V and W are G-representations and $x \in R(V)^G$ and $y \in R(W)^G$ are G-fixed-points that represent two classes in $\pi_0^G(R)$, then $[x] + [y]$ is represented by the G-fixed-point

$$\mu_{V,W}(x, y) \ \in \ R(V \oplus W) .$$

We write the pairing on the equivariant homotopy sets of R additively because we will mostly be interested in *commutative* orthogonal monoid spaces. Obviously, the additive notation is slightly dangerous for non-commutative orthogonal monoid spaces, because there the pairing need not be commutative.

The following properties of the operation '+' are direct consequences of the corresponding properties of the pairings '$\times$', compare Proposition 1.5.17; a direct proof from the explicit definition of the operation '+' above is also straightforward.

Corollary 2.2.2. *Let R be an orthogonal monoid space.*

(i) *For every compact Lie group G the binary operation $+$ makes the set $\pi_0^G(R)$ a monoid.*
(ii) *If the multiplication of R is commutative, then so is the operation $+$.*
(iii) *The restriction map $\alpha^* : \pi_0^G(R) \longrightarrow \pi_0^K(R)$ associated to a continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups is a monoid homomorphism.*

Now we turn to special features that happen for ultra-commutative monoids. If the multiplication on an orthogonal monoid space R is commutative, then this does not only imply commutativity of the monoids $\pi_0^G(R)$; strict commutativity of the multiplication also gives rise to additional *power operations* that we discuss now. An important special case later on will be the multiplicative ultra-commutative monoid $\Omega^\bullet R$ arising from an ultra-commutative ring spectrum R. In this situation the power operations satisfy further compatibility conditions with respect to the addition and the transfer maps on $\underline{\pi}_0(\Omega^\bullet R) = \underline{\pi}_0(R)$; altogether this structure makes the 0th equivariant homotopy groups of an ultra-commutative ring spectrum a *global power functor*.

Construction 2.2.3. We let R be an ultra-commutative monoid, G a compact Lie group and $m \geq 1$. We construct a natural *power operation*

$$[m] \ : \ \pi_0^G(R) \longrightarrow \pi_0^{\Sigma_m \wr G}(R) \tag{2.2.4}$$

that is an equivariant refinement of the map $x \longmapsto m \cdot x$.

We recall that the wreath product $\Sigma_m \wr G$ of a symmetric group Σ_m and a group G is the semidirect product

$$\Sigma_m \wr G \ = \ \Sigma_m \ltimes G^m$$

formed with respect to the action of Σ_m by permuting the factors of G^m. So the multiplication in $\Sigma_m \wr G$ is given by

$$(\sigma;\, g_1, \ldots, g_m) \cdot (\tau;\, k_1, \ldots, k_m) \ = \ (\sigma\tau;\, g_{\tau(1)}k_1, \ldots, g_{\tau(m)}k_m) \,.$$

For every G-space E, the wreath product $\Sigma_m \wr G$ acts on the space E^m by

$$(\sigma;\, g_1, \ldots, g_m) \cdot (e_1, \ldots, e_m) \ = \ (g_{\sigma^{-1}(1)}e_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(m)}e_{\sigma^{-1}(m)}) \,.$$

For every G-representation V, this action even makes V^m a $(\Sigma_m \wr G)$-representation. We let

$$\mu_{V,\ldots,V} \ : \ R(V) \times \cdots \times R(V) \longrightarrow R(V \oplus \cdots \oplus V)$$

denote the $(V, \ldots, V)$-component of the iterated multiplication map of R, and we observe that this map is $(\Sigma_m \wr G)$-equivariant because the multiplication of R is commutative. If $x \in R(V)^G$ is a G-fixed-point representing a class in $\pi_0^G(R)$, then $(x, \ldots, x) \in R(V)^m$ is a $(\Sigma_m \wr G)$-fixed-point. So its image under the map $\mu_{V,\ldots,V}$ is a $(\Sigma_m \wr G)$-fixed-point of $R(V^m)$, representing an element

$$[m]([x]) \ = \ \langle \mu_{V,\ldots,V}(x, \ldots, x) \rangle \ \in \ \pi_0^{\Sigma_m \wr G}(R) \,.$$

If we stabilize x along a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ to $R(\varphi)(x) \in R(W)^G$, then $\mu_{V,\ldots,V}(x, \ldots, x)$ changes into

$$\mu_{W,\ldots,W}(R(\varphi)(x), \ldots, R(\varphi)(x)) \ = \ R(\varphi^m)(\mu_{V,\ldots,V}(x, \ldots, x)) \ \in \ R(W^m)^{\Sigma_m \wr G} \,.$$

Since $\varphi^m : V^m \longrightarrow W^m$ is a $(\Sigma_m \wr G)$-equivariant linear isometric embedding, this element represents the same class in $\pi_0^{\Sigma_m \wr G}(R)$ as $\mu_{V,\dots,V}(x, \dots, x)$, so the class $[m]([x])$ only depends on the class of x in $\pi_0^G(R)$. We have thus constructed a well-defined power operation (2.2.4).

The power operations are clearly natural for homomorphisms $\varphi : R \longrightarrow S$ of ultra-commutative monoids, i.e., for every compact Lie group G, every $m \geq 0$ and all $x \in \pi_0^G(R)$ the relation

$$[m](\varphi_*(x)) \;=\; \varphi_*([m](x))$$

holds in $\pi_0^{\Sigma_m \wr G}(S)$.

The power operations $[m]$ satisfy various properties reminiscent of the map $x \mapsto m \cdot x$ in an abelian monoid. We formalize these properties into the concept of a *global power monoid*. In the definition we need certain homomorphisms between different wreath products, so we fix notation for these now. We use the plus symbol for the 'concatenation' group monomorphism

$$+ \;:\; \Sigma_i \times \Sigma_j \;\longrightarrow\; \Sigma_{i+j}$$

defined by

$$(\sigma + \sigma')(k) \;=\; \begin{cases} \sigma(k) & \text{for } 1 \leq k \leq i, \text{ and} \\ \sigma'(k-i) + i & \text{for } i+1 \leq k \leq i+j. \end{cases}$$

This operation is strictly associative, so we will leave out parentheses. The operation $+$ is *not* commutative, but the permutations $\sigma + \sigma'$ and $\sigma' + \sigma$ differ by conjugation with the (i, j)-shuffle. An embedding of a product of wreath products is now defined by

$$\begin{aligned} \Phi_{i,j} \;:\; (\Sigma_i \wr G) \times (\Sigma_j \wr G) \;&\longrightarrow\; \Sigma_{i+j} \wr G \\ ((\sigma;\, g_1, \dots, g_i), (\sigma';\, g_{i+1}, \dots, g_{i+j})) \;&\longmapsto\; (\sigma + \sigma';\, g_1, \dots, g_{i+j}) \,. \end{aligned} \tag{2.2.5}$$

Another group monomorphism

$$\natural \;:\; \Sigma_k \wr \Sigma_m \;\longrightarrow\; \Sigma_{km}$$

is defined by

$$(\sigma\natural(\tau_1, \dots, \tau_k))((i-1)m + j) \;=\; (\sigma(i) - 1)m + \tau_i(j)\,,$$

for $1 \leq i \leq k$ and $1 \leq j \leq m$. This yields an embedding of an iterated wreath product

$$\begin{aligned} \Psi_{k,m} \;:\; \Sigma_k \wr (\Sigma_m \wr G) \;&\longrightarrow\; \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, g^1), \dots, (\tau_k;\, g^k)) \;&\longmapsto\; (\sigma\natural(\tau_1, \dots, \tau_k);\, g^1 + \dots + g^k) \,. \end{aligned} \tag{2.2.6}$$

Here each $g^i = (g^i_1, \dots, g^i_m)$ is an m-tuple of elements of G, and

$$g^1 + \cdots + g^k = (g^1_1, \dots, g^1_m, g^2_1, \dots, g^2_m, \dots, g^k_1, \dots, g^k_m)$$

denotes the concatenation of the tuples.

Remark 2.2.7. The formula for the homomorphism $\natural : \Sigma_k \wr \Sigma_m \longrightarrow \Sigma_{km}$ may seems slightly ad hoc, but it can be motivated in a more conceptual way as a composite

$$\Sigma_k \wr \Sigma_m \longrightarrow \Sigma_{\{1,\dots,k\}\times\{1,\dots,m\}} \cong \Sigma_{km} .$$

The first monomorphism sends $(\sigma; \tau_1, \dots, \tau_k)$ to the permutation of the product set $\{1, \dots, k\} \times \{1, \dots, m\}$ defined by

$$(i, j) \longmapsto (\sigma(i), \tau_i(j)) .$$

The second isomorphism is conjugation by the lexicographic ordering

$$\{1, \dots, k\} \times \{1, \dots, m\} \cong \{1, \dots, km\} , \quad (i, j) \longmapsto (i-1)m + j .$$

The use of the lexicographic ordering (and hence the precise formula for the homomorphism $\natural$) is not essential here: if we use a different bijection between the sets $\{1, \dots, k\} \times \{1, \dots, m\}$ and $\{1, \dots, km\}$, then the homomorphisms $\natural$ and $\Psi_{k,m}$ change by inner automorphisms. So the conjugacy classes of $\natural$ and $\Psi_{k,m}$ (but not the actual homomorphisms) are canonical. Since we will always hit $\Psi_{k,m}$ with functors that are invariant under conjugation, this should motivate the construction being reasonably natural.

Definition 2.2.8. A *global power monoid* is a functor

$$M : \mathrm{Rep}^{\mathrm{op}} \longrightarrow \mathcal{A}bMon$$

from the opposite of the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms to the category of abelian monoids, equipped with monoid homomorphisms

$$[m] : M(G) \longrightarrow M(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Identity) The operation [1] is restriction along the preferred isomorphism $\Sigma_1 \wr G \cong G$, $(1; g) \mapsto g$.

(ii) (Naturality) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and every $m \geq 1$ the relation

$$[m] \circ \alpha^* = (\Sigma_m \wr \alpha)^* \circ [m]$$

holds as homomorphisms $M(G) \longrightarrow M(\Sigma_m \wr K)$.

(iii) (Transitivity) For all compact Lie groups G and all $k, m \geq 1$ the relation

$$\Psi^*_{k,m} \circ [km] \;=\; [k] \circ [m]$$

holds as homomorphisms $M(G) \longrightarrow M(\Sigma_k \wr (\Sigma_m \wr G))$, where $\Psi_{k,m}$ is the monomorphism (2.2.6).

(iv) (Additivity) For all compact Lie groups G, all $m > i > 0$ and all $x \in M(G)$ the relation

$$\Phi^*_{i,m-i}([m](x)) \;=\; p_1^*([i](x)) \;+\; p_2^*([m-i](x))$$

holds in $M((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G))$, where $\Phi_{i,m-i}$ is the monomorphism (2.2.5) and

$$p_1 : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_i \wr G \quad \text{and} \quad p_2 : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_{m-i} \wr G$$

are the two projections.

A *morphism* of global power monoids is a natural transformation of abelian monoid-valued functors that also commutes with the power operations $[m]$ for all $m \geq 1$.

Remark 2.2.9. In any abelian Rep-monoid M we can define external pairings

$$\oplus \;:\; M(G) \times M(K) \;\longrightarrow\; M(G \times K) \quad \text{by} \quad x \oplus y \;=\; p_G^*(x) + p_K^*(y)\,,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the two projections. In this notation, the additivity requirement in Definition 2.2.8 becomes the relation

$$\Phi^*_{i,m-i}([m](x)) \;=\; [i](x) \;\oplus [m-i](x)\,. \tag{2.2.10}$$

In a global power monoid, the power operations are also additive with respect to the external addition: for all compact Lie groups G and K and all $m \geq 1$, and all classes $x \in M(G)$ and $y \in M(K)$ the relation

$$[m](x \oplus y) \;=\; \Delta^*([m](x) \oplus [m](y))$$

holds in $M(\Sigma_m \wr (G \times K))$, where Δ is the 'diagonal' monomorphism

$$\begin{aligned} \Delta \;:\; \Sigma_m \wr (G \times K) \quad &\longrightarrow \quad (\Sigma_m \wr G) \times (\Sigma_m \wr K) \\ (\sigma;\, (g_1, k_1), \ldots, (g_m, k_m)) \;&\longmapsto\; ((\sigma;\, g_1, \ldots, g_m),\, (\sigma;\, k_1, \ldots, k_m))\,. \end{aligned} \tag{2.2.11}$$

Indeed, Δ factors as the composite

$$\begin{aligned} \Sigma_m \wr (G \times K) \xrightarrow{\Delta_{\Sigma_m \wr (G \times K)}} & \;(\Sigma_m \wr (G \times K)) \times (\Sigma_m \wr (G \times K)) \\ \xrightarrow{(\Sigma_m \wr p_G) \times (\Sigma_m \wr p_K)} & \;(\Sigma_m \wr G) \times (\Sigma_m \wr K)\,. \end{aligned}$$

So

$$\begin{aligned}
[m](x \oplus y) &= [m](p_G^*(x) + p_K^*(y)) \;=\; [m](p_G^*(x)) + [m](p_K^*(y)) \\
&= (\Sigma_m \wr p_G)^*([m](x)) + (\Sigma_m \wr p_K)^*([m](y)) \\
&= \Delta^*_{\Sigma_m \wr (G\times K)}((\Sigma_m \wr p_G)^*([m](x)) \oplus (\Sigma_m \wr p_K)^*([m](y))) \\
&= \Delta^*_{\Sigma_m \wr (G\times K)}(((\Sigma_m \wr p_G) \times (\Sigma_m \wr p_K))^*([m](x) \oplus [m](y))) \\
&= \Delta^*([m](x) \oplus [m](y)) \ .
\end{aligned}$$

The class $[m](x)$ is an equivariant refinement of $m\cdot x = x+\ldots+x$ (m summands) in the following sense. Applying the relation (2.2.10) repeatedly shows that $[m](x)$ restricts to the external m-fold sum

$$x \oplus \ldots \oplus x \;\in\; M(G^m)$$

on the normal subgroup G^m of $\Sigma_m \wr G$. Restricting further to the diagonal takes the m-fold external sum to $m \cdot x$ in $M(G)$.

We will soon discuss that the power operations of an ultra-commutative monoid define a global power monoid. One aspect of this is the additivity of the power operations, which could be shown directly from the definition. However, we will use this opportunity to establish a very general (and rather formal) additivity result that we will use several times in this book. We let C be a category with a zero object and finite coproducts. We let $X \vee Y$ be a coproduct of two objects X and Y with universal morphisms $i : X \longrightarrow X \vee Y$ and $j : Y \longrightarrow X \vee Y$. If $f : X \longrightarrow A$ and $g : Y \longrightarrow A$ are two morphisms with common target, we denote by $f + g : X \vee Y \longrightarrow A$ the unique morphism that satisfies $(f + g)i = f$ and $(f + g)j = g$.

We call a functor from C to the category of abelian monoids *reduced* if it takes every zero object in C to the trivial monoid. We call the functor F *additive* if for every pair of objects X, Y of C the map

$$(F(\mathrm{Id}_X + 0), F(0 + \mathrm{Id}_Y)) \;:\; F(X \vee Y) \;\longrightarrow\; F(X) \times F(Y)$$

is bijective (and hence an isomorphism of monoids).

Proposition 2.2.12. *Let C be a category with a zero object and finite coproducts and*

$$F, G \;:\; C \;\longrightarrow\; \mathcal{A}bMon$$

two reduced functors to the category of abelian monoids. Suppose that the functor G is additive. Then every natural transformation of set-valued functors from F to G is automatically additive.

Proof We let $\tau : F \longrightarrow G$ be a natural transformation of set-valued functors.

We consider two classes $x, y \in F(X)$. We let $i, j : X \longrightarrow X \vee X$ be the two inclusions into the coproduct. We claim that

$$\tau_{X\vee X}(F(i)(x) + F(j)(y)) \;=\; G(i)(\tau_X(x)) + G(j)(\tau_X(y)) \tag{2.2.13}$$

in the abelian monoid $G(X \vee X)$. Indeed,

$$\begin{aligned} G(\mathrm{Id}_X + 0)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) \;&=\; \tau_X(F(\mathrm{Id}_X + 0)(F(i)(x) + F(j)(y))) \\ &=\; \tau_X(F(\mathrm{Id}_X)(x) + F(0)(y)) \;=\; \tau_X(x) \\ &=\; G(\mathrm{Id}_X)(\tau_X(x)) + G(0)(\tau_X(x)) \\ &=\; G(\mathrm{Id}_X + 0)(G(i)(\tau_X(x)) + G(j)(\tau_X(y))) \end{aligned}$$

in $G(X)$. Similarly,

$$G(0 + \mathrm{Id}_X)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) \;=\; G(0 + \mathrm{Id}_X)(G(i)(\tau_X(x)) + G(j)(\tau_X(y)))\;.$$

Since G is additive, this shows the relation (2.2.13). We let $\nabla = (\mathrm{Id} + \mathrm{Id}) : X \vee X \longrightarrow X$ denote the fold morphism, so that

$$F(\nabla)(F(i)(x) + F(j)(y)) \;=\; F(\nabla i)(x) + F(\nabla j)(y) \;=\; x + y\;.$$

Then

$$\begin{aligned} \tau_X(x + y) \;&=\; \tau_X(F(\nabla)(F(i)(x) + F(j)(y))) \\ &=\; G(\nabla)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) \\ &\overset{(2.2.13)}{=}\; G(\nabla)(G(i)(\tau_X(x)) + G(j)(\tau_X(y))) \\ &=\; G(\nabla i)(\tau_X(x)) + G(\nabla j)(\tau_X(y)) \;=\; \tau_X(x) + \tau_X(y)\;. \end{aligned}$$

□

Proposition 2.2.14. *Let R be an ultra-commutative monoid. Then the binary operations* (2.2.1) *and the power operations* (2.2.4) *make the Rep-functor $\underline{\pi}_0(R)$ a global power monoid.*

Proof Corollary 2.2.2 shows that the binary operations (2.2.1) make the Rep-functor $\underline{\pi}_0(R)$ a functor to the category of abelian monoids. The coproduct of ultra-commutative monoids is given by the box product, so the two reduced functors

$$\pi_0^G\;,\; \pi_0^{\Sigma_m \wr G} \;:\; umon \;\longrightarrow\; \mathcal{A}bMon$$

are additive by Corollary 1.5.20. Since the power operation $[m] : \pi_0^G(R) \longrightarrow \pi_0^{\Sigma_m \wr G}(R)$ is natural in R, Proposition 2.2.12, applied to the category of ultra-commutative monoids, shows that $[m]$ is additive. The identity (i), naturality (ii), transitivity (iii) and additivity property (iv) in Definition 2.2.8 of global power monoids are straightforward, and we omit the proofs. □

Example 2.2.15. We let M be a commutative topological monoid. Then the constant orthogonal space $\underline{M}$ is naturally an ultra-commutative monoid. Moreover, the equivariant homotopy functor $\underline{\pi}_0(\underline{M})$ is constant with value $\pi_0(M)$, and monoid structure induced from the multiplication of M. The power operation

$$[m] \ : \ \pi_0(M) = \pi_0^G(\underline{M}) \longrightarrow \pi_0^{\Sigma_m \wr G}(\underline{M}) = \pi_0(M)$$

then sends an element x to $m \cdot x$.

Example 2.2.16 (Naive units of an orthogonal monoid space)**.** Every orthogonal monoid space R contains an interesting orthogonal monoid subspace $R^{n\times}$, the *naive units* of R. The value of $R^{n\times}$ at an inner product space V is the union of those path components of $R(V)$ that are taken to invertible elements, with respect to the monoid structure on $\pi_0(R)$, under the map

$$R(V) \ \longrightarrow \ \pi_0(R(V)) \ \longrightarrow \ \pi_0^e(R) \ .$$

In other words, a point $x \in R(V)$ belongs to $R^{n\times}(V)$ if and only if there is an inner product space W and a point $y \in R(W)$ such that

$$\mu_{V,W}(x,y) \ \in \ R(V \oplus W) \qquad \text{and} \qquad \mu_{W,V}(y,x) \ \in \ R(W \oplus V)$$

are in the same path component as the respective unit elements. We omit the verification that the subspaces $R^{n\times}(V)$ indeed form an orthogonal monoid subspace of R as V varies. The induced map

$$\underline{\pi}_0(R^{n\times}) \ \longrightarrow \ \underline{\pi}_0(R)$$

is also an inclusion, and the value $\pi_0^e(R^{n\times})$ at the trivial group is, by construction, the set of invertible elements of $\pi_0^e(R)$. For a general compact Lie group G,

$$\pi_0^G(R^{n\times}) \ = \ \{ x \in \pi_0^G(R) \mid \mathrm{res}^G_e(x) \text{ is invertible in } \pi_0^e(R)\}$$

is the submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. So contrary to what one might suspect at first sight, $\pi_0^G(R^{n\times})$ may contain non-invertible elements and the orthogonal monoid space $R^{n\times}$ is not necessarily group-like; this is why we use the adjective 'naive'.

Example 2.2.17 (Units of a global power monoid)**.** Every global power monoid M has a global power submonoid $M^\times$ of *units*. The value $M^\times(G)$ at a compact Lie group G is the subgroup of invertible elements of $M(G)$. Since the restriction maps and the power operations are homomorphisms, the sets $M^\times(G)$ are closed under restriction maps and power operations. So for varying G, the subgroups $M^\times(G)$ indeed form a global power submonoid of M.

We say a global power monoid N is *group-like* if the abelian monoid $N(G)$

is a group for every compact Lie group G. If $f : N \longrightarrow M$ is a homomorphism of global power monoids and N is group-like, then the image of f is contained in $M^\times$. So the functor $M \mapsto M^\times$ is right adjoint to the inclusion of the full subcategory of group-like global power monoids.

If R is an ultra-commutative monoid, then we introduce a global topological version $R^\times$ of the units in Construction 2.5.18 below. This construction comes with a homomorphism of ultra-commutative monoids $R^\times \longrightarrow R$ that realizes the inclusion of the units of $\underline{\pi}_0(R)$, compare Proposition 2.5.19.

Example 2.2.18 (Group completion of a global power monoid)**.** A morphism $j : M \longrightarrow M^\star$ of global power monoids is a *group completion* if for every group-like global power monoid N the map

$$j^* \ : \ (\text{global power monoids})(M^\star, N) \ \longrightarrow \ (\text{global power monoids})(M, N)$$

is bijective. Since the pair $(M^\star, j)$ represents a functor, it is unique up to preferred isomorphism under M. Every global power monoid M has a group completion, which can be constructed 'objectwise'. We define a global power monoid $M^\star$ at a compact Lie group G by letting $M^\star(G)$ be a group completion (Grothendieck construction) of the abelian monoid $M(G)$, with $j(G) : M(G) \longrightarrow M^\star(G)$ the universal homomorphism. Since the restriction maps $\alpha^* : M(G) \longrightarrow M(K)$ and the power operations $[m] : M(G) \longrightarrow M(\Sigma_m \wr G)$ are monoid homomorphisms, the universal property provides unique homomorphisms $\alpha^* : M^\star(G) \longrightarrow M^\star(K)$ and $[m] : M^\star(G) \longrightarrow M^\star(\Sigma_m \wr G)$ such that

$$\alpha^* \circ j(G) = j(K) \circ \alpha^* \qquad \text{and} \qquad [m] \circ j(G) = j(\Sigma_m \wr G) \circ [m] \, .$$

The functoriality of the restriction maps α^* and the additional relations required of a global power monoid are relations between monoid homomorphism; so they are inherited by $M^\star$ via the universal property of group completion of abelian monoids.

If R is an ultra-commutative monoid, then in Construction 2.5.20 below we introduce a global topological version $R^\star$ of the group completion. This construction comes with a homomorphism of ultra-commutative monoids $R \longrightarrow R^\star$ that realizes the algebraic group completion, compare Proposition 2.5.21.

Example 2.2.19 (Free ultra-commutative monoid of a global classifying space)**.** We look more closely at the free ultra-commutative monoid $\mathbb{P}(B_{\text{gl}}G)$ generated by the global classifying space $B_{\text{gl}}G$ of a compact Lie group G. For every G-representation V, Example 1.3.3 provides an isomorphism of orthogonal spaces

$$\mathbf{L}_{G,V}^{\boxtimes m} \ \cong \ \mathbf{L}_{G^m, V^m} \ .$$

At an inner product space W, the permutation action of Σ_m on the left-hand side becomes the action on

$$\mathbf{L}_{G^m,V^m}(W) \;=\; \mathbf{L}(V^m, W)/G^m$$

by permuting the summands in V^m. Passage to Σ_m-orbits gives an isomorphism

$$\mathbb{P}^m(\mathbf{L}_{G,V}) \;=\; \mathbf{L}_{G,V}^{\boxtimes m}/\Sigma_m \;\cong\; \mathbf{L}_{G^m,V^m}/\Sigma_m \;\cong\; \mathbf{L}_{\Sigma_m\wr G,V^m}\ .$$

Thus

$$\mathbb{P}(\mathbf{L}_{G,V}) \;=\; \coprod\nolimits_{m\geq 0} \mathbf{L}_{\Sigma_m\wr G,V^m}\ .$$

If G acts faithfully on V and $V \neq 0$, then the action of $\Sigma_m \wr G$ on V^m is again faithful. So in terms of global classifying spaces the free ultra-commutative monoid generated by $B_{\mathrm{gl}}G$ is given by

$$\mathbb{P}(B_{\mathrm{gl}}G) \;=\; \coprod\nolimits_{m\geq 0} B_{\mathrm{gl}}(\Sigma_m \wr G)\ . \tag{2.2.20}$$

The tautological class $u_G \in \pi_0^G(B_{\mathrm{gl}}G)$ is represented by the orbit of the identity of V in

$$(\mathbf{L}_{G,V}(V))^G \;=\; (\mathbf{L}(V,V)/G)^G\ ,$$

compare (1.5.11). So the class $[m](u_G) \in \pi_0^{\Sigma_m\wr G}(\mathbb{P}(B_{\mathrm{gl}}G))$ is represented by the orbit of the identity of V^m in

$$(\mathbf{L}_{\Sigma_m\wr G,V^m}(V^m))^G \;=\; (\mathbf{L}(V^m,V^m)/\Sigma_m \wr G)^{\Sigma_m\wr G}\ ;$$

so with respect to the identification (2.2.20) we have

$$[m](u_G) \;=\; u_{\Sigma_m\wr G}\ . \tag{2.2.21}$$

Example 2.2.22 (Coproducts of ultra-commutative monoids)**.** The category of ultra-commutative monoids is cocomplete; in particular, every family $\{R_i\}_{i\in I}$ of ultra-commutative monoids has a coproduct that we denote $\boxtimes'_{i\in I} R_i$. We claim that the functor

$$\underline{\pi}_0 \;:\; umon \;\longrightarrow\; (\text{global power monoids})$$

preserves coproducts. Indeed, if the indexing set I is finite, then the underlying orthogonal space of the coproduct $\boxtimes'_{i\in I} R_i$ is simply the iterated box product of the underlying orthogonal spaces. The functor $\underline{\pi}_0$ takes box products of orthogonal spaces to the objectwise product of Rep-functors, by Corollary 1.5.20. In the category of abelian monoids, finite products are also finite coproducts. Coproducts of global power monoids are formed objectwise, so a finite product of global power monoids is also a coproduct. This proves the claim whenever the indexing set I is finite.

In any category, an infinite coproduct is the filtered colimit of the finite coproducts. Moreover, filtered colimits of ultra-commutative monoids are formed on underlying orthogonal spaces, compare Corollary 2.1.4. So if the set I is infinite, then the underlying orthogonal space of the coproduct is the 'infinite box product' in the sense of Construction 1.5.21, i.e., the filtered colimit, formed over the poset of finite subsets of I, of the finite coproducts,

$$\boxtimes'_{i\in I} R_i \;\cong\; \operatorname{colim}_{J\subset I, J \text{ finite}} \boxtimes_{j\in J} R_j \ .$$

Proposition 1.5.24 thus provides a bijection

$$\prod\nolimits'_{i\in I} \pi_0^G(R_i) \;\longrightarrow\; \pi_0^G(\boxtimes'_{i\in I} R_i)$$

from the weak product of the abelian monoids $\pi_0^G(R_i)$ to the abelian monoid $\pi_0^G(\boxtimes'_{i\in I} R_i)$. For abelian monoids, the weak product is also the direct sum, i.e., the categorical coproduct. Since colimits of global power monoids are calculated objectwise, this proves the claim in general.

Here is another family of global power monoids, with underlying Rep-functor represented by an abelian compact Lie group A. In fact, the next proposition shows that $\mathrm{Rep}(-, A)$ is a free global power monoid subject to a specific set of explicit 'power relations'. In Construction 2.3.23 below we exhibit a multiplicative model for the global classifying space of A, i.e., an ultra-commutative monoid that realizes the global power monoid $\mathrm{Rep}(-, A)$ on $\underline{\pi}_0$.

Proposition 2.2.23. *For every abelian compact Lie group A, the Rep-functor* $\mathrm{Rep}(-, A)$ *has a unique structure of global power monoid. The monoid structure of* $\mathrm{Rep}(G, A)$ *is given by pointwise multiplication of homomorphisms. The power operation*

$$[m] \;:\; \mathrm{Rep}(G, A) \;\longrightarrow\; \mathrm{Rep}(\Sigma_m \wr G, A)$$

is given by

$$[m](\alpha) \;=\; p_m \circ (\Sigma_m \wr \alpha) \ ,$$

where $p_m : \Sigma_m \wr A \longrightarrow A$ *is the homomorphism defined by* $p_m(\sigma; a_1, \dots, a_m) = a_1 \cdot \ldots \cdot a_m$. *Moreover, for every global power monoid M the map*

$$(\text{global power monoids})(\mathrm{Rep}(-, A), M) \;\longrightarrow\; M(A) \ , \quad f \longmapsto f(A)(\mathrm{Id}_A)$$

is injective with image those $x \in M(A)$ *that satisfy* $[m](x) = p_m^*(x)$ *for all* $m \geq 1$.

Proof Since A is abelian, conjugate homomorphisms into A are already equal, i.e., we can ignore the difference between homomorphisms and conjugacy classes. We establish the monoid structure and power operations first, which

also shows the uniqueness. Since $\mathrm{Rep}(e, A)$ has only one element, it is the additive unit. Since restriction maps are monoid homomorphisms, the trivial homomorphism is the neutral element of $\mathrm{Rep}(G, A)$. The sum

$$q_1 + q_2 \ \in \ \mathrm{Rep}(A \times A, A)$$

of the two projections is a homomorphism from $A \times A$ to A whose restriction along the two maps $(-, 1), (1, -) : A \longrightarrow A \times A$ is the identity. The only such homomorphism is the multiplication $\mu : A \times A \longrightarrow A$, so we conclude that

$$q_1 + q_2 \ = \ \mu \ .$$

Naturality now gives

$$\alpha + \beta \ = \ (\alpha, \beta)^*(q_1) + (\alpha, \beta)^*(q_2) \ = \ (\alpha, \beta)^*(q_1 + q_2) \ = \ (\alpha, \beta)^*(\mu) \ = \ \mu \circ (\alpha, \beta) \ .$$

Since power operations refine power maps, the element

$$[m](\mathrm{Id}_A) \ \in \ \mathrm{Rep}(\Sigma_m \wr A, A)$$

restricts to the sum of the m projections on $A^m \leq \Sigma_m \wr A$. We let $1 : e \longrightarrow A$ be the unique homomorphism and claim that the composite

$$\Sigma_m \wr e \xrightarrow{\Sigma_m \wr 1} \Sigma_m \wr A \xrightarrow{[m](\mathrm{Id}_A)} A$$

is trivial. Indeed,

$$[m](\mathrm{Id}_A) \circ (\Sigma_m \wr 1) \ = \ (\Sigma_m \wr 1)^*([m](\mathrm{Id}_A)) \ = \ [m](1^*(\mathrm{Id}_A)) \ = \ [m](1) \ = \ 1 \ ,$$

since the operation $[m]$ is a monoid homomorphism. Thus

$$\begin{aligned} [m](\mathrm{Id}_A)(\sigma;\, a_1, \dots, a_m) \ &= \ [m](\mathrm{Id}_A)(\sigma;\, 1, \dots, 1) \cdot [m](\mathrm{Id}_A)(1;\, a_1, \dots, a_m) \\ &= \ a_1 \cdot \ldots \cdot a_m \ . \end{aligned}$$

In other words, $[m](\mathrm{Id}_A) = p_m$. Naturality now gives

$$\begin{aligned} [m](\alpha) \ = \ [m](\alpha^*(\mathrm{Id}_A)) \ &= \ (\Sigma_m \wr \alpha)^*([m](\mathrm{Id}_A)) \\ &= \ (\Sigma_m \wr \alpha)^*(p_m) \ = \ p_m \circ (\Sigma_m \wr \alpha) \ . \end{aligned}$$

It remains to show the existence of the global power monoid structure. Clearly, pointwise multiplication of homomorphisms makes $\mathrm{Rep}(G, A)$ into an abelian monoid (even an abelian group), and the monoid structure is contravariantly functorial in G. When we define $[m](\alpha)$ by the formula of the proposition, then the remaining axioms of a global power monoid (compare Definition 2.2.8) are similarly straightforward. The identity property (i) is clear, and naturality (ii) follows from the relation

$$[m](\alpha) \ = \ (\Sigma_m \wr \alpha)^*([m](\mathrm{Id}_A)) \ .$$

The transitivity relation (iii) holds by inspection:

$$\begin{aligned}
&\Psi^*_{k,m}([km](\alpha))(\sigma;(\tau_1;h^1),\dots,(\tau_k,h^k))\\
&\qquad = ([km](\alpha))(\Psi_{k,m}(\sigma;(\tau_1;h^1),\dots,(\tau_k,h^k)))\\
&\qquad = \prod_{i=1}^k\left(\prod_{j=1}^m h^i_j\right) = \prod_{i=1}^k([m](\alpha))(\tau_1;h^i))\\
&\qquad = ([k]([m](\alpha)))(\sigma;(\tau_1;h^1),\dots,(\tau_k,h^k)) ,
\end{aligned}$$

as does the additivity relation (iv):

$$\begin{aligned}
&\Phi^*_{i,m-i}([m](\alpha))((\sigma;\ g_1,\dots,g_i),(\sigma';\ g_{i+1},\dots,g_m))\\
&\qquad = ([m](\alpha))(\sigma+\sigma';\ g_1,\dots,g_m)\\
&\qquad = \alpha(g_1)\cdot\dots\cdot\alpha(g_m)\\
&\qquad = ([i](\alpha))(\sigma;\ g_1,\dots,g_i)\cdot([m-i](\alpha))(\sigma';\ g_i,\dots,g_m)\\
&\qquad = ([i](\alpha)\oplus[m-i](\alpha))((\sigma;\ g_1,\dots,g_i),(\sigma';\ g_{i+1},\dots,g_m)) .
\end{aligned}$$

It remains to identify the global power morphisms out of $\mathrm{Rep}(-,A)$. Since the class Id_A generates $\mathrm{Rep}(-,A)$ as a Rep-functor, the evaluation map in injective. For surjectivity we consider a class $x\in M(A)$ such that $[m](x) = p_m^*(x)$ for all $m\geq 1$. The Yoneda lemma then provides a unique morphism of Rep-functors

$$f\ :\ \mathrm{Rep}(-,A)\ \longrightarrow\ M$$

such that $f(A)(\mathrm{Id}_A) = x$, this morphism being given by $f(G)(\alpha) = \alpha^*(x)$, for $\alpha : G\longrightarrow A$. We need to show that f is a morphism of global power monoids, i.e., additive and compatible with power operations. For additivity we recall that

$$\alpha+\beta\ =\ \mu\circ(\alpha,\beta)\ =\ p_2\circ\mathrm{incl}_{A^2}^{\Sigma_2\wr A}\circ(\alpha,\beta) .$$

Hence

$$\begin{aligned}
f(G)(\alpha+\beta)\ &=\ (p_2\circ\mathrm{incl}_{A^2}^{\Sigma_2\wr A}\circ(\alpha,\beta))^*(x)\\
&=\ (\alpha,\beta)^*(\mathrm{res}_{A^2}^{\Sigma_2\wr A}(p_2^*(x)))\ =\ (\alpha,\beta)^*(\mathrm{res}_{A^2}^{\Sigma_2\wr A}([2](x)))\\
&=\ (\alpha,\beta)^*(x\oplus x)\ =\ \alpha^*(x)+\beta^*(x)\ =\ f(G)(\alpha)+f(G)(\beta) .
\end{aligned}$$

This shows that f is a morphism of abelian Rep-monoids. Finally, given a continuous homomorphism $\alpha : G\longrightarrow A$, we have

$$\begin{aligned}
[m](f(G)(\alpha))\ &=\ [m](\alpha^*(x))\ =\ (\Sigma_m\wr\alpha)^*([m](x))\ =\ (\Sigma_m\wr\alpha)^*(p_m^*(x))\\
&=\ (p_m\circ(\Sigma_m\wr\alpha))^*(x)\ =\ ([m](\alpha))^*(x)\ =\ f(\Sigma_m\wr G)([m](\alpha)) .
\end{aligned}$$

So the morphism f is also compatible with power operations. □

Now we are going to show that the restriction maps along continuous group homomorphisms and the power operations give all natural operations between equivariant homotopy sets of ultra-commutative monoids. The strategy is the same as in the analogous situation for orthogonal spaces, see Corollary 1.5.14: natural operations for ultra-commutative monoids from the functor π_0^G to the functor π_0^K biject with the K-equivariant homotopy set of $\mathbb{P}(B_{\mathrm{gl}}G)$, the free ultra-commutative monoids generated by a global classifying space of G. So ultimately we need to calculate $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$.

The tautological class u_G in $\pi_0^G(B_{\mathrm{gl}}G)$ was defined in (1.5.11). We set

$$u_G^{umon} \;=\; \eta_*(u_G) \;\in\; \pi_0^G(\mathbb{P}(B_{\mathrm{gl}}G)) ,$$

where $\eta : B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ is the adjunction unit, i.e., the inclusion of the homogeneous summand for $m = 1$. The next theorem says in particular that the global power monoid $\underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G))$ is freely generated by the element u_G^{umon}.

Theorem 2.2.24. *Let G and K be compact Lie groups.*

(i) *Every class in $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ is of the form $\alpha^*([m](u_G^{umon}))$ for a unique $m \geq 0$ and a unique conjugacy class of continuous homomorphisms $\alpha : K \longrightarrow \Sigma_m \wr G$.*

(ii) *For every global power monoid M and every $x \in M(G)$ there is a unique morphism of global power monoids $f : \underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M$ such that*

$$f(G)(u_G^{umon}) \;=\; x .$$

(iii) *Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set-valued functors on the category of ultra-commutative monoids is of the form $\alpha^* \circ [m]$ for a unique $m \geq 0$ and a unique conjugacy class of continuous group homomorphisms $\alpha : K \longrightarrow \Sigma_m \wr G$.*

Proof For the course of the proof we abbreviate $u = u_G^{umon}$.

(i) By (2.2.20) the underlying orthogonal space of $\mathbb{P}(B_{\mathrm{gl}}G)$ is the disjoint union of global classifying spaces for the wreath product groups $\Sigma_m \wr G$; moreover, the class $[m](u)$ lies in the mth summand of $\mathbb{P}(B_{\mathrm{gl}}G)$ and is a universal element for $B_{\mathrm{gl}}(\Sigma_m \wr G)$, by (2.2.21). So part (i) follows from Proposition 1.5.12 (ii) and the fact that π_0^K commutes with disjoint unions.

(ii) By (i) every element of $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ is of the form $\alpha^*([m](u))$; every morphism of global power monoids $f : \underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M$ satisfies

$$f(K)(\alpha^*([m](u))) \;=\; \alpha^*([m](f(G)(u))) ,$$

so f is determined by its value on the class u. This shows uniqueness.

Conversely, if $x \in M(G)$ is given, we define $f(K) : \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M(K)$

by the formula

$$f(K)(\alpha^*([m](u))) \;=\; \alpha^*([m](x))\;.$$

Then $f(G)(u) = x$. It remains to show that f is indeed a morphism of global power monoids. This is a routine – but somewhat lengthy – calculation as follows. Given $\alpha : K \longrightarrow \Sigma_m \wr G$ and $\bar{\alpha} : K \longrightarrow \Sigma_n \wr G$, we have

$$\begin{aligned}\alpha^*([m](u)) \;+\; \bar{\alpha}^*([n](u)) \;&=\; (\alpha,\bar{\alpha})^*(p_1^*([m](u)) + p_2^*([n](u)))\\ &=\; (\alpha,\bar{\alpha})^*(\Phi^*_{m,n}([m+n](u)))\\ &=\; (\Phi_{m,n}\circ(\alpha,\bar{\alpha}))^*([m+n](u))\;.\end{aligned}$$

So $f(K)$ is additive because

$$\begin{aligned}f(K)(\alpha^*([m](u)) + \bar{\alpha}^*([n](u))) \;&=\; f(K)((\Phi_{m,n}\circ(\alpha,\bar{\alpha}))^*([m+n](u)))\\ &=\; (\Phi_{m,n}\circ(\alpha,\bar{\alpha}))^*([m+n](x))\\ &=\; (\alpha,\bar{\alpha})^*(\Phi^*_{m,n}([m+n](x)))\\ &=\; (\alpha,\bar{\alpha})^*(p_1^*([m](x)) + p_2^*([n](x)))\\ &=\; \alpha^*([m](x)) \;+\; \bar{\alpha}^*([n](x))\\ &=\; f(K)(\alpha^*([m](u))) \;+\; f(K)(\bar{\alpha}^*([n](u)))\;.\end{aligned}$$

For a continuous homomorphism $\beta : L \longrightarrow K$ we have

$$\begin{aligned}(\beta^*\circ f(K))(\alpha^*([m](u))) \;&=\; \beta^*(\alpha^*([m](x))) \;=\; (\alpha\circ\beta)^*([m](x))\\ &=\; f(L)((\alpha\circ\beta)^*([m](u))) \;=\; (f(L)\circ\beta^*)(\alpha^*([m](u)))\;;\end{aligned}$$

so the homomorphisms $f(K)$ form a natural transformation of Rep-functors. For $k \geq 1$ we have

$$\begin{aligned}[k](\alpha^*([m](u))) \;&=\; (\Sigma_k\wr\alpha^*)([k]([m](u)))\\ &=\; (\Sigma_k\wr\alpha^*)(\Psi^*_{k,m}([km](u))) \;=\; (\Psi_{k,m}\circ(\Sigma_k\wr\alpha))^*([km](u))\;;\end{aligned}$$

hence

$$\begin{aligned}f(\Sigma_k\wr K)([k](\alpha^*([m](u)))) \;&=\; (\Psi_{k,m}\circ(\Sigma_k\wr\alpha))^*([km](x))\\ &=\; (\Sigma_k\wr\alpha)^*(\Psi^*_{k,m}([km](x))) \;=\; (\Sigma_k\wr\alpha)^*([k]([m](x)))\\ &=\; [k](\alpha^*([m](x))) \;=\; [k](f(K)(\alpha^*([m](u))))\;.\end{aligned}$$

So the homomorphisms $f(K)$ are compatible with power operations.

(iii) We apply the representability result of Proposition 1.5.13 to the category of ultra-commutative monoids and the free and forgetful adjoint functor pair:

$$\mathbb{P} \;:\; spc \rightleftarrows umon \;:\; U$$

If G is a compact Lie group, V a G-representation and W a non-zero faithful G-representation, then the restriction morphism $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,W}$

is a global equivalence between positive flat orthogonal spaces. We showed in the proof of Theorem 2.1.13 (ii) that the induced morphism of free ultra-commutative monoids $\mathbb{P}(\rho_{G,V,W}) : \mathbb{P}(\mathbf{L}_{G,V\oplus W}) \longrightarrow \mathbb{P}(\mathbf{L}_{G,W})$ is a global equivalence; in particular, the morphism of Rep-functors $\underline{\pi}_0(\mathbb{P}(\rho_{G,V,W}))$ is an isomorphism. So Proposition 1.5.13 applies and shows that evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{umon}(\pi_0^G, \pi_0^K) \longrightarrow \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G)) , \quad \tau \longmapsto \tau(u) .$$

Part (i) then completes the argument. □

Remark 2.2.25 (Natural n-ary operations)**.** By similar arguments as in the previous theorem we can also identify the natural n-ary operations on equivariant homotopy sets of ultra-commutative monoids. For every n-tuple $G_1, \dots, G_n$ of compact Lie groups the functor

$$\mathrm{Ho}(umon) \longrightarrow (\mathrm{sets}) , \quad X \longmapsto \pi_0^{G_1}(X) \times \cdots \times \pi_0^{G_n}(X)$$

is represented by the free ultra-commutative monoid $\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \ldots \amalg B_{\mathrm{gl}}G_n)$. So the set of natural transformations from the functor $\pi_0^{G_1} \times \cdots \times \pi_0^{G_n}$ to the functor π_0^K, for another compact Lie group K, bijects with the K-equivariant homotopy set of this representing object. Because

$$\begin{aligned}
\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \ldots \amalg B_{\mathrm{gl}}G_n) &\cong \mathbb{P}(B_{\mathrm{gl}}G_1) \boxtimes \cdots \boxtimes \mathbb{P}(B_{\mathrm{gl}}G_n) \\
&\cong \coprod\nolimits_{j_1,\dots,j_n \geq 0} B_{\mathrm{gl}}(\Sigma_{j_1} \wr G_1) \boxtimes \cdots \boxtimes B_{\mathrm{gl}}(\Sigma_{j_n} \wr G_n) \\
&\cong \coprod\nolimits_{j_1,\dots,j_n \geq 0} B_{\mathrm{gl}}((\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n)) ,
\end{aligned}$$

the group $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \ldots \amalg B_{\mathrm{gl}}G_n))$ bijects with the disjoint union of the sets

$$\pi_0^K(B_{\mathrm{gl}}((\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n))) \cong \mathrm{Rep}(K, (\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n)) .$$

So every natural operation from $\pi_0^{G_1} \times \cdots \times \pi_0^{G_n}$ to π_0^K is of the form

$$(x_1, \dots, x_n) \longmapsto \alpha^*([j_1](x_1) \oplus \cdots \oplus [j_n](x_n))$$

for a unique tuple $(j_1, \dots, j_n)$ of non-negative integers and a unique conjugacy class of continuous homomorphisms $\alpha : K \longrightarrow (\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n)$. In particular, the n-ary operations are generated by unary operations and external sum.

Construction 2.2.26. We describe an alternative (but isomorphic) way of organizing the book-keeping of the natural operations between the 0th equivariant homotopy sets of ultra-commutative monoids. We denote by Nat^{umon} the category whose objects are all compact Lie groups and where the morphism set $\mathrm{Nat}^{umon}(G, K)$ is the set of all natural transformations, of functors from the

ultra-commutative monoids to sets, from π_0^G to π_0^K. We define an isomorphic algebraic category $\mathbb{A}^+$, the *effective Burnside category*. Both Nat^{umon} and $\mathbb{A}^+$ are 'pre-preadditive' in the sense that all morphism sets are abelian monoids and composition is biadditive. In Nat^{umon}, the monoid structure is objectwise addition of natural transformations.

The category $\mathbb{A}^+$ has the same objects as Nat^{umon}, namely all compact Lie groups. In the effective Burnside category, the morphism set $\mathbb{A}^+(G,K)$ is the set of isomorphism classes of those K-G-spaces that are disjoint unions of finitely many free right G-orbits. This set is an abelian monoid via disjoint union of K-G-spaces. Composition is induced by the balanced product over K:

$$\circ \ : \ \mathbb{A}^+(K,L)\times\mathbb{A}^+(G,K) \longrightarrow \mathbb{A}^+(G,L)\ , \quad [T]\circ[S] \ = \ [T\times_K S]\ .$$

Here T has a left L-action and a commuting free right K-action, whereas S has a left K-action and a commuting free right G-action. The balanced product $T\times_K S$ then inherits a left L-action from T and a free right G-action from S. We define a functor

$$B \ : \ \mathrm{Nat}^{umon} \longrightarrow \mathbb{A}^+$$

as the identity on objects; on morphisms, the functor is given by

$$B \ : \ \mathrm{Nat}^{umon}(G,K) \longrightarrow \mathbb{A}^+(G,K)\ , \quad B(\alpha^*\circ[m]) \ = \ [\alpha^*(\{1,\dots,m\}\times G)_G]\ .$$

In the definition we use the characterization of the natural operations given by Theorem 2.2.24 (iii). Also, we consider $\{1,\dots,m\}\times G$ as a right G-space by translation; the wreath product $\Sigma_m\wr G$ acts from the left on $\{1,\dots,m\}\times G$ by

$$(\sigma;\, g_1,\dots,g_m)\cdot(i,\gamma) \ = \ (\sigma(i),\, g_i\cdot\gamma)\ . \tag{2.2.27}$$

Then we let K act by restriction of the $(\Sigma_m\wr G)$-action along α.

Proposition 2.2.28. *The functor $B:\mathrm{Nat}^{umon}\longrightarrow\mathbb{A}^+$ is additive and an isomorphisms of categories.*

Proof We start by showing that the map $B:\mathrm{Nat}^{umon}(G,K)\longrightarrow\mathbb{A}^+(G,K)$ is additive. The map

$$(\{1,\dots,k\}\times G) \ \amalg \ (\{1,\dots,m\}\times G) \ \longrightarrow \ \Phi^*_{k,m}(\{1,\dots,k+m\}\times G)$$

that is the inclusion on the first summand and given by $(j,g)\mapsto(k+j,g)$ on the second summand is an isomorphism of $((\Sigma_k\wr G)\times(\Sigma_m\wr G))$-$G$-bispaces. Restriction along the homomorphism $(\beta,\alpha):K\longrightarrow(\Sigma_k\wr G)\times(\Sigma_m\wr G)$ provides an isomorphism of K-G-spaces between

$$\beta^*(\{1,\dots,k\}\times G)_G \amalg \alpha^*(\{1,\dots,m\}\times G)_G \text{ and } (\Phi_{k,m}\circ(\beta,\alpha))^*(\{1,\dots,k+m\}\times G)_G\ .$$

This shows that

$$B((\beta^*\circ[k])+(\alpha^*\circ[m])) \ = \ B((\Phi_{k,m}\circ(\beta,\alpha))^*\circ[k+m]) \ = \ B(\beta^*\circ[k]) + B(\alpha^*\circ[m])\ .$$

The identity operation in $\mathrm{Nat}^{umon}(G,G)$ can be written as $\mathrm{Id}_G^*\circ[1]$; on the other hand, the G-bispace $\mathrm{Id}_G^*(\{1\}\times G)_G$ is isomorphic to G under left and right translation. Since the isomorphism class of ${}_GG_G$ is the identity of G in $\mathbb{A}^+$, the construction B preserves identities.

For the compatibility of B with composition we consider another operation $\beta^*\circ[k]\in\mathrm{Nat}^{umon}(K,M)$ and observe that

$$(\beta^*\circ[k])\circ(\alpha^*\circ[m]) \ = \ (\Psi_{k,m}\circ(\Sigma_k\wr\alpha)\circ\beta)^*\circ[km]\ .$$

Moreover, an isomorphism of M-G-spaces

$$\beta^*(\{1,\dots,k\}\times K)\times_K\alpha^*(\{1,\dots,m\}\times G) \ \cong \ (\Psi_{k,m}\circ(\Sigma_k\wr\alpha)\circ\beta)^*(\{1,\dots,km\}\times G)$$

is given by

$$[(i,\kappa),(j,\gamma)] \ \longmapsto \ ((i-1)m+\sigma(j)-1,\ g_j\cdot\gamma)\ ,$$

where

$$\alpha(\kappa) \ = \ (\sigma;\, g_1,\dots,g_m) \ \in \ \Sigma_m\wr G\ .$$

So B is a functor, which is bijective on objects by definition.

To see that B is full we let S be any K-G-space that is a disjoint union of m free right G-orbits. We choose a G-equivariant homeomorphism

$$\psi \ : \ S \ \longrightarrow \ \{1,\dots,m\}\times G\ .$$

We transport the left K-action from S to $\{1,\dots,m\}\times G$ along ψ, so that ψ becomes an isomorphism of K-G-spaces. The $(\Sigma_m\wr G)$-action on $\{1,\dots,m\}\times G$ defined in (2.2.27) identifies the wreath product with the group of G-equivariant automorphisms of $\{1,\dots,m\}\times G$; so the K-action on $\{1,\dots,m\}\times G$ corresponds to a continuous homomorphism $\alpha: K\longrightarrow\Sigma_m\wr G$. Altogether, S is isomorphic to $\alpha^*(\{1,\dots,m\}\times G)_G$.

If the K-G-spaces constructed from $\alpha^*\circ[m]$ and $\beta^*\circ[n]$ are isomorphic, then we must have $m=n$. Moreover, a K-G-isomorphism

$$\alpha^*(\{1,\dots,m\}\times G)_G \ \cong \ \beta^*(\{1,\dots,m\}\times G)_G$$

is given by the action of a unique element $\omega\in\Sigma_m\wr G$, and then the homomorphisms $\alpha,\beta: K\longrightarrow\Sigma_m\wr G$ are conjugate by ω. So the functor B is faithful. □

Now we define *transfer maps* $\mathrm{tr}_H^G : M(H)\longrightarrow M(G)$ in global power monoids, for every subgroup H of finite index in a compact Lie group G. As we will see in Proposition 2.2.30 below, the set of operations from π_0^G to π_0^K is a free abelian monoid with an explicit basis involving transfers.

Construction 2.2.29 (Transfer maps)**.** In the following we let M be a global power monoid, G a compact Lie group and H a closed subgroup of G of finite index m. We choose an 'H-basis' of G, i.e., an ordered m-tuple $\bar{g} = (g_1, \dots, g_m)$ of elements in disjoint H-orbits such that

$$G \;=\; \bigcup\nolimits_{i=1}^{m} g_i H \;.$$

The wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of G, by the formula

$$(g_1, \dots, g_m) \cdot (\sigma;\, h_1, \dots, h_m) \;=\; (g_{\sigma(1)} h_1, \dots, g_{\sigma(m)} h_m) \;.$$

We obtain a continuous homomorphism $\Psi_{\bar{g}} : G \longrightarrow \Sigma_m \wr H$ by requiring that

$$\gamma \cdot \bar{g} \;=\; \bar{g} \cdot \Psi_{\bar{g}}(\gamma) \;.$$

We define the transfer $\operatorname{tr}_H^G : M(H) \longrightarrow M(G)$ as the composite

$$M(H) \xrightarrow{[m]} M(\Sigma_m \wr H) \xrightarrow{\Psi_{\bar{g}}^*} M(G) \;.$$

Any other H-basis is of the form $\bar{g}\omega$ for a unique $\omega \in \Sigma_m \wr H$. We have $\Psi_{\bar{g}\omega} = c_\omega \circ \Psi_{\bar{g}}$ as maps $G \longrightarrow \Sigma_m \wr H$, where $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. Since inner automorphisms induce the identity in any Rep-functor, we conclude that

$$\Psi_{\bar{g}}^* = \Psi_{\bar{g}\omega}^* \;:\; M(\Sigma_m \wr H) \;\longrightarrow\; M(G) \;.$$

So the transfer tr_H^G does not depend on the choice of basis $\bar{g}$.

The various properties of the power operations imply certain properties of the transfer maps. Moreover, the last item of the following proposition shows that power operations in a global power monoid are determined by the transfer and restriction maps.

Proposition 2.2.30. *The transfer homomorphisms of a global power monoid M satisfy the following relations, where H is any subgroup of finite index in a compact Lie group G.*

(i) (Transitivity) *We have* $\operatorname{tr}_G^G = \mathrm{Id}_{M(G)}$ *and for nested subgroups $H \subseteq G \subseteq F$ of finite index the relation*

$$\operatorname{tr}_G^F \circ \operatorname{tr}_H^G \;=\; \operatorname{tr}_H^F$$

holds as maps $M(H) \longrightarrow M(F)$.

(ii) (Double coset formula) *For every subgroup K of G (not necessarily of finite index) the relation*

$$\operatorname{res}_K^G \circ \operatorname{tr}_H^G \;=\; \sum_{[g] \in K\backslash G/H} \operatorname{tr}_{K \cap {}^g H}^K \circ (c_g)^* \circ \operatorname{res}_{K^g \cap H}^H$$

holds as homomorphisms $M(H) \longrightarrow M(K)$. Here $[g]$ runs over a set of representatives of the finite set of K-H-double cosets.

(iii) (Inflation) *For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^* \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_L^K \circ (\alpha|_L)^*$$

holds as maps from $M(H)$ to $M(K)$, where $L = \alpha^{-1}(H)$.

(iv) *For every $m \geq 1$ the mth power operation can be recovered as*

$$[m] \;=\; \mathrm{tr}_K^{\Sigma_m \wr G} \circ q^* ,$$

where K is the subgroup of $\Sigma_m \wr G$ consisting of all $(\sigma;\, g_1, \dots, g_m)$ such that $\sigma(m) = m$, and $q : K \longrightarrow G$ is defined by $q(\sigma;\, g_1, \dots, g_m) = g_m$.

Proof (i) For $G = H$ we can choose the unit 1 as the G-basis, and with this choice $\Psi_1 : G \longrightarrow \Sigma_1 \wr G$ is the preferred isomorphism that sends g to $(1; g)$. The restriction of $[1](x)$ along this isomorphism is x, so we get $\mathrm{tr}_G^G(x) = x$.

For the second claim we choose a G-basis $\bar{f} = (f_1, \dots, f_k)$ of F and an H-basis $\bar{g} = (g_1, \dots, g_m)$ of G. Then

$$\bar{f}\bar{g} \;=\; (f_1 g_1, \dots, f_1 g_m,\; f_2 g_1, \dots, f_2 g_m,\; \dots,\; f_k g_1, \dots, f_k g_m)$$

is an H-basis of F. With respect to this basis, the homomorphism $\Psi_{\bar{f}\bar{g}} : F \longrightarrow \Sigma_{km} \wr H$ equals the composite

$$F \xrightarrow{\Psi_{\bar{f}}} \Sigma_k \wr G \xrightarrow{\Sigma_k \wr \Psi_{\bar{g}}} \Sigma_k \wr (\Sigma_m \wr H) \xrightarrow{\Psi_{k,m}} \Sigma_{km} \wr H$$

where the monomorphism $\Psi_{k,m}$ was defined in (2.2.6). So

$$\begin{aligned} \mathrm{tr}_H^F \;&=\; \Psi_{\bar{f}\bar{g}}^* \circ [km] \;=\; \Psi_{\bar{f}}^* \circ (\Sigma_k \wr \Psi_{\bar{g}})^* \circ \Psi_{k,m}^* \circ [km] \\ &=\; \Psi_{\bar{f}}^* \circ (\Sigma_k \wr \Psi_{\bar{g}})^* \circ [k] \circ [m] \;=\; \Psi_{\bar{f}}^* \circ [k] \circ \Psi_{\bar{g}}^* \circ [m] \;=\; \mathrm{tr}_G^F \circ \mathrm{tr}_H^G \; . \end{aligned}$$

(ii) We choose representatives $g_1, \dots, g_r$ for the K-H-double cosets in G. Then we choose, for each $1 \leq i \leq r$, a $(K \cap {}^{g_i}H)$-basis

$$\bar{k}^i \;=\; (k_1^i, \dots, k_{s_i}^i)$$

of K, where $s_i = [K : K \cap {}^{g_i}H]$. Then $s_1 + \cdots + s_r = m = [G : H]$ is the index of H in G, and this data provides an H-basis of G, namely

$$\bar{g} \;=\; (k_1^1 g_1,\, \dots, k_{s_1}^1 g_1,\, k_1^2 g_2,\, \dots, k_{s_2}^2 g_2,\, \dots, k_1^r g_r,\, \dots, k_{s_r}^r g_r) \, .$$

The following diagram of group homomorphisms then commutes:

$$\begin{tikzcd} K \arrow[r, "\mathrm{incl}"] \arrow[d, "(\Psi_{\bar{k}^1},\dots,\Psi_{\bar{k}^r})"'] & G \arrow[r, "\Psi_{\bar{g}}"] & \Sigma_m \wr H \\ \prod_{i=1}^r \Sigma_{s_i} \wr (K \cap {}^{g_i}H) \arrow[rr, "\prod \Sigma_{s_i} \wr c_{g_i}"'] & & \prod_{i=1}^r \Sigma_{s_1} \wr H \arrow[u, "\Phi_{s_1,\dots,s_r}"'] \end{tikzcd}$$

The right vertical morphism is the generalization of the embedding (2.2.5) to multiple factors. From here the double coset formula is straightforward:

$$\begin{aligned} \mathrm{res}^G_K \circ \mathrm{tr}^G_H &= \mathrm{res}^G_K \circ \Psi^*_{\bar{g}} \circ [m] \\ &= ((\Sigma_{s_1} \wr c_{g_1}) \circ \Psi_{\bar{k}^1}, \dots, (\Sigma_{s_r} \wr c_{g_r}) \circ \Psi_{\bar{k}^r})^* \circ \Phi^*_{s_1,\dots,s_r} \circ [m] \\ \overset{(2.2.10)}{=}& ((\Sigma_{s_1} \wr c_{g_1}) \circ \Psi_{\bar{k}^1}, \dots, (\Sigma_{s_r} \wr c_{g_r}) \circ \Psi_{\bar{k}^r})^* \circ ([s_1] \oplus \dots \oplus [s_r]) \\ &= \sum_{i=1}^r \Psi^*_{\bar{k}^i} \circ (\Sigma_{s_i} \wr c_{g_i})^* \circ [s_i] \\ &= \sum_{i=1}^r \Psi^*_{\bar{k}^i} \circ [s_i] \circ (c_{g_i})^* = \sum_{i=1}^r \mathrm{tr}^K_{K \cap {}^{g_i}H} \circ (c_{g_i})^* \circ \mathrm{res}^H_{K^{g_i} \cap H} . \end{aligned}$$

(iii) If $\bar{k} = (k_1, \dots, k_m)$ is an L-basis of K, then $\alpha(\bar{k}) = (\alpha(k_1), \dots, \alpha(k_m))$ is an H-basis of G. With respect to these bases we have

$$\Psi_{\alpha(\bar{k})} \circ \alpha = (\Sigma_m \wr (\alpha|_L)) \circ \Psi_{\bar{k}} : K \longrightarrow \Sigma_m \wr H .$$

So

$$\begin{aligned} \alpha^* \circ \mathrm{tr}^G_H = \alpha^* \circ \Psi^*_{\alpha(\bar{k})} \circ [m] &= \Psi^*_{\bar{k}} \circ (\Sigma_m \wr (\alpha|_L))^* \circ [m] \\ &= \Psi^*_{\bar{k}} \circ [m] \circ (\alpha|_L)^* = \mathrm{tr}^K_L \circ (\alpha|_L)^* . \end{aligned}$$

(iv) The subgroup K has index m in $\Sigma_m \wr G$ and a K-basis of $\Sigma_m \wr G$ is given by the elements $\tau_j = ((j, m);\ 1, \dots, 1)$ for $j = 1, \dots, m$. In order to determine the monomorphism $\Psi_{\bar{\tau}} : \Sigma_m \wr G \longrightarrow \Sigma_m \wr K$ associated with this K-basis, we consider any element $(\sigma;\ g_1, \dots, g_m)$ of $\Sigma_m \wr G$. The permutation $(\sigma(j), m) \cdot \sigma \cdot (j, m) \in \Sigma_m$ fixes m, so the element

$$l_j = \tau_{\sigma(j)} \cdot (\sigma;\ g_1, \dots, g_m) \cdot \tau_j \in \Sigma_m \wr G$$

in fact belongs to the subgroup K. Then

$$(\sigma;\ g_1, \dots, g_m) \cdot \tau_j = \tau_{\sigma(j)} \cdot l_j$$

in the group $\Sigma_m \wr G$, by definition. This means that

$$(\sigma;\ g_1, \dots, g_m) \cdot (\tau_1, \dots, \tau_m) = (\tau_1, \dots, \tau_m) \cdot (\sigma;\ l_1, \dots, l_m) ,$$

and hence

$$\Psi_{\bar{\tau}}(\sigma;\ g_1,\ldots,g_m) \ = \ (\sigma;\ l_1,\ldots,l_m)\ .$$

Because

$$(\Sigma_m \wr q)(\Psi_{\bar{\tau}}(\sigma;\ g_1,\ldots,g_m)) \ = \ (\Sigma_m \wr q)(\sigma;\ l_1,\ldots,l_m) \ = \ (\sigma;\ g_1,\ldots,g_m)\ ,$$

we conclude that the composite $(\Sigma_m \wr q) \circ \Psi_{\bar{\tau}}$ is the identity of $\Sigma_m \wr G$. So

$$\begin{aligned} \mathrm{tr}_K^{\Sigma_m \wr G} \circ q^* \ &= \ \Psi_{\bar{\tau}}^* \circ [m] \circ q^* \ = \ \Psi_{\bar{\tau}}^* \circ (\Sigma_m \wr q)^* \circ [m] \\ &= \ ((\Sigma_m \wr q) \circ \Psi_{\bar{\tau}})^* \circ [m] \ = \ [m]\ . \end{aligned}$$ □

Theorem 2.2.24 (iii) gives a description of the set of natural operations, on the category of ultra-commutative monoids, from π_0^G to π_0^K. The next proposition gives an alternative description that also captures the monoid structure given by pointwise addition of operations.

Proposition 2.2.31. *Let G and K be compact Lie groups. The monoid* $\mathrm{Nat}^{umon}(\pi_0^G, \pi_0^K)$ *is a free abelian monoid generated by the operations* $\mathrm{tr}_L^K \circ \alpha^*$ *where (L, α) runs over all $(K \times G)$-conjugacy classes of pairs consisting of*

- *a subgroup $L \leq K$ of finite index, and*
- *a continuous group homomorphism $\alpha : L \longrightarrow G$.*

Proof This a straightforward algebraic consequence of the calculation of the category Nat^{umon} given in Proposition 2.2.28. Every K-G-space with finitely many free G-orbits is the disjoint union of transitive K-G-spaces with the same property. So $\mathbb{A}^+(G, K)$ is a free abelian monoid with basis the isomorphism classes of the transitive K-G-spaces. A transitive K-G-space with finitely many free G-orbits is isomorphic to one of the form

$$K \times_\alpha G \ = \ (K \times G)/(kl, g) \sim (k, \alpha(l)g)$$

for a pair $(L, \alpha : L \longrightarrow G)$, with L of finite index in K. Moreover, $K \times_\alpha G$ is isomorphic to $K \times_{\alpha'} G$ if and only if (L, α) is conjugate to (L', α') by an element of $K \times G$. So $\mathbb{A}^+(G, K)$ is freely generated by the classes of the K-G-spaces $K \times_\alpha G$, where (L, α) runs through the $(K \times G)$-conjugacy classes of the relevant pairs.

The claim then follows from the verification that the isomorphism of categories $B : \mathrm{Nat}^{umon} \longrightarrow \mathbb{A}^+$ established in Proposition 2.2.28 takes the operation $\mathrm{tr}_L^K \circ \alpha^*$ to the class of $K \times_\alpha G$. Indeed, if $\bar{k} = (k_1, \ldots, k_m)$ is an L-basis of K and $\Psi_{\bar{k}} : K \longrightarrow \Sigma_m \wr L$ the classifying homomorphism, then

$$B(\mathrm{tr}_L^K \circ \alpha^*) \ = \ B(\Psi_{\bar{k}}^* \circ [m] \circ \alpha^*) \ = \ B(\Psi_{\bar{k}}^* \circ (\Sigma_m \wr \alpha)^* \circ [m])\ .$$

On the other hand, the map

$$((\Sigma_m \wr \alpha) \circ \Psi_{\bar{k}})^*(\{1,\dots,m\} \times G)_G \ \longrightarrow\ K \times_\alpha G\ , \quad (i,\gamma) \ \longmapsto\ [k_i,\gamma]$$

is an isomorphism of K-G-bispaces, so $B(\mathrm{tr}_K^K \circ \alpha^*) = [K \times_\alpha G]$ in $\mathbb{A}^+(G,K)$. $\square$

2.3 Examples

In this section we discuss various examples, mostly of a geometric nature, of ultra-commutative monoids, and several geometrically defined morphisms between them. We start with the ultra-commutative monoids **O** and **SO** (Example 2.3.6), **U** and **SU** (Example 2.3.7), **Sp** (Example 2.3.9) and **Spin** and **Spin**^{c} (Example 2.3.10), all made from the corresponding families of classical Lie groups. All of these are examples of 'symmetric monoid-valued orthogonal spaces' in the sense of Definition 2.3.4, a more general source of examples of ultra-commutative monoids. The additive Grassmannian **Gr** (Example 2.3.12), the oriented variant $\mathbf{Gr}^{\mathrm{or}}$ (Example 2.3.15) and the complex and quaternionic analogs $\mathbf{Gr}^{\mathbb{C}}$ and $\mathbf{Gr}^{\mathbb{H}}$ (Example 2.3.16) consist – as the names suggest – of Grassmannians with monoid structure arising from direct sum of subspaces. The multiplicative Grassmannian $\mathbf{Gr}_\otimes$ (Example 2.3.18) is globally equivalent as an orthogonal space to **Gr**, but the monoid structure arises from the tensor product of subspaces; the global projective space **P** is the ultra-commutative submonoid of $\mathbf{Gr}_\otimes$ consisting of lines (1-dimensional subspaces). The global projective space **P** is a multiplicative model of a global classifying space for the cyclic group of order 2, and Example 2.3.23 describes multiplicative models of global classifying spaces for all *abelian* compact Lie groups. Example 2.3.24 introduces the ultra-commutative monoid **F** of unordered frames, with monoid structure arising from disjoint union. The ultra-commutative 'multiplicative monoid of the sphere spectrum' $\Omega^\bullet\mathbb{S}$ is introduced in Example 2.3.26; this a special case of the multiplicative monoid of an ultra-commutative ring spectrum, and we return to the more general construction in Example 4.1.16.

Construction 2.3.1 (Orthogonal monoid spaces from monoid-valued orthogonal spaces)**.** Our first series of examples involves orthogonal spaces made from the infinite families of classical Lie groups, namely the orthogonal, unitary and symplectic groups, the special orthogonal and unitary groups, and the pin, pin^c, spin and spin^c groups. These orthogonal spaces have the special feature that they are *group-valued*; we will now explain that a group-valued (or even just a monoid-valued) orthogonal space automatically leads to an orthogonal monoid space. In the cases of the orthogonal, special orthogonal, unitary, special unitary, spin, spin^c and symplectic groups, these multiplications are symmetric, so those examples yield ultra-commutative monoids.

Definition 2.3.2. A *monoid-valued orthogonal space* is a monoid object in the category of orthogonal spaces.

Since orthogonal spaces are an enriched functor category, monoid-valued orthogonal spaces are the same thing as continuous functors from the category $\mathbf{L}$ to the category of topological monoids and continuous monoid homomorphisms (i.e., monoid objects in the category $\mathbf{T}$ of compactly generated spaces).

Now we let M be a monoid-valued orthogonal space. In (1.3.1) we introduced a lax symmetric monoidal natural transformation

$$\rho_{X,Y} \ : \ X \boxtimes Y \longrightarrow X \times Y$$

from the box product to the cartesian product of orthogonal spaces. For $X = Y = M$ we can form the composite

$$M \boxtimes M \xrightarrow{\rho_{M,M}} M \times M \xrightarrow{\text{multiplication}} M \tag{2.3.3}$$

with the objectwise multiplication of M. Since $\rho_{X,Y}$ is lax monoidal, this composite makes M into an orthogonal monoid space with unit the multiplicative unit $1 \in M(0)$. The morphism (2.3.3) corresponds, via the universal property of $\boxtimes$, to the bimorphism whose (V, W)-component is the composite

$$M(V) \times M(W) \xrightarrow{M(i_V)\times M(i_W)} M(V \oplus W) \times M(V \oplus W) \xrightarrow{\text{multiply}} M(V \oplus W) \ ,$$

where $i_V : V \longrightarrow V \oplus W$ and $i_W : W \longrightarrow V \oplus W$ are the direct summand embeddings. If $f : M \longrightarrow M'$ is a morphism of monoid-valued orthogonal spaces (i.e., a morphism of orthogonal spaces that is objectwise a monoid homomorphism), then f is also a homomorphism of orthogonal monoid spaces with respect to the multiplications (2.3.3).

If M is a commutative monoid-valued orthogonal space, then the associated $\boxtimes$-multiplication is also commutative, simply because the transformation $\rho_{X,Y}$ is *symmetric* monoidal. However, there is a more general condition that provides M with the structure of an ultra-commutative monoid.

Definition 2.3.4. A monoid-valued orthogonal space M is *symmetric* if for all inner product spaces V and W the images of the two homomorphisms

$$M(i_V) \ : \ M(V) \longrightarrow M(V \oplus W) \quad \text{and} \quad M(i_W) \ : \ M(W) \longrightarrow M(V \oplus W)$$

commute.

We emphasize that the objectwise multiplications in a symmetric monoid-valued orthogonal space need not be commutative – we'll discuss many interesting examples of this kind below. The proof of the following proposition is straightforward from the definitions, and we omit it.

Proposition 2.3.5. *Let M be a symmetric monoid-valued orthogonal space. Then the multiplication* (2.3.3) *makes M an ultra-commutative monoid.*

Example 2.3.6 (Orthogonal group ultra-commutative monoid)**.** We denote by $\mathbf{O}$ the orthogonal space that sends an inner product space V to its orthogonal group $O(V)$. A linear isometric embedding $\varphi : V \longrightarrow W$ induces a continuous group homomorphism $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$ by conjugation and the identity on the orthogonal complement of the image of φ. Construction 2.3.1 then gives $\mathbf{O}$ the structure of an orthogonal monoid space. The (V, W)-component of the bimorphism

$$\mu_{V,W} \ : \ \mathbf{O}(V) \times \mathbf{O}(W) \longrightarrow \ \mathbf{O}(V \oplus W)$$

is direct sum of orthogonal transformations. The unit element is the identity of the trivial vector space, the only element of $\mathbf{O}(0)$. So $\mathbf{O}$ is a symmetric group-valued orthogonal space, and hence it becomes an ultra-commutative monoid.

If G is a compact Lie group and V a G-representation, then the G-action on the group $\mathbf{O}(V)$ is by conjugation, so the fixed-points $\mathbf{O}(V)^G$ are the group of G-equivariant orthogonal automorphisms of V. Moreover, $\mathbf{O}(\mathcal{U}_G)$ is the orthogonal group of $\mathcal{U}_G$, i.e., $\mathbb{R}$-linear isometries of $\mathcal{U}_G$ (not necessarily G-equivariant) that are the identity on the orthogonal complement of some finite-dimensional subspace; the G-action is again by conjugation. Any G-equivariant isometry preserves the decomposition of $\mathcal{U}_G$ into isotypical summands, and the restriction to almost all of these isotypical summands must be the identity. The G-fixed subgroup is thus given by

$$\mathbf{O}(\mathcal{U}_G)^G \ = \ O^G(\mathcal{U}_G) \ = \ {\prod}'_{[\lambda]} O^G(\mathcal{U}_\lambda) \ ,$$

where the weak product is indexed by the isomorphism classes of irreducible orthogonal G-representations λ, and $\mathcal{U}_\lambda$ is the λ-isotypical summand. If the compact Lie group G is finite, then there are only finitely many isomorphism classes of irreducible G-representations, so in that case the weak product coincides with the product.

Irreducible orthogonal representations come in three different flavors, and the group $O^G(\mathcal{U}_\lambda)$ has one of three different forms. If λ is an irreducible orthogonal G-representation, then the endomorphism ring $\mathrm{Hom}^G_{\mathbb{R}}(\lambda, \lambda)$ is a finite-dimensional skew field extension of $\mathbb{R}$, so it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; the representation λ is accordingly called 'real', 'complex' or 'quaternionic', respectively. We have

$$O^G(\mathcal{U}_\lambda) \ \cong \ O^G(\lambda^\infty) \ \cong \ \begin{cases} O & \text{if } \lambda \text{ is real,} \\ U & \text{if } \lambda \text{ is complex, and} \\ Sp & \text{if } \lambda \text{ is quaternionic.} \end{cases}$$

So we conclude that the G-fixed-point space $\mathbf{O}(\mathcal{U}_G)^G$ is a weak product of copies of the infinite orthogonal, unitary and symplectic groups, indexed by the different types of irreducible orthogonal representations of G. Since the infinite unitary and symplectic groups are connected, only the 'real' factors contribute to $\pi_0(\mathbf{O}(\mathcal{U}_G)^G) = \pi_0^G(\mathbf{O})$, which is a weak product of copies of $\pi_0(O) = \mathbb{Z}/2$ indexed by the irreducible G-representations of real type.

There is a straightforward 'special orthogonal' analog: the property of having determinant 1 is preserved under conjugation by linear isometric embeddings and under direct sum of linear isometries, so the spaces $SO(V)$ form an ultra-commutative submonoid $\mathbf{SO}$ of $\mathbf{O}$. Here, $\mathbf{SO}(\mathcal{U}_G)$ is the group of $\mathbb{R}$-linear isometries of $\mathcal{U}_G$ (not necessarily G-equivariant) that have determinant 1 on some finite-dimensional subspace V of $\mathcal{U}_G$ and are the identity on the orthogonal complement of V.

Example 2.3.7 (Unitary group ultra-commutative monoid)**.** There is a straightforward unitary analog $\mathbf{U}$ of $\mathbf{O}$, defined as follows. We recall that

$$V_{\mathbb{C}} \;=\; \mathbb{C} \otimes_{\mathbb{R}} V$$

denotes the complexification of an inner product space V. The euclidean inner product $\langle -, - \rangle$ on V induces a hermitian inner product $(-, -)$ on $V_{\mathbb{C}}$, defined as the unique sesquilinear form that satisfies

$$(1 \otimes v, 1 \otimes w) \;=\; \langle v, w \rangle$$

for all $v, w \in V$. We now define an orthogonal space $\mathbf{U}$ by

$$\mathbf{U}(V) \;=\; U(V_{\mathbb{C}}) ,$$

the unitary group of the complexification of V. The complexification of every $\mathbb{R}$-linear isometric embedding $\varphi : V \longrightarrow W$ preserves the hermitian inner products, so we can define a continuous group homomorphism

$$\mathbf{U}(\varphi) \;:\; \mathbf{U}(V) \;\longrightarrow\; \mathbf{U}(W)$$

by conjugation with $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{C}}$. The $\boxtimes$-multiplication on $\mathbf{U}$ produced by Construction 2.3.1 is by direct sum of unitary transformations; this multiplication is symmetric, and hence ultra-commutative. There is a straightforward 'special unitary' ultra-commutative submonoid $\mathbf{SU}$ of $\mathbf{U}$; the value $\mathbf{SU}(V)$ is the group of unitary automorphisms of $V_{\mathbb{C}}$ of determinant 1.

If G is a compact Lie group, then the identification of the G-fixed-points of $\mathbf{U}$ also works much like the orthogonal analog. The outcome is an isomorphism between $\mathbf{U}(\mathcal{U}_G)^G$ and $U^G(\mathcal{U}_G^{\mathbb{C}})$. Here $\mathcal{U}_G^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{U}_G$ is the complexified complete universe for G, which happens to be a 'complete complex G-universe'

in the sense that every finite-dimensional complex G-representation embeds into it. The complete complex G-universe breaks up into (unitary) isotypical summands $\mathcal{U}_\lambda^{\mathbb{C}}$, indexed by the isomorphism classes of irreducible unitary G-representations λ, and the group $U^G(\mathcal{U}_G^{\mathbb{C}})$ breaks up accordingly as a weak product. In contrast to the orthogonal situation above, there is only one 'type' of irreducible unitary representation, and the group $U^G(\mathcal{U}_\lambda^{\mathbb{C}})$ is always isomorphic to the infinite unitary group U, independent of λ. So in the unitary context, we get a decomposition

$$\mathbf{U}(\mathcal{U}_G)^G \;=\; U^G(\mathcal{U}_G^{\mathbb{C}}) \;=\; {\prod}'_{[\lambda]} U^G(\mathcal{U}_\lambda^{\mathbb{C}}) \;\cong\; {\prod}'_{[\lambda]} U \,.$$

This weak product is indexed by the isomorphism classes of irreducible unitary G-representations. Since the unitary group U is connected, the set $\pi_0^G(\mathbf{U}) = \pi_0(\mathbf{U}(\mathcal{U}_G)^G)$ has only one element, and so $\mathbf{U}$ is globally connected.

The orthogonal monoid space $\mathbf{U}$ comes with an involution

$$\psi \;:\; \mathbf{U} \;\longrightarrow\; \mathbf{U}$$

by *complex conjugation* that is an automorphism of ultra-commutative monoids. The value of ψ at V is the map

$$\psi(V) \;:\; U(V_{\mathbb{C}}) \;\longrightarrow\; U(V_{\mathbb{C}}) \,, \quad A \;\longmapsto\; \psi_V \circ A \circ \psi_V \,,$$

where

$$\psi_V \;:\; V_{\mathbb{C}} \;\longrightarrow\; V_{\mathbb{C}} \,, \quad \lambda \otimes v \;\longmapsto\; \bar{\lambda} \otimes v$$

is the canonical $\mathbb{C}$-semilinear conjugation map on $V_{\mathbb{C}}$.

The *complexification morphism* is the homomorphism of ultra-commutative monoids given by

$$c \;:\; \mathbf{O} \;\longrightarrow\; \mathbf{U} \,, \quad c(V) \;:\; O(V) \;\longrightarrow\; U(V_{\mathbb{C}}) \,, \quad \varphi \;\longmapsto\; \varphi_{\mathbb{C}} \,.$$

Complexification is an isomorphism onto the ψ-invariant ultra-commutative submonoid of $\mathbf{U}$, and it takes $\mathbf{SO}$ to $\mathbf{SU}$.

Every hermitian inner product space W has an underlying $\mathbb{R}$-vector space equipped with the euclidean inner product defined by

$$\langle v, w \rangle \;=\; \mathrm{Re}(v, w) \,,$$

the real part of the given hermitian inner product. Every $\mathbb{C}$-linear isometric embedding is in particular an $\mathbb{R}$-linear isometric embedding of underlying euclidean vector spaces. So the unitary group $U(W)$ is a subgroup of the special orthogonal group of the underlying euclidean vector space of W. We can thus define the *realification morphism*

$$r \;:\; \mathbf{U} \;\longrightarrow\; \mathrm{sh}_{\mathbb{C}}^{\otimes}(\mathbf{SO}) \tag{2.3.8}$$

at V as the inclusion $r(V) : U(V_{\mathbb{C}}) \longrightarrow O(\mathbb{C} \otimes V)$. Here $\mathrm{sh}^{\otimes}_{\mathbb{C}}$ denotes the multiplicative shift by $\mathbb{C}$ as defined in Example 1.1.11.

Example 2.3.9 (Symplectic group ultra-commutative monoid)**.** There is also a quaternionic analog of **O** and **U**, the ultra-commutative monoid **Sp** made from symplectic groups. Given an $\mathbb{R}$-inner product space V, we denote by

$$V_{\mathbb{H}} \ = \ \mathbb{H} \otimes_{\mathbb{R}} V$$

the extension of scalars to the skew field $\mathbb{H}$ of quaternions. The extension comes with a $\mathbb{H}$-sesquilinear form

$$[-,-] \ : \ V_{\mathbb{H}} \times V_{\mathbb{H}} \ \longrightarrow \ \mathbb{H} \quad \text{characterized by} \quad [1 \otimes v, 1 \otimes w] \ = \ \langle v, w \rangle$$

for all $v, w \in V$. The symplectic group

$$\mathbf{Sp}(V) \ = \ Sp(V_{\mathbb{H}})$$

is the compact Lie group of $\mathbb{H}$-linear automorphisms $A : V_{\mathbb{H}} \longrightarrow V_{\mathbb{H}}$ that leave the form invariant, i.e., such that

$$[Ax, Ay] \ = \ [x, y]$$

for all $x, y \in V_{\mathbb{H}}$. The $\mathbb{H}$-linear extension $\varphi_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} \varphi : V_{\mathbb{H}} \longrightarrow W_{\mathbb{H}}$ of an $\mathbb{R}$-linear isometric embedding $\varphi : V \longrightarrow W$ preserves the new inner products, so we can define a continuous group homomorphism

$$\mathbf{Sp}(\varphi) \ : \ \mathbf{Sp}(V) \ \longrightarrow \ \mathbf{Sp}(W)$$

by conjugation with $\varphi_{\mathbb{H}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{H}}$. As for **O** and **U**, the $\boxtimes$-multiplication on **Sp** produced by Construction 2.3.1 is by direct sum of symplectic automorphisms; so **Sp** is symmetric, hence ultra-commutative.

If G is a compact Lie group, then the identification of the G-fixed-points of **Sp** also works much like the orthogonal case. Quaternionic representations decompose canonically into isotypical summands, and this results in a product decomposition for the G-fixed subgroup

$$\mathbf{Sp}(\mathcal{U}_G)^G \ = \ (Sp(\mathcal{U}_G^{\mathbb{H}}))^G \ = \ {\prod}'_{[\lambda]} (Sp(\mathcal{U}_\lambda^{\mathbb{H}}))^G \ ,$$

where the weak product is indexed by the isomorphism classes of irreducible quaternionic G-representations λ, and $\mathcal{U}_\lambda^{\mathbb{H}}$ is the λ-isotypical summand in $\mathcal{U}_G^{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} \mathcal{U}_G$. As in the real case, irreducible quaternionic representations λ come in three different flavors, depending on whether $\mathrm{Hom}^G_{\mathbb{H}}(\lambda, \lambda)$ – again a finite-dimensional skew field extension of $\mathbb{R}$ – is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. As for **O**, the G-fixed-point space $\mathbf{Sp}(\mathcal{U}_G)^G$ is a weak product of copies of the infinite orthogonal, unitary and symplectic groups, indexed by the different types of irreducible quaternionic representations of G.

Example 2.3.10 (Pin and Spin group orthogonal spaces)**.** Given a real inner product space V we denote by $\mathrm{Cl}(V)$ the Clifford algebra of the negative-definite quadratic form on V, i.e., the quotient of the $\mathbb{R}$-tensor algebra of V by the ideal generated by $v \otimes v + |v|^2 \cdot 1$ for all $v \in V$. The Clifford algebra is $\mathbb{Z}/2$-graded with the even (or odd) part generated by an even (or odd) number of vectors from V. The composite

$$V \xrightarrow{\text{linear summand}} TV \xrightarrow{\text{proj}} \mathrm{Cl}(V)$$

is $\mathbb{R}$-linear and injective, and we denote it by $v \mapsto [v]$.

We recall that orthogonal vectors of V anti-commute in the Clifford algebra: given $v, \bar{v} \in V$ with $\langle v, \bar{v}\rangle = 0$, then

$$\begin{aligned}[v][\bar{v}] + [\bar{v}][v] \;&=\; (|v|^2 \cdot 1 + [v]^2) + [v][\bar{v}] + [\bar{v}][v] + (|\bar{v}^2| \cdot 1 + [\bar{v}]^2) \\ &=\; |v + \bar{v}|^2 \cdot 1 + [v + \bar{v}]^2 \;=\; 0\ .\end{aligned}$$

In $\mathrm{Cl}(V)$ every unit vector $v \in S(V)$ satisfies $[v]^2 = -1$, so all unit vectors of V become units in $\mathrm{Cl}(V)$. The *pin group* of V is the subgroup

$$\mathrm{Pin}(V) \;\subset\; \mathrm{Cl}(V)^\times$$

generated inside the multiplicative group of $\mathrm{Cl}(V)$ by -1 and all unit vectors of V. A linear isometric embedding $\varphi : V \longrightarrow W$ induces a morphism of $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras $\mathrm{Cl}(\varphi) : \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(W)$ that restricts to φ on V. So $\mathrm{Cl}(\varphi)$ restricts to a continuous homomorphism

$$\mathrm{Pin}(\varphi) \;:\; \mathrm{Cl}(\varphi)|_{\mathrm{Pin}(V)} \;:\; \mathrm{Pin}(V) \;\longrightarrow\; \mathrm{Pin}(W)$$

between the pin groups. The map $\mathrm{Pin}(\varphi)$ depends continuously on φ and satisfies $\mathrm{Pin}(\psi) \circ \mathrm{Pin}(\varphi) = \mathrm{Pin}(\psi \circ \varphi)$, so we have defined a group-valued orthogonal space **Pin**. Construction 2.3.1 then gives **Pin** the structure of an orthogonal monoid space.

Since the group $\mathrm{Pin}(V)$ is generated by homogeneous elements of the Clifford algebra, all of its elements are homogeneous. The $\mathbb{Z}/2$-grading of $\mathrm{Cl}(V)$ provides a continuous homomorphism $\mathrm{Pin}(V) \longrightarrow \mathbb{Z}/2$ whose kernel

$$\mathrm{Spin}(V) \;=\; \mathrm{Cl}(V)_{\mathrm{ev}} \cap \mathrm{Pin}(V)$$

is the *spin group* of V. The map $\mathrm{Pin}(\varphi)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is homogeneous, so it restricts to a homomorphism

$$\mathrm{Spin}(\varphi) \;=\; \mathrm{Pin}(\varphi)|_{\mathrm{Spin}(V)} \;:\; \mathrm{Spin}(V) \;\longrightarrow\; \mathrm{Spin}(W)$$

between the spin groups. The spin groups thus form an orthogonal monoid subspace **Spin** of **Pin**. We claim that **Spin** is symmetric, i.e., for all inner product

spaces V and W the images of the two group homomorphisms

$$\mathrm{Spin}(V) \xrightarrow{\mathrm{Spin}(i_V)} \mathrm{Spin}(V \oplus W) \xleftarrow{\mathrm{Spin}(i_W)} \mathrm{Spin}(W)$$

commute. Indeed, $\mathrm{Spin}(V)$ is generated by -1 and the elements $[v][v']$ for $v, v' \in S(V)$, and similarly for W. The elements -1 map to the central element -1 in $\mathrm{Spin}(V \oplus W)$, and

$$[v,0] \cdot [v',0] \cdot [0,w] \cdot [0,w'] \;=\; [0,w] \cdot [0,w'] \cdot [v,0] \cdot [v',0]$$

because $[v,0]$ and $[v',0]$ anti-commute with $[0,w]$ and $[0,w']$. So **Spin** is an ultra-commutative monoid with respect to the $\boxtimes$-multiplication of Construction 2.3.1.

In contrast to **Spin**, the group-valued orthogonal space **Pin** is *not* symmetric; equivalently, the continuous map

$$\mu_{V,W} \;:\; \mathrm{Pin}(V) \times P(W) \longrightarrow \mathrm{Pin}(V \oplus W)$$

is *not* a group homomorphism. The issue is that for $v \in S(V)$ and $w \in S(W)$ the elements $[v,0]$ and $[0,w]$ *anti-commute* in $\mathrm{Pin}(V \oplus W)$.

Now we turn to the groups pin^c and spin^c, the complex variations on the pin and spin groups. These arise from the complexified Clifford algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$, where V is again a euclidean inner product space. The complexified Clifford algebra is again $\mathbb{Z}/2$-graded, and functorial for $\mathbb{R}$-linear isometric embeddings. The *pin^c group* of V is the subgroup

$$\mathrm{Pin}^c(V) \;\subset\; (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V))^\times$$

generated inside the multiplicative group by the unit scalars $\lambda \otimes 1$ for all $\lambda \in U(1)$ and the elements $1 \otimes [v]$ for all unit vectors $v \in S(V)$. The pin^c groups form a group-valued orthogonal space $\mathbf{Pin}^c$, and hence an orthogonal monoid space, in much the same way as do the pin groups above. As for **Pin**, the monoid-valued orthogonal space $\mathbf{Pin}^c$ is *not* symmetric, so the associated $\boxtimes$-multiplication of $\mathbf{Pin}^c$ is *not* commutative.

Since the group $\mathrm{Pin}^c(V)$ is generated by homogeneous elements of the complexified Clifford algebra, all of its elements are homogeneous. So the $\mathbb{Z}/2$-grading of $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$ provides a continuous homomorphism $\mathrm{Pin}^c(V) \longrightarrow \mathbb{Z}/2$ whose kernel

$$\mathrm{Spin}^c(V) \;=\; (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V))_{\mathrm{ev}} \cap \mathrm{Pin}^c(V)$$

is the *spin^c group* of V. As V varies, the spin^c groups from a group-valued orthogonal subspace $\mathbf{Spin}^c$ of $\mathbf{Pin}^c$. As for **Spin**, the images of the homomorphisms $\mathrm{Spin}^c(i_V)$ and $\mathrm{Spin}^c(i_W)$ commute, so $\mathbf{Spin}^c$ is even an ultra-commutative monoid.

We embed the real Clifford algebra $\mathrm{Cl}(V)$ as an $\mathbb{R}$-subalgebra of its complexification by

$$\iota(V) = 1 \otimes - \ : \ \mathrm{Cl}(V) \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V) .$$

This homomorphism restricts to embeddings

$$\iota(V) \ : \ \mathrm{Pin}(V) \longrightarrow \mathrm{Pin}^c(V) \quad \text{and} \quad \iota(V) \ : \ \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) ;$$

as V varies, these maps form morphisms of group-valued orthogonal spaces $\iota : \mathbf{Pin} \longrightarrow \mathbf{Pin}^c$ and $\iota : \mathbf{Spin} \longrightarrow \mathbf{Spin}^c$, and hence morphisms of orthogonal monoid spaces. These morphisms extend to isomorphisms

$$\mathbf{Pin} \times_{\{\pm 1\}} U(1) \cong \mathbf{Pin}^c \qquad \text{and} \qquad \mathbf{Spin} \times_{\{\pm 1\}} U(1) \cong \mathbf{Spin}^c$$

of orthogonal monoid spaces.

We let $\alpha : \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(V)$ denote the unique $\mathbb{R}$-algebra automorphism of the Clifford algebra such that $\alpha[v] = -[v]$ for all $v \in V$. The map α is the grading involution, i.e., it is the identity on the even part and -1 on the odd part of the Clifford algebra. We also denote by α the automorphism of the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$ obtained by complexifying the real version. For every element $x \in \mathrm{Pin}^c(V)$ the twisted conjugation map

$$c_x \ : \ \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V) \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V) , \quad c_x(y) = \alpha(x) y x^{-1}$$

is an automorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras. We let $v \in S(V)$ be a unit vector. Then twisted conjugation by $[v]$ takes $[v]$ to $-[v]$, and it fixes the elements $[w]$ for all $w \in V$ that are orthogonal to v. So twisted conjugation by $[v]$ 'is' reflection in the hyperplane orthogonal to v, hence a linear isometry of V of determinant -1. The complex scalars are central in $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$, so conjugation by elements $\lambda \otimes 1$ with $\lambda \in U(1)$ is the identity. So for all elements $x \in \mathrm{Pin}^c(V)$, the conjugation map c_x restricts to a linear isometry on V, in the sense that there is a unique $\mathrm{ad}(x) \in O(V)$ satisfying

$$\alpha(x)[v]x^{-1} = [\mathrm{ad}(x)(v)]$$

for all $v \in V$. We thus obtain a continuous group homomorphism

$$\mathrm{ad}(V) \ : \ \mathrm{Pin}^c(V) \longrightarrow O(V) , \quad x \longmapsto \mathrm{ad}(x) ,$$

the *twisted adjoint representation*. The kernel of the twisted adjoint representation is the subgroup $U(1) \cdot 1$, compare [4, Thm. 3.17]. This homomorphism takes the spin^c group to the special orthogonal group, and restricts to the *adjoint representation*

$$\mathrm{ad}(V) \ : \ \mathrm{Spin}^c(V) \longrightarrow SO(V) . \tag{2.3.11}$$

These homomorphisms form morphisms of group-valued orthogonal spaces

$$\mathrm{ad} \; : \; \mathbf{Pin}^c \longrightarrow \mathbf{O} \qquad \text{and} \qquad \mathrm{ad} \; : \; \mathbf{Spin}^c \longrightarrow \mathbf{SO}.$$

Via Construction 2.3.1 we can interpret these as morphisms of orthogonal monoid spaces.

There is yet another interesting morphism of group-valued orthogonal spaces

$$l \; : \; \mathbf{U} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Spin}^c)$$

that lifts the forgetful realification morphism (2.3.8) through

$$\mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathrm{ad}) \; : \; \mathrm{sh}^{\otimes}(\mathbf{Spin}^c) \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{SO}) \; .$$

The definition of $l(V_{\mathbb{C}})$ is a coordinate-free description of the homomorphism $U(n) \longrightarrow \mathrm{Spin}^c(2n)$ that is defined for example in [4, §3, p.10]. Since we don't need this, we won't go into any details. Since both $\mathbf{U}$ and $\mathbf{Spin}^c$ are *symmetric* group-valued orthogonal spaces, $l \, : \, \mathbf{U} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Spin}^c)$ is also a homomorphism of ultra-commutative monoids. The morphism l takes the special unitary group $SU(V_{\mathbb{C}})$ to the group $\mathrm{Spin}(\mathbb{C} \otimes V)$, so it restricts to a morphism of ultra-commutative monoids $l : \mathbf{SU} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}} \, \mathbf{Spin}$.

Most of the examples discussed so far can be summarized in the commutative diagram of orthogonal monoid spaces:

$$\begin{tikzcd}
\mathbf{SU} \arrow[r, "\mathrm{incl}"] \arrow[d, dashed, "l"'] & \mathbf{U} \arrow[d, dashed, "l"'] \arrow[dr, dashed, "r"] & \\
\mathbf{Spin} \arrow[r, "\iota"] \arrow[d, "\mathrm{incl}"'] & \mathbf{Spin}^c \arrow[r, "\mathrm{ad}"'] \arrow[d, "\mathrm{incl}"] & \mathbf{SO} \arrow[d, "\mathrm{incl}"] \\
\mathbf{Pin} \arrow[r, "\iota"'] & \mathbf{Pin}^c \arrow[r, "\mathrm{ad}"'] & \mathbf{O}
\end{tikzcd}$$

The two dotted arrows mean that the actual morphism goes to a multiplicative shift of the target. With the exception of **Pin** and $\mathbf{Pin}^c$, all the orthogonal monoid spaces are ultra-commutative.

Example 2.3.12 (Additive Grassmannian)**.** We define an ultra-commutative monoid **Gr**, the *additive Grassmannian*. The value of **Gr** at an inner product space V is

$$\mathbf{Gr}(V) \; = \; \coprod\nolimits_{m \geq 0} Gr_m(V) \, ,$$

the disjoint union of all Grassmannians in V. The structure map induced by a linear isometric embedding $\varphi \, : \, V \longrightarrow W$ is given by $\mathbf{Gr}(\varphi)(L) = \varphi(L)$. A commutative multiplication on **Gr** is given by direct sum:

$$\mu_{V,W} \; : \; \mathbf{Gr}(V) \times \mathbf{Gr}(W) \longrightarrow \mathbf{Gr}(V \oplus W) \, , \quad (L, L') \longmapsto L \oplus L'$$

The unit is the only point $\{0\}$ in $\mathbf{Gr}(0)$. The orthogonal space $\mathbf{Gr}$ is naturally $\mathbb{N}$-graded, with the degree m part given by $\mathbf{Gr}^{[m]}(V) = Gr_m(V)$. The multiplication is graded in that it sends $\mathbf{Gr}^{[m]}(V) \times \mathbf{Gr}^{[n]}(W)$ to $\mathbf{Gr}^{[m+n]}(V \oplus W)$.

As an orthogonal space, $\mathbf{Gr}$ is the disjoint union of global classifying spaces of the orthogonal groups. Indeed, the homeomorphisms

$$\mathbf{L}(\mathbb{R}^m, V)/O(m) \cong \mathbf{Gr}^{[m]}(V) , \quad \varphi \cdot O(m) \longmapsto \varphi(\mathbb{R}^m)$$

show that the summand $\mathbf{Gr}^{[m]}$ is isomorphic to the semifree orthogonal space $\mathbf{L}_{O(m),\mathbb{R}^m}$. Since the tautological action of $O(m)$ on $\mathbb{R}^m$ is faithful, this is a global classifying space for the orthogonal group. So, as orthogonal spaces,

$$\mathbf{Gr} = \coprod_{m\geq 0} B_{\mathrm{gl}}O(m) .$$

Proposition 1.5.12 (ii) identifies the equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}O(m))$ with the set of conjugacy classes of continuous homomorphisms from G to $O(m)$; by restricting the tautological $O(m)$-representation on $\mathbb{R}^m$, this set bijects with the set of isomorphism classes of m-dimensional G-representations. In the union over all $m \geq 0$, this becomes a bijection between $\pi_0^G(\mathbf{Gr})$ and $\mathbf{RO}^+(G)$, the set of isomorphism classes of orthogonal G-representations.

We make the bijection more explicit, showing at the same time that it is an isomorphism of monoids. We let V be a finite-dimensional orthogonal G-representation. The G-fixed-points of $\mathbf{Gr}(V)$ are the G-invariant subspaces of V, i.e., the G-subrepresentations. We define a map

$$\mathbf{Gr}(V)^G = \coprod_{m\geq 0} (Gr_m(V))^G \longrightarrow \mathbf{RO}^+(G)$$

from this fixed-point space to the monoid of isomorphism classes of G-representations by sending $L \in \mathbf{Gr}(V)^G$ to its isomorphism class. The isomorphism class of L only depends on the path component of L in $\mathbf{Gr}(V)^G$, and the resulting maps $\pi_0(\mathbf{Gr}(V)^G) \longrightarrow \mathbf{RO}^+(G)$ are compatible as V runs through the finite-dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{Gr}) = \operatorname{colim}_{V\in s(\mathcal{U}_G)} \pi_0(\mathbf{Gr}(V)^G) \longrightarrow \mathbf{RO}^+(G) , \tag{2.3.13}$$

and this map is an isomorphism of monoids with respect to the direct sum of representations on the target. Moreover, the isomorphism is compatible with restriction maps, and it takes the transfer maps induced by the commutative multiplication of $\mathbf{Gr}$ to induction of representations on the right-hand side; so as G varies, the maps (2.3.13) form an isomorphism of global power monoids.

An interesting morphism of ultra-commutative monoids

$$\tau : \mathbf{Gr} \longrightarrow \mathbf{O} \tag{2.3.14}$$

is defined at an inner product space V as the map

$$\tau(V) : \mathbf{Gr}(V) \longrightarrow \mathbf{O}(V) , \quad L \longmapsto p_{L^\perp} - p_L ,$$

sending a subspace $L \subset V$ to the difference of the orthogonal projection onto $L^\perp = V - L$ and the orthogonal projection onto L. Put differently, $\tau(V)(L)$ is the linear isometry that is multiplication by -1 on L and the identity on the orthogonal complement $L^\perp$. We omit the straightforward verification that these maps do define a morphism of ultra-commutative monoids. The induced monoid homomorphism

$$\pi_0^G(\tau) \ : \ \pi_0^G(\mathbf{Gr}) \ \longrightarrow \ \pi_0^G(\mathbf{O})$$

is easily calculated. The isomorphism (2.3.13) identifies the source with the monoid $\mathbf{RO}^+(G)$ of isomorphism classes of orthogonal G-representations, under direct sum. By Example 2.3.6 the group $\pi_0^G(\mathbf{O})$ is a direct sum of copies of $\mathbb{Z}/2$, indexed by the isomorphism classes of irreducible orthogonal G-representations of real type. If λ is any irreducible orthogonal G-representation, then $\pi_0^G(\tau)$ sends its class to the automorphism $-\mathrm{Id}_\lambda$. The group $\mathbf{O}(\lambda)^G$ is isomorphic to $O(1)$, $U(1)$ or $Sp(1)$ depending on whether λ is of real, complex or quaternionic type. In the real case, the map $-\mathrm{Id}_\lambda$ lies in the non-identity path component; in the complex and quaternionic cases, the group $\mathbf{O}(\lambda)^G$ is path connected. So under the previous isomorphisms, $\pi_0^G(\tau)$ becomes the homomorphism

$$\mathbf{RO}^+(G) \ \longrightarrow \ \bigoplus_{[\lambda] \text{ real}} \mathbb{Z}/2$$

that sends the class of λ to the generator of the λ-summand if λ is of real type, and to the trivial element if λ is of complex or quaternionic type. Since the classes of irreducible representations freely generate $\mathbf{RO}^+(G)$ as an abelian monoid, this determines the morphism $\pi_0^G(\tau)$. Moreover, this also shows that $\pi_0^G(\mathbf{Gr}) \longrightarrow \pi_0^G(\mathbf{O})$ is surjective.

Example 2.3.15 (Oriented Grassmannian)**.** A variant of the previous example is the orthogonal monoid space $\mathbf{Gr}^{\mathrm{or}}$ of *oriented Grassmannians*. The value of $\mathbf{Gr}^{\mathrm{or}}$ at an inner product space V is

$$\mathbf{Gr}^{\mathrm{or}}(V) \ = \ \coprod\nolimits_{m\geq 0} Gr_m^{\mathrm{or}}(V) \,,$$

the disjoint union of all oriented Grassmannians in V. Here a point in $Gr_m^{\mathrm{or}}(V)$ is a pair $(L, [b_1, \dots, b_m])$ consisting of an $L \in Gr_m(V)$ and an orientation, i.e., a $GL^+(L)$-equivalence class $[b_1, \dots, b_m]$ of bases of L. The structure map induced by $\varphi : V \longrightarrow W$ sends $(L, [b_1, \dots, b_m])$ to $(\varphi(L), [\varphi(b_1), \dots \varphi(b_m)])$. A multiplication on $\mathbf{Gr}^{\mathrm{or}}$ is given by direct sum:

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{Gr}^{\mathrm{or}}(V) \times \mathbf{Gr}^{\mathrm{or}}(W) \ &\longrightarrow \ \mathbf{Gr}^{\mathrm{or}}(V \oplus W) \\ ((L, [b_1, \dots, b_m]), \ (L', [b'_1, \dots, b'_n])) \ &\longmapsto \\ &(L \oplus L', \ [(b_1, 0), \dots, (b_m, 0), (0, b'_1), \dots, (0, b'_n)]) \,. \end{aligned}$$

The unit is the only point $(\{0\}, \emptyset)$ in $\mathbf{Gr}^{\mathrm{or}}(0)$.

As an orthogonal space, $\mathbf{Gr}^{\mathrm{or}}$ is the disjoint union of global classifying spaces of the special orthogonal groups, via the homeomorphisms

$$\begin{aligned}(B_{\mathrm{gl}}SO(m))(V) \ = \ \mathbf{L}(\mathbb{R}^m, V)/SO(m) \ &\cong \quad Gr_m^{\mathrm{or}}(V) \\ \varphi \cdot SO(m) \quad &\longmapsto \ (\varphi(\mathbb{R}^m), [\varphi(e_1), \dots \varphi(e_m)]) \ .\end{aligned}$$

The multiplication of $\mathbf{Gr}^{\mathrm{or}}$ is *not* commutative. The issue is that when pushing a pair around the two ways of the square

$$\begin{tikzcd} Gr_m^{\mathrm{or}}(V) \times Gr_n^{\mathrm{or}}(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\mathrm{twist}"'] & Gr_{m+n}^{\mathrm{or}}(V \oplus W) \arrow[d, "Gr_{m+n}^{\mathrm{or}}(\tau_{V,W})"] \\ Gr_n^{\mathrm{or}}(W) \times Gr_m^{\mathrm{or}}(V) \arrow[r, "\mu_{W,V}"'] & Gr_{n+m}^{\mathrm{or}}(W \oplus V) \end{tikzcd}$$

then we end up with the same subspaces of $W \oplus V$, but they come with *different* orientations if m and n are both odd.

We can arrange commutativity of the multiplication by passing to the orthogonal submonoid $\mathbf{Gr}^{\mathrm{or,ev}}$ of *even-dimensional* oriented Grassmannians, defined as

$$\mathbf{Gr}^{\mathrm{or,ev}}(V) \ = \ \coprod\nolimits_{n \geq 0} Gr_{2n}^{\mathrm{or}}(V) \ ;$$

the multiplication of $\mathbf{Gr}^{\mathrm{or,ev}}$ is then commutative. Moreover, the forgetful map $\mathbf{Gr}^{\mathrm{or,ev}} \longrightarrow \mathbf{Gr}$ to the additive Grassmannian is a homomorphism of ultra-commutative monoids.

Example 2.3.16 (Complex and quaternionic Grassmannians)**.** The *complex additive Grassmannian* $\mathbf{Gr}^{\mathbb{C}}$ and the *quaternionic additive Grassmannian* $\mathbf{Gr}^{\mathbb{H}}$ are two more ultra-commutative monoids, the complex and quaternionic analogs of Example 2.3.12. The underlying orthogonal spaces send an inner product space V to

$$\mathbf{Gr}^{\mathbb{C}}(V) \ = \ \coprod\nolimits_{m \geq 0} Gr_m^{\mathbb{C}}(V_{\mathbb{C}}) \quad \text{and} \quad \mathbf{Gr}^{\mathbb{H}}(V) \ = \ \coprod\nolimits_{m \geq 0} Gr_m^{\mathbb{H}}(V_{\mathbb{H}}) \ ,$$

the disjoint union of all complex (or quaternionic) Grassmannians in the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ (or in $V_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} V$). As in the real analog, the structure maps are given by taking images under (complexified or quaternified) linear isometric embeddings; direct sum of subspaces, plus identification along the isomorphism $V_{\mathbb{C}} \oplus W_{\mathbb{C}} \cong (V \oplus W)_{\mathbb{C}}$ (or $V_{\mathbb{H}} \oplus W_{\mathbb{H}} \cong (V \oplus W)_{\mathbb{H}}$), provides an ultra-commutative multiplication on $\mathbf{Gr}^{\mathbb{C}}$ and on $\mathbf{Gr}^{\mathbb{H}}$.

The homogeneous summand $\mathbf{Gr}^{\mathbb{C},[m]}$ is a global classifying space for the unitary group $U(m)$. Indeed, in Construction 1.3.10 we introduced the ‘com-

plex semifree' orthogonal space $\mathbf{L}^{\mathbb{C}}_{U(m),\mathbb{C}^m}$ associated to the tautological $U(m)$-representation on $\mathbb{C}^m$. This orthogonal space is isomorphic to $\mathbf{Gr}^{\mathbb{C},[m]}$ via

$$\mathbf{L}^{\mathbb{C}}(\mathbb{C}^m, V_{\mathbb{C}})/U(m) \cong \mathbf{Gr}^{\mathbb{C},[m]}(V) , \quad \varphi \cdot U(m) \longmapsto \varphi(\mathbb{C}^m) .$$

Proposition 1.3.11 (i) then exhibits a global equivalence

$$B_{\mathrm{gl}}U(m) = \mathbf{L}_{U(m),u(\mathbb{C}^m)} \xrightarrow{\sim} \mathbf{L}^{\mathbb{C}}_{U(m),\mathbb{C}^m} \cong \mathbf{Gr}^{\mathbb{C},[m]} .$$

Although Proposition 1.3.11 (i) does not literally apply to $\mathbf{Gr}^{\mathbb{H},[m]}$, it has an $\mathbb{H}$-linear analog for symplectic representations, showing that the homogeneous summand $\mathbf{Gr}^{\mathbb{H},[m]}$ is a global classifying space for the symplectic group $Sp(m)$.

The ultra-commutative monoid $\mathbf{Gr}^{\mathbb{C}}$ comes with an involutive automorphism

$$\psi : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}$$

given by complex conjugation. Here we exploit the fact that the complexification of an $\mathbb{R}$-vector space V comes with a preferred $\mathbb{C}$-semilinear morphism

$$\psi_V : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} , \quad \lambda \otimes v \longmapsto \bar{\lambda} \otimes v .$$

The value of ψ at V takes a $\mathbb{C}$-subspace $L \subset V_{\mathbb{C}}$ to the conjugate subspace $\bar{L} = \psi_V(L)$. Complexification of subspaces defines a morphism of ultra-commutative monoids

$$c : \mathbf{Gr} \longrightarrow \mathbf{Gr}^{\mathbb{C}} , \quad \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}^{\mathbb{C}}(V) , \quad L \longmapsto L_{\mathbb{C}}$$

from the real to the complex additive Grassmannian. A complex subspace of $V_{\mathbb{C}}$ is invariant under ψ_V if and only if it is the complexification of an $\mathbb{R}$-subspace of V (namely the ψ_V-fixed subspace of V). So the morphism c is an isomorphism of $\mathbf{Gr}$ onto the ψ-invariant ultra-commutative submonoid $(\mathbf{Gr}^{\mathbb{C}})^{\psi}$.

Realification defines a morphism of ultra-commutative monoids

$$r : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathrm{sh}^{\mathbb{C}}_{\otimes}(\mathbf{Gr}^{\mathrm{or,ev}})$$

to the multiplicative shift (see Example 1.1.11) of the even part of the oriented Grassmannian of Example 2.3.15. The value $r(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow \mathbf{Gr}^{\mathrm{or,ev}}(V_{\mathbb{C}})$ takes a complex subspace of $V_{\mathbb{C}}$ to the underlying real vector space, endowed with the preferred orientation $[x_1, ix_1, \ldots, x_n, ix_n]$, where $(x_1, \ldots, x_n)$ is any complex basis.

The isomorphism (2.3.13) between $\pi_0^G(\mathbf{Gr})$ and $\mathbf{RO}^+(G)$ has an obvious complex analog. For every compact Lie group G an isomorphism of commutative monoids

$$\pi_0^G(\mathbf{Gr}^{\mathbb{C}}) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} \pi_0(\mathbf{Gr}^{\mathbb{C}}(V)^G) \cong \mathbf{RU}^+(G) \tag{2.3.17}$$

is given by sending the class of a G-fixed-point in $\mathbf{Gr}^{\mathbb{C}}(V)^G$, i.e., a complex

G-subrepresentation of $V_{\mathbb{C}}$, to its isomorphism class. The isomorphism is compatible with restriction maps, takes the involution $\pi_0^G(\psi)$ of $\pi_0^G(\mathbf{Gr}^{\mathbb{C}})$ to the complex conjugation involution of $\mathbf{RU}^+(G)$, and takes the transfer maps induced by the commutative multiplication of $\mathbf{Gr}^{\mathbb{C}}$ to induction of representations. So as G varies, the maps form an isomorphism of global power monoids $\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \cong \mathbf{RU}^+$. The isomorphisms are also compatible with complexification and realification, in the sense of the commutative diagram:

$$\begin{tikzcd}
\underline{\pi}_0(\mathbf{Gr}) \arrow[r, "\underline{\pi}_0(c)"] \arrow[d, "\cong", "(2.3.13)"'] & \underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \arrow[r, "\underline{\pi}_0(r)"] \arrow[d, "\cong"', "(2.3.17)"] & \underline{\pi}_0(\mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}^{\mathrm{or,ev}})) \arrow[r, "\cong"] & \underline{\pi}_0(\mathbf{Gr}^{\mathrm{or,ev}}) \arrow[r, "\mathrm{forget}"] & \underline{\pi}_0(\mathbf{Gr}) \arrow[d, "\cong"', "(2.3.13)"] \\
\mathbf{RO}^+ \arrow[r, "c"'] & \mathbf{RU}^+ \arrow[rrr, "r"'] & & & \mathbf{RO}^+
\end{tikzcd}$$

The lower horizontal maps are complexification and realification of representations. The isomorphism in the upper row is inverse to the one induced by the homomorphism $\mathbf{Gr}^{\mathrm{or,ev}} \circ i : \mathbf{Gr}^{\mathrm{or,ev}} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}^{\mathrm{or,ev}})$, i.e., pre-composition with the natural linear isometric embedding

$$i_V \ : \ V \ \longrightarrow \ V_{\mathbb{C}} \ , \quad v \ \longmapsto \ 1 \otimes v \ ;$$

the morphism $\mathbf{Gr}^{\mathrm{or,ev}} \circ i$ is a global equivalence by Theorem 1.1.10.

Example 2.3.18 (Multiplicative Grassmannians)**.** We define an ultra-commutative monoid $\mathbf{Gr}_{\otimes}$, the *multiplicative Grassmannian*. We let

$$\mathrm{Sym}(V) \ = \ \bigoplus\nolimits_{i\geq 0} \mathrm{Sym}^i(V) \ = \ \bigoplus\nolimits_{i\geq 0} V^{\otimes i}/\Sigma_i$$

denote the symmetric algebra of an inner product space V. If W is another inner product space, then the two direct summand inclusions induce algebra homomorphisms

$$\mathrm{Sym}(V) \ \longrightarrow \ \mathrm{Sym}(V \oplus W) \ \longleftarrow \ \mathrm{Sym}(W) \ .$$

We use the commutative multiplication on $\mathrm{Sym}(V \oplus W)$ to combine these into an $\mathbb{R}$-algebra isomorphism

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(W) \ \cong \ \mathrm{Sym}(V \oplus W) \ . \tag{2.3.19}$$

These isomorphisms are natural for linear isometric embeddings in V and W. The value of $\mathbf{Gr}_{\otimes}$ at an inner product space V is then

$$\mathbf{Gr}_{\otimes}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr_n(\mathrm{Sym}(V)) \ ,$$

the disjoint union of all Grassmannians in the symmetric algebra of V. The structure map $\mathbf{Gr}_{\otimes}(\varphi) : \mathbf{Gr}_{\otimes}(V) \longrightarrow \mathbf{Gr}_{\otimes}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is given by

$$\mathbf{Gr}_{\otimes}(\varphi)(L) \ = \ \mathrm{Sym}(\varphi)(L) \ ,$$

where $\mathrm{Sym}(\varphi) : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(W)$ is the induced map of symmetric algebras. A commutative multiplication on $\mathbf{Gr}_\otimes$ is given by tensor product, i.e.,

$$\mu_{V,W} \ : \ \mathbf{Gr}_\otimes(V) \times \mathbf{Gr}_\otimes(W) \ \longrightarrow \ \mathbf{Gr}_\otimes(V \oplus W)$$

sends $(L, L') \in \mathbf{Gr}_\otimes(V) \times \mathbf{Gr}_\otimes(W)$ to the image of $L \otimes L'$ under the isomorphism (2.3.19). The multiplicative unit is the point $\mathbb{R}$ in $\mathbf{Gr}_\otimes(0) = \mathbb{R}$. As the additive Grassmannian $\mathbf{Gr}$, the multiplicative Grassmannian $\mathbf{Gr}_\otimes$ is $\mathbb{N}$-graded, with degree n part given by $\mathbf{Gr}_\otimes^{[n]}(V) = Gr_n(\mathrm{Sym}(V))$. The multiplication sends $\mathbf{Gr}_\otimes^{[m]}(V) \times \mathbf{Gr}_\otimes^{[n]}(W)$ to $\mathbf{Gr}_\otimes^{[m\cdot n]}(V \oplus W)$.

Regarded as orthogonal spaces, the additive and multiplicative Grassmannians are globally equivalent. For an inner product space V we let $i : V \longrightarrow \mathrm{Sym}(V)$ be the embedding as the linear summand of the symmetric algebra. Then as V varies, the maps

$$\mathbf{Gr}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr_n(V) \ \longrightarrow \ \coprod\nolimits_{n\geq 0} Gr_n(\mathrm{Sym}(V)) \ = \ \mathbf{Gr}_\otimes(V)$$

sending L to $i(L)$ form a global equivalence $\mathbf{Gr} \longrightarrow \mathbf{Gr}_\otimes$. Indeed, for each $n \geq 0$, $\mathbf{Gr}_\otimes^{[n]}(V)$ is a sequential colimit, along closed embeddings, of a sequence of orthogonal spaces

$$\mathbf{Gr}^{[n]} \ \longrightarrow \ \mathbf{Gr}_{\leq 1}^{[n]} \ \longrightarrow \ \mathbf{Gr}_{\leq 2}^{[n]} \ \longrightarrow \ \ldots \ \longrightarrow \ \mathbf{Gr}_{\leq k}^{[n]} \ \longrightarrow \ \ldots ,$$

where $\mathbf{Gr}_{\leq k}^{[n]}(V) \ = \ Gr_n(\bigoplus_{i=0}^k \mathrm{Sym}^i(V))$. Each of the morphisms $\mathbf{Gr}^{[n]} \longrightarrow \mathbf{Gr}_{\leq k}^{[n]}$ is a global equivalence by Theorem 1.1.10; so all morphisms in the sequence are global equivalences, hence so is the map from $\mathbf{Gr}^{[n]}$ to the colimit $\mathbf{Gr}_\otimes^{[n]}$, by Proposition 1.1.9 (ix). This global equivalence induces a bijection

$$\pi_0^G(\mathbf{Gr}) \ \cong \ \pi_0^G(\mathbf{Gr}_\otimes)$$

for every compact Lie group G, hence both are isomorphic to the set $\mathbf{RO}^+(G)$ of isomorphism classes of orthogonal G-representations. The commutative monoid structures and transfer maps induced by the products of $\mathbf{Gr}$ and $\mathbf{Gr}_\otimes$ are quite different though: the monoid structure of $\pi_0^G(\mathbf{Gr})$ corresponds to direct sum of representations, and the transfer maps are additive transfers; the monoid structure of $\pi_0^G(\mathbf{Gr}_\otimes)$ corresponds to tensor product of representations, and the transfer maps are multiplicative transfers, also called norm maps.

The orthogonal subspace $\mathbf{P} = \mathbf{Gr}_\otimes^{[1]}$ of the multiplicative Grassmannian $\mathbf{Gr}_\otimes$ is closed under the product and contains the multiplicative unit, hence $\mathbf{P}$ is an ultra-commutative monoid in its own right. Because

$$\mathbf{P}(V) \ = \ \mathbf{Gr}_\otimes^{[1]}(V) \ = \ P(\mathrm{Sym}(V))$$

is the projective space of the symmetric algebra of V, we use the symbol $\mathbf{P}$ and refer to it as the *global projective space*. The multiplication is given by tensor

product of lines, and application of the isomorphism (2.3.19). Since $\mathbf{P} = \mathbf{Gr}_{\otimes}^{[1]}$ is globally equivalent to the additive variant $\mathbf{Gr}^{[1]}$, it is a global classifying space for the group $O(1)$, a cyclic group of order 2:

$$\mathbf{P} \ \simeq \ \mathbf{Gr}^{[1]} \ \simeq \ B_{\mathrm{gl}}O(1) \ = \ B_{\mathrm{gl}}C_2 \ .$$

In other words, $\mathbf{P}$ is an ultra-commutative multiplicative model for $B_{\mathrm{gl}}C_2$.

There is a straightforward complex analog of the multiplicative Grassmannian, namely the ultra-commutative monoid $\mathbf{Gr}_{\otimes}^{\mathbb{C}}$ with value at V given by

$$\mathbf{Gr}_{\otimes}^{\mathbb{C}}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr_n^{\mathbb{C}}(\mathrm{Sym}(V)_{\mathbb{C}}) \ ,$$

the disjoint union of all Grassmannians in the complexified symmetric algebra of V. The structure maps and multiplication (by tensor product) are as in the real case. The orthogonal subspace $\mathbf{P}^{\mathbb{C}} = \mathbf{Gr}_{\otimes}^{\mathbb{C},[1]}$ consisting of 1-dimensional subspaces is closed under the product and contains the multiplicative unit; hence $\mathbf{P}^{\mathbb{C}}$ is an ultra-commutative monoid in its own right, the *complex global projective space*. As an orthogonal space, $\mathbf{P}^{\mathbb{C}}$ is globally equivalent to the additive variant $\mathbf{Gr}^{\mathbb{C},[1]}$, and hence a global classifying space for the group $U(1)$,

$$\mathbf{P}^{\mathbb{C}} \ \simeq \ \mathbf{Gr}^{\mathbb{C},[1]} \ \simeq \ B_{\mathrm{gl}}U(1) \ . \tag{2.3.20}$$

Since the multiplication in the skew field of quaternions $\mathbb{H}$ is not commutative, there is no tensor product of $\mathbb{H}$-vector spaces; so there is no multiplicative version of the quaternionic Grassmannian $\mathbf{Gr}^{\mathbb{H}}$.

Construction 2.3.21 (Bar construction)**.** For the next class of examples we quickly recall the *bar construction* of a topological monoid M. This is the geometric realization of the simplicial space $M^{\bullet}$ whose space of n-simplices is M^n, the n-fold cartesian power of M. For $n \geq 1$ and $0 \leq i \leq n$, the face map $d_i^* : M^n \longrightarrow M^{n-1}$ is given by

$$d_i^*(x_1, \dots x_n) \ = \begin{cases} (x_2, \dots, x_n) & \text{for } i = 0, \\ (x_1, \dots, x_{i-1}, x_i \cdot x_{i+1}, x_{i+2}, \dots, x_n) & \text{for } 0 < i < n, \\ (x_1, \dots, x_{n-1}) & \text{for } i = n. \end{cases}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ the degeneracy map $s_i^* : M^{n-1} \longrightarrow M^n$ is given by

$$s_i^*(x_1, \dots, x_{n-1}) \ = \ (x_1, \dots, x_i, 1, x_{i+1}, \dots, x_{n-1}) \ .$$

The bar construction is the geometric realization (see Construction A.32)

$$BM \ = \ |M^{\bullet}| \ = \ |[n] \mapsto M^n|$$

of this simplicial space; the construction $M \mapsto BM$ is functorial in continuous

monoid homomorphisms. The bar construction commutes with products in the sense that for a pair of topological monoids M and N, the canonical map

$$(Bp_M, Bp_N) \ : \ B(M \times N) \ \longrightarrow \ BM \times BN \tag{2.3.22}$$

is a homeomorphism, where $p_M : M \times N \longrightarrow M$ and $p_N : M \times N \longrightarrow N$ are the projections. Indeed, this map factors as the composite of two maps

$$|(M \times N)^\bullet| \ \xrightarrow{|(p_M^\bullet, p_N^\bullet)|} \ |M^\bullet \times N^\bullet| \ \xrightarrow{(|p_{M^\bullet}|, |p_{N^\bullet}|)} \ |M^\bullet| \times |N^\bullet| \ .$$

The first map is the realization of an isomorphism of simplicial spaces, given level-wise by shuffling the factors. The second map is a homeomorphism because realization commutes with products, see Proposition A.37 (ii).

Construction 2.3.23 (Multiplicative global classifying spaces)**.** As we discuss now, all *abelian* compact Lie groups admit multiplicative models of their global classifying spaces. We use the bar construction, giving a non-equivariant classifying space, followed by the cofree functor R (see Construction 1.2.25). The cofree functor takes a space A to the orthogonal space RA with values

$$(RA)(V) \ = \ \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) \ .$$

We endow the cofree functor with a lax symmetric monoidal transformation

$$\mu_{A,B} \ : \ RA \boxtimes RB \ \longrightarrow \ R(A \times B) \ .$$

To construct $\mu_{A,B}$ we start from the continuous maps

$$\begin{aligned} \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) \times \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), B) \ &\xrightarrow{\times} \ \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(W, \mathbb{R}^\infty), A \times B) \\ &\xrightarrow{(\mathrm{res}_{V,W})^*} \ \mathrm{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), A \times B) \end{aligned}$$

that constitute a bimorphism from (RA, RB) to $R(A \times B)$. Here

$$\mathrm{res}_{V,W} \ : \ \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(W, \mathbb{R}^\infty)$$

restricts an embedding of $V \oplus W$ to the summands V and W. The morphism $\mu_{A,B}$ is associated to this bimorphism via the universal property of the box product.

The bar construction preserves products in the sense that for every pair of compact Lie groups G and K the canonical map

$$(Bp_G, Bp_K) \ : \ B(G \times K) \ \longrightarrow \ BG \times BK$$

is a homeomorphism, compare (2.3.22). So the composite $G \mapsto R(BG)$ is a lax symmetric monoidal functor via the morphism of orthogonal spaces

$$R(BG) \boxtimes R(BK) \ \xrightarrow{\mu_{BG,BK}} \ R(BG \times BK) \ \cong \ R(B(G \times K)) \ .$$

The bar construction is functorial in continuous group homomorphisms, so for an abelian compact Lie group A the composite

$$R(BA) \boxtimes R(BA) \longrightarrow R(B(A \times A)) \xrightarrow{R(B\mu_A)} R(BA)$$

is an ultra-commutative and associative multiplication on the orthogonal space $R(BA)$, where $\mu_A : A \times A \longrightarrow A$ is the multiplication of A. Theorem 1.2.32 shows that for abelian A the cofree orthogonal space $R(BA)$ is a global classifying space for A. In particular, the Rep-functor $\underline{\pi}_0(R(BA))$ is representable by A. We saw in Proposition 2.2.23 that there is then a *unique* structure of a global power monoid on $\underline{\pi}_0(R(BA))$, and the power operations are characterized by the relation

$$[m](u_A) = p_m^*(u_A)$$

where $u_A \in \pi_0^A(R(BA))$ is a tautological class and $p_m : \Sigma_m \wr A \longrightarrow A$ is the homomorphism defined by

$$p_m(\sigma; a_1, \ldots, a_m) = a_1 \cdot \ldots \cdot a_m .$$

Example 2.3.24 (Unordered frames)**.** The ultra-commutative monoid $\mathbf{F}$ of *unordered frames* sends an inner product space V to

$$\mathbf{F}(V) = \{A \subset V \mid A \text{ is orthonormal}\} ,$$

the space of all unordered frames in V, i.e., subsets of V that consist of pairwise orthogonal unit vectors. Since V is finite-dimensional, such a subset is necessarily finite. The topology on $\mathbf{F}(V)$ is as the disjoint union, over the cardinality of the sets, of quotient spaces of Stiefel manifolds. The structure map induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is given by $\mathbf{F}(\varphi)(A) = \varphi(A)$. A commutative multiplication on $\mathbf{F}$ is given, essentially, by disjoint union:

$$\mu_{V,W} : \mathbf{F}(V) \times \mathbf{F}(W) \longrightarrow \mathbf{F}(V \oplus W) , \quad (A, A') \longmapsto i_V(A) \cup i_W(A') ;$$

here $i_V : V \longrightarrow V \oplus W$ and $i_W : W \longrightarrow V \oplus W$ are the direct summand embeddings. The unit is the empty set, the only point in $\mathbf{F}(0)$. The orthogonal space $\mathbf{F}$ is naturally $\mathbb{N}$-graded, with degree m part $\mathbf{F}^{[m]}$ given by the unordered frames of cardinality m; the multiplication sends $\mathbf{F}^{[m]}(V) \times \mathbf{F}^{[n]}(W)$ to $\mathbf{F}^{[m+n]}(V \oplus W)$.

As an orthogonal space, $\mathbf{F}$ is the disjoint union of global classifying spaces of the symmetric groups. We let Σ_m act on $\mathbb{R}^m$ by permuting the coordinates, which is also the permutation representation of the tautological Σ_m-action on $\{1, \ldots, m\}$. This Σ_m-action is faithful, so the semifree orthogonal space $\mathbf{L}_{\Sigma_m,\mathbb{R}^m}$ is a global classifying space for the symmetric group. The homeomorphisms

$$\mathbf{L}(\mathbb{R}^m, V)/\Sigma_m \cong \mathbf{F}^{[m]}(V) , \quad \varphi \cdot \Sigma_m \longmapsto \{\varphi(e_1), \ldots, \varphi(e_m)\}$$

show that the homogeneous summand $\mathbf{F}^{[m]}$ is isomorphic to $\mathbf{L}_{\Sigma_m, \mathbb{R}^m} = B_{\mathrm{gl}}\Sigma_m$; here $e_1, \dots, e_m$ is the canonical basis of $\mathbb{R}^m$. So as orthogonal spaces,

$$\mathbf{F} \;=\; \coprod\nolimits_{m \geq 0} B_{\mathrm{gl}}\Sigma_m \ .$$

Proposition 1.5.12 (ii) identifies the equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}\Sigma_m)$ with the set of conjugacy classes of continuous homomorphisms from G to Σ_m; by restricting the tautological Σ_m-representation on $\{1, \dots, m\}$, this set bijects with the set of isomorphism classes of finite G-sets of cardinality m.

As m varies, this gives an isomorphism of monoids from $\pi_0^G(\mathbf{F})$ to the set $\mathbb{A}^+(G)$ of isomorphism classes of finite G-sets that we make explicit now. We let V be a G-representation. An unordered frame $A \in \mathbf{F}(V)$ is a G-fixed-point if and only if it is G-invariant. So for such frames, the G-action restricts to an action on A, making it a finite G-set. We define a map

$$\mathbf{F}(V)^G \;\longrightarrow\; \mathbb{A}^+(G)\ , \quad A \;\longmapsto\; [A]$$

from this fixed-point space to the monoid of isomorphism classes of finite G-sets. The isomorphism class of A as a G-set only depends on the path component of A in $\mathbf{F}(V)^G$, and the resulting maps $\pi_0(\mathbf{F}(V)^G) \longrightarrow \mathbb{A}^+(G)$ are compatible as V runs through the finite-dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{F}) \;=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, \pi_0(\mathbf{F}(V)^G) \;\longrightarrow\; \mathbb{A}^+(G)\ , \tag{2.3.25}$$

and this map is a monoid isomorphism with respect to the disjoint union of G-sets on the target. Moreover, the isomorphisms are compatible with restriction maps, and they take the transfer maps induced by the commutative multiplication of $\mathbf{F}$ to induction of equivariant sets on the right-hand side.

It goes without saying that actions of compact Lie groups are required to be *continuous* and that the use of the term 'set' (as opposed to 'space') implies the discrete topology on the set; so the identity path component G° acts trivially on every G-set. Hence the monoids $\pi_0^G(\mathbf{F})$ and $\mathbb{A}^+(G)$ only see the finite group $\pi_0(G) = G/G^\circ = \bar{G}$ of path components, i.e., for every compact Lie group G, the inflation maps

$$p^* \;:\; \pi_0^{\bar{G}}(\mathbf{F}) \;\longrightarrow\; \pi_0^G(\mathbf{F}) \qquad \text{and} \qquad p^* \;:\; \mathbb{A}^+(\bar{G}) \;\longrightarrow\; \mathbb{A}^+(G)$$

along the projection $p : G \longrightarrow \bar{G}$ are isomorphisms. So if G has positive dimension, then the group completion of the monoid $\mathbb{A}^+(G)$ need not be isomorphic to what is sometimes called the Burnside ring of G (the 0th G-equivariant stable stem).

A morphism of $\mathbb{N}$-graded ultra-commutative monoids

$$\mathrm{span} \;:\; \mathbf{F} \;\longrightarrow\; \mathbf{Gr} \qquad \text{is defined by} \qquad \mathrm{span}(V)(A) \;=\; \mathrm{span}(A)\ ,$$

i.e., a frame is sent to its linear span. The induced morphism of global power monoids is linearization: the square of monoid homomorphisms

$$\begin{tikzcd} \pi_0^G(\mathbf{F}) \arrow[r, "\pi_0^G(\mathrm{span})"] \arrow[d, "(2.3.25)"', "\cong"] & \pi_0^G(\mathbf{Gr}) \arrow[d, "\cong"', "(2.3.13)"] \\ \mathbb{A}^+(G) \arrow[r, "{[S]\longmapsto[\mathbb{R}S]}"'] & \mathbf{RO}^+(G) \end{tikzcd}$$

commutes, where the lower map sends the class of a G-set to the class of its permutation representation.

Example 2.3.26 (Multiplicative monoid of the sphere spectrum)**.** We define an ultra-commutative monoid $\Omega^\bullet\mathbb{S}$, the 'multiplicative monoid of the sphere spectrum'. The notation and terminology indicate that this is a special case of a more general construction that associates with an ultra-commutative ring spectrum R its multiplicative monoid $\Omega^\bullet R$, see Example 4.1.16 below.

The values of the orthogonal space $\Omega^\bullet\mathbb{S}$ are given by

$$(\Omega^\bullet\mathbb{S})(V) \;=\; \mathrm{map}_*(S^V, S^V)\ ,$$

the space of continuous based self-maps of the sphere S^V. A linear isometric embedding $\varphi : V \longrightarrow W$ acts by conjugation and extension by the identity, i.e., the map

$$(\Omega^\bullet\mathbb{S})(\varphi) \;:\; \mathrm{map}_*(S^V, S^V) \;\longrightarrow\; \mathrm{map}_*(S^W, S^W)$$

sends a continuous based map $f : S^V \longrightarrow S^V$ to the composite

$$S^W \cong\; S^V \wedge S^{W-\varphi(V)} \xrightarrow{\;f\wedge S^{W-\varphi(V)}\;} S^V \wedge S^{W-\varphi(V)} \;\cong\; S^W\ .$$

The two unnamed homeomorphisms between $S^V \wedge S^{W-\varphi(V)}$ and S^W use the map φ on the factor S^V. In particular, the orthogonal group $O(V)$ acts on $\mathrm{map}_*(S^V, S^V)$ by conjugation.

The multiplication of $\Omega^\bullet\mathbb{S}$ is by smash product, i.e., the map

$$\mu_{V,W} \;:\; (\Omega^\bullet\mathbb{S})(V) \times (\Omega^\bullet\mathbb{S})(W) \;\longrightarrow\; (\Omega^\bullet\mathbb{S})(V \oplus W)$$

smashes a self-map of S^V with a self-map of S^W and conjugates with the canonical homeomorphism between $S^V \wedge S^W$ and $S^{V\oplus W}$. The unit is the identity of S^V.

The equivariant homotopy set $\pi_0^G(\Omega^\bullet\mathbb{S})$ is equal to the stable G-equivariant 0-stem $\pi_0^G(\mathbb{S})$, compare Construction 4.1.6 below. The monoid structure on $\pi_0^G(\Omega^\bullet\mathbb{S})$ arising from the multiplication on $\Omega^\bullet\mathbb{S}$ is the *multiplicative* (rather than the additive) monoid structure of $\pi_0^G(\mathbb{S})$. The set $\pi_0^G(\Omega^\bullet\mathbb{S})$ thus bijects with

the underlying set of the Burnside ring $\mathbb{A}(G)$ of the group G (compare Example 4.2.7), which is additively a free abelian group with basis the conjugacy classes of closed subgroups of G with finite Weyl group. The multiplication on $\pi_0^G(\Omega^\bullet\mathbb{S})$ corresponds to the multiplication (not the addition!) in the Burnside ring $\mathbb{A}(G)$. When G is finite, $\pi_0^G(\Omega^\bullet\mathbb{S})$ bijects with the underlying set of the Grothendieck group of finite G-sets, and the multiplication corresponds to the product of G-sets. The power operations in $\underline{\pi}_0(\Omega^\bullet\mathbb{S})$ are thus represented by 'raising a G-set to the cartesian power', and the transfer maps are known as 'norm maps' or 'multiplicative induction'.

Example 2.3.27 (Exponential homomorphisms)**.** The classical J-homomorphism fits in nicely here, in the form of a global refinement to a morphism of ultra-commutative monoids

$$J \;:\; \mathbf{O} \;\longrightarrow\; \Omega^\bullet\mathbb{S}$$

defined at an inner product space V as the map

$$J(V) \;:\; \mathbf{O}(V) \;\longrightarrow\; \mathrm{map}_*(S^V, S^V)$$

sending a linear isometry $\varphi : V \longrightarrow V$ to its one-point compactification $S^\varphi : S^V \longrightarrow S^V$. The fact that these maps are multiplicative and compatible with the structure maps is straightforward. The induced map

$$\pi_0^G(J) \;:\; \pi_0^G(\mathbf{O}) \;\longrightarrow\; \pi_0^G(\Omega^\bullet\mathbb{S}) \;=\; \pi_0^G(\mathbb{S})\,,$$

for G a compact Lie group, can be described as follows. By Example 2.3.6, the group $\pi_0^G(\mathbf{O})$ is a direct sum of copies of $\mathbb{Z}/2$, indexed by the isomorphism classes of irreducible orthogonal G-representations of real type. If λ is such an irreducible G-representation, then the image of the λ-indexed copy of $\mathbb{Z}/2$ is represented by the antipodal map of S^λ.

In (2.3.14) we defined a morphism of ultra-commutative monoids $\tau : \mathbf{Gr} \longrightarrow \mathbf{O}$. The composite morphism

$$\mathbf{Gr} \xrightarrow{\tau} \mathbf{O} \xrightarrow{J} \Omega^\bullet\mathbb{S} \tag{2.3.28}$$

realizes an 'exponential' homomorphism from the real representation ring to the multiplicative group of the Burnside ring of a compact Lie group G. The exponential homomorphism was studied by tom Dieck and is sometimes called the 'tom Dieck exponential map'. Tom Dieck's definition of the exponential homomorphism in [178, 5.5.9] is completely algebraic: we start from the homomorphism

$$s \;:\; \mathbf{RO}^+(G) \;\longrightarrow\; \mathrm{Cl}(G,\mathbb{Z})^\times\,, \quad s[V](H) \;=\; (-1)^{\dim(V^H)}$$

that sends an orthogonal representation V to the class function $s(V)$ that records

the parities of the dimensions of the fixed-point spaces. The Burnside ring embeds into the ring of class functions by 'fixed-point counting'

$$\Phi \ : \ \mathbb{A}(G) \ \longrightarrow \ \mathrm{Cl}(G,\mathbb{Z}) \ , \ \Phi[S](H) \ = \ |S^H| \ ,$$

i.e., a (virtual) G-set S is sent to the class function that counts the number of fixed-points. This map is injective, and for finite groups the image can be characterized by an explicit system of congruences [178, Prop. 1.3.5]. The image of s satisfies the congruences, so there is a unique map $\exp : \mathbf{RO}^+(G) \longrightarrow \mathbb{A}(G)^\times$ such that $\Phi \circ \exp \ = \ s$. The map s sends the direct sum of representations to the product of the parity functions, so exp is a homomorphism of abelian monoids, and thus extends to a homomorphism from the orthogonal representation ring $\mathbf{RO}(G)$. The morphism of ultra-commutative monoids (2.3.28) realizes the exponential morphism in the sense that the following diagram of monoid homomorphisms commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{Gr}) \arrow[r, "\pi_0^G(J\circ\tau)"] \arrow[d, "(2.3.13)"', "\cong"] & \pi_0^G(\Omega^\bullet\mathbb{S}) \arrow[r, equal] & (\pi_0^G(\mathbb{S}))^\times \\
\mathbf{RO}^+(G) \arrow[rr, "\exp"'] & & \mathbb{A}(G)^\times \arrow[u, "\alpha", "\cong"']
\end{tikzcd}$$

To show the commutativity of this diagram we may compose with the degree monomorphism

$$\deg \ : \ (\pi_0^G(\mathbb{S}))^\times \longrightarrow \mathrm{Cl}(G,\mathbb{Z}) \ , \quad \deg[f](H) \ = \ \deg(f^H : S^{V^H} \longrightarrow S^{V^H})$$

that takes the class of an equivariant self-map $f : S^V \longrightarrow S^V$ of a representation sphere to the class function that records the fixed-point dimensions. The composite $\deg\circ\alpha : \mathbb{A}(G) \longrightarrow \mathrm{Cl}(G,\mathbb{Z})$ coincides with the fixed-point counting map Φ, so

$$\deg\circ\alpha\circ\exp \ = \ \Phi\circ\exp \ = \ s \ : \mathbf{RO}^+(G) \ \longrightarrow \ \mathrm{Cl}(G,\mathbb{Z}) \ .$$

On the other hand, the map $\pi_0^G(J\circ\tau)$ sends the class of a G-representation V to the involution $S^{-\mathrm{Id}_V} : S^V \longrightarrow S^V$. The diagram thus commutes because

$$\deg\left((S^{-\mathrm{Id}_V})^H\right) \ = \ (-1)^{\dim(V^H)} \ = \ s[V](H) \ .$$

2.4 Global forms of *BO*

In this section we discuss different orthogonal monoid spaces whose underlying non-equivariant homotopy type is BO, a classifying space for the infinite orthogonal group. Each example is interesting in its own right, and as a whole,

the global forms of BO are a great illustration of how non-equivariant homotopy types ‘fold up’ into many different global homotopy types. The different forms of BO have associated orthogonal Thom spectra with underlying non-equivariant stable homotopy type MO; we will return to these Thom spectra in Section 6.1. The examples we discuss here all come with multiplications, some ultra-commutative, but some only E_∞-commutative; so our case study also illustrates the different degrees of commutativity that arise ‘in nature’.

We can name five different global homotopy types that all have the same underlying non-equivariant homotopy type, namely that of a classifying space of the infinite orthogonal group:

- the constant orthogonal space $\underline{BO}$ with value a classifying space of the infinite orthogonal group;
- the ‘full Grassmannian model’ $\mathbf{BO}$, the degree 0 part of the periodic global Grassmannian $\mathbf{BOP}$ (Example 2.4.1);
- the bar construction model $\mathbf{B}^\circ\mathbf{O}$ (Construction 2.4.14);
- the ‘restricted Grassmannian model’ $\mathbf{bO}$ that is also a sequential homotopy colimit of the global classifying spaces $B_{\mathrm{gl}}O(n)$ (Example 2.4.18); and
- the cofree orthogonal space $R(BO)$ associated with a classifying space of the infinite orthogonal group (Construction 1.2.25).

These global homotopy types are related by weak morphisms of orthogonal spaces:

$$\begin{tikzcd} \underline{BO} \arrow[r] \arrow[d] & \mathbf{bO} \arrow[d] & \\ \mathbf{B}^\circ\mathbf{O} \arrow[r] & \mathbf{BO} \arrow[r] & R(BO) \end{tikzcd}$$

The orthogonal spaces $\mathbf{B}^\circ\mathbf{O}$ and $\mathbf{BO}$ come with ultra-commutative multiplications. The global homotopy type of $R(BO)$ also admits an ultra-commutative multiplication; we will not elaborate this point, but one way to see it is to extend the cofree functor R to a lax symmetric monoidal functor on the category of orthogonal spaces, so that $R(\mathbf{BO})$ is an ultra-commutative monoid within this global homotopy type. The orthogonal spaces $\underline{BO}$ and $\mathbf{bO}$ admit E_∞-multiplications; for $\underline{BO}$ this is a consequence of the non-equivariant E_∞-structure of BO. All weak morphisms above can be arranged to preserve the E_∞-multiplications, so they induce additive maps of abelian monoids on π_0^G for every compact Lie group G.

As we explain in Example 2.4.17, the bar construction model makes sense more generally for monoid-valued orthogonal spaces; in particular, applying the bar construction to the ultra-commutative monoids made from the families of classical Lie groups discussed in the previous section provides ultra-

commutative monoids $\mathbf{B}^\circ\mathbf{SO}$, $\mathbf{B}^\circ\mathbf{U}$, $\mathbf{B}^\circ\mathbf{SU}$ $\mathbf{B}^\circ\mathbf{Sp}$, $\mathbf{B}^\circ\mathbf{Spin}$ and $\mathbf{B}^\circ\mathbf{Spin}^c$. Example 2.4.33 introduces the complex and quaternionic analogs of **BO** and **bO**, i.e., the ultra-commutative monoids **BU** and **BSp** and the E_∞-orthogonal monoid spaces **bU** and **bSp**.

Example 2.4.1 (Periodic Grassmannian)**.** We define an ultra-commutative monoid **BOP** that is a global refinement of the non-equivariant homotopy type $\mathbb{Z}\times BO$, and at the same time a global group completion of the additive Grassmannian **Gr** introduced in Example 2.3.12. The orthogonal space **BOP** comes with tautological vector bundles whose Thom spaces form the periodic Thom spectrum **MOP**, discussed in Example 6.1.7 below.

For an inner product space V we set

$$\mathbf{BOP}(V) \ = \ \coprod\nolimits_{m\geq 0} Gr_m(V^2)\ ,$$

the disjoint union of the Grassmannians of m-dimensional subspaces in $V^2 = V\oplus V$. The structure map associated with a linear isometric embedding $\varphi : V \longrightarrow W$ is given by

$$\mathbf{BOP}(\varphi)(L) \ = \ \varphi^2(L) \ + \ ((W-\varphi(V))\oplus 0)\ ,$$

the internal orthogonal sum of the image of L under $\varphi^2 : V^2 \longrightarrow W^2$ and the orthogonal complement of the image of $\varphi : V \longrightarrow W$, viewed as sitting in the first summand of $W^2 = W\oplus W$. In particular, the orthogonal group $O(V)$ acts on $\mathbf{BOP}(V)$ through its diagonal action on V^2.

We make the orthogonal space **BOP** an ultra-commutative monoid by endowing it with multiplication maps

$$\mu_{V,W} \ : \ \mathbf{BOP}(V)\times\mathbf{BOP}(W) \ \longrightarrow \ \mathbf{BOP}(V\oplus W)\ , \quad (L,L') \ \longmapsto \ \kappa_{V,W}(L\oplus L')\ ,$$

where

$$\kappa_{V,W} \ : \ V^2\oplus W^2 \ \cong \ (V\oplus W)^2 \quad \text{is defined by} \quad \kappa_{V,W}(v,v',w,w') = (v,w,v',w')\ .$$

The unit is the unique element $\{0\}$ of $\mathbf{BOP}(0)$.

The orthogonal space **BOP** is naturally $\mathbb{Z}$-graded: for $m\in\mathbb{Z}$ we let

$$\mathbf{BOP}^{[m]}(V) \ \subset \ \mathbf{BOP}(V)$$

be the path component consisting of all subspaces $L\subset V^2$ such that $\dim(L) = \dim(V)+m$. For fixed m the spaces $\mathbf{BOP}^{[m]}(V)$ form a subfunctor of **BOP**, i.e., $\mathbf{BOP}^{[m]}$ is an orthogonal subspace of **BOP**. The multiplication is graded in the sense that $\mu_{V,W}$ takes $\mathbf{BOP}^{[m]}(V)\times\mathbf{BOP}^{[n]}(W)$ to $\mathbf{BOP}^{[m+n]}(V\oplus W)$. We write $\mathbf{BO} = \mathbf{BOP}^{[0]}$ for the homogeneous summand of **BOP** of degree 0, which is thus an ultra-commutative monoid in its own right. The underlying non-equivariant homotopy type of **BO** is that of a classifying space of the infinite orthogonal group.

While **BOP** and the additive Grassmannian **Gr** are both made from Grassmannians, one should beware of the different nature of their structure maps. There is a variation $\mathbf{Gr}'$ of the additive Grassmannian with values $\mathbf{Gr}'(V) = \coprod_{n\geq 0} Gr_n(V^2)$ and structure maps $\mathbf{Gr}'(\varphi)(L) = \varphi^2(L)$. This orthogonal space is a 'multiplicative shift' of **Gr** in the sense of Example 1.1.11, it admits a commutative multiplication in much the same way as **Gr**, and the maps

$$\mathbf{Gr}(V) \longrightarrow \mathbf{Gr}'(V) , \quad L \longmapsto L \oplus 0$$

form a global equivalence of ultra-commutative monoids by Theorem 1.1.10. A source of possible confusion is the fact that $\mathbf{Gr}'(V)$ and $\mathbf{BOP}(V)$ are *equal* as spaces, but they come with very different structure maps making them into two different global homotopy types.

Example 2.4.2 (Gr versus BOP). In Example 2.3.12 we explained that for every compact Lie group G, the monoid $\pi_0^G(\mathbf{Gr})$ is isomorphic to the monoid $\mathbf{RO}^+(G)$, under direct sum, of isomorphism classes of orthogonal G-representations. In Theorem 2.4.13 we will identify the monoid $\pi_0^G(\mathbf{BOP})$ with the orthogonal representation ring $\mathbf{RO}(G)$. The latter is the algebraic group completion of the former, and this group completion is realized by a morphism of ultra-commutative monoids

$$i \; : \; \mathbf{Gr} \longrightarrow \mathbf{BOP} . \tag{2.4.3}$$

The morphism i is given at V by the map

$$\mathbf{Gr}(V) \; = \; \coprod\nolimits_{m\geq 0} Gr_m(V) \; \longrightarrow \; \coprod\nolimits_{n\geq 0} Gr_n(V^2) = \mathbf{BOP}(V) , \quad L \longmapsto V \oplus L .$$

The morphism is homogeneous in that it takes $\mathbf{Gr}^{[m]}$ to $\mathbf{BOP}^{[m]}$.

As we will now show, the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ induces an algebraic group completion of abelian monoids upon taking equivariant homotopy sets from any equivariant space. This fact is the algebraic shadow of a more refined relationship: as we will show in Theorem 2.5.33 below, the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is a group completion in the world of ultra-commutative monoids, i.e., 'homotopy initial', in the category of ultra-commutative monoids, among morphisms from **Gr** to group-like ultra-commutative monoids.

We recall from Definition 1.5.1 the equivariant homotopy set

$$[A, R]^G \; = \; \operatorname{colim}_{V\in s(\mathcal{U}_G)} [A, R(V)]^G ,$$

where R is an orthogonal space, G is a compact Lie group and A a G-space. If R is an ultra-commutative monoid, then this set inherits an abelian monoid structure defined as follows. We let $\alpha : A \longrightarrow R(V)$ and $\beta : A \longrightarrow R(W)$ be two G-maps that represent classes in $[A, R]^G$. Then their sum is defined as

$$[\alpha] + [\beta] \; = \; [\mu_{V,W}(\alpha, \beta)] , \tag{2.4.4}$$

where $\mu_{V,W} : R(V) \times R(W) \longrightarrow R(V \oplus W)$ is the (V, W)-component of the multiplication of R. The monoid structure is contravariantly functorial for G-maps in A, and covariantly functorial for morphisms of ultra-commutative monoids in R. When $A = *$ is a one-point G-space, then $[A, R]^G$ becomes $\pi_0^G(R)$, and the addition (2.4.4) reduces to the addition as previously defined in (2.2.1).

Proposition 2.4.5. *For every compact Lie group G and every G-space A, the homomorphism*

$$[A, i]^G \ : \ [A, \mathbf{Gr}]^G \ \longrightarrow \ [A, \mathbf{BOP}]^G$$

is a group completion of abelian monoids.

Proof We start by showing that the abelian monoid $[A, \mathbf{BOP}]^G$ is a group. We consider a G-representation V. For a linear subspace $L \subseteq V^2$ we consider the 1-parameter family of linear isometric embeddings

$$H_L \ : \ [0, 1] \times L^\perp \ \longrightarrow \ V^2 \oplus V^2 \ , \quad (t, x) \ \longmapsto \ (t \cdot x, \ \sqrt{1 - t^2} \cdot x) \ .$$

For every $t \in [0, 1]$, the image of $H_L(t, -)$ is isomorphic to $L^\perp$ and orthogonal to the space $L \oplus 0 \oplus 0$. We can thus define a G-equivariant homotopy

$$K \ : \ [0, 1] \times Gr(V^2) \ \longrightarrow \ Gr(V^2 \oplus V^2) \quad \text{by} \quad K(t, L) \ = \ (L \oplus 0 \oplus 0) + H_L(t, L^\perp) \ .$$

Then

$$K(0, L) \ = \ (L \oplus 0 \oplus 0) + H_L(0, L^\perp) \ = \ (L \oplus 0) + (0 \oplus L^\perp) \ = \ L \oplus L^\perp$$

and

$$\begin{aligned} K(1, L) \ &= \ (L \oplus 0 \oplus 0) + H_L(1, L^\perp) \\ &= \ (L \oplus 0 \oplus 0) + (L^\perp \oplus 0 \oplus 0) \ = \ V \oplus V \oplus 0 \oplus 0 \ . \end{aligned}$$

We recall that the multiplication of **BOP** is given by

$$\mu_{V,V}^{\mathbf{BOP}} \ : \ \mathbf{BOP}(V) \times \mathbf{BOP}(V) \ \longrightarrow \mathbf{BOP}(V \oplus V) \ , \quad \mu_{V,V}^{\mathbf{BOP}}(L, L') \ = \ \kappa_{V,V}(L \oplus L') \ ,$$

where $\kappa_{V,V}(v, v', w, w') = (v, w, v', w')$. Therefore the equivariant homotopy $Gr(\kappa_{V,V}) \circ K$ interpolates between the composite

$$\mathbf{BOP}(V) \xrightarrow{(\mathrm{Id}, (-)^\perp)} \mathbf{BOP}(V) \times \mathbf{BOP}(V) \xrightarrow{\mu_{V,V}^{\mathbf{BOP}}} \mathbf{BOP}(V \oplus V)$$

and the constant map with value

$$\kappa_{V,V}(V \oplus V \oplus 0 \oplus 0) \ = \ V \oplus 0 \oplus V \oplus 0 \ .$$

The subspace $V \oplus 0 \oplus V \oplus 0$ lies in the same path component of $\mathbf{BOP}(V^2)^G = (Gr(V^2 \oplus V^2))^G$ as the subspace $V \oplus V \oplus 0 \oplus 0$. So altogether this shows that the composite $\mu_{V,V}^{\mathbf{BOP}} \circ (\mathrm{Id}, (-)^\perp)$ is G-equivariantly homotopic to the constant map with value $V \oplus V \oplus 0 \oplus 0$.

Let $\alpha : A \longrightarrow \mathbf{BOP}(V)$ be a G-map, representing a class in $[A, \mathbf{BOP}]^G$. The composite of α and the orthogonal complement map $(-)^\perp : \mathbf{BOP}(V) \longrightarrow \mathbf{BOP}(V)$ represents another class in $[A, \mathbf{BOP}]^G$, and

$$[\alpha] + [(-)^\perp \circ \alpha] \ = \ [\mu^{\mathbf{BOP}}_{V,V} \circ (\mathrm{Id}, (-)^\perp) \circ \alpha] \ = \ [c_{V\oplus V\oplus 0\oplus 0} \circ \alpha] \ = \ 0$$

in the monoid structure of $[A, \mathbf{BOP}]^G$, because the subspace $V \oplus V \oplus 0 \oplus 0$ is the neutral element in $\mathbf{BOP}(V \oplus V)$. So the class $[\alpha]$ has an additive inverse, and this concludes the proof that the abelian monoid $[A, \mathbf{BOP}]^G$ is a group.

To show that the homomorphism $[A, i]^G$ is a group completion we show two separate statements that amount to the surjectivity, and injectivity, respectively, of the extension of $[A, i]^G$ to a homomorphism on the Grothendieck group of the monoid $[A, \mathbf{Gr}]^G$.

(a) We show that every class in $[A, \mathbf{BOP}]^G$ is the difference of two classes in the image of $i_* = [A, i]^G : [A, \mathbf{Gr}]^G \longrightarrow [A, \mathbf{BOP}]^G$. To see this, we represent a given class $x \in [A, \mathbf{BOP}]^G$ by a G-map $\alpha : A \longrightarrow \mathbf{BOP}(V)$, for some G-representation V. Because $\mathbf{BOP}(V) = \mathbf{Gr}(V \oplus V)$, the same map α also represents a class in $[A, \mathbf{Gr}]^G$; to emphasize the different role, we write this map as $\alpha^\sharp : A \longrightarrow \mathbf{Gr}(V \oplus V)$. We let $c_V : A \longrightarrow \mathbf{Gr}(V)$ denote the constant map with value V and $\chi : V^4 \longrightarrow V^4$ the linear isometry defined by

$$\chi(v_1, v_2, v_3, v_4) \ = \ (v_2, v_3, v_1, v_4) \ .$$

We observe that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{BOP}(V) \arrow[rr, "(i(V)\circ c_V, \mathrm{Id})"] \arrow[d, equal] & & \mathbf{BOP}(V) \times \mathbf{BOP}(V) \arrow[d, "\mu^{\mathbf{BOP}}_{V,V}"] \\
\mathbf{Gr}(V \oplus V) \arrow[r, "i(V\oplus V)"'] & \mathbf{BOP}(V \oplus V) \arrow[r, "Gr(\chi)"'] & \mathbf{BOP}(V \oplus V)
\end{tikzcd}$$

Since χ is equivariantly homotopic, through linear isometries, to the identity, this shows that the composite $\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, \mathrm{Id})$ is G-equivariantly homotopic to $i(V \oplus V)$. Thus

$$\begin{aligned} i_*[c_V] + x \ &= \ [\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, \alpha)] \\ &= \ [\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, \mathrm{Id}) \circ \alpha] \ = \ [i(V \oplus V) \circ \alpha^\sharp] \ = \ i_*[\alpha^\sharp] \ . \end{aligned}$$

Thus $x = i_*[\alpha^\sharp] - i_*[c_V]$, which shows the claim.

(b) Now we consider two classes $a, b \in [A, \mathbf{Gr}]^G$ such that $i_*(a) = i_*(b)$ in $[A, \mathbf{BOP}]^G$. We show that there exist another class $c \in [A, \mathbf{Gr}]^G$ such that $c + a = c + b$. We can represent a and b by two G-maps $\alpha : A \longrightarrow \mathbf{Gr}(V)$ and $\beta : A \longrightarrow \mathbf{Gr}(V)$ such that the two composites

$$i(V) \circ \alpha \ , \quad i(V) \circ \beta \ : \ A \ \longrightarrow \ \mathbf{BOP}(V)$$

are equivariantly homotopic. As before we let $c_V : A \longrightarrow \mathbf{Gr}(V)$ be the constant map with value V. The map $i(V) : \mathbf{Gr}(V) \longrightarrow \mathbf{BOP}(V) = \mathbf{Gr}(V \oplus V)$ factors as the composite

$$\mathbf{Gr}(V) \xrightarrow{(c_V,\mathrm{Id})} \mathbf{Gr}(V) \times \mathbf{Gr}(V) \xrightarrow{\mu^{\mathbf{Gr}}_{V,V}} \mathbf{Gr}(V \oplus V) ,$$

so

$$\begin{aligned}[c_V] + a \ &= \ [c_V] + [\alpha] \ = \ [\mu^{\mathbf{Gr}}_{V,V} \circ (c_V, \alpha)] \\ &= \ [\mu_{V,V} \circ (c_V, \mathrm{Id}) \circ \alpha] \ = \ [i(V) \circ \alpha] \ .\end{aligned}$$

Similarly, $[c_V]+b = [i(V)\circ\beta]$. So $[c_V]+a = [c_V]+b$ in $[A, \mathbf{Gr}]^G$, as claimed. □

Our next aim is to show that the ultra-commutative monoid **Gr** represents equivariant vector bundles, and **BOP** represents equivariant K-theory, at least for compact G-spaces.

Construction 2.4.6. We let G be a compact Lie group and A a G-space. We recall that a *G-vector bundle* over A consists of a vector bundle $\xi : E \longrightarrow A$ and a continuous G-action on the total space E such that

- the bundle projection $\xi : E \longrightarrow A$ is a G-map,
- for every $g \in G$ and $a \in A$ the map $g \cdot - : E_a \longrightarrow E_{ga}$ is $\mathbb{R}$-linear.

We let $\mathrm{Vect}_G(A)$ be the commutative monoid, under Whitney sum, of isomorphism classes of G-vector bundles over A. We define a homomorphism of monoids

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, \ \mathbf{Gr}(V)]^G \ \longrightarrow \ \mathrm{Vect}_G(A) \qquad (2.4.7)$$

that will turn out to be an isomorphism for compact A and that specializes to the isomorphism (2.3.13) from $\pi_0^G(\mathbf{Gr})$ to $\mathbf{RO}^+(G)$ when A is a one-point G-space. We let $f : A \longrightarrow \mathbf{Gr}(V)$ be a continuous G-map, for some G-representation V. We pull back the tautological G-vector bundle γ_V over $\mathbf{Gr}(V)$ and obtain a G-vector bundle $f^\star(\gamma_V) : E \longrightarrow A$ over A with total space

$$E \ = \ \{(v, a) \in V \times A \mid v \in f(a)\} \ .$$

The G-action and bundle structure are as a G-subbundle of the trivial bundle $V \times A$. Since the base $\mathbf{Gr}(V)$ of the tautological bundle is a disjoint union of compact spaces, the isomorphism class of the bundle $f^\star(\gamma_V)$ depends only on the G-homotopy class of f, see for example [151, Prop. 1.3]. So the construction yields a well-defined map

$$[A, \ \mathbf{Gr}(V)]^G \ \longrightarrow \ \mathrm{Vect}_G(A) \ , \quad [f] \ \longmapsto \ [f^\star(\gamma_V)] \ .$$

If $\varphi : V \longrightarrow W$ is a linear isometric embedding of G-representations, then the

restriction along $\mathbf{Gr}(\varphi) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(W)$ of the tautological G-vector bundle over $\mathbf{Gr}(W)$ is isomorphic to the tautological G-vector bundle over $\mathbf{Gr}(V)$. So the two G-vector bundles $f^\star(\gamma_V)$ and $(\mathbf{Gr}(\varphi) \circ f)^\star(\gamma_W)$ over A are isomorphic. We can thus pass to the colimit over the poset $s(\mathcal{U}_G)$ and get a well-defined map (2.4.7). The map (2.4.7) is a monoid homomorphism because all additions in sight arise from direct sum of inner product spaces.

Now we 'group complete' the picture. We denote by $\mathbf{KO}_G(A)$ the G-equivariant K-group of A, i.e., the group completion (Grothendieck group) of the abelian monoid $\mathrm{Vect}_G(A)$. The composite

$$[A, \mathbf{Gr}]^G \xrightarrow{\langle - \rangle} \mathrm{Vect}_G(A) \longrightarrow \mathbf{KO}_G(A)$$

of (2.4.7) and the group completion map is a monoid homomorphism into an abelian group. The morphism $[A, i]^G : [A, \mathbf{Gr}]^G \longrightarrow [A, \mathbf{BOP}]^G$ is a group completion of abelian monoids by Proposition 2.4.5, where $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ was defined in (2.4.3). So there is a unique homomorphism of abelian groups

$$\langle - \rangle \ : \ [A, \mathbf{BOP}]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, \ \mathbf{BOP}(V)]^G \ \longrightarrow \ \mathbf{KO}_G(A) \qquad (2.4.8)$$

such that the following square commutes:

$$\begin{tikzcd}
{[A, \mathbf{Gr}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,i]^G}"'] & \mathrm{Vect}_G(A) \arrow[d] \\
{[A, \mathbf{BOP}]^G} \arrow[r, "\langle - \rangle"'] & \mathbf{KO}_G(A)
\end{tikzcd} \qquad (2.4.9)$$

We can make the homomorphism (2.4.8) more explicit as follows. We let $f : A \longrightarrow \mathbf{BOP}(V)$ be a G-map for some G-representation V. We pull back the tautological G-vector bundle over $\mathbf{BOP}(V) = Gr(V^2)$ and obtain a G-vector bundle $f^\star(\gamma_{V^2}) : E \longrightarrow A$ over A with total space

$$E \ = \ \{(v, a) \in V^2 \times A \mid v \in f(a)\} \ .$$

Again the G-action and bundle structure are as a G-subbundle of the trivial bundle $V^2 \times A$. The homomorphism (2.4.8) then sends the G-homotopy class of f to the class of the virtual G-vector bundle

$$[f^\star(\gamma_{V^2})] - [V \times A] \ \in \ \mathbf{KO}_G(A) \ .$$

In contrast to the definition of the earlier map (2.4.7), we now subtract the class of the trivial G-vector bundle $V \times A$ over A. The class in $\mathbf{KO}_G(A)$ only depends on the class of f in $[A, \mathbf{BOP}]^G$, and this recipe defines the map (2.4.8).

Theorem 2.4.10. *For every compact Lie group G and every compact G-space A the homomorphisms*

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^G \ \longrightarrow \ \mathrm{Vect}_G(A) \quad \textit{and} \quad \langle - \rangle \ : \ [A, \mathbf{BOP}]^G \ \longrightarrow \ \mathbf{KO}_G(A)$$

defined in (2.4.7) *and* (2.4.8)*, respectively, are isomorphisms. As G varies, the isomorphisms are compatible with restriction along continuous homomorphisms.*

Proof The Grassmannian $\mathbf{Gr}$ is the disjoint union of the homogeneous pieces $\mathbf{Gr}^{[n]}$, and the latter is isomorphic to the semifree orthogonal space $\mathbf{L}_{O(n),\mathbb{R}^n}$, via

$$\mathbf{L}(\mathbb{R}^n, V)/O(n) \ \longrightarrow \ \mathbf{Gr}^{[n]}(V) \ , \quad \varphi \cdot O(n) \ \longmapsto \ \varphi(\mathbb{R}^n) \ .$$

Since the tautological action of $O(n)$ on $\mathbb{R}^n$ is faithful, $\mathbf{L}_{O(n),\mathbb{R}^n}$ is a global classifying space for $O(n)$; Example 1.5.4 thus provides a bijection

$$[A, \mathbf{Gr}^{[n]}]^G \ \longrightarrow \ \mathrm{Prin}_{(G,O(n))}(A)$$

to the set of isomorphism classes of G-equivariant principal $O(n)$-bundles over A, by pulling back the $(G, O(n))$-principal bundle $\mathbf{L}(\mathbb{R}^n, V) \longrightarrow \mathbf{Gr}^{[n]}(V)$, the frame bundle of the tautological vector bundle over $\mathbf{Gr}^{[n]}(V)$. On the other hand, we can consider the map

$$\mathrm{Prin}_{(G,O(n))}(A) \ \longrightarrow \ \mathrm{Vect}_G^{[n]}(A)$$

to the set of isomorphism classes of G-vector bundles of rank n over A, sending a $(G, O(n))$-bundle $\gamma : E \longrightarrow A$ to the associated G-vector bundle with total space $E \times_{O(n)} \mathbb{R}^n$. Since A is compact, every G-vector bundle admits a G-invariant euclidean inner product, so it arises from a $(G, O(n))$-bundle; hence the latter map is bijective as well. Altogether this shows that map

$$[A, \mathbf{Gr}^{[n]}]^G \ \longrightarrow \ \mathrm{Vect}_G^{[n]}(A) \tag{2.4.11}$$

given by pulling back the tautological vector bundles is bijective.

A general G-vector bundle need not have constant rank, so it remains to assemble the results for varying n. We let ξ be any G-vector bundle over A, not necessarily of constant rank. Then the subset

$$A_{(n)} \ = \ \{a \in A \mid \dim(\xi_a) = n\}$$

of points over which ξ is n-dimensional is open by local triviality of vector bundles. So A is the disjoint union of the sets $A_{(n)}$ for $n \geq 0$, and each subset $A_{(n)}$ is also closed and hence compact in the subspace topology. Moreover, $A_{(n)}$ is G-invariant, so the restriction $\xi_{(n)}$ of the bundle to $A_{(n)}$ is classified by a G-map $f_{(n)} : A_{(n)} \longrightarrow \mathbf{Gr}^{[n]}(V_n)$ for some finite-dimensional G-representation

V_n. Since A is compact, almost all $A_{(n)}$ are empty, so by increasing the representations, if necessary, we can assume that the classifying maps have target in $\mathbf{Gr}^{[n]}(V)$ for a fixed finite-dimensional G-representation V, independent of n. Then

$$\amalg_{n\geq 0} f_{(n)} \ : \ \amalg_{n\geq 0} A_{(n)} \ = \ A \ \longrightarrow \ \amalg_{n\geq 0} \mathbf{Gr}^{[n]}(V) \ = \ \mathbf{Gr}(V)$$

is a classifying G-map for the original bundle ξ. This shows that the map (2.4.7) is surjective.

The argument for injectivity is similar. Any pair of classes in $[A, \mathbf{Gr}]^G$ can be represented by G-maps $f, \bar{f} : A \longrightarrow \mathbf{Gr}(V)$ for some finite-dimensional G-representation V. Since $\mathbf{Gr}(V)$ is the disjoint union of the subspaces $\mathbf{Gr}^{[n]}(V)$ for $n \geq 0$, their inverse images under f and $\bar{f}$ provide disjoint union decompositions of A by fiber dimension. If the bundles $f^\star(\gamma_V)$ and $\bar{f}^\star(\gamma_V)$ are isomorphic, the decompositions of A induced by f and $\bar{f}$ must be the same. The rank n summands $f_{(n)}, \bar{f}_{(n)} : A_{(n)} \longrightarrow \mathbf{Gr}^{[n]}(V)$ become equivariantly homotopic after increasing the representation V, because the map (2.4.11) is injective. Moreover, almost all summands are empty, once again by compactness. So there is a single finite-dimensional representation W and a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ such that $f, \bar{f} : A \longrightarrow \mathbf{Gr}(V)$ become equivariantly homotopic after composition with $\mathbf{Gr}(\varphi) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(W)$. Hence f and $\bar{f}$ represent the same class in $[A, \mathbf{Gr}]^G$, so the map (2.4.7) is injective. This completes the proof that the map is an isomorphism for compact A.

The left vertical map in the commutative square (2.4.9) is a group completion by Proposition 2.4.5, and the right vertical map is a group completion by definition. So the lower horizontal map (2.4.8) is also an isomorphism. □

We take the time to specialize Theorem 2.4.10 to the one-point G-space. This special case identifies the global power monoid $\underline{\pi}_0(\mathbf{BOP})$ with the global power monoid $\mathbf{RO}$ of orthogonal representation rings. For every compact Lie group G the abelian monoid $\mathbf{RO}(G)$ is the Grothendieck group, under direct sum, of finite-dimensional G-representations. The restriction maps are induced by restriction of representations, and the power operation $[m] : \mathbf{RO}(G) \longrightarrow \mathbf{RO}(\Sigma_m \wr G)$ takes the class of a G-representation V to the class of the $(\Sigma_m \wr G)$-representation V^m. The resulting transfer $\mathrm{tr}_H^G : \mathbf{RO}(H) \longrightarrow \mathbf{RO}(G)$ of Construction 2.2.29, for H of finite index in G, is then the transfer (or induction), sending the class of an H-representation V to the class of the induced G-representation $\mathrm{map}^H(G, V)$. A G-vector bundle over a one-point space 'is' a G-representation and the map

$$\mathbf{RO}(G) \ \longrightarrow \ \mathbf{KO}_G(*)$$

that considers a (virtual) representation as a (virtual) vector bundle is an iso-

morphism of groups and compatible with restriction along continuous homomorphisms of compact Lie groups.

For easier reference we spell out the isomorphism (2.4.8) in the special case $A = *$ more explicitly. We let V be a finite-dimensional orthogonal G-representation. The G-fixed-points of $\mathbf{BOP}(V)$ are the G-invariant subspaces of V^2, i.e., the G-subrepresentations W of V^2. Representations of compact Lie groups are discrete (compare the example after [151, Prop. 1.3]), so two fixed-points in the same path component of $\mathbf{BOP}(V)^G$ are isomorphic as G-representations. Hence we obtain a well-defined map

$$\pi_0(\mathbf{BOP}(V)^G) \longrightarrow \mathbf{RO}(G)$$

by sending $W \in \mathbf{BOP}(V)^G$ to $[W] - [V]$, the formal difference in $\mathbf{RO}(G)$ of the classes of W and V. These maps are compatible as V runs through the finite-dimensional G-subrepresentations of $\mathcal{U}_G$, so they assemble into a map

$$\pi_0^G(\mathbf{BOP}) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} \pi_0(\mathbf{BOP}(V)^G) \longrightarrow \mathbf{RO}(G) . \qquad (2.4.12)$$

Theorem 2.4.13. *For every compact Lie group G the map* (2.4.12) *is an isomorphism of groups. As G varies, these isomorphisms form an isomorphism of global power monoids*

$$\underline{\pi}_0(\mathbf{BOP}) \cong \mathbf{RO} .$$

Proof The special case $A = *$ of Theorem 2.4.10 shows that the map (2.4.12) is an isomorphism and compatible with restriction along continuous homomorphisms. We have to argue that in addition, the maps (2.4.12) are also compatible with transfers (or equivalently, with power operations). The compatibility with transfers can either be deduced directly from the definitions; equivalently it can be formally deduced from the compatibility of the isomorphisms $\underline{\pi}_0(\mathbf{Gr}) \cong \mathbf{RO}^+$ with transfers by the universal property of a group completion. □

The bijection (2.4.12) sends elements of $\pi_0^G(\mathbf{BOP}^{[k]})$ to virtual representations of dimension k, so we can also identify the global power monoid of the homogeneous degree 0 part $\mathbf{BO} = \mathbf{BOP}^{[0]}$. Indeed, the map (2.4.12) restricts to an isomorphism of abelian groups

$$\pi_0^G(\mathbf{BO}) \cong \mathbf{IO}(G)$$

to the augmentation ideal $\mathbf{IO}(G) \subset \mathbf{RO}(G)$ of the orthogonal representation ring, compatible with restriction maps, power operations and transfer maps.

Example 2.4.14 (Bar construction model $\mathbf{B}^\circ\mathbf{O}$)**.** Using the functorial bar construction we define another global refinement $\mathbf{B}^\circ\mathbf{O}$ of the classifying space of

the infinite orthogonal group. This ultra-commutative monoid is globally connected, and it comes with a weak homomorphism to **BO** that 'picks out' the path components of the neutral element in the G-fixed-point spaces $\mathbf{BO}(\mathcal{U}_G)^G$.

We define $\mathbf{B}^\circ\mathbf{O}$ by applying the bar construction (see Construction 2.3.21) objectwise to the monoid-valued orthogonal space **O** of Example 2.3.6. So the value at an inner product space V is

$$(\mathbf{B}^\circ\mathbf{O})(V) \;=\; B(O(V)) ,$$

the bar construction of the orthogonal group of V. The structure map of a linear isometric embedding $\varphi : V \longrightarrow W$ is obtained by applying the bar construction to the continuous homomorphism $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$. We make $\mathbf{B}^\circ\mathbf{O}$ an ultra-commutative monoid by endowing it with multiplication maps

$$\mu_{V,W} \;:\; (\mathbf{B}^\circ\mathbf{O})(V) \times (\mathbf{B}^\circ\mathbf{O})(W) \;\longrightarrow\; (\mathbf{B}^\circ\mathbf{O})(V \oplus W)$$

defined as the composite

$$B(O(V)) \times B(O(W)) \;\xrightarrow{\cong}\; B(O(V) \times O(W)) \;\xrightarrow{B\oplus}\; B(O(V \oplus W)) ,$$

where the first map is inverse to the homeomorphism (2.3.22).

Now we let G be a compact Lie group and V an orthogonal G-representation. Taking fixed-points commutes with geometric realization (see Proposition B.1 (iv)) and with products, so

$$((\mathbf{B}^\circ\mathbf{O})(V))^G \;=\; |O(V)^\bullet|^G \;\cong\; |(O(V)^G)^\bullet| \;=\; B(O^G(V)) .$$

Taking colimit over the poset $s(\mathcal{U}_G)$ gives

$$((\mathbf{B}^\circ\mathbf{O})(\mathcal{U}_G))^G \;\cong\; \operatorname{colim}_{V \in s(\mathcal{U}_G)} B(O^G(V)) \;\cong\; B(O^G(\mathcal{U}_G)) \;\cong\; {\prod_{[\lambda]}}' B(O^G(\mathcal{U}_\lambda)) .$$

Here the last weak product is indexed by isomorphism classes of irreducible G-representations, and each of the groups $O^G(\mathcal{U}_\lambda)$ is either an infinite orthogonal, unitary or symplectic group, depending on the type of the irreducible representation, compare Example 2.3.6. In particular, the space $((\mathbf{B}^\circ\mathbf{O})(\mathcal{U}_G))^G$ is connected, so the equivariant homotopy set $\pi_0^G(\mathbf{B}^\circ\mathbf{O})$ has one element for every compact Lie group G; the global power monoid structure is then necessarily trivial. In particular, $\mathbf{B}^\circ\mathbf{O}$ is *not* globally equivalent to **BO**.

However, the difference seen by $\underline{\pi}_0$ is the only difference between $\mathbf{B}^\circ\mathbf{O}$ and **BO**, as we shall now explain. We construct a weak morphism of ultra-commutative monoids that exhibits $\mathbf{B}^\circ\mathbf{O}$ as the 'globally connected component' of **BO**. We define an ultra-commutative monoid $\mathbf{B}'\mathbf{O}$ by combining the constructions of $\mathbf{B}^\circ\mathbf{O}$ (bar construction) and **BO** (Grassmannians) into one definition. The value of $\mathbf{B}'\mathbf{O}$ at an inner product space V is

$$(\mathbf{B}'\mathbf{O})(V) \;=\; |B_\bullet(\mathbf{L}(V, V^2), \mathbf{O}(V), *)| ,$$

the two-sided bar construction (homotopy orbit construction) of the right $O(V)$-action on the space $\mathbf{L}(V, V^2)$ by pre-composition. Here $B_\bullet(\mathbf{L}(V, V^2), \mathbf{O}(V), *)$ is the simplicial space whose space of n-simplices is $\mathbf{L}(V, V^2) \times \mathbf{O}(V)^n$. For $n \geq 1$ and $0 \leq i \leq n$, the face map d_i^* is given by

$$d_i^*(\varphi, A_1, \dots A_n) = \begin{cases} (\varphi \circ A_1, A_2, \dots, A_n) & \text{for } i = 0, \\ (\varphi, A_1, \dots, A_{i-1}, A_i \circ A_{i+1}, A_{i+2}, \dots, A_n) & \text{for } 0 < i < n, \\ (\varphi, A_1, \dots, A_{n-1}) & \text{for } i = n. \end{cases}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ the degeneracy map s_i^* is given by

$$s_i^*(\varphi, A_1, \dots, A_{n-1}) = (\varphi, A_1, \dots, A_i, \mathrm{Id}, A_{i+1}, \dots, A_{n-1}) .$$

Then $(\mathbf{B'O})(V)$ is the realization of the simplicial space $B_\bullet(\mathbf{L}(V, V^2), \mathbf{O}(V), *)$.

To define the structure map associated with a linear isometric embedding $\varphi : V \longrightarrow W$ we recall that the structure map $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$ of the orthogonal space $\mathbf{O}$ is given by conjugation by φ and direct sum with the identity on $W - \varphi(V)$. We define a continuous map

$$\varphi_\sharp : \mathbf{L}(V, V^2) \longrightarrow \mathbf{L}(W, W^2)$$

by

$$(\varphi_\sharp\psi)(\varphi(v) + w) = \varphi^2(\psi(v) + (w, 0)) ;$$

here $v \in V$ and $w \in W - \varphi(V)$ is orthogonal to $\varphi(V)$. The map $\varphi_\sharp$ is compatible with the actions of the orthogonal groups, i.e., the following square commutes:

$$\begin{tikzcd} \mathbf{L}(V, V^2) \times \mathbf{O}(V) \arrow[r, "\varphi_\sharp \times \mathbf{O}(\varphi)"] \arrow[d, "\circ"'] & \mathbf{L}(W, W^2) \times \mathbf{O}(W) \arrow[d, "\circ"] \\ \mathbf{L}(V, V^2) \arrow[r, "\varphi_\sharp"'] & \mathbf{L}(W, W^2) \end{tikzcd}$$

This equivariance property of $\varphi_\sharp$ ensures that it passes to the two-sided bar construction, i.e., we can define the structure map $(\mathbf{B'O})(\varphi) : (\mathbf{B'O})(V) \longrightarrow (\mathbf{B'O})(W)$ as the geometric realization of the morphism of simplicial spaces

$$B_\bullet(\varphi_\sharp, \mathbf{O}(\varphi), *) : B_\bullet(\mathbf{L}(V, V^2), \mathbf{O}(V), *) \longrightarrow B_\bullet(\mathbf{L}(W, W^2), \mathbf{O}(W), *) .$$

A commutative multiplication

$$\mu_{V,W} : (\mathbf{B'O})(V) \times (\mathbf{B'O})(W) \longrightarrow (\mathbf{B'O})(V \oplus W)$$

is obtained by combining the multiplications of $\mathbf{B}^\circ\mathbf{O}$ and $\mathbf{BO}$. The construction

comes with two collections of continuous maps:

$$(\mathbf{B}^\circ\mathbf{O})(V) \xleftarrow{\alpha(V)} B(\mathbf{L}(V,V^2),\mathbf{O}(V),*) = (\mathbf{B}'\mathbf{O})(V)$$
$$\xrightarrow{\beta(V)} \mathbf{L}(V,V^2)/\mathbf{O}(V) = \mathbf{BO}(V)$$

The left map $\alpha(V)$ is defined by applying the bar construction to the unique map from $\mathbf{L}(V,V^2)$ to the one-point space. The right map $\beta(V)$ is the canonical map from homotopy orbits to strict orbits. As V varies, the α and β maps form morphisms of ultra-commutative monoids

$$\mathbf{B}^\circ\mathbf{O} \xleftarrow{\alpha} \mathbf{B}'\mathbf{O} \xrightarrow{\beta} \mathbf{BO}\,,$$

essentially by construction. As we shall now see, the morphism α is a global equivalence; so we can view the chain as a weak morphism of ultra-commutative monoids from $\mathbf{B}^\circ\mathbf{O}$ to $\mathbf{BO}$. The ultra-commutative monoid $\mathbf{B}^\circ\mathbf{O}$ is globally connected, whereas $\mathbf{BO}$ is not, so the morphism β cannot be a global equivalence. However, the second part of the next proposition shows that it is as close to a global equivalence as it can be.

Proposition 2.4.15. (i) *The morphism* $\alpha : \mathbf{B}'\mathbf{O} \longrightarrow \mathbf{B}^\circ\mathbf{O}$ *is a global equivalence of ultra-commutative monoids.*

(ii) *For every compact Lie group G, the morphism* $\beta : \mathbf{B}'\mathbf{O} \longrightarrow \mathbf{BO}$ *induces a weak equivalence from the G-fixed-point space* $(\mathbf{B}'\mathbf{O}(\mathcal{U}_G))^G$ *to the path component of the unit element in the G-fixed-point space* $(\mathbf{BO}(\mathcal{U}_G))^G$.

Proof (i) We let V be a representation of a compact Lie group G and compare the two-sided bar constructions for the $O(V)$-equivariant map from $\mathbf{L}(V,V^2)$ to the one-point space

$$\tilde\alpha(V)\ :\ |B_\bullet(\mathbf{L}(V,V^2),O(V),O(V))| \longrightarrow |B_\bullet(*,O(V),O(V))| = EO(V)\,.$$

The group G acts by conjugation on $\mathbf{L}(V,V^2)$ and on $O(V)$. The group $O(V)$ acts freely from the right on the last factor in the bar construction, and this right $O(V)$-action then commutes with the G-action. The map $\tilde\alpha(V)$ is $(G\times O(V))$-equivariant. The map $\alpha(V) : \mathbf{B}'\mathbf{O}(V) \longrightarrow \mathbf{BO}(V)$ is obtained from $\tilde\alpha(V)$ by passage to $O(V)$-orbits. So Proposition B.17 allows us to analyze and compare the G-fixed-points of $\alpha(V)$. Indeed, the proposition shows that the G-fixed-points of $\mathbf{B}'\mathbf{O}(V) = |B_\bullet(\mathbf{L}(V,V^2),O(V),O(V))|/O(V)$ are a disjoint union, indexed by conjugacy classes of continuous homomorphisms $\gamma : G \longrightarrow O(V)$ of the spaces

$$|B_\bullet(\mathbf{L}(V,V^2),O(V),O(V))|^{\Gamma(\gamma)}/C(\gamma)\,,$$

where $\Gamma(\gamma)$ is the graph of γ. Since fixed-points commute with geometric realization (see Proposition B.1 (iv)) and with products, we have

$$|B(\mathbf{L}(V, V^2), O(V), O(V))|^{\Gamma(\gamma)} = |B_\bullet(\mathbf{L}^G(V, V^2), O^G(V), \mathbf{L}^G(\gamma^*(V), V))| .$$

If $\gamma^*(V)$ and V are not isomorphic as G-representations, then $\mathbf{L}^G(\gamma^*(V), V)$ and hence also the bar construction is empty. So there is in fact only one summand in the disjoint union decomposition of $(\mathbf{B}'\mathbf{O}(V))^G$, namely the one indexed by the representation homomorphism $G \longrightarrow O(V)$ that specifies the given G-action on V. We conclude that the inclusions of G-fixed-points

$$\mathbf{L}^G(V, V^2) \longrightarrow \mathbf{L}(V, V^2) \qquad \text{and} \qquad O^G(V) \longrightarrow O(V)$$

induce a homeomorphism

$$|B_\bullet(\mathbf{L}^G(V, V^2), O^G(V), *)| \xrightarrow{\cong} |B_\bullet(\mathbf{L}(V, V^2), O(V), *)|^G = (\mathbf{B}'\mathbf{O}(V))^G .$$

The same argument identifies the G-fixed-points of the bar construction $B(O(V)) = \mathbf{B}^\circ\mathbf{O}(V)$, and we arrive at a commutative square

$$\begin{tikzcd}
|B_\bullet(\mathbf{L}^G(V, V^2), O^G(V), *)| \arrow[r] \arrow[d, "\cong"'] & B(O^G(V)) \arrow[d, "\cong"] \\
(\mathbf{B}'\mathbf{O}(V))^G \arrow[r, "\alpha(V)^G"'] & (\mathbf{B}^\circ\mathbf{O}(V))^G
\end{tikzcd}$$

in which both vertical maps are homeomorphisms. The space $\mathbf{L}^G(V, V^2)$ becomes arbitrarily highly connected as V exhausts the complete G-universe $\mathcal{U}_G$. So the upper horizontal quotient map also becomes arbitrarily highly connected as V grows. Hence the map $\alpha(V)^G$ becomes an equivalence

$$\mathrm{tel}_i\ \alpha(V_i)^G \ : \ \mathrm{tel}_i\,(\mathbf{B}'\mathbf{O}(V_i))^G \longrightarrow \mathrm{tel}_i\,(\mathbf{B}^\circ\mathbf{O}(V_i))^G$$

on the mapping telescopes over an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations. fixed-points commute with mapping telescopes, so we conclude that the map

$$\mathrm{tel}_i\ \alpha(V_i) \ : \ \mathrm{tel}_i\ \mathbf{B}'\mathbf{O}(V_i) \longrightarrow \mathrm{tel}_i\ \mathbf{B}^\circ\mathbf{O}(V_i)$$

is a G-weak equivalence. The mapping telescope criterion of Proposition 1.1.7 thus shows that the morphism $\alpha : \mathbf{B}'\mathbf{O} \longrightarrow \mathbf{B}^\circ\mathbf{O}$ is a global equivalence.

(ii) We let V be a G-representation, specified by a continuous homomorphism $\rho : G \longrightarrow O(V)$. We show that the map

$$\beta(V)^G \ : \ (\mathbf{B}'\mathbf{O}(V))^G = |B(\mathbf{L}(V, V^2), O(V), *)|^G \longrightarrow \mathbf{BO}(V)^G$$

is a weak equivalence of the source onto the path component of $\mathbf{BO}(V)^G$ that

contains the neutral element of the addition. The claim then follows by passing to colimits over V in $s(\mathcal{U}_G)$.

We showed in part (i) that the inclusions of G-fixed-points $\mathbf{L}^G(V, V^2) \longrightarrow \mathbf{L}(V, V^2)$ and $O^G(V) \longrightarrow O(V)$ induce a homeomorphism

$$|B_\bullet(\mathbf{L}^G(V, V^2), O^G(V), *)| \xrightarrow{\cong} |B_\bullet(\mathbf{L}(V, V^2), O(V), *)|^G = (\mathbf{B}'\mathbf{O}(V))^G .$$

Since the pre-composition action of $O^G(V)$ on $\mathbf{L}^G(V, V^2)$ is free, and $\mathbf{L}^G(V, V^2)$ is cofibrant as an $O^G(V)$-space, the homotopy orbits map by a weak equivalence to the strict orbits,

$$(\mathbf{B}'\mathbf{O}(V))^G = |B(\mathbf{L}(V, V^2), O(V), *)|^G \xrightarrow{\simeq} \mathbf{L}^G(V, V^2)/O^G(V) .$$

The map $\beta(V)^G$ factors as the composite

$$(\mathbf{B}'\mathbf{O}(V))^G \xrightarrow{\simeq} \mathbf{L}^G(V, V^2)/O^G(V) \longrightarrow \left(\mathbf{L}(V, V^2)/O(V)\right)^G = \mathbf{BO}(V)^G ,$$

where the second map is induced by the inclusion $\mathbf{L}^G(V, V^2) \longrightarrow \mathbf{L}(V, V^2)$. The space $\mathbf{BO}(V)^G$ is the Grassmannian of G-invariant subspaces of V^2 of the same dimension as V, and the space $\mathbf{L}^G(V, V^2)/O^G(V)$ consists of those subspaces that are G-isomorphic to V. This is precisely the path component of $\mathbf{BO}(V)^G$ containing the neutral element. □

We have now identified the G-equivariant path components of the three ultra-commutative monoids $\mathbf{B}^\circ\mathbf{O}$, $\mathbf{BO}$ and $\mathbf{BOP}$, and they are isomorphic to the trivial group, the augmentation ideal $\mathbf{IO}(G)$ and the real representation ring $\mathbf{RO}(G)$, respectively. Now we determine the entire homotopy types of the G-fixed-point spaces of the three ultra-commutative monoids $\mathbf{B}^\circ\mathbf{O}$, $\mathbf{BO}$ and $\mathbf{BOP}$.

Corollary 2.4.16. *Let G be a compact Lie group.*

(i) *The G-fixed-point space of* $\mathbf{B}^\circ\mathbf{O}$ *is a classifying space of the group* $O^G(\mathcal{U}_G)$ *of G-equivariant orthogonal isometries of the complete G-universe:*

$$(\mathbf{B}^\circ\mathbf{O}(\mathcal{U}_G))^G \simeq B(O^G(\mathcal{U}_G))$$

(ii) *The G-fixed-point space of* $\mathbf{BO}$ *is a disjoint union, indexed by the augmentation ideal* $\mathbf{IO}(G)$*, of classifying spaces of the group* $O^G(\mathcal{U}_G)$*:*

$$(\mathbf{BO}(\mathcal{U}_G))^G \simeq \mathbf{IO}(G) \times B(O^G(\mathcal{U}_G))$$

(iii) *The G-fixed-point space of* $\mathbf{BOP}$ *is a disjoint union, indexed by* $\mathbf{RO}(G)$*, of classifying spaces of the group* $O^G(\mathcal{U}_G)$*:*

$$(\mathbf{BOP}(\mathcal{U}_G))^G \simeq \mathbf{RO}(G) \times B(O^G(\mathcal{U}_G))$$

Proof (i) This is almost a tautology. Since G-fixed-points commute with geometric realization (see Proposition B.1 (iv)) and with products, they commute with the bar construction. So the space $(\mathbf{B}^\circ\mathbf{O}(V))^G$ is homeomorphic to $B(O^G(V))$ for every finite-dimensional G-representation V. Since G-fixed-points also commute with the filtered colimit at hand (see Proposition B.1 (ii)), we have

$$\begin{aligned}(\mathbf{B}^\circ\mathbf{O}(\mathcal{U}_G))^G &= (\mathrm{colim}_{V\in s(\mathcal{U}_G)}\,\mathbf{B}^\circ\mathbf{O}(V))^G\\ &\cong \mathrm{colim}_{V\in s(\mathcal{U}_G)}\,(\mathbf{B}^\circ\mathbf{O}(V))^G\\ &\cong \mathrm{colim}_{V\in s(\mathcal{U}_G)}\,B\Big(O^G(V)\Big)\\ &\cong B\Big(\mathrm{colim}_{V\in s(\mathcal{U}_G)}\,O^G(V)\Big) \;=\; B\Big(O^G(\mathcal{U}_G)\Big)\ .\end{aligned}$$

(ii) As we explained in Remark 2.1.2, the commutative multiplication of $\mathbf{BO}$ makes the G-space $\mathbf{BO}(\mathcal{U}_G)$ an E_∞-G-space; so the fixed-points $\mathbf{BO}(\mathcal{U}_G)^G$ come with the structure of a non-equivariant E_∞-space. The abelian monoid of path components $\pi_0(\mathbf{BO}(\mathcal{U}_G)^G)$ is isomorphic to $\pi_0^G(\mathbf{BO}) \cong \mathbf{IO}(G)$, hence an abelian group. So all path components of the space $\mathbf{BO}(\mathcal{U}_G)^G$ are homotopy equivalent. Proposition 2.4.15 identifies the zero path component of $\mathbf{BO}(\mathcal{U}_G)^G$ with $\mathbf{B}^\circ\mathbf{O}(\mathcal{U}_G)^G$, so part (i) finishes the proof.

The proof of part (iii) is the same as for part (ii), the only difference being that $\underline{\pi}_0(\mathbf{BOP}(\mathcal{U}_G)^G)$ is isomorphic to the abelian group $\mathbf{RO}(G)$. □

The fixed-point spaces described in Corollary 2.4.16 can be decomposed even further. As we explained in Example 2.3.6, the group $O^G(\mathcal{U}_G)$ is a weak product of infinite orthogonal, unitary and symplectic groups, indexed by the isomorphism classes of irreducible G-representations λ. The classifying space construction commutes with weak products, which gives a weak equivalence

$$B(O^G(\mathcal{U}_G)) \;\simeq\; \prod{}' B(O^G(\mathcal{U}_\lambda))\ .$$

Moreover, the group $O^G(\mathcal{U}_\lambda)$ is isomorphic to an infinite orthogonal, unitary or symplectic group, depending on the type of the irreducible representation λ.

Example 2.4.17 (More bar construction models)**.** Since the bar construction is functorial and continuous for continuous homomorphisms between topological monoids, we can apply it objectwise to every monoid-valued orthogonal space M in the sense of Definition 2.3.2; the result is an orthogonal space $\mathbf{B}^\circ M$. The bar construction is symmetric monoidal, so if M is symmetric (and hence an ultra-commutative monoid), then $\mathbf{B}^\circ M$ inherits an ultra-commutative multiplication. By the same argument as for $\mathbf{B}^\circ\mathbf{O}$, the orthogonal space $\mathbf{B}^\circ M$ is globally connected.

So the ultra-commutative monoid $\mathbf{B}^\circ\mathbf{O}$ has variations with $\mathbf{O}$ replaced by

SO, **U**, **SU**, **Sp**, **Pin**, **Spin**, $\mathbf{Pin}^c$ and $\mathbf{Spin}^c$. We can also apply the bar construction to morphisms of monoid-valued orthogonal spaces, i.e., morphism of orthogonal spaces that are objectwise monoid homomorphisms. So hitting all the previous examples with the bar construction yields a commutative diagram of globally connected orthogonal spaces:

$$\begin{tikzcd}
\mathbf{B}^\circ\mathbf{SU} \arrow[r, "\mathbf{B}^\circ\mathrm{incl}"] \arrow[d, dashed, "\mathbf{B}^\circ l"'] & \mathbf{B}^\circ\mathbf{U} \arrow[d, dashed, "\mathbf{B}^\circ l"'] \arrow[dr, dashed, "\mathbf{B}^\circ r"] & \\
\mathbf{B}^\circ\mathbf{Spin} \arrow[r, "\mathbf{B}^\circ\iota"] \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"'] & \mathbf{B}^\circ\mathbf{Spin}^c \arrow[r, "\mathbf{B}^\circ\,\mathrm{ad}"'] \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"'] & \mathbf{B}^\circ\mathbf{SO} \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"] \\
\mathbf{B}^\circ\mathbf{Pin} \arrow[r, "\mathbf{B}^\circ\iota"'] & \mathbf{B}^\circ\mathbf{Pin}^c \arrow[r, "\mathbf{B}^\circ\,\mathrm{ad}"'] & \mathbf{B}^\circ\mathbf{O}
\end{tikzcd}$$

As before, the two dotted arrows mean that the actual morphism goes to a multiplicative shift of the target. With the exception of $\mathbf{B}^\circ\mathbf{Pin}$ and $\mathbf{B}^\circ\mathbf{Pin}^c$, all these orthogonal spaces inherit ultra-commutative multiplications.

Example 2.4.18. We define an orthogonal space **bO**; its values come with tautological vector bundles whose Thom spaces form the global Thom spectrum **mO**, compare Example 6.1.24 below. For finite and abelian compact Lie groups G, the equivariant homotopy groups of **mO** are isomorphic to the bordism groups of smooth closed G-manifolds, compare Theorem 6.2.33; so **bO** and **mO** are also geometrically relevant. In Remark 2.4.25 we define an E_∞-multiplication on **bO** and show, using power operations, that this E_∞-multiplication cannot be refined to an ultra-commutative multiplication.

For an inner product space V of dimension n we set

$$\mathbf{bO}(V) \;=\; Gr_n(V \oplus \mathbb{R}^\infty)\,,$$

the Grassmannian of n-dimensional subspaces in $V \oplus \mathbb{R}^\infty$. The structure map $\mathbf{bO}(\varphi) : \mathbf{bO}(V) \longrightarrow \mathbf{bO}(W)$ is given by

$$\mathbf{bO}(\varphi)(L) \;=\; (\varphi \oplus \mathbb{R}^\infty)(L) + ((W - \varphi(V)) \oplus 0)\,,$$

the internal orthogonal sum of the image of L under $\varphi \oplus \mathbb{R}^\infty : V \oplus \mathbb{R}^\infty \longrightarrow W \oplus \mathbb{R}^\infty$ and the orthogonal complement of the image of $\varphi : V \longrightarrow W$, viewed as sitting in the first summand of $W \oplus \mathbb{R}^\infty$.

We want to describe the equivariant homotopy sets $\pi_0^G(\mathbf{bO})$ and the homotopy types of the fixed-point spaces $\mathbf{bO}(\mathcal{U}_G)^G$, for every compact Lie group G. We denote by $\mathbf{RO}^\sharp(G)$ the abelian submonoid of $\mathbf{RO}^+(G)$ consisting of the isomorphism classes of G-representations with trivial G-fixed-points. We let V be a G-representation. The G-fixed-points of $\mathbf{bO}(V)$ are the G-subrepresentations

L of $V \oplus \mathbb{R}^\infty$ of the same dimension as V. Since G acts trivially on $\mathbb{R}^\infty$, the 'non-trivial summand' $L^\perp = L - L^G$ is contained in $V^\perp = V - V^G$. So $V^\perp - L^\perp$ is a G-representation with trivial fixed-points. We can thus define a map

$$(\mathbf{bO}(V))^G \ = \ (Gr_{|V|}(V \oplus \mathbb{R}^\infty))^G \ \longrightarrow \ \mathbf{RO}^\sharp(G)$$

from this fixed-point space by sending $L \in \mathbf{bO}(V)^G$ to $[V^\perp - L^\perp]$. As before, the isomorphism type of L only depends on the path component of L in $\mathbf{bO}(V)^G$. Moreover, for every linear isometric embedding $\varphi : V \longrightarrow W$ the relation

$$\begin{aligned}(\mathbf{bO}(\varphi)(L))^\perp \ &= \ ((\varphi \oplus \mathbb{R}^\infty)(L) + (W - \varphi(V)) \oplus 0)^\perp \\ &= \ \left(\varphi(L^\perp) + (W^\perp - \varphi(V^\perp))\right) \oplus 0 \ = \ \left(W^\perp - \varphi(V^\perp - L^\perp)\right) \oplus 0\end{aligned}$$

shows that

$$[W^\perp - (\mathbf{bO}(\varphi)(L))^\perp] \ = \ [\varphi(V^\perp - L^\perp)] \ = \ [V^\perp - L^\perp] \, .$$

So the class in $\mathbf{RO}^\sharp(G)$ depends only on the class of L in $\pi_0^G(\mathbf{bO})$, and the assignments assemble into a well-defined map

$$\pi_0^G(\mathbf{bO}) \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \ \pi_0 \left(\mathbf{bO}(V)^G\right) \ \longrightarrow \ \mathbf{RO}^\sharp(G) \, . \tag{2.4.19}$$

For the description of the G-fixed-points of $\mathbf{bO}$ we introduce the abbreviation

$$Gr_j^{G,\perp} \ = \ \left(Gr_j(\mathcal{U}_G^\perp)\right)^G$$

for the space of j-dimensional G-invariant subspaces of $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$. The space $Gr_j^{G,\perp}$ can be decomposed further: before taking G-fixed-points, $Gr_j(\mathcal{U}_G^\perp)$ is G-equivariantly homeomorphic to $\mathbf{L}(\mathbb{R}^j, \mathcal{U}_G^\perp)/O(j)$. So Proposition B.17 provides a decomposition of $Gr_j^{G,\perp}$ as the disjoint union, indexed over conjugacy classes of continuous homomorphisms $\alpha : G \longrightarrow O(j)$, of the spaces

$$\mathbf{L}^G(\alpha^*(\mathbb{R}^j), \mathcal{U}_G^\perp)/C(\alpha) \, ,$$

where $C(\alpha)$ is the centralizer, in $O(j)$, of the image of α. Conjugacy classes of homomorphisms from G to $O(j)$ biject – by restriction of the tautological $O(j)$-representation – with isomorphism classes of j-dimensional G-representations. If $V = \alpha^*(\mathbb{R}^j)$ is such a G-representation, then the space $\mathbf{L}^G(V, \mathcal{U}_G^\perp)$ is empty if V has non-trivial G-fixed-points, and contractible otherwise. Moreover, the centralizer $C(\alpha)$ is precisely the group of G-equivariant linear self-isometries of V, which acts freely on $\mathbf{L}^G(V, \mathcal{U}_G^\perp)$. So if $V^G = 0$, then the orbit space $\mathbf{L}^G(V, \mathcal{U}_G^\perp)/C(\alpha)$ is a classifying space for the group $\mathbf{L}^G(V, V) = O^G(V)$. So altogether,

$$Gr_j^{G,\perp} \ \simeq \ \coprod_{[V] \in \mathbf{RO}^\sharp(G),\ |V| = j} B(O^G(V)) \, . \tag{2.4.20}$$

Every G-representation V is the direct sum of its isotypical components V_λ, indexed by the isomorphism classes of irreducible orthogonal G-representations. If $V^G = 0$, then only the non-trivial irreducibles occur, and the group $O^G(V)$ decomposes accordingly as a product

$$O^G(V) \cong \prod\nolimits_{[\lambda]} O^G(V_\lambda) ,$$

indexed by non-trivial irreducible G-representations. The irreducibles come in three flavors (real, complex or quaternionic), and so the group $O^G(V_\lambda)$ is isomorphic to one of the groups $O(m)$, $U(m)$, and $Sp(m)$, where m is the multiplicity of λ in V. So altogether, $Gr_j^{G,\perp}$ is a disjoint union of products of classifying spaces of orthogonal, unitary and symplectic groups.

Proposition 2.4.21. *Let G be a compact Lie group.*

(i) *The G-fixed-point space $\mathbf{bO}(\mathcal{U}_G)^G$ is weakly equivalent to the space*

$$\coprod\nolimits_{j\geq 0} Gr_j^{G,\perp} \times BO .$$

(ii) *The map* (2.4.19) *is a bijection from $\pi_0^G(\mathbf{bO})$ to $\mathbf{RO}^\sharp(G)$.*

(iii) *If U is a G-representation with trivial fixed-points, then the path component of $\mathbf{bO}(\mathcal{U}_G)^G$ indexed by U is a classifying space for the group $O^G(U) \times O$.*

Proof (i) We let V be a G-representation with $V^G = 0$ and W a trivial G-representation. Then every G-invariant subspace L of $V \oplus W \oplus \mathbb{R}^\infty$ is the internal direct sum of the fixed part L^G (which is contained in $W \oplus \mathbb{R}^\infty$) and its orthogonal complement $L^\perp = L - L^G$ (which is contained in the summand V). This canonical decomposition provides a homeomorphism

$$\mathbf{bO}(V\oplus W)^G \;=\; (Gr_{|V|+|W|}(V\oplus W\oplus\mathbb{R}^\infty))^G \;\cong\; \coprod_{j=0,\dots,|V|} \Big(Gr_j(V)\Big)^G \times Gr_{j+|W|}(W\oplus\mathbb{R}^\infty)$$

sending L to the pair $(V - L^\perp,\ L^G)$. Every G-invariant subspace of $\mathcal{U}_G$ is the direct sum of its fixed-points and their orthogonal complement, so the poset $s(\mathcal{U}_G)$ is the product of the two posets $s(\mathcal{U}_G^\perp)$ and $s((\mathcal{U}_G)^G)$. We can thus calculate the colimit over $s(\mathcal{U}_G)$ in two steps. For fixed W, passing to the colimit over $s(\mathcal{U}_G^\perp)$ gives a homeomorphism

$$\operatorname{colim}_{V\in s(\mathcal{U}_G^\perp)} \mathbf{bO}(V \oplus W)^G \;\cong\; \coprod\nolimits_{j\geq 0} Gr_j^{G,\perp} \times Gr_{j+|W|}(W \oplus \mathbb{R}^\infty) .$$

The factor $Gr_{j+|W|}(W \oplus \mathbb{R}^\infty)$ is a classifying space for the group $O(j + |W|)$.

Passing to the colimit over $s((\mathcal{U}_G)^G)$ then provides a weak equivalence

$$\begin{aligned}\mathbf{bO}(\mathcal{U}_G)^G \;&=\; \operatorname{colim}_{W\in s((\mathcal{U}_G)^G)} \operatorname{colim}_{V\in s(\mathcal{U}_G^\perp)} \mathbf{bO}(V\oplus W)^G \\ &\simeq\; \operatorname{colim}_{W\in s((\mathcal{U}_G)^G)} \left(\coprod\nolimits_{j\geq 0} Gr_j^{G,\perp}\times BO(j+|W|)\right) \\ &\simeq\; \coprod\nolimits_{j\geq 0} Gr_j^{G,\perp}\times BO\ .\end{aligned}$$

(ii) Since the orthogonal space $\mathbf{bO}$ is closed, we can calculate $\pi_0^G(\mathbf{bO})$ as the set of path components of the space $\mathbf{bO}(\mathcal{U}_G)^G$, by Corollary 1.5.7 (i). Since the space BO is path connected, part (i) allows us to identify $\pi_0(\mathbf{bO}(\mathcal{U}_G)^G)$ with the disjoint union, over $j\geq 0$, of the path components of $Gr_j^{G,\perp}$. As we explained in the weak equivalence (2.4.20), the set $\pi_0(Gr_j^{G,\perp})$ bijects with the set of isomorphism classes of j-dimensional G-representations with trivial fixed-points. Altogether this identifies $\pi_0(\mathbf{bO}(\mathcal{U}_G)^G)$ with $\mathbf{RO}^\sharp(G)$, and unraveling all definitions shows that the combined bijection between $\pi_0^G(\mathbf{bO})$ and $\mathbf{RO}^\sharp(G)$ is the map (2.4.19).

(iii) This is a direct consequence of part (i) and the description (2.4.20) of the path components of the space $Gr_j^{G,\perp}$. □

If H is a closed subgroup of a compact Lie group G and V a G-representation with $V^G=0$, then V may have non-zero H-fixed-points. So the restriction homomorphism $\operatorname{res}_H^G : \mathbf{RO}^+(G)\longrightarrow \mathbf{RO}^+(H)$ does not in general take $\mathbf{RO}^\sharp(G)$ to $\mathbf{RO}^\sharp(H)$. So the monoids $\mathbf{RO}^\sharp(G)$ do *not* form a sub Rep-functor of $\mathbf{RO}$, and Proposition 2.4.21 does not describe $\underline{\pi}_0(\mathbf{bO})$ as a Rep functor. We will give a description of $\underline{\pi}_0(\mathbf{bO})$ as sub-Rep monoid of the augmentation ideal global power monoid $\mathbf{IO}$ in Proposition 2.4.29 below.

As we shall now explain, the global homotopy type of the orthogonal space $\mathbf{bO}$ is that of a sequential homotopy colimit, in the category of orthogonal spaces, of global classifying spaces of the orthogonal groups $O(m)$:

$$\mathbf{bO} \;\simeq\; \operatorname{hocolim}_{m\geq 1}\; B_{\mathrm{gl}}O(m)$$

The homotopy colimit is taken over morphisms $B_{\mathrm{gl}}O(m)\longrightarrow B_{\mathrm{gl}}O(m+1)$ that classify the homomorphisms $O(m)\longrightarrow O(m+1)$ given by $A\mapsto A\oplus\mathbb{R}$. To make this relation rigorous, we define a filtration

$$* \;\cong\; \mathbf{bO}_{(0)} \;\subset\; \mathbf{bO}_{(1)} \;\subset\; \ldots \;\subset\; \mathbf{bO}_{(m)} \;\subset\; \ldots \qquad (2.4.22)$$

of $\mathbf{bO}$ by orthogonal subspaces. At an inner product space V we define

$$\mathbf{bO}_{(m)}(V) \;=\; Gr_{|V|}(V\oplus\mathbb{R}^m)\ ; \qquad (2.4.23)$$

here we consider $\mathbb{R}^m$ as the subspace of vectors of the form $(x_1,\ldots,x_m,0,0,\ldots)$ in $\mathbb{R}^\infty$. The inclusion of $\mathbf{bO}_{(m)}$ into $\mathbf{bO}_{(m+1)}$ is a closed embedding, so the global

invariance property of Proposition 1.1.9 (viii) entitles us to view the union $\mathbf{bO}$ as a global homotopy colimit of the filtration.

The tautological action of $O(m)$ on $\mathbb{R}^m$ is faithful, so the semifree orthogonal space $B_{\mathrm{gl}}O(m) = \mathbf{L}_{O(m),\mathbb{R}^m}$ is a global classifying space for $O(m)$. We define a morphism $\gamma_m : B_{\mathrm{gl}}O(m) \longrightarrow \mathbf{bO}_{(m)}$ by

$$\begin{aligned} \gamma_m(V) \ : \ \mathbf{L}(\mathbb{R}^m, V)/O(m) \ &\longrightarrow \ Gr_{|V|}(V \oplus \mathbb{R}^m) \ = \ \mathbf{bO}_{(m)}(V)\ , \\ \varphi \cdot O(m) \ &\longmapsto \ (V - \varphi(\mathbb{R}^m)) \oplus \mathbb{R}^m\ . \end{aligned}$$

We omit the straightforward verification that these maps indeed form a morphism of orthogonal spaces. The semifree orthogonal space $B_{\mathrm{gl}}O(m)$ comes with a tautological class $u_{O(m),\mathbb{R}^m}$, defined in (1.5.11), which freely generates the Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}O(m))$. We denote by

$$u_m \ = \ (\gamma_m)_*(u_{O(m),\mathbb{R}^m}) \ \in \ \pi_0^{O(m)}(\mathbf{bO}_{(m)})$$

the image in $\mathbf{bO}_{(m)}$ of the tautological class. The following proposition justifies the claim that $\mathbf{bO}$ is a homotopy colimit of the orthogonal spaces $B_{\mathrm{gl}}O(m)$.

Proposition 2.4.24. *For every $m \geq 0$ the morphism*

$$\gamma_m \ : \ B_{\mathrm{gl}}O(m) \ \longrightarrow \ \mathbf{bO}_{(m)}$$

is a global equivalence of orthogonal spaces. The inclusion $\mathbf{bO}_{(m)} \longrightarrow \mathbf{bO}_{(m+1)}$ takes the class u_m to the class

$$\mathrm{res}^{O(m+1)}_{O(m)}(u_{m+1}) \ \in \ \pi_0^{O(m)}(\mathbf{bO}_{(m+1)})\ .$$

Proof The morphism γ_m factors as the composite of two morphisms of orthogonal spaces

$$\mathbf{L}_{O(m),\mathbb{R}^m} \ \xrightarrow{\mathbf{L}_{O(m),\mathbb{R}^m}\circ i} \ \mathrm{sh}_\oplus^{\mathbb{R}^m}(\mathbf{L}_{O(m),\mathbb{R}^m}) \ \xrightarrow[\cong]{(-)^\perp} \ \mathbf{bO}_{(m)}\ .$$

For the first morphism we let $i : V \longrightarrow V \oplus \mathbb{R}^m$ denote the embedding of the first summand; furthermore, $\mathrm{sh}_\oplus^{\mathbb{R}^m}$ is the additive shift by $\mathbb{R}^m$ as defined in Example 1.1.11. The first morphism is a global equivalence by Theorem 1.1.10. At an inner product space V, the second morphism is the map

$$\begin{aligned} (\mathrm{sh}_\oplus^{\mathbb{R}^m}\mathbf{L}_{O(m),\mathbb{R}^m})(V) \ = \ \mathbf{L}(\mathbb{R}^m, V \oplus \mathbb{R}^m)/O(m) \ &\longrightarrow \ Gr_{|V|}(V \oplus \mathbb{R}^m)\ , \\ \varphi \cdot O(m) \ &\longmapsto \ \varphi(\mathbb{R}^m)^\perp\ , \end{aligned}$$

the orthogonal complement of the image. This is a homeomorphism, so the second morphism is an isomorphism. Altogether this shows that γ_m is a global equivalence.

The second claim is also reasonably straightforward from the definitions, but it needs one homotopy. We let $j : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ denote the linear isometric

embedding defined by $j(x_1, \dots, x_m) = (x_1, \dots, x_m, 0)$. The tautological class $u_{O(m),\mathbb{R}^m}$ is defined as the path component of the point

$$\mathrm{Id}_{\mathbb{R}^m} \cdot O(m) \in (\mathbf{L}(\mathbb{R}^m, \mathbb{R}^m)/O(m))^{O(m)} ,$$

the $O(m)$-orbit of the identity of $\mathbb{R}^m$. So the classes

$$\mathrm{incl}_*(u_m) \qquad \text{and} \qquad \mathrm{res}^{O(m)}_{O(m+1)}(u_{m+1}) \quad \text{in } \pi_0^{O(m)}(\mathbf{bO}_{(m+1)})$$

are represented by the two subspaces

$$(\mathbb{R}^{m+1} - j(\mathbb{R}^m)) \oplus j(\mathbb{R}^m) \quad \text{and} \quad 0 \oplus \mathbb{R}^{m+1}$$

of $\mathbb{R}^{m+1} \oplus \mathbb{R}^{m+1}$. These two representatives are *not* the same. However, the $O(m+1)$-action on $\mathbf{bO}_{(m+1)}(\mathbb{R}^{m+1})$, and hence also the restricted $O(m)$-action, is through the first copy of $\mathbb{R}^{m+1}$ (and *not* diagonally!). So there is a path of $O(m)$-invariant subspaces of $\mathbb{R}^{m+1} \oplus \mathbb{R}^{m+1}$ connecting the two representatives. The two points thus represent the same class in $\pi_0^{O(m)}(\mathbf{bO}_{(m+1)})$, and this proves the second claim. □

Remark 2.4.25 (Commutative versus E_∞-orthogonal monoid spaces)**.** Non-equivariantly, every E_∞-multiplication on an orthogonal monoid space can be rigidified to a strictly commutative multiplication. We will now see that this is not the case globally, with power operations being an obstruction.

To illustrate the difference between a strictly commutative multiplication and an E_∞-multiplication, we take a closer look at the orthogonal space **bO**. If we try to define a multiplication on **bO** in a similar way as for **BO**, we run into the problem that $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ is different from $\mathbb{R}^\infty$; even worse, although $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ and $\mathbb{R}^\infty$ are isometrically isomorphic, there is no preferred isomorphism. The standard way out is to use *all* isomorphisms at once, i.e., to parametrize the multiplications by the E_∞-operad of linear isometric self-embeddings of $\mathbb{R}^\infty$. We recall that the nth space of the linear isometries operad is

$$\mathcal{L}(n) = \mathbf{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty) ,$$

with operad structure by direct sum and composition of linear isometric embeddings (see for example [112, Def. 1.2] for details). For all $n \geq 0$ and all inner product spaces $V_1, \dots, V_n$ we define a linear isometry

$$\kappa : (V_1 \oplus \mathbb{R}^\infty) \oplus \dots \oplus (V_n \oplus \mathbb{R}^\infty) \cong V_1 \oplus \dots \oplus V_n \oplus (\mathbb{R}^\infty)^n$$

by shuffling the summands, i.e.,

$$\kappa(v_1, x_1, \dots, v_n, x_n) = (v_1, \dots, v_n, x_1, \dots, x_n) .$$

We can then define a continuous map

$$\mu_n : \mathcal{L}(n) \times \mathbf{bO}(V_1) \times \dots \times \mathbf{bO}(V_n) \longrightarrow \mathbf{bO}(V_1 \oplus \dots \oplus V_n)$$

by

$$\mu_n(\varphi, L_1, \dots, L_n) \;=\; ((V_1 \oplus \dots \oplus V_n) \oplus \varphi) \circ \kappa)(L_1 \oplus \dots \oplus L_n)\,.$$

For fixed φ these maps form a multi-morphism, so the universal property of the box product produces a morphism of orthogonal spaces

$$\mu_n(\varphi, -) \;:\; \mathbf{bO} \boxtimes \dots \boxtimes \mathbf{bO} \;\longrightarrow\; \mathbf{bO}\,.$$

For varying φ, these maps define a morphism of orthogonal spaces

$$\mu_n \;:\; \mathcal{L}(n) \times (\mathbf{bO} \boxtimes \dots \boxtimes \mathbf{bO}) \;\longrightarrow\; \mathbf{bO}\,.$$

As n varies, all these morphism together make the orthogonal space **bO** an algebra (with respect to the box product) over the linear isometries operad $\mathcal{L}$. Since the linear isometries operad is an E_∞-operad, we call **bO**, endowed with this $\mathcal{L}$-action, an E_∞-orthogonal monoid space.

Now we explain why the E_∞-structure on **bO** *cannot* be refined to an ultra-commutative multiplication. An E_∞-structure gives rise to abelian monoid structures on the equivariant homotopy sets. In more detail, we let R be any E_∞-orthogonal monoid space, such as for example **bO**. We obtain binary pairings

$$\pi_0^G(R) \times \pi_0^G(R) \;\xrightarrow{\;\times\;}\; \pi_0^G(R \boxtimes R) \;\xrightarrow{\pi_0^G(\mu_2(\varphi,-))}\; \pi_0^G(R)\,,$$

where $\varphi \in \mathcal{L}(2)$ is any linear isometric embedding of $(\mathbb{R}^\infty)^2$ into $\mathbb{R}^\infty$. The second map (and hence the composite) is independent of φ because the space $\mathcal{L}(2)$ is contractible. In the same way as for strict multiplications in (2.2.1), this binary operation makes $\pi_0^G(R)$ an abelian monoid for every compact Lie group G, such that all restriction maps are homomorphisms. In other words, the E_∞-structure provides a lift of the Rep-functor $\underline{\pi}_0(R)$ to an abelian Rep-monoid, i.e., a functor

$$\underline{\pi}_0(R) \;:\; \mathrm{Rep}^{\mathrm{op}} \;\longrightarrow\; \mathcal{A}b\mathcal{M}on\,.$$

This structure is natural for homomorphisms of E_∞-orthogonal monoid spaces.

An ultra-commutative monoid can be viewed as an E_∞-orthogonal monoid space by letting every element of $\mathcal{L}(n)$ act as the iterated multiplication. Equivalently: we let the linear isometries operad act along the unique homomorphism to the terminal operad (whose algebras, with respect to the box product, are the ultra-commutative monoids). For E_∞-orthogonal monoid spaces arising in this way from ultra-commutative monoids, the products on $\underline{\pi}_0$ defined here coincide with those originally defined in (2.2.1). For ultra-commutative monoids R, the abelian Rep-monoid $\underline{\pi}_0(R)$ is underlying a global power monoid, i.e., it comes with power operations and transfer maps that satisfy various relations. We show in Proposition 2.4.29 below that the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid whatsoever; hence **bO**

is not globally equivalent, as an E_∞-orthogonal monoid space, to any ultra-commutative monoid. A curious fact, however, is that *after global group completion* the E_∞-multiplication of **bO** can be refined to an ultra-commutative multiplication, compare Remark 2.5.36.

We compare the E_∞-orthogonal monoid space **bO** to the ultra-commutative monoid **BO** in the most highly structured way possible. Every ultra-commutative monoid can be viewed as an E_∞-orthogonal monoid space, and we now define a 'weak E_∞-morphism' from **bO** to **BO**. The zigzag of morphisms passes through the orthogonal space $\mathbf{BO}'$ with values

$$\mathbf{BO}'(V) \;=\; Gr_{|V|}(V^2 \oplus \mathbb{R}^\infty) \,. \tag{2.4.26}$$

The structure maps of $\mathbf{BO}'$ are a mixture of those for **bO** and **BO**, i.e.,

$$\mathbf{BO}'(\varphi)(L) \;=\; (\varphi^2 \oplus \mathbb{R}^\infty)(L) \oplus ((W - \varphi(V)) \oplus 0 \oplus 0) \,,$$

where now the orthogonal complement of the image of φ is viewed as sitting in the first summand of $W \oplus W \oplus \mathbb{R}^\infty$. The linear isometries operad acts on $\mathbf{BO}'$ in much the same way as for **bO**, making it an E_∞-orthogonal monoid space. Post-Composition with the direct summand embeddings

$$V \oplus \mathbb{R}^\infty \xrightarrow{(v,x)\mapsto(v,0,x)} V^2 \oplus \mathbb{R}^\infty \xleftarrow{(v,v',0)\leftarrow\!\shortmid(v,v')} V^2$$

induces maps of Grassmannians

$$\mathbf{bO}(V) \xrightarrow{a(V)} \mathbf{BO}'(V) \xleftarrow{b(V)} \mathbf{BO}(V)$$

that form morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bO} \xrightarrow{a} \mathbf{BO}' \xleftarrow{b} \mathbf{BO} \,. \tag{2.4.27}$$

Proposition 2.4.28. *The morphism $b : \mathbf{BO} \longrightarrow \mathbf{BO}'$ is a global equivalence of orthogonal spaces.*

Proof We define an exhaustive filtration

$$\mathbf{BO} \;=\; \mathbf{BO}'_{(0)} \subset \mathbf{BO}'_{(1)} \subset \ldots \subset \mathbf{BO}'_{(m)} \subset \ldots$$

of $\mathbf{BO}'$ by orthogonal subspaces by setting

$$\mathbf{BO}'_{(m)}(V) \;=\; Gr_{|V|}(V^2 \oplus \mathbb{R}^m) \,.$$

We denote by $\mathrm{sh} = \mathrm{sh}^\oplus_\mathbb{R}$ the additive shift functor defined in Example 1.1.11, and by $i_X : X \longrightarrow \mathrm{sh}\, X$ the morphism of orthogonal spaces given by applying X to the direct summand embeddings $V \longrightarrow V \oplus \mathbb{R}$. The morphism i_X is a global

equivalence for every orthogonal space X, by Theorem 1.1.10. We define a morphism

$$j \ : \ \mathbf{BO}'_{(m+1)} \longrightarrow \mathrm{sh}(\mathbf{BO}'_{(m)})$$

at an inner product space V by

$$\begin{aligned} j(V) \ : \ \mathbf{BO}'_{(m+1)}(V) \ &= \ Gr_{|V|}(V \oplus V \oplus \mathbb{R}^{m+1}) \\ &\longrightarrow \ Gr_{|V|+1}(V \oplus \mathbb{R} \oplus V \oplus \mathbb{R} \oplus \mathbb{R}^m) = \mathrm{sh}(\mathbf{BO}'_{(m)})(V) \end{aligned}$$

by applying the linear isometric embedding

$$\begin{aligned} V \oplus V \oplus \mathbb{R}^{m+1} \quad &\longrightarrow \quad V \oplus \mathbb{R} \oplus V \oplus \mathbb{R} \oplus \mathbb{R}^m \\ (v, v', (x_1, \dots, x_{m+1})) \ &\longmapsto \ (v, 0, v', x_1, (x_2, \dots, x_{m+1})) \end{aligned}$$

and adding the first copy of $\mathbb{R}$ (the orthogonal complement of this last embedding). Then the following diagram commutes:

$$\begin{tikzcd} \mathbf{BO}'_{(m)} \arrow[r, "\mathrm{incl}"] \arrow[dr, "i_{\mathbf{BO}'_{(m)}}"'] & \mathbf{BO}'_{(m+1)} \arrow[d, "j"] \arrow[dr, "i_{\mathbf{BO}'_{(m+1)}}"] & \\ & \mathrm{sh}(\mathbf{BO}'_{(m)}) \arrow[r, "\mathrm{sh(incl)}"'] & \mathrm{sh}(\mathbf{BO}'_{(m+1)}) \end{tikzcd}$$

The two diagonal morphisms are global equivalences, hence so is the inclusion $\mathbf{BO}'_{(m)} \longrightarrow \mathbf{BO}'_{(m+1)}$, by the 2-out-of-6 property for global equivalences (Proposition 1.1.9 (iii)). The inclusion of $\mathbf{BO}'_{(m)}$ into $\mathbf{BO}'_{(m+1)}$ is also objectwise a closed embedding, so the inclusion

$$\mathbf{BO} \ = \ \mathbf{BO}'_{(0)} \ \longrightarrow \ \bigcup\nolimits_{m \geq 0} \mathbf{BO}'_{(m)} \ = \ \mathbf{BO}'$$

is a global equivalence, by Proposition 1.1.9 (ix). □

We recall that $\mathbf{IO}(G)$ denotes the augmentation ideal in the real representation ring $\mathbf{RO}(G)$ of a compact Lie group G. In the same way as for $\mathbf{BO}$ we define a monoid homomorphism

$$\gamma \ : \ \pi_0^G(\mathbf{BO}') \ \longrightarrow \ \mathbf{IO}(G)$$

by sending the path component of $W \in \mathbf{BO}'(V)^G$ to the class $[W] - [V]$. Then the triangle of monoid homomorphisms on the right of following diagram commutes:

$$\begin{tikzcd} \pi_0^G(\mathbf{bO}) \arrow[r, "a_*"] \arrow[dr] & \pi_0^G(\mathbf{BO}') \arrow[d, "\gamma", "\cong"'] & \pi_0^G(\mathbf{BO}) \arrow[l, "b_*"', "\cong"] \arrow[dl, "\cong", "(2.4.12)"'] \\ & \mathbf{IO}(G) & \end{tikzcd}$$

The two maps on the right are isomorphisms, hence so is the map γ.

The following proposition says that $\pi_0^G(\mathbf{bO})$ is isomorphic to the free abelian submonoid of $\mathbf{IO}(G)$ generated by $\dim(\lambda)\cdot 1 - [\lambda]$ as λ runs over all isomorphism classes of non-trivial irreducible G-representations. We emphasize that for non-trivial groups G, the monoid $\pi_0^G(\mathbf{bO})$ does not have inverses, so $\mathbf{bO}$ is *not* group-like.

Proposition 2.4.29. *The composite morphism of abelian* Rep*-monoids*

$$\underline{\pi}_0(\mathbf{bO}) \xrightarrow{a_*} \underline{\pi}_0(\mathbf{BO}') \xrightarrow{\gamma} \mathbf{IO}$$

is a monomorphism. For every compact Lie group G the image of $\pi_0^G(\mathbf{bO})$ in the augmentation ideal $\mathbf{IO}(G)$ consists of the submonoid of elements of the form $\dim(U)\cdot 1 - [U]$, for G-representations U. The abelian Rep*-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid.*

Proof If $L \subset V \oplus \mathbb{R}^\infty$ is a G-invariant subspace of the same dimension as V, then

$$\begin{aligned}[L]-[V] &= (\dim(L)-\dim(L^\perp))\cdot 1 + [L^\perp] - (\dim(V)-\dim(V^\perp))\cdot 1 - [V^\perp]\\ &= \dim(V^\perp - L^\perp)\cdot 1 - [V^\perp - L^\perp]\end{aligned}$$

in the group $\mathbf{IO}(G)$. This shows that the following square commutes:

$$\begin{tikzcd} \pi_0^G(\mathbf{bO}) \arrow[r, "a_*"] \arrow[d, "(2.4.19)"'] & \pi_0^G(\mathbf{BO}') \arrow[d, "\gamma"] \\ \mathbf{RO}^\sharp(G) \arrow[r, "{[U]\mapsto \dim(U)\cdot 1-[U]}"'] & \mathbf{IO}(G) \end{tikzcd}$$

The left vertical map is bijective by Proposition 2.4.21 (ii). The right vertical map is an isomorphism as explained above. This shows the first two claims because the lower horizontal map is injective and has the desired image.

Now we argue, by contradiction, that the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid. The additional structure would in particular specify a transfer map

$$\mathrm{tr}_{A_3}^{\Sigma_3} \ : \ \pi_0^{A_3}(\mathbf{bO}) \longrightarrow \pi_0^{\Sigma_3}(\mathbf{bO}) \tag{2.4.30}$$

from the alternating group A_3 to the symmetric group Σ_3. Since the monoid $\pi_0^e(\mathbf{bO})$ has only one element, the double coset formula shows that

$$\mathrm{res}^{\Sigma_3}_{(12)} \circ \mathrm{tr}_{A_3}^{\Sigma_3} \ = \ \mathrm{tr}_e^{(12)} \circ \mathrm{res}^{A_3}_e \ = \ 0 \ : \ \pi_0^{A_3}(\mathbf{bO}) \longrightarrow \pi_0^{(12)}(\mathbf{bO})\ .$$

The group Σ_3 has two non-trivial irreducible orthogonal representations, the 1-dimensional sign representation σ and the 2-dimensional reduced natural

representation ν. So $\pi_0^{\Sigma_3}(\mathbf{bO})$ 'is' (via $\gamma \circ a_*$) the free abelian submonoid of $\mathbf{IO}(\Sigma_3)$ generated by

$$1 - \sigma \qquad \text{and} \qquad 2 - \nu\ .$$

We abuse notation and also write σ for the 1-dimensional sign representation of the cyclic subgroup of Σ_3 generated by the transposition (1 2). Then

$$\mathrm{res}^{\Sigma_3}_{(12)}(1-\sigma) \ = \ 1-\sigma \quad \text{and} \quad \mathrm{res}^{\Sigma_3}_{(12)}(2-\nu) \ = \ 2-(1+\sigma) \ = \ 1-\sigma\ ;$$

so the only element of $\pi_0^{\Sigma_3}(\mathbf{bO})$ that restricts to 0 in $\pi_0^{(12)}(\mathbf{bO})$ is the zero element. Hence the transfer map (2.4.30) must be the zero map. However, another instance of the double coset formula is

$$\mathrm{res}^{\Sigma_3}_{A_3} \circ \mathrm{tr}^{\Sigma_3}_{A_3} \ = \ \mathrm{Id} + (c_{(12)})^* \ = \ 2 \cdot \mathrm{Id} \ : \ \pi_0^{A_3}(\mathbf{bO}) \ \longrightarrow \ \pi_0^{A_3}(\mathbf{bO})\ .$$

The second equality uses that conjugation by the transposition (12) is the non-trivial automorphism of A_3, which acts trivially on $\mathbf{RO}(A_3)$. Since $\mathbf{bO}(A_3)$ is a non-trivial free abelian monoid, this contradicts the relation $\mathrm{tr}^{\Sigma_3}_{A_3} = 0$. So the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be endowed with transfer maps that satisfy the double coset formulas. □

Example 2.4.31 (Periodic **bO**)**.** The E_∞-orthogonal monoid space **bO** also has a straightforward 'periodic' variant **bOP** that we briefly discuss. For an inner product space V we set

$$\mathbf{bOP}(V) \ = \ \coprod\nolimits_{n \geq 0} Gr_n(V \oplus \mathbb{R}^\infty)\ ,$$

the disjoint union of all the Grassmannians in $V \oplus \mathbb{R}^\infty$. The structure maps of **bOP** are defined in exactly the same way as for **bO**. The orthogonal space **bOP** is naturally $\mathbb{Z}$-graded: for $m \in \mathbb{Z}$ we let

$$\mathbf{bOP}^{[m]}(V) \ \subset \ \mathbf{bOP}(V)$$

be the path component consisting of all subspaces $L \subset V \oplus \mathbb{R}^\infty$ such that $\dim(L) = \dim(V) + m$. For fixed m these spaces form an orthogonal subspace $\mathbf{bOP}^{[m]}$ of **bOP**. The E_∞-multiplication of **bO** extends naturally to a $\mathbb{Z}$-graded E_∞-multiplication on **bOP**, taking $\mathbf{bOP}^{[m]} \boxtimes \mathbf{bOP}^{[n]}$ to $\mathbf{bOP}^{[m+n]}$. Moreover, $\mathbf{bO} = \mathbf{bOP}^{[0]}$, the homogeneous summand of **bOP** of degree 0.

We offer two descriptions of the global homotopy type of the orthogonal space **bOP** in terms of other global homotopy types previously discussed. As we show in Proposition 2.4.32 below, each of the homogeneous summands $\mathbf{bOP}^{[m]}$ is in fact globally equivalent to the degree 0 summand **bO**, and hence

$$\mathbf{bOP} \ \simeq \ \mathbb{Z} \times \mathbf{bO}$$

globally as orthogonal spaces. When combined with Proposition 2.4.21, this yields a description of the homotopy types of the fixed-point spaces $\mathbf{bOP}(\mathcal{U}_G)^G$.

To compare the different summands of **bOP** we choose a linear isometric embedding $\psi : \mathbb{R}^\infty \oplus \mathbb{R} \longrightarrow \mathbb{R}^\infty$ and define an endomorphism $\psi_\sharp : \mathbf{bOP} \longrightarrow \mathbf{bOP}$ of the orthogonal space **bOP** at an inner product space V as the map

$$\psi_\sharp(V) \;:\; \mathbf{bOP}(V) \;\longrightarrow \mathbf{bOP}(V)\,, \quad L \;\longmapsto\; (V \oplus \psi)(L \oplus \mathbb{R})\,.$$

The morphism $\psi_\sharp$ increases the dimension of subspaces by 1, so it takes the summand $\mathbf{bOP}^{[k]}$ to the summand $\mathbf{bOP}^{[k+1]}$. Any two linear isometric embeddings from $\mathbb{R}^\infty \oplus \mathbb{R}$ to $\mathbb{R}^\infty$ are homotopic through linear isometric embeddings, so the homotopy class of $\psi_\sharp$ is independent of the choice of ψ.

Proposition 2.4.32. *For every linear isometric embedding $\psi : \mathbb{R}^\infty \oplus \mathbb{R} \longrightarrow \mathbb{R}^\infty$ the morphism of orthogonal spaces $\psi_\sharp : \mathbf{bOP} \longrightarrow \mathbf{bOP}$ is a global equivalence. Hence for every $m \in \mathbb{Z}$ the restriction is a global equivalence*

$$\psi_\sharp \;:\; \mathbf{bOP}^{[m]} \;\longrightarrow \mathbf{bOP}^{[m+1]}\,.$$

Proof We let $\mathrm{sh} = \mathrm{sh}_\oplus^{\mathbb{R}}$ denote the additive shift of an orthogonal space as defined in Example 1.1.11, with $i : \mathrm{Id} \longrightarrow - \oplus \mathbb{R}$ the natural transformation given by the embedding of the first summand. The morphism $\psi_\sharp$ factors as the composite

$$\mathbf{bOP} \xrightarrow{\mathbf{bOP}\circ i} \mathrm{sh}\,\mathbf{bOP} \xrightarrow{\psi_!} \mathbf{bOP}\,.$$

The second morphism is defined at V as the map

$$(\mathrm{sh}\,\mathbf{bOP})(V) \;=\; \mathbf{bOP}(V \oplus \mathbb{R}) \;\longrightarrow\; \mathbf{bOP}(V)$$

that sends a subspace of $V \oplus \mathbb{R} \oplus \mathbb{R}^\infty$ to its image under the linear isometric embedding

$$V \oplus \mathbb{R} \oplus \mathbb{R}^\infty \;\longrightarrow\; V \oplus \mathbb{R}^\infty\,, \quad (v, x, y) \;\longmapsto\; (v, \psi(y, x))\,.$$

The morphism $\mathbf{bOP} \circ i : \mathbf{bOP} \longrightarrow \mathrm{sh}\,\mathbf{bOP}$ is a global equivalence by Theorem 1.1.10. Any two linear isometric embeddings from $\mathbb{R}^\infty \oplus \mathbb{R}$ to $\mathbb{R}^\infty$ are homotopic through linear isometric embeddings; in particular, the linear isometric embedding ψ is homotopic to a linear isometric isomorphism. Thus $\psi_!$ is homotopic to an isomorphism, hence a global equivalence. Since both $\mathbf{bOP} \circ i$ and $\psi_!$ are global equivalences, so is the composite $\psi_\sharp$. □

Another description of the global homotopy type of **bOP** is as a global homotopy colimit of a sequence of self-maps of **Gr**:

$$\mathbf{bOP} \;\simeq\; \mathrm{hocolim}_{m \geq 1}\; \mathbf{Gr}\,.$$

The homotopy colimit is taken over iterated instances of a morphism $\mathbf{Gr} \longrightarrow$

Gr that classifies the map 'add a trivial summand $\mathbb{R}$'. To make this relation rigorous, we use the filtration

$$* \cong \mathbf{bOP}_{(0)} \subset \mathbf{bOP}_{(1)} \subset \ldots \subset \mathbf{bOP}_{(m)} \subset \ldots$$

of **bOP** by orthogonal subspaces, analogous to the filtration (2.4.22) for **bO**. In other words,

$$\mathbf{bOP}_{(m)}(V) = \coprod_{n\geq 0} Gr_n(V \oplus \mathbb{R}^m) ;$$

as before we consider $\mathbb{R}^m$ as the subspace of $\mathbb{R}^\infty$ of all vectors of the form $(x_1, \ldots, x_m, 0, 0, \ldots)$.

We write sh^m for $\mathrm{sh}_\oplus^{\mathbb{R}^m}$, the additive shift by $\mathbb{R}^m$ in the sense of Example 1.1.11. We define a morphism $\gamma_m : \mathrm{sh}^m\, \mathbf{Gr} \longrightarrow \mathbf{bOP}_{(m)}$ by

$$\begin{aligned} \gamma_m(V) \ : \ (\mathrm{sh}^m\, \mathbf{Gr})(V) \ = \ \mathbf{Gr}(V \oplus \mathbb{R}^m) \ &\longrightarrow \ \mathbf{bOP}_{(m)}(V) \\ L \qquad &\longmapsto \ L^\perp = (V \oplus \mathbb{R}^m) - L \ , \end{aligned}$$

the orthogonal complement of L inside $V \oplus \mathbb{R}^m$. We omit the straightforward verification that these maps indeed form a morphism of orthogonal spaces. Since each map $\gamma_m(V)$ is a homeomorphism, the morphism γ_m is in fact an isomorphism of orthogonal spaces.

The morphism

$$i_m \ : \ \mathbf{Gr} \ \longrightarrow \ \mathrm{sh}^m\, \mathbf{Gr}$$

induced by the direct summand embedding $V \longrightarrow V \oplus \mathbb{R}^m$ is a global equivalence by Theorem 1.1.10. The inclusion of $\mathbf{bOP}_{(m)}$ into $\mathbf{bOP}_{(m+1)}$ is a closed embedding, so the global invariance property of Proposition 1.1.9 (ix) entitles us to view the union **bOP** as a global homotopy colimit of the filtration. This justifies the interpretation of **bOP** as a global homotopy colimit of a sequence of copies of **Gr**. For this description to be useful we should identify the global homotopy classes of the morphisms in the sequence, i.e., the weak morphisms

$$\mathbf{Gr} \ \xrightarrow[\sim]{\gamma_m \circ i_m} \ \mathbf{bOP}_{(m)} \ \xrightarrow{\text{incl}} \ \mathbf{bOP}_{(m+1)} \ \xleftarrow[\sim]{\gamma_{m+1} \circ i_{m+1}} \ \mathbf{Gr} \ .$$

As is straightforward from the definition, this weak morphism models 'adding a summand $\mathbb{R}$ with trivial action'.

Finally, we compare the E_∞-orthogonal monoid space **bOP** to the ultra-commutative monoid **BOP**. In analogy with the non-periodic version in (2.4.26), we introduce the orthogonal space $\mathbf{BOP}'$ with values

$$\mathbf{BOP}'(V) \ = \ \coprod_{n\geq 0} Gr_n(V^2 \oplus \mathbb{R}^\infty) \ .$$

The structure maps of $\mathbf{BOP}'$ are defined in the same way as for $\mathbf{BO}'$, and they

mix the structure maps for **bOP** and **BOP**. Post-Composition with the direct summand embeddings

$$V \oplus \mathbb{R}^\infty \xrightarrow{(v,x)\mapsto(v,0,x)} V^2 \oplus \mathbb{R}^\infty \xleftarrow{(v,v',0)\leftarrow(v,v')} V^2$$

induces morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bOP} \xrightarrow{a} \mathbf{BOP'} \xleftarrow{b} \mathbf{BOP} .$$

These morphisms preserve the $\mathbb{Z}$-grading; the restrictions to the homogeneous degree 0 summand are precisely the morphisms with the same names introduced in (2.4.27). The same argument as in Proposition 2.4.28 also shows that the morphism $b : \mathbf{BOP} \longrightarrow \mathbf{BOP'}$ is a global equivalence of orthogonal spaces.

We define a monoid homomorphism

$$\gamma \ : \ \pi_0^G(\mathbf{BOP'}) \ \longrightarrow \ \mathbf{RO}(G)$$

by sending the path component of $W \in \mathbf{BOP'}(V)^G$ to the class $[W] - [V]$. Then the triangle of monoid homomorphisms on the right of the following diagram commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{bOP}) \arrow[r, "a_*"] \arrow[dr] & \pi_0^G(\mathbf{BOP'}) \arrow[d, "\gamma", "\cong"'] & \pi_0^G(\mathbf{BOP}) \arrow[l, "b_*", "\cong"'] \arrow[dl, "\cong"', "(2.4.12)"] \\
& \mathbf{RO}(G) &
\end{tikzcd}$$

The two maps on the right are isomorphisms, hence so is the map γ. The same argument as in Proposition 2.4.29 shows that the composite morphism of abelian Rep-monoids $\gamma \circ a_* : \underline{\pi}_0(\mathbf{bOP}) \longrightarrow \mathbf{RO}$ is a monomorphism, and for every compact Lie group G the image of $\pi_0^G(\mathbf{bOP})$ in the representation ring $\mathbf{RO}(G)$ consists of the submonoid of elements of the form $n \cdot 1 - [U]$, for $n \in \mathbb{Z}$ and U any G-representation.

Example 2.4.33 (Complex and quaternionic periodic Grassmannians)**.** The ultra-commutative monoids **BO** and **BOP** and the E_∞-orthogonal monoid spaces **bO** and **bOP** have straightforward complex and quaternionic analogs; we quickly give the relevant definitions for the sake of completeness. We define the periodic Grassmannians **BUP** and **BSpP** by

$$\mathbf{BUP}(V) \ = \ \coprod\nolimits_{m\geq 0} Gr_m^{\mathbb{C}}(V_{\mathbb{C}}^2) \quad \text{and} \quad \mathbf{BSpP}(V) \ = \ \coprod\nolimits_{m\geq 0} Gr_m^{\mathbb{H}}(V_{\mathbb{H}}^2) \, ,$$

the disjoint union of the respective Grassmannians, with structure maps as for **BOP**. External direct sum of subspaces defines a $\mathbb{Z}$-graded ultra-commutative multiplication on **BUP** and on **BSpP**, again in much the same way as for **BOP**.

The homogeneous summands of degree zero are closed under the multiplication and form ultra-commutative monoids $\mathbf{BU} = \mathbf{BUP}^{[0]}$ and $\mathbf{BSp} = \mathbf{BSpP}^{[0]}$. More explicitly,

$$\mathbf{BU}(V) = Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}}) \qquad \text{and} \qquad \mathbf{BSp}(V) = Gr^{\mathbb{H}}_{|V|}(V^2_{\mathbb{H}}) .$$

The complex and quaternionic analogues of Theorem 2.4.13 provide isomorphisms of global power monoids

$$\underline{\pi}_0(\mathbf{BUP}) \cong \mathbf{RU} \qquad \text{and} \qquad \underline{\pi}_0(\mathbf{BSpP}) \cong \mathbf{RSp} ;$$

here $\mathbf{RU}(G)$ and $\mathbf{RSp}(G)$ are the Grothendieck groups, under direct sum, of isomorphism classes of unitary and symplectic G-representations, respectively. The isomorphisms above match the $\mathbb{Z}$-grading of **BUP** and **BSpP** with the grading by virtual dimension of representations, so they restrict to isomorphisms of global power monoids from $\underline{\pi}_0(\mathbf{BU})$ and $\underline{\pi}_0(\mathbf{BSp})$ to the augmentation ideal global power monoids inside **RU** and **RSp**.

Theorem 2.4.10 also generalizes to natural group isomorphisms, compatible with restrictions,

$$\langle - \rangle \ : \ \mathbf{BUP}_G(A) \longrightarrow \mathbf{KU}_G(A) \quad \text{and} \quad \langle - \rangle \ : \ \mathbf{BSpP}_G(A) \longrightarrow \mathbf{KSp}_G(A)$$

to the equivariant unitary and symplectic K-groups, where G is any compact Lie group and A a compact G-space.

Example 2.4.18 can be modified to define E_∞-orthogonal monoid spaces **bU** and **bSp** with values

$$\mathbf{bU}(V) = Gr^{\mathbb{C}}_{|V|}(V_{\mathbb{C}} \oplus \mathbb{C}^\infty) \quad \text{and} \quad \mathbf{bSp}(V) = Gr^{\mathbb{H}}_{|V|}(V_{\mathbb{H}} \oplus \mathbb{H}^\infty) .$$

The structure maps and E_∞-multiplication are defined as for **bO**. As orthogonal spaces, **bU** and **bSp** are global homotopy colimits of the sequence of global classifying spaces $B_{\mathrm{gl}}U(m)$ and $B_{\mathrm{gl}}Sp(m)$. Periodic versions, **bUP** and **bSpP**, are defined by taking the full Grassmannian inside $V_{\mathbb{C}} \oplus \mathbb{C}^\infty$ and $V_{\mathbb{H}} \oplus \mathbb{H}^\infty$, respectively, as in the real case in Example 2.4.31. The periodic versions are, viewed as orthogonal spaces, global homotopy colimits of iterated instances of the self-maps of $\mathbf{Gr}^{\mathbb{C}}$ and $\mathbf{Gr}^{\mathbb{H}}$, respectively, that represent ‘adding a trivial 1-dimensional representation’.

2.5 Global group completion and units

For every orthogonal monoid space R and every compact Lie group G, the operation (2.2.1) makes the equivariant homotopy set $\pi_0^G(R)$ a monoid, and this multiplication is natural with respect to restriction maps in G. If the multiplication of R is commutative, then so is the multiplication of $\pi_0^G(R)$. In this section

we look more closely at the *group-like* ultra-commutative monoids, i.e., the ones where all these monoid structures have inverses. There are two universal ways of making an ultra-commutative monoid group-like: the 'global units' (Construction 2.5.18) are universal from the left and the 'global group completion' (Construction 2.5.20 and Corollary 2.5.31) is universal from the right. In the homotopy category of ultra-commutative monoids, these constructions are right adjoint or left adjoint to the inclusion of group-like objects. On π_0^G both constructions have the expected effect: the global units pick out the invertible elements of π_0^G (see Proposition 2.5.19), and the effect of global group completion is group completion of the abelian monoids π_0^G (see Proposition 2.5.21). Naturally occurring examples of global group completions are the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ from the additive Grassmannian to the periodic global version of BO, and its complex and quaternionic versions, see Theorem 2.5.33. At the end of this section we use global group completion to prove a global, highly structured version of Bott periodicity: Theorem 2.5.41 shows that **BUP** is globally equivalent, as an ultra-commutative monoid, to $\Omega\mathbf{U}$.

The category of ultra-commutative monoids is pointed, and product and box product are the categorical product and coproduct, respectively, in the category of ultra-commutative monoids. These descend to product and coproduct in the homotopy category Ho(*umon*) of ultra-commutative monoids, with respect to the global model structure of Theorem 2.1.15. The morphism $\rho_{R,S} : R \boxtimes S \longrightarrow R \times S$ is a global equivalence by Theorem 1.3.2 (i), so in Ho(*umon*) the canonical morphism from a coproduct to a product is an isomorphism. Various features of units and group completions only depend on these formal properties, and work just as well in any pointed model category in which coproducts and products coincide up to weak equivalence. So we develop large parts of the theory in this generality.

Construction 2.5.1. Let $\mathcal{D}$ be a category which has finite products and a zero object. We write $A \times B$ for any product of the objects A and B and leave the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$ implicit. Given morphisms $f : T \longrightarrow A$ and $g : T \longrightarrow B$ we write $(f, g) : T \longrightarrow A \times B$ for the unique morphism that projects to f and g. We write 0 for any morphism that factors through a zero object.

We call the category $\mathcal{D}$ *pre-additive* if 'finite products are coproducts'; more precisely, we require that every product $A \times B$ of two objects A and B is also a *co*-product, with respect to the morphisms

$$i_1 = (\mathrm{Id}_A, 0) : A \longrightarrow A \times B \quad \text{and} \quad i_2 = (0, \mathrm{Id}_B) : B \longrightarrow A \times B .$$

In other words, we demand that for every object X the map

$$\mathcal{D}(A \times B, X) \longrightarrow \mathcal{D}(A, X) \times \mathcal{D}(B, X) , \quad f \mapsto (fi_1, fi_2)$$

is bijective. The main example we care about is $\mathcal{D} = \mathrm{Ho}(umon)$, the homotopy category of ultra-commutative monoids.

In this situation we can define a binary operation on the morphism set $\mathcal{D}(A, X)$ for every pair of objects A and X. Given morphisms $a, b : A \longrightarrow X$ we let $a \perp b : A \times A \longrightarrow X$ be the unique morphism such that $(a \perp b)i_1 = a$ and $(a \perp b)i_2 = b$. Then we define

$$a + b \ = \ (a \perp b)\Delta \ : \ A \longrightarrow X ,$$

where $\Delta = (\mathrm{Id}_A, \mathrm{Id}_A) : A \longrightarrow A \times A$ is the diagonal morphism.

The next proposition is well known, but I do not know a convenient reference.

Proposition 2.5.2. *Let $\mathcal{D}$ be a pre-additive category. For every pair of objects A and X of $\mathcal{D}$ the binary operation $+$ makes the set $\mathcal{D}(A, X)$ of morphisms an abelian monoid with the zero morphism as neutral element. Moreover, the monoid structure is natural for all morphisms in both variables, or, equivalently, composition is biadditive.*

Proof The proof is lengthy, but completely formal. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ and $(a + b) + c$ are the two outer composites around the diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \times A \arrow[d, "\mathrm{Id} \times \Delta"'] & & A \times A \arrow[d, "\Delta \times \mathrm{Id}"] \\
A \times (A \times A) \arrow[rr, "\alpha"] \arrow[dr, "a \perp (b \perp c)"'] & & (A \times A) \times A \arrow[dl, "(a \perp b) \perp c"] \\
 & X &
\end{tikzcd}$$

Here α is the associativity isomorphism. The upper part of the diagram commutes because the diagonal morphism is coassociative. The lower triangle then commutes since the two morphisms

$$a \perp (b \perp c) \ , \ ((a \perp b) \perp c) \circ \alpha \ : A \times (A \times A) \ \longrightarrow \ X$$

have the same 'restrictions', namely a, b and c.

The commutativity is a consequence of two elementary facts: first, $b \perp a = (a \perp b)\tau$ where $\tau : A \times A \longrightarrow A \times A$ is the automorphism that interchanges the two factors; this follows from $\tau i_1 = i_2$ and $\tau i_2 = i_1$. Second, the diagonal morphism is cocommutative, i.e., $\tau \Delta = \Delta : A \longrightarrow A \times A$. Altogether we get

$$a + b \ = \ (a \perp b)\Delta \ = \ (a \perp b)\tau\Delta \ = \ (b \perp a)\Delta \ = \ b + a \ .$$

As before we denote by $0 \in \mathcal{D}(A, X)$ the unique morphism that factors through a zero object. We let $p_1 : A \times A \longrightarrow A$ be the projection to the first factor. Then

$$(a\perp 0)i_1 \;=\; a \;=\; ap_1(\mathrm{Id}, 0) \;=\; ap_1 i_1 \quad \text{and}$$
$$(a\perp 0)i_2 \;=\; 0 \;=\; ap_1(0, \mathrm{Id}) \;=\; ap_1 i_2 \;.$$

So we have $a\perp 0 = ap_1$ in $\mathcal{D}(A \times A, X)$. Hence $a + 0 = (a\perp 0)\Delta = ap_1\Delta = a$; by commutativity we also have $0 + a = a$.

Now we verify naturality of the addition. To check $(a + b)c = ac + bc$ for $a, b : A \longrightarrow X$ and $c : A' \longrightarrow A$ we consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \times A \arrow[d, "a\perp b"] \\
A' \times A' \arrow[r, "c\times c"] \arrow[rr, bend right, "ac\perp bc"'] & A \times A \arrow[r, "a\perp b"] & X
\end{tikzcd}$$

in which the composite through the upper right corner is $(a + b)c$. We have

$$(a\perp b)(c \times c)i_1 \;=\; (a\perp b)(c, 0) \;=\; ac \;=\; (ac\perp bc)i_1$$

and similarly for i_2 instead of i_1. So $(a\perp b)(c \times c) = ac\perp bc$ since both sides have the same 'restrictions' to the two factors of $A' \times A'$. Since the composite through the lower left corner is $ac + bc$, we have shown $(a + b)c = ac + bc$. Naturality in X is even easier. For a morphism $d : X \longrightarrow Y$ we have $d(a\perp b) = da\perp db : A\times A \longrightarrow Y$ since both sides have the same 'restrictions' da and db to the two factors of $A \times A$. Thus $d(a + b) = da + db$ by the definition of '+'. □

Now we introduce the group-like objects in a pre-additive category.

Proposition 2.5.3. *Let $\mathcal{D}$ be a pre-additive category. For every object A of $\mathcal{D}$ the following two conditions are equivalent:*

(a) *The shearing morphism $\Delta\perp i_2 = (\Delta p_1) + i_2 p_2 : A \times A \longrightarrow A \times A$ is an isomorphism.*
(b) *The identity of A has an inverse in the abelian monoid $\mathcal{D}(A, A)$.*

We call A group-like *if it satisfies* (a) *and* (b). *If A is group-like, then moreover for every object X of $\mathcal{D}$ the abelian monoids $\mathcal{D}(A, X)$ and $\mathcal{D}(X, A)$ have inverses.*

Proof (a)$\Longrightarrow$(b) Since the shearing map is an isomorphism, there is a morphism $(k, j) : A \longrightarrow A \times A$ such that

$$(\mathrm{Id}_A, 0) \;=\; (\Delta\perp i_2) \circ (k, j) \;=\; (k, k + j) \;.$$

So $k = \mathrm{Id}_A$ and $\mathrm{Id}_A + j = 0$, i.e., j is an additive inverse of the identity of A.

(b)$\Longrightarrow$(a) If $j \in \mathcal{D}(A, A)$ is an inverse of the identity of A, then the morphism

$$(p_1, j \perp \mathrm{Id}_A) \ = \ (\mathrm{Id}_A, j) \perp i_2 \ : \ A \times A \ \longrightarrow \ A \times A$$

is a two-sided inverse to the shearing morphism, which is thus an isomorphism.

If $j \in \mathcal{D}(A, A)$ is an additive inverse to the identity of A, then for all $f \in \mathcal{D}(X, A)$

$$f + (jf) \ = \ (\mathrm{Id}_A \circ f) + (j \circ f) \ = \ (\mathrm{Id}_A + j) \circ f \ = \ 0 \circ f \ = \ 0 \ ;$$

so $j \circ f$ is inverse to f. Similarly, for every $g \in \mathcal{D}(A, X)$ the morphism $g \circ j$ is additively inverse to g. □

In the next definition and in what follows, we denote by $M^\times$ the subgroup of invertible elements in an abelian monoid M.

Definition 2.5.4. Let $\mathcal{D}$ be a pre-additive category. A morphism $u : R^\times \longrightarrow R$ is a *unit morphism* if for every object T the map

$$\mathcal{D}(T, u) \ : \ \mathcal{D}(T, R^\times) \ \longrightarrow \ \mathcal{D}(T, R)$$

is injective with image the subgroup $\mathcal{D}(T, R)^\times$ of invertible elements. A morphism $i : R \longrightarrow R^\star$ in $\mathcal{D}$ is a *group completion* if for every object T the map

$$\mathcal{D}(i, T) \ : \ \mathcal{D}(R^\star, T) \ \longrightarrow \ \mathcal{D}(R, T)$$

is injective with image the subgroup $\mathcal{D}(R, T)^\times$ of invertible elements.

Remark 2.5.5. If $u : R^\times \longrightarrow R$ is a unit morphism then the abelian monoid $\mathcal{D}(R^\times, R^\times)$ is a group, by the defining property; so the object $R^\times$ is in particular group-like. Since the pair $(R^\times, u)$ represents the functor

$$\mathcal{D} \ \longrightarrow \ (\text{sets}) \ , \quad T \ \longmapsto \ \mathcal{D}(T, R)^\times \ ,$$

it is unique up to preferred isomorphism. A formal consequence is that if we choose a unit morphism $u_R : R^\times \longrightarrow R$ for every object R, then this extends canonically to a functor

$$(-)^\times \ : \ \mathcal{D} \ \longrightarrow \ \mathcal{D}$$

and a natural transformation $u : (-)^\times \longrightarrow \mathrm{Id}$. Since the functor $(-)^\times$ takes values in group-like objects, it is effectively a right adjoint to the inclusion of the full subcategory of group-like objects.

A category $\mathcal{D}$ is pre-additive if and only if its opposite category $\mathcal{D}^{\mathrm{op}}$ is pre-additive. Moreover, in that situation $\mathcal{D}(A, X)$ and $\mathcal{D}^{\mathrm{op}}(X, A)$ are not only the same set (by definition), but they also have the same monoid structure. Thus the concepts of unit morphism and group completion are ‘dual’ (or ‘opposite’) to each other: a morphism is a unit morphism in $\mathcal{D}$ if and only if it is a group

completion in $\mathcal{D}^{\mathrm{op}}$. This is why many properties of unit maps have a corresponding 'dual' property for group completions, and why most proofs for unit maps have 'dual' proofs for group completions. Since the identity of any object of $\mathcal{D}$ is also the identity of the same object in $\mathcal{D}^{\mathrm{op}}$, part (b) of Proposition 2.5.3 shows that 'group-like' is a self-dual property: an object is group-like in $\mathcal{D}$ if and only if it is group-like in $\mathcal{D}^{\mathrm{op}}$.

So the above properties of unit morphisms dualize: if $i : R \longrightarrow R^\star$ is a group completion, then $R^\star$ is in particular group-like. The pair $(R^\star, i)$ is unique up to preferred isomorphism, and if we choose a group completion $i_R : R \longrightarrow R^\star$ for every object R, then this extends canonically to a functor

$$(-)^\star \ : \ \mathcal{D} \ \longrightarrow \ \mathcal{D}$$

and a natural transformation $i : \mathrm{Id} \longrightarrow (-)^\star$, producing a left adjoint to the inclusion of group-like objects.

Example 2.5.6 (Unit morphisms and group completion for abelian monoids)**.** The category of abelian monoids is the prototypical example of a pre-additive category, and the general theory of units and group completions is an abstraction of this special case. So we take the time to convince ourselves that the concepts of 'unit morphism' and 'group completion' have their familiar meanings in the motivating example.

A basic observation is that in the category of abelian monoids, the abstract addition of morphism as in Proposition 2.5.2 is simply pointwise addition of homomorphisms. So an abelian monoid is group-like in the abstract sense of Proposition 2.5.3 if and only if every element has an inverse; so the group-like objects are precisely the abelian groups.

A given homomorphism $f : M \longrightarrow N$ of abelian monoids is invertible if and only if it is pointwise invertible in N, which is the case if any only if the image of f lies in the subgroup $N^\times$ of invertible elements. So the inclusion $u : N^\times \longrightarrow N$ of the subgroup of invertible elements is a unit morphism in the sense of Definition 2.5.4.

We recall the *Grothendieck group* of an abelian monoid M. An equivalence relation $\sim$ on M^2 is defined by declaring (x, y) equivalent to (x', y') if and only if there is an element $z \in M$ with

$$x + y' + z \ = \ x' + y + z \, .$$

The componentwise addition on M^2 is well-defined on equivalence classes, so the set of equivalence classes

$$M^\star \ = \ M^2/\sim$$

inherits an abelian monoid structure. We write $[x, y]$ for the equivalence class

in $M^\star$ of a pair (x, y). The pair $(x + y, y + x)$ is equivalent to $(0, 0)$, so

$$[x, y] + [y, x] \;=\; [x + y, y + x] \;=\; 0$$

in the monoid $M^\star$. This shows that every element of $M^\star$ has an inverse, and $M^\star$ is an abelian group. We claim that the monoid homomorphism

$$i \;:\; M \longrightarrow M^\star\;, \quad i(x) \;=\; [x, 0]$$

is a group completion in the sense of Definition 2.5.4. Indeed, given a monoid homomorphism $h : M \longrightarrow N$ that is pointwise invertible, then we can define $f : M^\star \longrightarrow N$ by

$$f[x, y] \;=\; h(x) - h(y)\;.$$

A routine verification shows that f is indeed a well-defined homomorphism and that sending h to f is inverse to the restriction map

$$\mathcal{A}bMon(i, N) \;:\; \mathcal{A}bMon(M^\star, N) \;\longrightarrow\; \mathcal{A}bMon(M, N)^\times\;.$$

A slightly different way to summarize the construction of the Grothendieck group is to say that the group completion of an abelian monoid M is a cokernel, in the category of commutative monoids, of the diagonal morphism $\Delta : M \longrightarrow M \times M$.

We observe that

$$[x, y] \;=\; [x, 0] + [0, y] \;=\; i(x) - i(y)\;,$$

so every element in $M^\star$ is the difference of two elements in the image of $i : M \longrightarrow M^\star$. Moreover, if $x, x' \in M$ satisfy $i(x) = i(x')$, then the pairs $(x, 0)$ and $(x', 0)$ are equivalent, which happens if and only if there is an element $z \in M$ such that $x + z = x' + z$. Conversely, these properties of the Grothendieck construction characterize group completions of abelian monoids: a homomorphism $j : M \longrightarrow N$ of abelian monoids is a group completion if and only if the following three conditions are satisfied:

- the monoid N is a group;
- every element in N is the difference of two elements in the image of j; and
- if $x, x' \in M$ satisfy $j(x) = j(x')$, then there is an element $z \in M$ such that $x + z = x' + z$.

Indeed, the first condition guarantees that j extends (necessarily uniquely) to a homomorphism $M^\star \longrightarrow N$; the second and third conditions guarantee that the extension is surjective and injective, respectively, and hence an isomorphism.

We mostly care about the situation where $\mathcal{D} = \mathrm{Ho}(\mathcal{C})$ is the homotopy category of a pointed model category $\mathcal{C}$, such as the category of ultra-commutative

monoids. As we shall now proceed to prove, in this situation units and group completions always exist.

We consider two composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in a pointed category $\mathcal{D}$. We recall that f is a *kernel* of g if $gf = 0$ and for every morphism $\beta : T \longrightarrow B$ such that $g\beta = 0$, there is a unique morphism $\alpha : T \longrightarrow A$ such that $f\alpha = \beta$. Dually, g is a *cokernel* of f if $gf = 0$ and for every morphism $\beta : B \longrightarrow Y$ such that $\beta f = 0$, there is a unique morphism $\gamma : C \longrightarrow Y$ such that $\gamma g = \beta$.

Proposition 2.5.7. *Let R be an object of a pre-additive category $\mathcal{D}$.*

(i) *Let $e : R^\times \longrightarrow R \times R$ be a kernel of the codiagonal morphism* $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$. *Then the composite*

$$u = (\mathrm{Id} \perp 0) \circ e : R^\times \longrightarrow R$$

is a unit morphism and

$$e = (u, -u) : R^\times \longrightarrow R \times R .$$

Conversely, if $u : R^\times \longrightarrow R$ is a unit morphism, then the morphism $(u, -u) : R^\times \longrightarrow R \times R$ is a kernel of the codiagonal morphism $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$.

(ii) *Let $d : R \times R \longrightarrow R^\star$ be a cokernel of the diagonal morphism* $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$. *Then the composite*

$$i = d \circ (\mathrm{Id}, 0) : R \longrightarrow R^\star$$

is a group completion and

$$d = i \perp (-i) : R \times R \longrightarrow R^\star .$$

Conversely, if $i : R \longrightarrow R^\star$ is a group completion, then the morphism $i \perp (-i) : R \times R \longrightarrow R^\star$ is a cokernel of the diagonal morphism $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$.

Proof We prove part (ii). Part (i) is dual, i.e., equivalent to part (ii) in the opposite category $\mathcal{D}^{\mathrm{op}}$. We let T be any object of $\mathcal{D}$. Then the map

$$\{(f, g) \in \mathcal{D}(R, T)^2 \mid f + g = 0\} \longrightarrow \mathcal{D}(R, T)^\times , \quad (f, g) \longmapsto f$$

is bijective because inverses in abelian monoids, if they exist, are unique. A cokernel of the diagonal morphism is a morphism $d : R \times R \longrightarrow R^\star$ that represents the left-hand side of this bijection; a group completion is a morphism $i : R \longrightarrow R^\star$ that represents the right-hand side of this bijection. Hence $d : R \times R \longrightarrow R^\star$ is a cokernel of the diagonal if and only if $d \circ (\mathrm{Id}, 0)$ is a group completion.

The relation $d \circ (\mathrm{Id}, 0) = i$ holds by definition. The relation

$$(d \circ (0, \mathrm{Id})) + i \;=\; d \circ ((0, \mathrm{Id}) + (\mathrm{Id}, 0)) \;=\; d \circ (\mathrm{Id}, \mathrm{Id}) \;=\; 0$$

holds in the monoid $\mathcal{D}(R, R^\star)$, and thus $d \circ (0, \mathrm{Id}) = -i$. This shows that $d = i \perp (-i)$. □

The previous characterization of unit morphisms as certain kernels and group completions as certain cokernels implies the following corollary.

Corollary 2.5.8. *Let $F : \mathcal{D} \longrightarrow \mathcal{E}$ be a functor between pre-additive categories that preserves products.*

(i) *If F preserves kernels of splittable epimorphisms, then for every unit morphism $u : R^\times \longrightarrow R$ in $\mathcal{D}$, the morphism $Fu : F(R^\times) \longrightarrow FR$ is a unit morphism.*
(ii) *If F preserves cokernels of splittable monomorphisms, then for every group completion $i : R \longrightarrow R^\star$ in $\mathcal{D}$, the morphism $Fi : FR \longrightarrow F(R^\star)$ is a group completion.*

Now we consider a pointed model category C whose homotopy category is pre-additive. The main example we have in mind is $C = umon$, the category of ultra-commutative monoids with the global model structure of Theorem 2.1.15. The homotopy category $\mathrm{Ho}(C)$ then comes with an adjoint functor pair (Σ, Ω) of suspension and loop, compare [134, I.2].

Proposition 2.5.9. *Let C be a pointed model category whose homotopy category is pre-additive.*

(i) *For every object R of C, the loop object ΩR and the suspension ΣR are group-like in* $\mathrm{Ho}(C)$.
(ii) *If $u : R^\times \longrightarrow R$ is a unit morphism, then its loop $\Omega u : \Omega(R^\times) \longrightarrow \Omega R$ is an isomorphism in* $\mathrm{Ho}(C)$.
(iii) *If $i : R \longrightarrow R^\star$ is a group completion, then its suspension $\Sigma i : \Sigma R \longrightarrow \Sigma(R^\star)$ is an isomorphism in* $\mathrm{Ho}(C)$.

Proof (i) This is a version of the Eckmann-Hilton argument. For every object T of C, the set $[T, \Omega R]$ has one abelian monoid structure via Construction 2.5.1, coming from the fact that $\mathrm{Ho}(C)$ is pre-additive. A second binary operation on the set $[T, \Omega R]$ arises from the fact that ΩR is a group object in $\mathrm{Ho}(C)$, compare [134, I.2]. This operation makes $[T, \Omega R]$ a group. The monoid structure of Construction 2.5.1 is natural for morphisms in the second variable ΩR, in particular for the group structure morphism $\Omega R \times \Omega R \longrightarrow \Omega R$. This means that the two binary operations satisfy the interchange law. Since they also share

the same neutral element, they coincide. Since one of the two operations has inverses, so does the other.

The argument that ΣR is group-like is dual, beause ΣR is the loop object of R in $\mathrm{Ho}(\mathcal{C})^{\mathrm{op}} = \mathrm{Ho}(\mathcal{C}^{\mathrm{op}})$, and 'group-like' is a self-dual property.

(ii) Since the functor $\Omega : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ is right adjoint to Σ, it preserves products and kernels. So $\Omega u : \Omega(R^\times) \longrightarrow \Omega R$ is a unit morphism by Corollary 2.5.8. Since ΩR is already group-like by part (i), Ωu is an isomorphism. Part (iii) is dual to part (ii); so it admits the dual proof, or can be obtained by applying part (ii) to the opposite model category. □

Proposition 2.5.10. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

in a pointed model category $\mathcal{C}$ such that the object C is weakly contractible.

(i) *If the square is homotopy cartesian and g admits a section in* $\mathrm{Ho}(\mathcal{C})$*, then the morphism f is a kernel of g in* $\mathrm{Ho}(\mathcal{C})$.

(ii) *If the square is homotopy cocartesian and f admits a retraction in* $\mathrm{Ho}(\mathcal{C})$*, then the morphism g is a cokernel of f in* $\mathrm{Ho}(\mathcal{C})$.

Proof We prove part (i). Part (ii) can be proved by dualizing the argument or by applying part (i) to the opposite category with the opposite model structure. Since the square is homotopy cartesian and C is weakly contractible, the object A is weakly equivalent to the homotopy fiber, in the abstract model category sense, of the morphism g. As Quillen explains in Section I.3 of [134], there is an action map (up to homotopy)

$$A \times (\Omega D) \longrightarrow A ,$$

by an abstract version of 'fiber transport'. For every other object T of $\mathcal{C}$, Proposition 4 of [134, I.3] provides a sequence of based sets

$$[T, \Omega B] \xrightarrow{[T,\Omega g]} [T, \Omega D] \xrightarrow{[T,\partial]} [T, A] \xrightarrow{[T,f]} [T, B] \xrightarrow{[T,g]} [T, D]$$

that is exact in the sense explained in [134, I.3 Prop. 4], where $[-,-]$ denotes the morphism sets in the homotopy category of $\mathcal{C}$. In particular, the image of $[T, f]$ is equal to the preimage of the zero morphism under the map $[T, g]$.

So in order to show that f is a kernel of g it remains to check that the map $[T, f]$ is injective. So we consider two morphisms $\alpha_1, \alpha_2 \in [T, A]$ such that $f \circ \alpha_1 = f \circ \alpha_2$. Then by Proposition 4 (ii) of [134, I.3], there is an element $\lambda \in [A, \Omega D]$ such that $\alpha_2 = \alpha_1 \cdot \lambda$. Since the morphism $g : B \longrightarrow D$ has

a section, so does the morphism $\Omega g : \Omega B \longrightarrow \Omega D$. So there is a morphism $\bar{\lambda} \in [T, \Omega B]$ such that $\lambda = (\Omega g) \circ \bar{\lambda}$. But all elements in the image of $[T, \Omega g]$ act trivially on $[T, A]$, so then

$$\alpha_2 \ = \ \alpha_1 \cdot \lambda \ = \ \alpha_1 \cdot ((\Omega g) \circ \bar{\lambda}) \ = \ \alpha_1 \ . \qquad \square$$

Theorem 2.5.11. *Let C be a pointed model category whose homotopy category is pre-additive.*

(i) *Every object of C has a unit morphism and a group completion in* $\mathrm{Ho}(C)$.

(ii) *If C is right proper, then every object R admits a C-morphism $u : R^\times \longrightarrow R$ that becomes a unit morphism in the homotopy category* $\mathrm{Ho}(C)$.

(iii) *If C is left proper, then every object R admits a C-morphism $i : R \longrightarrow R^\star$ that becomes a group completion in the homotopy category* $\mathrm{Ho}(C)$.

Proof (i) We let R be any object of C. It suffices to show, by Proposition 2.5.7, that the codiagonal morphism $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$ has a kernel in $\mathrm{Ho}(C)$ and the diagonal morphism $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$ has a cokernel in $\mathrm{Ho}(C)$. The arguments are again dual to each other, so we only show the first one.

We can assume without loss of generality that R is cofibrant and fibrant. Then the fold map $\nabla : R \amalg R \longrightarrow R$ in the model category C becomes the codiagonal morphism of R in $\mathrm{Ho}(C)$. We factor $\nabla = q \circ j$ for some weak equivalence $j : R \amalg R \xrightarrow{\sim} Q$ followed by a fibration $q : Q \longrightarrow R$. Then we choose a pullback, so that we arrive at the homotopy cartesian square:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d, "q"] \\ * \arrow[r] & R \end{tikzcd}$$

The morphism q still becomes a codiagonal morphism in $\mathrm{Ho}(C)$, and so it has a section. By Proposition 2.5.10 (i) the morphism f becomes a kernel of q in $\mathrm{Ho}(C)$. So the codiagonal morphism of R has a kernel.

(ii) We choose a weak equivalence $q : R \longrightarrow \bar{R}$ to a fibrant object. A unit morphism $\bar{R}^\times \longrightarrow \bar{R}$ exists in $\mathrm{Ho}(C)$ by part (i). By replacing the source $\bar{R}^\times$ by a weakly equivalent object, if necessary, we can assume that it is cofibrant as an object in the model category C. Every morphism in $\mathrm{Ho}(C)$ from a cofibrant to a fibrant object is the image of some C-morphism under the localization functor, i.e., there is a C-morphism $\bar{u} : \bar{R}^\times \longrightarrow \bar{R}$ that becomes a unit morphism in $\mathrm{Ho}(C)$. By factoring $\bar{u}$ as a weak equivalence followed by a fibration we can moreover assume without loss of generality that $\bar{u}$ is a fibration. We form a

pullback

$$\begin{tikzcd} R^\times \arrow[r, "u"] \arrow[d, "p"'] & R \arrow[d, "q", "\sim"'] \\ \bar{R}^\times \arrow[r, "\bar{u}"'] & \bar{R} \end{tikzcd}$$

Since q is a weak equivalence, $\bar{u}$ a fibration and C is right proper, the base change p of q is also a weak equivalence. So u is isomorphic to $\bar{u}$ in the arrow category in $\mathrm{Ho}(C)$, hence u is also a unit morphism of $\mathrm{Ho}(C)$. Part (iii) is dual to part (ii). □

Remark 2.5.12. We claim that unit morphisms and group completions also behave nicely on derived mapping spaces. We explain this in detail for unit morphisms, the other case being dual, one more time. Model categories have derived mapping spaces (i.e., simplicial sets) $\mathrm{map}^h(-,-)$, giving well-defined homotopy types such that

$$\pi_0(\mathrm{map}^h(T,R)) \;\cong\; \mathrm{Ho}(C)(T,R)\,, \tag{2.5.13}$$

compare [80, Sec. 5.4] or [78, Ch. 18]. We let $u : R^\times \longrightarrow R$ be a C-morphism that becomes a unit morphism in $\mathrm{Ho}(C)$, and T any other object of C. Because of the bijection (2.5.13) the map

$$u_* \;:\; \mathrm{map}^h(T,R^\times) \;\longrightarrow\; \mathrm{map}^h(T,R)$$

lands in the subspace $\mathrm{map}^{h,\times}(T,R)$, defined as the union of those path components that represent invertible elements in the monoid $\mathrm{Ho}(C)(T,R)$. We claim that u_* is a weak equivalence onto the subspace $\mathrm{map}^{h,\times}(T,R)$. To see this we exploit the fact that both $\mathrm{map}^h(T,R^\times)$ and $\mathrm{map}^{h,\times}(T,R)$ are group-like H-spaces, the multiplication arising from the fact T is a comonoid object up to homotopy. Moreover, the map u_* is an H-map and bijection on path components (by the universal property of unit morphisms and the bijection (2.5.13)). So it suffices to show that the restriction of u to the identity path components is a weak equivalence. For this it suffices in turn to show that the looped map

$$\Omega(u_*) \;:\; \Omega(\mathrm{map}^h(T,R^\times)) \;\longrightarrow\; \Omega(\mathrm{map}^{h,\times}(T,R))$$

is a weak equivalence. But this map is weakly equivalent to

$$(\Omega u)_* \;:\; \mathrm{map}^h(T,\Omega(R^\times)) \;\longrightarrow\; \mathrm{map}^h(T,\Omega R)\,.$$

Since Ωu is an isomorphism in $\mathrm{Ho}(C)$ (by Proposition 2.5.9 (ii)) it is a weak equivalence in C, hence so is the induced map on derived mapping spaces.

The next proposition will be used to show that loops on the bar construction provide functorial global group completions of ultra-commutative monoids.

Proposition 2.5.14. *Let C be a pointed model category whose homotopy category is pre-additive. Suppose that for every group-like object R of C the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism in* $\mathrm{Ho}(C)$*. Then for every object R of C the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is a group completion.*

Proof We let $i : R \longrightarrow R^\star$ be a group completion, which exists by Theorem 2.5.11. In the commutative square in $\mathrm{Ho}(C)$

$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[d, "i"'] & \Omega(\Sigma R) \arrow[d, "\Omega(\Sigma i)", "\cong"'] \\ R^\star \arrow[r, "\cong", "\eta_{R^\star}"'] & \Omega(\Sigma(R^\star)) \end{tikzcd}$$

the lower horizontal morphism is an isomorphism by hypothesis because $R^\star$ is group-like. The morphism $\Sigma i : \Sigma R \longrightarrow \Sigma(R^\star)$ is an isomorphism by Proposition 2.5.9 (iii), hence the right vertical morphism $\Omega(\Sigma i)$ is also an isomorphism. So η_R is isomorphic, as an object in the comma category $R \downarrow \mathrm{Ho}(C)$, to i, and hence also a group completion. □

The previous proposition also has a dual statement (with the dual proof): if for every group-like object R of C the adjunction counit $\epsilon : \Sigma(\Omega R) \longrightarrow R$ is an isomorphism in $\mathrm{Ho}(C)$, then ϵ is a unit morphism. In practice, however, this dual formulation is less useful: in the important examples that arise 'in nature', for example for ultra-commutative monoids, the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism for all group-like objects R, whereas the adjunction counit $\epsilon : \Sigma(\Omega R) \longrightarrow R$ is *not* always an isomorphism.

Now we specialize the theory of units and group completions to ultra-commutative monoids. We recall that the category of ultra-commutative monoids has the trivial monoid as zero object, and the canonical morphism $\rho_{R,S} : R \boxtimes S \longrightarrow R \times S$ from the coproduct to the product of two ultra-commutative monoids is a global equivalence by Theorem 1.3.2 (i). So the homotopy category $\mathrm{Ho}(umon)$ is pre-additive.

Definition 2.5.15. An ultra-commutative monoid R is *group-like* if it is group-like as an object of the pre-additive category $\mathrm{Ho}(umon)$. A morphism $u : R^\times \longrightarrow R$ of ultra-commutative monoids is a *global unit morphism* if it is a unit morphism in the pre-additive category $\mathrm{Ho}(umon)$. A morphism $i : R \longrightarrow R^\star$ of ultra-commutative monoids is a *global group completion* if it is a group completion in the pre-additive category $\mathrm{Ho}(umon)$.

The global model structure on the category of ultra-commutative monoids is proper, see Theorem 2.1.15. Theorem 2.5.11 thus guarantees that every ultra-

commutative monoid R admits a global unit morphism $u : R^\times \longrightarrow R$ and a global group completion $i : R \longrightarrow R^\star$.

As a reality check we show that for ultra-commutative monoids R, the abstract definition of 'group-like' is equivalent to the requirement that all the abelian monoids $\pi_0^G(R)$ are groups. This part works more generally for all orthogonal monoid spaces, not necessarily commutative. A monoid M (not necessarily abelian) is a group if and only if the shearing map

$$\chi \ : \ M^2 \ \longrightarrow \ M^2 \ , \quad (x,y) \ \longmapsto \ (x,xy)$$

is bijective. Indeed, if M is a group, then the map $(x,z) \mapsto (x, x^{-1}z)$ is inverse to χ. Conversely, if χ is bijective, then for every $x \in M$ there is a $y \in M$ such that $\chi(x,y) = (x,1)$, i.e., with $xy = 1$. Then $\chi(x,yx) = (x,xyx) = (x,x) = \chi(x,1)$, so $yx = 1$ by injectivity of χ. Thus y is a two-sided inverse for x.

For orthogonal monoid spaces R (not necessarily commutative), the group-like condition has a similar characterization as follows. The *shearing morphism* is the morphism of orthogonal spaces

$$\chi = (\rho_1, \mu) \ : \ R \boxtimes R \ \longrightarrow \ R \times R$$

whose first component is the projection ρ_1 to the first factor and whose second component is the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$.

The multiplication morphism $\mu : R \boxtimes R \longrightarrow R$, and hence the shearing morphism χ, is a homomorphism of orthogonal monoid spaces only if R is commutative.

Proposition 2.5.16. *Let R be an orthogonal monoid space. Then the following two conditions are equivalent:*

(i) *The shearing morphism $\chi : R \boxtimes R \longrightarrow R \times R$ is a global equivalence of orthogonal spaces.*
(ii) *For every compact Lie group G the monoid $\pi_0^G(R)$ is a group.*

For commutative orthogonal monoid spaces, conditions (i) *and* (ii) *are moreover equivalent to being group-like in the pre-additive homotopy category of ultra-commutative monoids.*

Proof (i)$\Longrightarrow$(ii) The vertical maps in the commutative diagram

$$\begin{tikzcd}
\pi_0^G(R \boxtimes R) \arrow[r, "\pi_0^G(\chi)"] \arrow[d, "(\pi_0^G(\rho_1){,}\pi_0^G(\rho_2))"', "\cong"] & \pi_0^G(R \times R) \arrow[d, "\cong"', "(\pi_0^G(p_1){,}\pi_0^G(p_2))"] \\
\pi_0^G(R) \times \pi_0^G(R) \arrow[r, "(x{,}y) \longmapsto (x{,}xy)"'] & \pi_0^G(R) \times \pi_0^G(R)
\end{tikzcd} \tag{2.5.17}$$

are bijective by Corollary 1.5.20. If the shearing morphism is a global equivalence, then the map $\pi_0^G(\chi)$ is bijective, hence so is the algebraic shearing map of the monoid $\pi_0^G(R)$. This monoid is thus a group.

(ii)$\Longrightarrow$(i) Now we assume that all the monoids $\pi_0^G(R)$ are groups. We assume first that R is flat as an orthogonal space; then $R \boxtimes R$ is also flat, by Proposition 1.4.12 (i) for the global family of all compact Lie groups. The product $R \times R$ is also flat, by Proposition 1.3.9. Since $R \boxtimes R$ and $R \times R$ are flat, they are also closed as orthogonal spaces by Proposition 1.2.11 (iii). We may thus show that for every compact Lie group G the continuous map

$$\chi^G \ = \ \chi(\mathcal{U}_G)^G \ : \ (R \boxtimes R)(\mathcal{U}_G)^G \ \longrightarrow \ (R \times R)(\mathcal{U}_G)^G \ = \ R(\mathcal{U}_G)^G \times R(\mathcal{U}_G)^G$$

is a weak equivalence, compare Proposition 1.1.17. Since the monoid $\pi_0^G(R)$ has inverses, the shearing morphism $\chi : R \boxtimes R \longrightarrow R \times R$ induces a bijection on π_0^G, by the commutative diagram (2.5.17) with vertical bijections. On path components we have

$$\pi_0^G(R \boxtimes R) \ \cong \ \pi_0((R \boxtimes R)(\mathcal{U}_G)^G) \quad \text{and} \quad \pi_0^G(R \times R) \ \cong \ \pi_0((R \times R)(\mathcal{U}_G)^G) \ ,$$

compare Corollary 1.5.7. So we conclude that the map χ^G induces a bijection on path components.

Now we show that χ^G also induces bijections on homotopy groups in positive dimensions. We consider a point $x \in ((R \boxtimes R)(\mathcal{U}_G))^G$ and $k \geq 1$. We let $\varphi : \mathcal{U}_G^2 \longrightarrow \mathcal{U}_G$ be any G-equivariant linear isometric embedding. As we explained in Remark 2.1.2, this map induces an H-space structure on $(R \boxtimes R)(\mathcal{U}_G)^G$, and hence a continuous map

$$\varphi_*(-, x)^G \ : \ (R \boxtimes R)(\mathcal{U}_G)^G \ \longrightarrow \ (R \boxtimes R)(\mathcal{U}_G)^G \ .$$

Since the unit element 1 is a homotopy unit for the H-space structure, the element $x' = \varphi_*(1, x)$ belongs to the same path component as x. Since the monoid $\pi_0(R(\mathcal{U}_G)^G)$ is isomorphic to the group $\pi_0^G(R)$, the H-space structure has inverses. So the map $\varphi_*(-, x)^G$ is a homotopy equivalence. The same argument applies to $R \times R$ instead of $R \boxtimes R$, and we obtain a commutative diagram

$$\begin{tikzcd}
\pi_k^G((R \boxtimes R)(\mathcal{U}_G)^G, 1) \arrow[r, "\pi_k(\chi^G, 1)"] \arrow[d, "\pi_k(\varphi_*(-,x)^G)"', "\cong"] & \pi_k^G((R \times R)(\mathcal{U}_G)^G, 1) \arrow[d, "\cong"', "\pi_k(\varphi_*(-,\chi^G(x))^G)"] \\
\pi_k^G((R \boxtimes R)(\mathcal{U}_G)^G, x') \arrow[r, "\pi_k(\chi^G, x')"'] & \pi_k^G((R \times R)(\mathcal{U}_G)^G, \chi^G(x'))
\end{tikzcd}$$

in which both vertical maps are bijective. So to show that the map χ^G induces bijections of homotopy groups based at x, it suffices to show this for the special case $x = 1$ of the unit element.

Now the map $\pi_k(\chi^G, 1)$ is a group homomorphism such that the composite

$$(\pi_k(R(\mathcal{U}_G)^G, 1))^2 \xrightarrow{(x,y)\mapsto x\times y} \pi_k((R \boxtimes R)(\mathcal{U}_G)^G, 1)$$
$$\xrightarrow{\pi_k(\chi^G,1)} \pi_k((R \times R)(\mathcal{U}_G)^G, 1) \xrightarrow{((\rho_1)_*,(\rho_2)_*)} (\pi_k(R(\mathcal{U}_G)^G, 1))^2$$

sends (x, y) to $(x, \mu_*(x \times y))$, where $\mu : R \boxtimes R \longrightarrow R$ is the multiplication map. By the Eckmann-Hilton argument, $\mu_*(x \times y) = xy$, the product with respect to the group structure of $\pi_k(R(\mathcal{U}_G)^G, 1)$. The first and third maps are bijective, and so is the composite (because $\pi_k(R(\mathcal{U}_G)^G, 1)$ is a group). So the middle map is bijective. Altogether this shows that the map χ^G is a weak equivalence.

For general R we choose a global equivalence $f : R' \longrightarrow R$ of orthogonal monoid spaces such that R' is flat as an orthogonal space. One way to arrange this is by cofibrant replacement in the global model structure of orthogonal monoid spaces (Corollary 1.4.15 (ii) with $R = *$ and $\mathcal{F} = \mathcal{A}ll$). Then $f \boxtimes f$ is a global equivalence by Theorem 1.3.2 and $f \times f$ is a global equivalence by Proposition 1.1.9 (vi). Since $\chi^{R'}$ is a global equivalence by the previous paragraph and $\chi^R \circ (f \boxtimes f) = (f \times f) \circ \chi^{R'}$, the morphism χ^R is also a global equivalence.

Finally, if R is ultra-commutative, then the point-set level shearing morphism χ becomes the shearing morphism in the sense of Proposition 2.5.3 (a) in the pre-additive homotopy category Ho(*umon*). So χ is a global equivalence if and only if the shearing morphism in Ho(*umon*) is an isomorphism, i.e., precisely when R is group-like. □

Now we look more closely at global unit morphisms, and we give an explicit, functorial point-set level construction. For elements in an abelian monoid M, left inverses are automatically right inverses, and they are unique (if they exist). So the subgroup of invertible elements of an abelian monoid is isomorphic to the kernel of the multiplication map, by

$$M^\times \cong \ker(+ : M^2 \longrightarrow M) , \quad x \longmapsto (x, -x) .$$

Proposition 2.5.7 (i) gives an abstract formulation of this and explains how an abstract kernel of the multiplication map gives rise to a unit morphism. The proof of Theorem 2.5.11 then shows that the homotopy fiber of the multiplication map, formed at the model category level, constructs such a kernel. If we make all this explicit for the model category of ultra-commutative monoids, we arrive at the following construction.

Construction 2.5.18 (Units of an ultra-commutative monoid)**.** We introduce a functorial point-set level construction of the global units of an ultra-commutative monoid R. We define $R^\times$ as the homotopy fiber, over the additive unit ele-

ment 0, of the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$, i.e.,

$$R^\times \ = \ F(\mu) \ = \ (R \boxtimes R) \times_\mu R^{[0,1]} \times_R \{0\} \ .$$

So at an inner product space V, we have

$$(R^\times)(V) \ = \ (R \boxtimes R)(V) \times_{\mu(V)} R(V)^{[0,1]} \times_{R(V)} \{0\} \ ,$$

the space of pairs (x, ω) consisting of a point $x \in (R \boxtimes R)(V)$ and a path $\omega : [0,1] \longrightarrow R(V)$ such that $\mu(V)(x) = \omega(0)$ and $\omega(1) = 0$, the unit element in $R(V)$. Since limits and cotensors with topological spaces of ultra-commutative monoids are formed on underlying orthogonal spaces, this homotopy fiber inherits a preferred structure of ultra-commutative monoid.

We claim that the composite

$$u \ : \ R^\times \xrightarrow{\ p\ } R \boxtimes R \xrightarrow{\ \rho_1\ } R$$

is a global unit morphism, where p is the projection onto the first factor. Indeed, the commutative square

$$\begin{tikzcd}
R^\times \arrow[r, "p"] \arrow[d, "q"'] & R \boxtimes R \arrow[d, "\mu"] \\
R^{[0,1]} \times_R \{0\} \arrow[r, "\mathrm{ev}_0"'] & R
\end{tikzcd}$$

is a pullback of ultra-commutative monoids, by definition, and both horizontal morphisms are strong level fibrations, where q denotes the projection to the second factor. So the square is homotopy cartesian. The multiplication morphism μ has a section, so Proposition 2.5.10 (i) shows that the morphism $p : R^\times \longrightarrow R \boxtimes R$ becomes a kernel of the multiplication morphism in the homotopy category Ho(*umon*). So u is a unit morphism by Proposition 2.5.7 (i).

We recall from Example 2.2.17 that every global power monoid M has a global power submonoid $M^\times$ of *units*; the value $M^\times(G)$ at a compact Lie group G is the group of invertible elements of $M(G)$. The next proposition verifies that global unit morphisms have the expected behavior on $\underline{\pi}_0$.

Proposition 2.5.19. *Let $u : R^\times \longrightarrow R$ be a unit morphism of ultra-commutative monoids. Then the morphism of global power monoids*

$$\underline{\pi}_0(u) \ : \ \underline{\pi}_0(R^\times) \ \longrightarrow \ \underline{\pi}_0(R)$$

is an isomorphism onto the global power submonoid $(\underline{\pi}_0(R))^\times$ of units of $\underline{\pi}_0(R)$.

Proof This is a formal consequence of the fact that the functor π_0^G from the homotopy category of ultra-commutative monoids is representable. We let G be a compact Lie group and V a non-zero faithful G-representation. Then the

global classifying space $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ supports the tautological class $u_{G,V} \in \pi_0^G(B_{\mathrm{gl}}G)$, compare (1.5.11). We recall that

$$u_G^{umon} \;=\; \eta_*(u_{G,V}) \;\in \pi_0^G(\mathbb{P}(B_{\mathrm{gl}}G))$$

where $\eta : B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ is the inclusion of the linear summand. We claim that evaluation at u_G^{umon} is an isomorphism of abelian monoids

$$\mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), T) \;\cong\; \pi_0^G(T)\,, \quad [f] \;\longmapsto\; f_*(u_G^{umon})$$

for every ultra-commutative monoid T. Indeed, both sides take global equivalence in T to isomorphisms, so we may assume that T is fibrant in the global model structure of ultra-commutative monoids, hence positively static. Now we consider the composite

$$\begin{aligned}\pi_0(T(V)^G) \;\cong\; \pi_0(\mathrm{map}^{umon}(\mathbb{P}(B_{\mathrm{gl}}G), T)) &\xrightarrow{\gamma_*} \\ \mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), T) &\xrightarrow{[f]\mapsto f_*(u_G^{umon})} \pi_0^G(T)\end{aligned}$$

with the adjunction bijection and the map induced by the localization functor $\gamma : umon \longrightarrow \mathrm{Ho}(umon)$. Since V is non-zero and faithful, $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ is positively flat, so $\mathbb{P}(B_{\mathrm{gl}}G)$ is cofibrant in the global model structure of ultra-commutative monoids. Since $\mathbb{P}(B_{\mathrm{gl}}G)$ is cofibrant and T is fibrant, the middle map is bijective. The composite is the canonical map $\pi_0(T(V)^G) \longrightarrow \pi_0^G(T)$ to the colimit, which is bijective because T is positively static.

The evaluation isomorphism is natural in the second variable, so we arrive at a commutative square of abelian monoids

$$\begin{tikzcd}
\mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), R^\times) \arrow[r, "\cong"] \arrow[d, "u_*"'] & \pi_0^G(R^\times) \arrow[d, "u_*"] \\
\mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), R) \arrow[r, "\cong"'] & \pi_0^G(R)
\end{tikzcd}$$

in which both horizontal maps are bijective. The left vertical map is injective with image the subgroup of invertible elements; hence the same is true for the right vertical map. □

Now we turn to global group completions of ultra-commutative monoids. Every ultra-commutative monoid has a group completion in the homotopy category, by Theorem 2.5.11 (i). Even better: since the model category of ultra-commutative monoids is left proper, every ultra-commutative monoid is the source of a global group completion in the model category of ultra-commutative monoids, by Theorem 2.5.11 (iii). Now we discuss two functorial point-set level constructions of global group completions. The first one is dual to Construction 2.5.18 of global units.

Construction 2.5.20 (Global group completion of an ultra-commutative monoid)**.** We let R be an ultra-commutative monoid that is cofibrant in the global model structure of Theorem 2.1.15. We define the cone of R as a pushout in the category of ultra-commutative monoids:

$$\begin{tikzcd} R \otimes \{0\} \arrow[r] \arrow[d, "R \otimes \mathrm{incl}"'] & * \arrow[d] \\ R \otimes [0,1] \arrow[r] & CR \end{tikzcd}$$

Here $\otimes$ is the tensor of an ultra-commutative monoid with a topological space, as explained in Construction 2.1.6 (not to be confused with the objectwise product of an orthogonal space with a space). So the cone is $R \triangleright [0,1]$, the tensor of R with the based space $([0,1],0)$, as defined more generally in (2.1.9). Since R is cofibrant and the global model structure is topological, the left vertical morphism is an acyclic cofibration, and so the cone CR is globally equivalent to the zero monoid.

We can then construct a global group completion as a homotopy cofiber of the diagonal morphism $\Delta : R \longrightarrow R \times R$, i.e., a pushout in the category of ultra-commutative monoids:

$$\begin{tikzcd} R \arrow[r, "\Delta"] \arrow[d] & R \times R \arrow[d, "d"] \\ R \triangleright [0,1] \arrow[r] & R^\star \end{tikzcd}$$

The left vertical morphism is induced by $1 \in [0,1]$, and it is a cofibration since R is cofibrant. We claim that the composite

$$i \ : \ R \xrightarrow{(\mathrm{Id},0)} R \times R \xrightarrow{d} R^\star$$

is a global group completion. Indeed, the square above is homotopy cocartesian by construction and the diagonal morphism has a retraction. So Proposition 2.5.10 (ii) shows that the morphism $d : R \times R \longrightarrow R^\star$ becomes a cokernel of the diagonal morphism in the homotopy category Ho(*umon*). So i is a global group completion by Proposition 2.5.7 (ii).

The previous construction of the global group completion can be rewritten as a two-sided bar construction as follows. The unit interval $[0,1] = \Delta^1$ is also the topological 1-simplex, and hence canonically homeomorphic to the geometric realization of the simplicial 1-simplex $\Delta[1]$, by

$$[0,1] \longrightarrow |\Delta[1]| \, , \quad t \longmapsto [\mathrm{Id}_{[1]}, t] \, .$$

So Proposition 2.1.10 shows that $CR = R \triangleright [0,1]$ is effectively a bar construction of R with respect to the box product, i.e., the internal realization of

the simplicial object of ultra-commutative monoids $B(R, \Delta[1])$, where $\Delta[1]$ is pointed by the vertex 0. Since $\Delta[1]$ has $(m+1)$ non-basepoint simplices of dimension m, expanding this yields

$$B_m(R, \Delta[1]) \;=\; R \triangleright \Delta[1]_m \;\cong\; R^{\boxtimes(m+1)} ,$$

with simplicial structure induced by that of $\Delta[1]$.

Since the internal realization of simplicial ultra-commutative monoids is a coend, it commutes with pushouts. So the defining pushout for $R^\star$ can be rewritten as the realization of a simplicial ultra-commutative monoid, the two-sided bar construction with respect to the box product:

$$R^\star \;=\; (R \triangleright [0,1]) \boxtimes_R (R \times R) \;\cong\; B(R, \Delta[1]) \boxtimes_R (R \times R) \;\cong\; B^{\boxtimes}(*, R, R \times R) .$$

In simplicial dimension m, this bar construction is given by

$$B^{\boxtimes}_m(*, R, R \times R) \;=\; R^{\boxtimes m} \boxtimes (R \times R) ;$$

the simplicial face morphisms are given by projections away from the first factor (for d_0), the multiplication of R on two adjacent factors (for $d_1, \dots, d_{m-1}$) and the action of R on $R \times R$ through the diagonal. The simplicial degeneracy morphisms are inserting the unit of R. In the context of topological monoids, this bar construction of a group completion for ‘sufficiently homotopy commutative’ monoids is sketched by Segal on p. 305 of [153].

The next proposition is a reality check, showing that global group completion has the expected effect on equivariant homotopy sets.

Proposition 2.5.21. *Let $i : R \longrightarrow R^\star$ be a global group completion of ultra-commutative monoids. Then for every compact Lie group G the map*

$$\pi_0^G(i) \;:\; \pi_0^G(R) \;\longrightarrow\; \pi_0^G(R^\star)$$

is an algebraic group completion of abelian monoids and

$$\underline{\pi}_0(i) \;:\; \underline{\pi}_0(R) \;\longrightarrow\; \underline{\pi}_0(R^\star)$$

is a group completion of global power monoids.

Proof We may assume that R is cofibrant in the global model structure of ultra-commutative monoids of Theorem 2.1.15. As we explained in Construction 2.5.20, a group completion $R^\star$ can then be constructed as the realization of a certain simplicial ultra-commutative monoid, the two-sided bar construction $B^{\boxtimes}(*, R, R \times R)$, where R acts on $R \times R$ through the diagonal morphism. By Proposition 2.1.7, the realization can equivalently be taken internal to the

category of ultra-commutative monoids, or in the underlying category of orthogonal spaces. The 'underlying' realization is the sequential colimit of partial realizations $B^{[n]}$, i.e., 'skeleta' in the simplicial direction, defined as the coend

$$B^{[n]} = \int^{[m]\in\mathbf{\Delta}_{\leq n}} B^{\boxtimes}_m(*, R, R\times R)\times\Delta^m$$

of the restriction to the full subcategory $\mathbf{\Delta}_{\leq n}$ of $\mathbf{\Delta}$ with objects all $[m]$ with $m\leq n$. Since $\mathbf{\Delta}_{\leq n}$ is contained in $\mathbf{\Delta}_{\leq n+1}$, there is a canonical morphism $B^{[n]}\longrightarrow B^{[n+1]}$ and the realization $B^{\boxtimes}(*, R, R\times R)$ is the colimit of the sequence of orthogonal spaces

$$R\times R = B^{[0]} \longrightarrow B^{[1]} \longrightarrow \cdots \longrightarrow B^{[n]} \longrightarrow \cdots .$$

The 1-skeleton $B^{[1]}$ is the pushout of orthogonal spaces:

$$\begin{tikzcd} R\boxtimes(R\times R)\times\{0,1\} \arrow[r, "\mathrm{incl}"] \arrow[d] & R\boxtimes(R\times R)\times[0,1] \arrow[d] \\ R\times R \arrow[r] & B^{[1]} \end{tikzcd}$$

The orthogonal space $R\boxtimes(R\times R)\times\{0,1\}$ is the disjoint union of two copies of $R\boxtimes(R\times R)$, and the left vertical map is projection to $R\times R$ on one copy; on the other copy the morphism has the two components

$$\mu\circ(R\boxtimes p_1)\,,\ \mu\circ(R\boxtimes p_2)\ : R\boxtimes(R\times R) \longrightarrow R\,,$$

where $\mu : R\boxtimes R\longrightarrow R$ is the multiplication and $p_1, p_2 : R\times R\longrightarrow R$ are the two projections. The functor π_0^G takes both the box product and the product of orthogonal spaces to products of sets, by Corollary 1.5.20. So for a compact Lie group G, the set $\pi_0^G(B^{[1]})$ is the coequalizer of the two maps

$$\pi_0^G(R)\times\pi_0^G(R)\times\pi_0^G(R) \underset{b}{\overset{a}{\rightrightarrows}} \pi_0^G(R)\times\pi_0^G(R)$$

given by

$$a(x,y,z) = (y,z) \qquad\text{and}\qquad b(x,y,z) = (x+y, x+z)\,.$$

The equivalence relation on the set $\pi_0^G(R)\times\pi_0^G(R)$ generated by declaring $a(x,y,z)$ equivalent to $b(x,y,z)$ for all $(x,y,z)\in\pi_0^G(R)$ is precisely the equivalence relation that constructs the Grothendieck group of $\pi_0^G(R)$, compare Example 2.5.6. So $\pi_0^G(B^{[1]})$ bijects with the algebraic group completion of $\pi_0^G(R)$. For $n\geq 1$ the passage from $B^{[n]}$ to $B^{[n+1]}$ involves attaching simplices of dimension at least 2 along their boundary, and this process does not change the path components of G-fixed-points of the values at any G-representation. So the morphism $B^{[1]}\longrightarrow B^{\boxtimes}(*, R, R\times R)$ induces a bijection on π_0^G. This proves

that the morphism $R \longrightarrow B^{\boxtimes}(*, R, R\times R) = R^\star$ is a group completion of abelian monoids.

The second claim then follows because group completions of global power monoids are calculated 'groupwise', compare Example 2.2.18. □

For topological monoids, the loop space of the bar construction (see Construction 2.3.21) provides a functorial group completion. We will now explain that a similar construction also provides global group completion for ultra-commutative monoids, before passing to the homotopy category. Part of this works for arbitrary orthogonal monoid spaces, not necessarily commutative. We refer to Construction 1.2.34 for generalities about the realization of simplicial objects, in particular simplicial orthogonal spaces.

If R is an orthogonal monoid space, then the *bar construction* is the simplicial object of orthogonal spaces

$$B_\bullet(R) \ = \ ([n] \mapsto \ R^{\boxtimes n}) \ .$$

The simplicial face morphisms are induced by the multiplication in R, and the degeneracy morphisms are induced by the unit morphism of R, much like for the bar construction with respect to cartesian product (as opposed to box product) in Construction 2.3.21. The geometric realization in the category of orthogonal spaces is then the orthogonal space

$$B(R) \ = \ |B_\bullet(R)| \ . \qquad (2.5.22)$$

Geometric realization of orthogonal spaces is 'objectwise', i.e., for an inner product space V we have

$$B(R)(V) \ = \ |B_\bullet(R)(V)| \ ,$$

the realization of the simplicial space $[n] \mapsto R^{\boxtimes n}(V)$.

The next proposition shows that the bar construction of orthogonal monoid spaces preserves global equivalences under a mild non-degeneracy condition on the unit.

Definition 2.5.23. An orthogonal monoid space R has a *flat unit* if the unit morphism $* \longrightarrow R$ is a flat cofibration of orthogonal spaces.

The condition that the unit morphism $* \longrightarrow R$ is a flat cofibration is equivalent to the requirements that the underlying orthogonal space of R is flat and the unit map $* \longrightarrow R(0)$ is a cofibration of spaces.

We recall from Definition 1.2.36 that a simplicial orthogonal space X is *Reedy flat* if the latching morphism $l_n^\Delta : L_n^\Delta(X) \longrightarrow X_n$ in the simplicial direction is a flat cofibration of orthogonal spaces for every $n \geq 0$.

Proposition 2.5.24. (i) *For every orthogonal monoid space R with flat unit the simplicial orthogonal space $B_\bullet(R)$ is Reedy flat.*

(ii) *Let $f : R \longrightarrow S$ be a morphism of orthogonal monoid spaces with flat units. If f is a global equivalence, so is the morphism $B(f) : B(R) \longrightarrow B(S)$.*

Proof (i) The nth latching morphism $L_n^\Delta(B_\bullet(R)) \longrightarrow B_n(R)$ in the simplicial direction is the iterated pushout product

$$i^{\Box n} : Q^n(i) \longrightarrow R^{\boxtimes n}$$

with respect to the unit morphism $* \longrightarrow R$. Since this unit morphism is a flat cofibration, the pushout product property of the flat cofibrations shows that $i^{\Box n}$, and hence the latching morphism, is a flat cofibration for all $n \geq 0$.

(ii) Since R and S have flat units, the simplicial orthogonal spaces $B_\bullet(R)$ and $B_\bullet(S)$ are Reedy flat by part (i). Moreover, the morphism $B_n(f) = f^{\boxtimes n} : R^{\boxtimes n} \longrightarrow S^{\boxtimes n}$ is a global equivalence since f is and because the box product is homotopical for global equivalences (by Theorem 1.3.2). So the claim follows from the global invariance of realizations between Reedy flat simplicial orthogonal spaces (Proposition 1.2.37 (ii)). □

Remark 2.5.25 (Comparing bar constructions)**.** We let M be a monoid-valued orthogonal space in the sense of Definition 2.3.2. Then we have two bar constructions available: on the one hand we can take the bar construction objectwise as in Example 2.4.17, resulting in the orthogonal space $\mathbf{B}^\circ M$. On the other hand, we can first pass to the associated orthogonal monoid space as in (2.3.3), and then perform the bar construction with respect to the $\boxtimes$-multiplication as in (2.5.22). There is a natural comparison map: the symmetric monoidal transformation $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ defined in (1.3.1) has an analog for any finite number of factors, and for varying n, the morphisms

$$\rho_{M,\dots,M} : M^{\boxtimes n} \longrightarrow M^n$$

form a morphism of simplicial orthogonal spaces

$$\rho_\bullet : B_\bullet^\boxtimes(M) \longrightarrow B_\bullet^\times(M)$$

from the $\boxtimes$-bar construction to the $\times$-bar construction. If M has a flat unit, then the simplicial orthogonal spaces $B_\bullet^\boxtimes(M)$ and $B_\bullet^\times(M)$ are both Reedy flat. Indeed, for the former, this is Proposition 2.5.24 (i), and for the latter the same proof works because the categorical product of orthogonal spaces also satisfies the pushout product property with respect to flat cofibrations (Proposition 1.3.9). So $\rho_\bullet$ is a morphism between Reedy flat simplicial orthogonal spaces that is a global equivalence in every simplicial degree (by Theorem 1.3.2 (i)). The

induced morphism $|\rho_\bullet| : B(M) \longrightarrow \mathbf{B}^\circ M$ between the realizations is then a global equivalence by Proposition 1.2.37 (ii).

The canonical morphism

$$R \times [0,1] \;=\; B_1(R) \times \Delta^1 \;\longrightarrow\; |B_\bullet(R)| \;=\; B(R)$$

takes $R \times \{0,1\}$ to the basepoint, so it factors over a morphism of orthogonal spaces

$$R \wedge ([0,1]/\{0,1\}) \;\longrightarrow\; B(R)\,.$$

We let $u : S^1 \longrightarrow [0,1]/\{0,1\}$ be the composite homeomorphism

$$S^1 \xrightarrow{\;c\;} U(1) \xrightarrow{\;\log\;} [0,1]/\{0,1\}\,, \tag{2.5.26}$$

of the Cayley transform

$$c \;:\; S^1 \longrightarrow U(1)\,, \quad x \longmapsto (x+i)(x-i)^{-1}\,,$$

and the logarithm, i.e., the inverse of the exponential homeomorphism

$$[0,1]/\{0,1\} \;\cong\; U(1)\,, \quad t \longmapsto e^{2\pi i t}\,.$$

This yields a composite morphism

$$R \wedge S^1 \xrightarrow[\cong]{R \wedge u} R \wedge ([0,1]/\{0,1\}) \longrightarrow B(R)$$

which is adjoint to a morphism of orthogonal spaces

$$\eta_R \;:\; R \longrightarrow \Omega B(R)\,. \tag{2.5.27}$$

Proposition 2.5.28. *Let R be an orthogonal monoid space with flat unit. If for every compact Lie group G the monoid $\pi_0^G(R)$ is a group, then the morphism $\eta_R : R \longrightarrow \Omega B(R)$ is a global equivalence.*

Proof Since R has a flat unit, the simplicial orthogonal space $B_\bullet(R)$ is Reedy flat by Proposition 2.5.24 (i), so the underlying orthogonal space of $B(R)$ is flat by Proposition 1.2.37 (i). As a flat orthogonal space, $B(R)$ is in particular closed. The loop space functor preserves closed inclusions by [96, Prop. 7.7], so the pointwise loop space $\Omega B(R)$ is also closed as an orthogonal space. Since R and $\Omega B(R)$ are both closed orthogonal spaces, we can detect global equivalences on G-fixed-points, see Proposition 1.1.17.

So we let G be a compact Lie group. Then $((\Omega B(R))(\mathcal{U}_G))^G$ is homeomorphic to $\Omega\left((B(R)(\mathcal{U}_G))^G\right)$, which is in turn homeomorphic to the loop space of the geometric realization of the simplicial space

$$[n] \longmapsto (R^{\boxtimes n}(\mathcal{U}_G))^G\,. \tag{2.5.29}$$

We define $i_k : [1] \longrightarrow [n]$ by $i_k(0) = k - 1$ and $i_k(1) = k$. Then the morphism

$$(i_1^*, \dots, i_n^*) \ : \ R^{\boxtimes n} \ = \ B_n(R) \ \longrightarrow \ (B_1(R))^n \ = \ R^n$$

is precisely the morphism $\rho_{R,\dots,R} : R^{\boxtimes n} \longrightarrow R^n$, and hence a global equivalence by Theorem 1.3.2 (i). Since $\pi_0^G(R)$ is a group, the simplicial space (2.5.29) satisfies the hypotheses of Segal's theorem [153, Prop. 1.5]; so the adjoint of the canonical map

$$R(\mathcal{U}_G)^G \wedge S^1 \ \longrightarrow \ |[n] \mapsto ((R^{\boxtimes n})(\mathcal{U}_G))^G| \ \cong \ (B(R)(\mathcal{U}_G))^G$$

is a weak equivalence. Here we have used again the fact that fixed-points commute with geometric realization, see Proposition B.1 (iv). This adjoint is precisely the underlying map of G-fixed-points of the morphism $\eta_R : R \longrightarrow \Omega B(R)$. □

The previous proposition works for general orthogonal monoid spaces, not necessarily commutative; in that generality the bar construction $B(R)$ is an orthogonal space, but it does not have any natural multiplication. When we apply the bar construction to ultra-commutative monoids, then something special happens: since the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ is then a homomorphism of ultra-commutative monoids, $B_\bullet(R)$ is a simplicial object in the category of ultra-commutative monoids, i.e., a simplicial ultra-commutative monoid. The geometric realization $B(R)$ is then canonically an ultra-commutative monoid, and it coincides with the realization of $B_\bullet(R)$ internal to the category of ultra-commutative monoids, compare Proposition 2.1.7.

Moreover, we claim that for ultra-commutative monoids, the bar construction $B(R)$ is naturally isomorphic to $R \rhd S^1$, the 'suspension' of R internal to the category of ultra-commutative monoids. To see this we consider the 'simplicial circle' $\mathbf{S}^1$, the simplicial set given by

$$(\mathbf{S}^1)_n \ = \ \{0, 1, \dots, n\} \ ,$$

with face maps $d_i^* : (\mathbf{S}^1)_n \longrightarrow (\mathbf{S}^1)_{n-1}$ given by

$$d_i^*(j) \ = \ \begin{cases} j - 1 & \text{for } i < j, \text{ and} \\ j & \text{for } i \geq j \text{ and } j \neq n, \\ 0 & \text{for } i = j = n, \end{cases}$$

and degeneracy maps $s_i^* : (\mathbf{S}^1)_n \longrightarrow (\mathbf{S}^1)_{n+1}$ given by

$$s_i^*(j) \ = \ \begin{cases} j + 1 & \text{for } i < j, \text{ and} \\ j & \text{for } i \geq j. \end{cases}$$

The simplicial set $\mathbf{S}^1$ is based by 0; it is isomorphic to the simplicial 1-simplex

modulo its boundary, and its realization is homeomorphic to a circle, whence the name.

The 'obvious' isomorphisms

$$p_n \ : \ R^{\boxtimes n} \xrightarrow{\cong} R \triangleright \{0, 1, \ldots, n\} \ = \ R \triangleright (\mathbf{S}^1)_n \ = \ B_n(R, \mathbf{S}^1) \ ,$$

are compatible with the simplicial structure maps as n varies, so they define an isomorphism of simplicial ultra-commutative monoids

$$p_\bullet \ : \ B_\bullet(R) \ \cong \ B_\bullet(R, \mathbf{S}^1) \ .$$

When we specialize Proposition 2.1.10 to $A = \mathbf{S}^1$, we obtain an isomorphism of ultra-commutative monoids

$$R \triangleright |\mathbf{S}^1| \cong \ B(R) \ .$$

The homeomorphism $u \ : \ S^1 \ \longrightarrow \ [0, 1]/\{0, 1\}$ from (2.5.26) and the homeomorphism

$$[0, 1]/\{0, 1\} \xrightarrow{\cong} |\mathbf{S}^1| \ , \quad t \longmapsto [1, t]$$

turn this into an isomorphism of ultra-commutative monoids

$$R \triangleright S^1 \ \underset{\cong}{\xrightarrow{R \triangleright u}} \ R \triangleright ([0, 1]/\{0, 1\}) \ \cong \ R \triangleright |\mathbf{S}^1| \ \cong \ B(R) \tag{2.5.30}$$

whose adjoint $R \longrightarrow \Omega B(R)$ is the morphism η_R of (2.5.27).

Corollary 2.5.31. *For every ultra-commutative monoid with flat unit R the adjunction unit*

$$\eta_R \ : \ R \ \longrightarrow \ \Omega(R \triangleright S^1)$$

is a global group completion.

Proof We let R be a group-like cofibrant ultra-commutative monoid. Then R has a flat unit by Theorem 2.1.15 (ii a). Since ultra-commutative monoids form a topological model category, $R \triangleright S^1$ is an abstract suspension of R. The isomorphism (2.5.30) transforms the adjunction unit $R \longrightarrow \Omega(R \triangleright S^1)$ into the morphism $\eta_R : R \longrightarrow \Omega B(R)$ defined in (2.5.27). Proposition 2.5.28 shows that this adjunction unit is a global equivalence for every cofibrant group-like ultra-commutative monoid R. In the homotopy category Ho(*umon*) this implies that for every group-like ultra-commutative monoid R the derived adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism. So Proposition 2.5.14 shows that for every ultra-commutative monoid R the derived adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is a group completion in the pre-additive category Ho(*umon*). For cofibrant R the point-set level adjunction unit $R \longrightarrow \Omega(R \triangleright S^1)$ realizes the derived unit, hence the claim follows for every ultra-commutative monoid that is cofibrant in the global model structure of Theorem 2.1.15.

In the general case we choose a cofibrant replacement $q : R^c \longrightarrow R$ in the global model structure of Theorem 2.1.15, i.e., a global equivalence of ultra-commutative monoids with cofibrant source. Since R^c and R have flat units, the induced morphism of bar constructions $B(q) : B(R^c) \longrightarrow B(R)$ is a global equivalence by Proposition 2.5.24 (ii). Hence the morphism $q \triangleright S^1 : R^c \triangleright S^1 \longrightarrow R \triangleright S^1$ is a global equivalence, and so is $\Omega(q \triangleright S^1) : \Omega(R^c \triangleright S^1) \longrightarrow \Omega(R \triangleright S^1)$. The morphism $\eta_{R^c} : R^c \longrightarrow \Omega(R^c \triangleright S^1)$ is a global group completion by the previous paragraph. Since $\eta_R : R \longrightarrow \Omega(R \triangleright S^1)$ is isomorphic to the morphism η_{R^c} in the homotopy category Ho(*umon*), the morphism η_R is also a global group completion. □

An example of a global group completion that comes up naturally is the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ introduced in Example 2.4.2. The verification of the group completion property will be done through a homological criterion. For that purpose we define the homology groups of an orthogonal space Y as

$$H_*(Y^G; \mathbb{Z}) \ = \ \operatorname{colim}_{V \in s(\mathcal{U}_G)} \ H_*(Y(V)^G; \mathbb{Z}) \ .$$

Every global equivalence induces isomorphisms on $H_*((-)^G; \mathbb{Z})$ for all compact Lie groups G. Indeed, the functor $H_*((-)^G; \mathbb{Z})$ takes strong level equivalences to isomorphisms, which reduces the claim (by cofibrant approximation in the strong level model structure) to global equivalences $f : X \longrightarrow Y$ between flat orthogonal spaces. Flat orthogonal spaces are closed, so the global equivalence induces weak equivalences $f(\mathcal{U}_G)^G : X(\mathcal{U}_G)^G \longrightarrow Y(\mathcal{U}_G)^G$ on G-fixed-points. The poset $s(\mathcal{U}_G)$ is filtered, so homology commutes with this colimit, i.e.,

$$H_*(Y^G; \mathbb{Z}) \ \cong \ H_*(Y(\mathcal{U}_G)^G; \mathbb{Z}) \ .$$

Thus the morphism f also induces an isomorphism on $H_*((-)^G; \mathbb{Z})$.

The multiplication of an orthogonal monoid space R induces a graded multiplication on the homology groups $H_*(R^G; \mathbb{Z})$, by simultaneous passage to colimits in both variables of the maps

$$\begin{aligned} H_m(R(V)^G; \mathbb{Z}) \otimes H_n(R(W)^G; \mathbb{Z}) \ &\xrightarrow{\ \times\ } H_{m+n}(R(V)^G \times R(W)^G; \mathbb{Z}) \\ &\xrightarrow{((\mu_{V,W})^G)_*} H_{m+n}(R(V \oplus W)^G; \mathbb{Z}) \ . \end{aligned}$$

Assigning to a path component its homology class is a map

$$\pi_0(R(V)^G) \ \longrightarrow \ H_0(R(V)^G; \mathbb{Z})$$

compatible with increasing V. On colimits over $s(\mathcal{U}_G)$ this provides a map

$$\pi_0^G(R) \ \longrightarrow \ H_0(R^G; \mathbb{Z}) \ .$$

This map takes the addition in R to the multiplication in $H_0(R^G;\mathbb{Z})$, so its image is a multiplicative subset of $H_0(R^G;\mathbb{Z})$. If the multiplication of R is commutative, then the product of $H_*(R^G;\mathbb{Z})$ is commutative in the graded sense. In particular, the multiplicative subset of $\pi_0^G(R)$ is then automatically central.

Proposition 2.5.32. *A morphism $i : R \longrightarrow R^\star$ of ultra-commutative monoids is a global group completion if and only if the following two conditions are satisfied.*

(i) *The ultra-commutative monoid $R^\star$ is group-like, and*
(ii) *for every compact Lie group G the map of graded commutative rings*

$$H_*(i^G;\mathbb{Z}) \ : \ H_*(R^G;\mathbb{Z}) \ \longrightarrow \ H_*((R^\star)^G;\mathbb{Z})$$

is a localization at the multiplicative subset $\pi_0^G(R)$ of $H_0(R^G;\mathbb{Z})$.

Proof We start by showing that a global group completion satisfies properties (i) and (ii). Property (i) holds by definition of 'group completion'. We give two alternative proofs for why a global group completion satisfies property (ii), based on the two different bar construction models in Construction 2.5.20 and Corollary 2.5.31.

The first argument uses the loop space of the bar construction $B(R)$, which is isomorphic to internal suspension $R \triangleright S^1$. By Corollary 2.5.31 it suffices to show that for every cofibrant ultra-commutative monoid R the morphism $\eta_R : R \longrightarrow \Omega(R \triangleright S^1) = \Omega(BR)$ has property (ii). Since R is cofibrant and the global model structure of ultra-commutative monoids is topological (Theorem 2.1.15), the basepoint inclusion of S^1 induces a cofibration

$$R \otimes \mathrm{incl} \ : \ R \otimes \{\infty\} \ \longrightarrow \ R \otimes S^1 .$$

The cobase change is the unique morphism $* \longrightarrow R \triangleright S^1$ from the terminal ultra-commutative monoid to the reduced tensor, and this is thus a cofibration. In other words, $B(R) = R \triangleright S^1$ is again cofibrant as an ultra-commutative monoid. Since R and $B(R)$ are cofibrant as ultra-commutative monoids, Theorem 2.1.15 (ii) shows that their underlying orthogonal spaces are flat, hence closed. The loop space functor preserves closed inclusions by [96, Prop. 7.7], so the pointwise loop space $\Omega B(R)$ is also closed as an orthogonal space.

Since R and $\Omega B(R)$ are closed as orthogonal spaces, it suffices to show that for every compact Lie group G the map

$$H_*(R(\mathcal{U}_G)^G;\mathbb{Z}) \ \longrightarrow \ H_*((\Omega B(R))(\mathcal{U}_G)^G;\mathbb{Z})$$

is a localization at the multiplicative subset $\pi_0(R(\mathcal{U}_G)^G)$ of the source. Since the H-space structure of $R(\mathcal{U}_G)^G$ comes from the action of an E_∞-operad, the graded ring $H_*(R(\mathcal{U}_G)^G;\mathbb{Z})$ is graded commutative. We can thus apply

Quillen's group completion theorem from the unpublished, but widely circulated preprint *'On the group completion of a simplicial monoid'*. Quillen's manuscript was later published as Appendix Q of the Friedlander-Mazur paper [56], where the relevant theorem appears on page 104 in Section Q.9.

The second, alternative, argument first reduces to cofibrant ultra-commutative monoids by cofibrant approximation in the global model of Theorem 2.1.15. As explained in Construction 2.5.20, a group completion $R^\star$ can then be constructed as the homotopy cofiber of the diagonal $\Delta : R \longrightarrow R \times R$, which is concretely given by the geometric realization of the simplicial ultra-commutative monoid $B^{\boxtimes}(*, R, R \times R)$, the bar construction with respect to the box product. Segal [153, p. 305 f] sketches an argument why the resulting morphism $R \longrightarrow B^{\boxtimes}(*, R, R \times R)$ is localization on homology with field coefficients. The argument is reproduced in more detail in the proof of [47, Lemma 3.2.2.1].

Now we prove the reverse implication. We let $i : R \longrightarrow R^\star$ be a morphism of ultra-commutative monoids that satisfies properties (i) and (ii); we need to show that i is a global group completion. We assume first that both R and $R^\star$ are cofibrant in the global model structure of ultra-commutative monoids of Theorem 2.1.15. Then the unit morphisms of R and $R^\star$ are flat cofibrations of underlying orthogonal spaces by Theorem 2.1.15 (ii). So the morphism $\eta_R : R \longrightarrow \Omega(R \triangleright S^1) = \Omega B(R)$ is a global group completion by Corollary 2.5.31. Since $R^\star$ is group-like, the morphism $\eta_{R^\star}$ is a global equivalence by Proposition 2.5.28.

We claim that the morphism $B(i) : B(R) \longrightarrow B(R^\star)$ is a global equivalence. For every coefficient system L on $(B(R^\star))(\mathcal{U}_G)^G$ we compare the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{H_*(R^G;k)}(k, L) \implies H_*((B(R))(\mathcal{U}_G)^G; L)$$

(obtained by filtering the bar construction by simplicial skeleta) with the analogous one for the homology of $(B(R^\star))(\mathcal{U}_G)^G$. The localization hypothesis implies that the map of Tor groups

$$\mathrm{Tor}_p^{H_*(R^G;k)}(k, L) \longrightarrow \mathrm{Tor}_p^{H_*((R^\star)^G;k)}(k, L)$$

is an isomorphism, see for example [140, Prop. 7.17] or [187, Prop. 3.2.9]. So we have a morphism of first quadrant spectral sequences that is an isomorphism of E^2-terms; the map on abutments is then an isomorphism as well. This shows that $B(i) : B(R) \longrightarrow B(R^\star)$ is a global equivalence. Since looping preserves global equivalences, the morphism $\Omega B(i)$ is a global equivalence.

Now we contemplate the commutative square

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "\eta_R"'] & R^\star \arrow[d, "\eta_{R^\star}"] \\ \Omega B(R) \arrow[r, "\Omega B(i)"'] & \Omega B(R^\star) \end{tikzcd}$$

The left vertical morphism is a global group completion, and the lower horizontal and right vertical morphisms are global equivalences. So the upper horizontal morphism i is global group completion.

Now we reduce the general case to the special case by cofibrant replacement. We choose a cofibrant replacement $q : R^c \longrightarrow R$ in the global model structure of ultra-commutative monoids of Theorem 2.1.15, and then factor the morphism $iq : R^c \longrightarrow R^\star$ as a cofibration $i^c : R^c \longrightarrow R^\dagger$ followed by a global equivalence $\varphi : R^\dagger \longrightarrow R^\star$. Properties (i) and (ii) are invariant under global equivalences of pairs, so the morphism $i^c : R^c \longrightarrow R^\dagger$ satisfies (i) and (ii). Since R^c and $R^\dagger$ are both cofibrant, the morphism i^c is a global group completion by the special case above. So the morphism i is also a global group completion. □

We showed in Theorem 2.4.13 that the ultra-commutative monoid **BOP** is group-like and that its equivariant homotopy sets $\underline{\pi}_0(\mathbf{BOP})$ realize the orthogonal representation rings additively. In Example 2.4.2 we introduced a morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ of ultra-commutative monoids from the additive Grassmannian and showed in Proposition 2.4.5 that for every compact Lie group G and every G-space A, the homomorphism

$$[A, i]^G \;:\; [A, \mathbf{Gr}]^G \;\longrightarrow\; [A, \mathbf{BOP}]^G$$

is a group completion of abelian monoids. In particular, the map $\pi_0^G(i) : \pi_0^G(\mathbf{Gr}) \longrightarrow \pi_0^G(\mathbf{BOP})$ is an algebraic group completion. In much the same way we can define morphisms of ultra-commutative monoids

$$i \;:\; \mathbf{Gr}^{\mathbb{C}} \;\longrightarrow\; \mathbf{BUP} \qquad \text{and} \qquad i \;:\; \mathbf{Gr}^{\mathbb{H}} \;\longrightarrow\; \mathbf{BSpP}$$

by replacing $\mathbb{R}$-subspaces in V by $\mathbb{C}$-subspaces in $V_{\mathbb{C}}$, and $\mathbb{H}$-subspaces in $V_{\mathbb{H}}$.

Theorem 2.5.33. *The morphisms $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$, $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ and $i : \mathbf{Gr}^{\mathbb{H}} \longrightarrow \mathbf{BSpP}$ are global group completions of ultra-commutative monoids.*

Proof We prove the real case in detail and leave the complex and quaternionic cases to the reader. We verify the localization criterion of Proposition 2.5.32. To this end we define a bi-orthogonal space, i.e., a functor

$$\mathbf{Gr}^\sharp \;:\; \mathbf{L} \times \mathbf{L} \;\longrightarrow\; \mathbf{T}$$

on objects by

$$\mathbf{Gr}^\sharp(U,V) \;=\; \mathbf{Gr}(U\oplus V)\;.$$

For linear isometric embeddings $\varphi : U \longrightarrow \bar U$ and $\psi : V \longrightarrow \bar V$, the induced map is

$$\mathbf{Gr}^\sharp(\varphi,\psi) \;:\; \mathbf{Gr}^\sharp(U,V) \;\longrightarrow\; \mathbf{Gr}^\sharp(\bar U,\bar V)\;,\quad L \longmapsto (\varphi\oplus\psi)(L) + ((\bar U - \varphi(U))\oplus 0)\;.$$

We emphasize that the behavior on objects is *not* symmetric in the two variables, and in the first variable it is not just applying φ.

Now we fix a compact Lie group G and consider the colimit of the bi-orthogonal space $\mathbf{Gr}^\sharp$ over the poset $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$. Since the diagonal is cofinal in the poset $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$, this 'double colimit' is also a colimit over the restriction to the diagonal. But the diagonal of $\mathbf{Gr}^\sharp$ is precisely the orthogonal space $\mathbf{BOP}$, and so

$$\operatorname{colim}_{(U,V)\in s(\mathcal{U}_G)^2} \mathbf{Gr}^\sharp(U,V) \;=\; \operatorname{colim}_{W\in s(\mathcal{U}_G)} \mathbf{BOP}(W) \;=\; \mathbf{BOP}(\mathcal{U}_G)\;.$$

On the other hand, if we fix an inner product space U as the first variable, then $\mathbf{Gr}^\sharp(U,-)$ is isomorphic to the additive U-shift (in the sense of Example 1.1.11) of the Grassmannian $\mathbf{Gr}$. Hence for fixed U,

$$\operatorname{colim}_{V\in s(\mathcal{U}_G)} \mathbf{Gr}^\sharp(U,V) \;=\; \mathbf{Gr}(U\oplus\mathcal{U}_G)\;.$$

A colimit over $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$ can be calculated in two steps, first in one variable and then in the other, so we conclude that

$$\mathbf{BOP}(\mathcal{U}_G) \;=\; \operatorname{colim}^\sharp_{U\in s(\mathcal{U}_G)} \mathbf{Gr}(U\oplus\mathcal{U}_G)\;; \tag{2.5.34}$$

under this identification, the map $i(\mathcal{U}_G) : \mathbf{Gr}(\mathcal{U}_G) \longrightarrow \mathbf{BOP}(\mathcal{U}_G)$ becomes the canonical morphism

$$i^\sharp \;:\; \mathbf{Gr}(\mathcal{U}_G) \;\longrightarrow\; \operatorname{colim}^\sharp_{U\in s(\mathcal{U}_G)} \mathbf{Gr}(U\oplus\mathcal{U}_G)$$

to the colimit, for $U = 0$.

The decoration '$\sharp$' is meant to emphasize that the structure maps in this colimit system come from the functoriality of $\mathbf{Gr}^\sharp$ in the first variable, so they are *not* the maps obtained by applying $\mathbf{Gr}(-\oplus\mathcal{U}_G)$ to an inclusion $U \subset \bar U$. For example, the maps in the colimit (2.5.34) do *not* preserve the $\mathbb{N}$-grading by dimension. So one should not confuse the colimit (2.5.34) with the space $\mathbf{Gr}(\mathcal{U}_G\oplus\mathcal{U}_G)$, which is G-homeomorphic to $\mathbf{Gr}(\mathcal{U}_G)$ by a choice of equivariant linear isometry $\mathcal{U}_G\oplus\mathcal{U}_G \cong \mathcal{U}_G$.

We claim that the map

$$H_*((i^\sharp)^G) \;:\; H_*(\mathbf{Gr}(\mathcal{U}_G)^G) \;\longrightarrow\; H_*(\operatorname{colim}^\sharp_{U\in s(\mathcal{U}_G)} \mathbf{Gr}(U\oplus\mathcal{U}_G)^G) \tag{2.5.35}$$

is a localization at the multiplicative subset $\pi_0^G(\mathbf{Gr})$, where homology stands for singular homology with integer coefficients. To see this we observe that all the maps in the colimit system are closed embeddings; so singular homology commutes with this particular colimit.

For $U \in s(\mathcal{U}_G)$ we denote by $j_U : \mathbf{Gr}(\mathcal{U}_G)^G \longrightarrow \mathbf{Gr}(U \oplus \mathcal{U}_G)^G$ the map induced by applying the direct summand inclusion $\mathcal{U}_G \longrightarrow U \oplus \mathcal{U}_G$. The map j_U is a homotopy equivalence because $\mathcal{U}_G$ is a complete G-universe. For all $U \subset V$ in $s(\mathcal{U}_G)$ the following square commutes

$$\begin{tikzcd} H_*(\mathbf{Gr}(\mathcal{U}_G)^G) \arrow[r, "\cdot[V-U]"] \arrow[d, "H_*(j_U)"', "\cong"] & H_*(\mathbf{Gr}(\mathcal{U}_G)^G) \arrow[d, "\cong"', "H_*(j_V)"] \\ H_*(\mathbf{Gr}(U \oplus \mathcal{U}_G)^G) \arrow[r, "H_*(i^\sharp_{U,V})"'] & H_*(\mathbf{Gr}(V \oplus \mathcal{U}_G)^G) \end{tikzcd}$$

and the vertical maps are isomorphisms. So the target of (2.5.35) is the colimit of the functor on $s(\mathcal{U}_G)$ that takes all objects to the ring $H_*(\mathbf{Gr}(\mathcal{U}_G)^G; \mathbb{Z})$ and an inclusion $U \subset V$ to multiplication by the class $[V - U]$ in the multiplicative subset under consideration. Hence the map (2.5.35) is indeed a localization as claimed. Since the ultra-commutative monoid **BOP** is group-like, the criteria of Proposition 2.5.32 are satisfied, and so the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is a global group completion. □

Remark 2.5.36. We had earlier defined an E_∞-orthogonal monoid space $\mathbf{BOP}'$ as a mixture of **bOP** and **BOP**: the value at an inner product space V is

$$\mathbf{BOP}'(V) \;=\; \coprod\nolimits_{m \geq 0} Gr_m(V^2 \oplus \mathbb{R}^\infty)\,.$$

The structure maps and an E_∞-multiplication can be defined in essentially the same way as for $\mathbf{BO}'$, which was defined in (2.4.26). So $\mathbf{BOP}'$ becomes the $\mathbb{Z}$-graded periodic analog of the orthogonal space $\mathbf{BO}'$. In the same way as for the homogeneous degree 0 summands in (2.4.27), we defined two morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bOP} \xrightarrow{\ a\ } \mathbf{BOP}' \underset{\simeq}{\xleftarrow{\ b\ }} \mathbf{BOP}\,.$$

The same arguments as in Proposition 2.4.22 show that the morphism b is a global equivalence. A very similar argument to that in Theorem 2.5.33 shows that the morphism a is a global group completion in the homotopy category of E_∞-orthogonal monoid spaces. Strictly speaking we would first have to justify that the homotopy category is pre-additive (which we won't do), so that the formalism of group completions applies.

As we argued in Proposition 2.4.29, the E_∞-structure on **bO** *cannot* be refined to an ultra-commutative multiplication. The argument was based on an

algebraic obstruction that exists in the same way in $\underline{\pi}_0(\mathbf{bOP})$, so **bOP** cannot be refined to an ultra-commutative monoid either. The fact that **bOP** has an ultra-commutative group completion can be interpreted as saying that in this particular case 'global group completion kills to obstruction to ultra-commutativity'.

Bott periodicity is traditionally seen as a homotopy equivalence between the space $\mathbb{Z} \times BU$ and the loop space of the infinite unitary group U. Since a loop space only sees the basepoint component, and the loop space of BU is weakly equivalent to U, the 2-fold periodicity then takes the form of a chain of weak equivalences:

$$\Omega^2(\mathbb{Z} \times BU) \simeq \Omega(\Omega(BU)) \simeq \Omega U \simeq \mathbb{Z} \times BU .$$

We are going to prove a highly structured version of complex Bott periodicity, in the form of a global equivalence of ultra-commutative monoids between **BUP** and $\Omega\mathbf{U}$. Bott periodicity has been elucidated from many different angles, and before we start, we put our approach into perspective. Since Bott's original geometric argument [23] a large number of different proofs have become available, see for example [79] for an overview. In essence, our proof of global Bott periodicity is an adaptation of Harris' proof [70] of complex Bott periodicity. The reviewer for Math Reviews praises Harris proof as 'a beautiful well-motivated proof of the complex Bott periodicity theorem using only two essential properties of the complex numbers'. Suslin [169] calls this the 'trivial proof' of Bott periodicity and extends it to a 'Real' (i.e., C_2-equivariant) context. I also think that for readers with a homotopy theory background, Harris' proof may be particularly accessible and appealing.

Harris' argument uses two main ingredients. On the one hand, the group completion theorem is used to identify the loop space of the bar construction of $\amalg_{n\geq 0} Gr_n$ (under the monoid structure induced by orthogonal direct sum) with $\mathbb{Z} \times BU$. On the other hand, Harris exhibits an explicit homeomorphism between the bar construction of $\amalg_{n\geq 0} Gr_n$ and the infinite unitary group, essentially the inverse to the eigenspace decomposition of a unitary matrix. Together these two ingredients provide a chain of weak equivalences

$$\mathbb{Z} \times BU \simeq \Omega(B(\amalg_{n\geq 0} Gr_n)) \simeq \Omega U .$$

Our global proof is analogous: Theorem 2.5.33 above shows that **BUP** is a global group completion of $\mathbf{Gr}^{\mathbb{C}}$, essentially by applying the group completion theorem to all fixed-point spaces. This part of the argument works just as well for the real and symplectic versions of $\mathbf{Gr}^{\mathbb{C}}$ and **BUP**. Theorem 2.5.40 below shows that $\Omega\mathbf{U}$ is also a global group completion of $\mathbf{Gr}^{\mathbb{C}}$, by globally identifying the bar construction of $\mathbf{Gr}^{\mathbb{C}}$ (with respect to the box product of orthogonal spaces) with **U**, using Harris' homeomorphism between the re-

alization $|\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}(W)|$ and $U(W_{\mathbb{C}})$. Two global group completions of the same ultra-commutative monoid are necessarily globally equivalent, which yields the global version of complex Bott periodicity of Theorem 2.5.41.

Construction 2.5.37 (Global Bott periodicity)**.** After this outline, we now provide the necessary details. The ultra-commutative monoid $\mathbf{U}$ of unitary groups was defined in Example 2.3.7. The orthogonal space $\Omega\mathbf{U}$ inherits an ultra-commutative multiplication by pointwise multiplication of loops, where Ω means objectwise continuous based maps from S^1. We define a morphism of ultra-commutative monoids

$$\beta \; : \; \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U} \tag{2.5.38}$$

at an inner product space V by

$$\beta(V)(L)(x) \; = \; c(x) \cdot p_L \; + \; p_{L^\perp} \; .$$

Here L is a complex subspace of $V_{\mathbb{C}}$, $x \in S^1$,

$$c \; : \; S^1 \longrightarrow U(1) \, , \quad x \longmapsto \frac{x+i}{x-i} \, ,$$

is the Cayley transform, and p_L and $p_{L^\perp}$ denote the orthogonal projections to L and to its orthogonal complement. In other words, L and $L^\perp$ are the eigenspaces of $\beta(V)(L)(x)$, for the eigenvalues $c(x)$ and 1, respectively. Then

$$\beta(V)(0)(x) \; = \; p_{V_{\mathbb{C}}} \; = \; \mathrm{Id}_{V_{\mathbb{C}}} \; ;$$

so $\beta(V)(0)$ is the constant loop at the identity, which is the unit element of $\Omega\mathbf{U}(V)$. Now we consider subspaces $L \in \mathbf{Gr}^{\mathbb{C}}(V)$ and $L' \in \mathbf{Gr}^{\mathbb{C}}(W)$. Then

$$\begin{aligned} \beta(V \oplus W)(L \oplus L')(x) \; &= \; (c(x) \cdot p_{L \oplus L'}) \; + \; p_{L^\perp \oplus (L')^\perp} \\ &= \; ((c(x) \cdot p_L) \; + \; p_{L^\perp}) \oplus ((c(x) \cdot p_{L'}) \; + \; p_{(L')^\perp}) \\ &= \; \beta(V)(L)(x) \oplus \beta(W)(L')(x) \; . \end{aligned} \tag{2.5.39}$$

In other words, the square

$$\begin{tikzcd} \mathbf{Gr}^{\mathbb{C}}(V) \times \mathbf{Gr}^{\mathbb{C}}(W) \arrow[r, "\beta(V)\times\beta(W)"] \arrow[d, "\oplus"'] & \Omega\mathbf{U}(V) \times \Omega\mathbf{U}(W) \arrow[d, "\mu^{\Omega\mathbf{U}}_{V,W}"] \\ \mathbf{Gr}^{\mathbb{C}}(V \oplus W) \arrow[r, "\beta(V\oplus W)"'] & \Omega\mathbf{U}(V \oplus W) \end{tikzcd}$$

commutes, i.e., β is compatible with the multiplications on both sides. Since β respects multiplication and unit, it also respects the structure maps. The upshot is that β is a morphism of ultra-commutative monoids.

The category of ultra-commutative monoids is tensored and cotensored over based spaces, so the functor of taking objectwise loops is right adjoint to the

functor $-\triangleright S^1$ defined in (2.1.9). For an ultra-commutative monoid R, the based tensor $R \triangleright S^1$ is isomorphic to the bar construction $B(R)$, compare (2.5.30).

Theorem 2.5.40. *The adjoint* $\beta^\flat : B(\mathbf{Gr}^{\mathbb{C}}) = \mathbf{Gr}^{\mathbb{C}} \triangleright S^1 \longrightarrow \mathbf{U}$ *of the morphism* $\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ *is a global equivalence of ultra-commutative monoids. The morphism* $\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ *is a global group completion of ultra-commutative monoids.*

Proof We factor $\beta^\flat$ as a composite of two morphisms of ultra-commutative monoids

$$B(\mathbf{Gr}^{\mathbb{C}}) \;=\; |B_\bullet(\mathbf{Gr}^{\mathbb{C}})| \;\xrightarrow[\simeq]{\zeta}\; |\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}| \;\xrightarrow[\cong]{\epsilon}\; \mathbf{U}\;;$$

then we show that the morphism ζ is a global equivalence and the morphism ϵ is an isomorphism. Together this shows the first claim.

The middle object is the realization of a simplicial ultra-commutative monoid $\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}$, and the first morphism is the realization of a simplicial morphism. The object of n-simplices $\mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}$ is the ultra-commutative monoid of n-tuples of pairwise orthogonal complex subspaces, i.e.,

$$\mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}(V) \;=\; \{(L_1,\dots,L_n) \in (Gr^{\mathbb{C}}(V_{\mathbb{C}}))^n \;:\; L_i \text{ is orthogonal to } L_j \text{ for } i \neq j\}\,.$$

For varying n, the ultra-commutative monoids $\mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}$ assemble into a simplicial ultra-commutative monoid: the face morphisms

$$d_i^* \;:\; \mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle} \;\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}_{\langle n-1\rangle}$$

are given by

$$d_i^*(L_1,\dots,L_n) \;=\; \begin{cases} (L_2,\dots,L_n) & \text{for } i=0,\\ (L_1,\dots,L_{i-1},L_i\oplus L_{i+1},L_{i+2},\dots,L_n) & \text{for } 0<i<n,\\ (L_1,\dots,L_{n-1}) & \text{for } i=n. \end{cases}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ the degeneracy morphisms are given by

$$s_i^*(L_1,\dots,L_{n-1}) \;=\; (L_1,\dots,L_i,0,L_{i+1},\dots,L_{n-1})\,.$$

The direct sum maps

$$\begin{aligned} \mathbf{Gr}^{\mathbb{C}}(V_1)\times\cdots\times\mathbf{Gr}^{\mathbb{C}}(V_n) \;&\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}(V_1\oplus\cdots\oplus V_n)\\ (L_1,\dots,L_n) \;&\longmapsto\; (i_1(L_1),\dots,i_n(L_n)) \end{aligned}$$

form a multi-morphism, where $i_k : (V_k)_{\mathbb{C}} \longrightarrow (V_1\oplus\cdots\oplus V_n)_{\mathbb{C}}$ is the embedding as the kth summand. The universal property of the box product turns this multi-morphism into a morphism of orthogonal spaces

$$\zeta_n \;:\; B_n(\mathbf{Gr}^{\mathbb{C}}) \;=\; (\mathbf{Gr}^{\mathbb{C}})^{\boxtimes n} \;\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}\,.$$

The morphisms ζ_n are compatible with the simplicial face and degeneracy

maps, since these are given by orthogonal direct sum and insertion of 0 on both sides. So for varying n, they form a morphism of simplicial ultra-commutative monoids

$$\zeta_\bullet \ : \ B_\bullet(\mathbf{Gr}^{\mathbb{C}}) \longrightarrow \mathbf{Gr}^{\mathbb{C}}_{\langle \bullet \rangle} .$$

We claim that ζ_n is a global equivalence for every $n \geq 0$. Since $\mathbf{Gr}^{\mathbb{C}} = \coprod_{j\geq 0} \mathbf{Gr}^{\mathbb{C},[j]}$ and the box product distributes over disjoint unions, $(\mathbf{Gr}^{\mathbb{C}})^{\boxtimes n}$ is the disjoint union of the orthogonal spaces

$$\mathbf{Gr}^{\mathbb{C},[j_1]} \boxtimes \cdots \boxtimes \mathbf{Gr}^{\mathbb{C},[j_n]}$$

indexed over all tuples $(j_1, \ldots, j_n) \in \mathbb{N}^n$. The orthogonal space $\mathbf{Gr}^{\mathbb{C}}_{\langle n \rangle}$ has an analogous decomposition, where $\mathbf{Gr}^{\mathbb{C},[j_1,\ldots,j_n]}_{\langle n \rangle}$ consists of those tuples $(L_1, \ldots, L_n)$ with $\dim(L_i) = j_i$. The morphism ζ_n respects the decomposition, i.e., it matches the two summands indexed by the same tuple $(j_1, \ldots, j_n)$. A disjoint union of global equivalences is a global equivalence (Proposition 1.1.9 (v)), so we are reduced to showing that each of the morphisms

$$\zeta_{j_1,\ldots,j_n} \ : \ \mathbf{Gr}^{\mathbb{C},[j_1]} \boxtimes \cdots \boxtimes \mathbf{Gr}^{\mathbb{C},[j_n]} \longrightarrow \mathbf{Gr}^{\mathbb{C},[j_1,\ldots,j_n]}_{\langle n \rangle}$$

is a global equivalence. This is in fact a restatement of an earlier result about box products of orthogonal spaces 'represented' by unitary representations. In Construction 1.3.10 we defined an orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ from a unitary representation W of a compact Lie group G. The value at a euclidean inner product space V is

$$\mathbf{L}^{\mathbb{C}}_{G,W}(V) = \mathbf{L}^{\mathbb{C}}(W, V_{\mathbb{C}})/G .$$

Here $\mathbf{L}^{\mathbb{C}}$ is the space of $\mathbb{C}$-linear maps that preserve the hermitian inner products. In the special case of the tautological $U(n)$-representation on $\mathbb{C}^n$, the homeomorphisms

$$\mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V_{\mathbb{C}})/U(n) \longrightarrow Gr^{\mathbb{C}}_n(V_{\mathbb{C}}) = \mathbf{Gr}^{\mathbb{C},[n]}(V) , \quad \varphi \cdot U(n) \longmapsto \varphi(\mathbb{C}^n)$$

form an isomorphism of orthogonal spaces $\mathbf{L}^{\mathbb{C}}_{U(n),\mathbb{C}^n} \cong \mathbf{Gr}^{\mathbb{C},[n]}$. Similarly, passage to images provides an isomorphism of orthogonal spaces

$$\mathbf{L}^{\mathbb{C}}_{U(j_1)\times\cdots\times U(j_n),\mathbb{C}^{j_1}\oplus\cdots\oplus\mathbb{C}^{j_n}} \cong \mathbf{Gr}^{\mathbb{C},[j_1,\ldots,j_n]}_{\langle n \rangle} .$$

Under these identifications, the morphism $\zeta_{j_1,\ldots,j_n}$ becomes the morphism

$$\zeta_{U(j_1),\ldots,U(j_n);\mathbb{C}^{j_1},\ldots,\mathbb{C}^{j_n}} \ : \ \mathbf{L}^{\mathbb{C}}_{U(j_1),\mathbb{C}^{j_1}} \boxtimes \cdots \boxtimes \mathbf{L}^{\mathbb{C}}_{U(j_n),\mathbb{C}^{j_n}} \longrightarrow \mathbf{L}^{\mathbb{C}}_{U(j_1)\times\cdots\times U(j_n),\mathbb{C}^{j_1}\oplus\cdots\oplus\mathbb{C}^{j_n}} ,$$

the iterate of the morphism discussed in Proposition 1.3.12. Proposition 1.3.12 thus shows that the morphism $\zeta_{j_1,\ldots,j_n}$ is a global equivalence. This completes the proof that the morphism $\zeta_n : (\mathbf{Gr}^{\mathbb{C}})^{\boxtimes n} \longrightarrow \mathbf{Gr}^{\mathbb{C}}_{\langle n \rangle}$ is a global equivalence.

Now we observe that the underlying simplicial orthogonal spaces of source

and target of $\zeta_\bullet$ are Reedy flat in the sense of Definition 1.2.36, i.e., all latching morphisms (in the simplicial direction) are flat cofibrations of orthogonal spaces. Indeed, the unit of the ultra-commutative monoid $\mathbf{Gr}^{\mathbb{C}}$ is the inclusion of the summand $\mathbf{Gr}^{\mathbb{C},[0]}$ into the disjoint union of all $\mathbf{Gr}^{\mathbb{C},[n]}$. Since the orthogonal space $\mathbf{Gr}^{\mathbb{C},[n]}$ is isomorphic to $\mathbf{L}^{\mathbb{C}}_{U(n),\mathbb{C}^n}$, it is flat by Proposition 1.3.11 (ii). So $\mathbf{Gr}^{\mathbb{C}}$ has a flat unit, and its bar construction is Reedy flat by Proposition 2.5.24 (i). Since $\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}$ is not the bar construction of any orthogonal monoid space, we must show Reedy flatness directly. Each simplicial degeneracy morphism of $\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}$ inserts the zero vector space in one slot; so the degeneracy morphisms are embeddings of summands in a disjoint union. The latching morphism

$$L^{\Delta}_m(\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}) \longrightarrow \mathbf{Gr}^{\mathbb{C}}_{\langle m\rangle}$$

is then also the inclusion of certain summands, namely those $\mathbf{Gr}^{\mathbb{C},[j_1,\dots,j_n]}_{\langle n\rangle}$ for which $j_i = 0$ for at least one i. Since the summand $\mathbf{Gr}^{\mathbb{C},[j_1,\dots,j_n]}_{\langle n\rangle}$ is isomorphic to $\mathbf{L}^{\mathbb{C}}_{U(j_1)\times\dots\times U(j_n),\mathbb{C}^{j_1}\oplus\dots\oplus\mathbb{C}^{j_n}}$, it is flat by Proposition 1.3.11 (ii). This verifies the Reedy flatness condition for $\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}$. Since source and target of the morphism $\zeta_\bullet$ are Reedy flat as simplicial orthogonal spaces, and $\zeta_\bullet$ is a global equivalence in every simplicial dimension, the induced morphism of realizations

$$\zeta = |\zeta_\bullet| : B(\mathbf{Gr}^{\mathbb{C}}) = |B_\bullet(\mathbf{Gr}^{\mathbb{C}})| \longrightarrow |\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}|$$

is a global equivalence by Proposition 1.2.37 (ii).

The isomorphism of ultra-commutative monoids $\epsilon : |\mathbf{Gr}^{\mathbb{C}}_{\langle\bullet\rangle}| \cong \mathbf{U}$ is taken from Harris [70, Sec. 2, Thm.], and we recall it in some detail for the convenience of the reader. We let V be an inner product space and consider the continuous map

$$\begin{aligned} \epsilon_n : \mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}(V)\times\Delta^n &\longrightarrow U(V_{\mathbb{C}}) \\ (L_1,\dots,L_n;\, t_1,\dots,t_n) &\longmapsto \prod\nolimits_{j=1}^{n} \exp(2\pi i t_j\cdot p_{L_j})\,, \end{aligned}$$

where $p_L : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ is the orthogonal projection onto L. In other words, $\epsilon_n(L_1,\dots,L_n;\, t_1,\dots,t_n)$ is the unitary automorphism of $V_{\mathbb{C}}$ that has L_j as eigenspace with eigenvalue $\exp(2\pi i t_j)$, and is the identity on the orthogonal complement of all the L_j. We have the relations

$$\epsilon_n(L_1,\dots,L_n;\, 0,t_1,\dots,t_{n-1}) = \epsilon_{n-1}(L_2,\dots,L_n;\, t_1,\dots,t_{n-1})$$

and

$$\epsilon_n(L_1,\dots,L_n;\, t_1,\dots,t_{n-1},1) = \epsilon_{n-1}(L_1,\dots,L_{n-1};\, t_1,\dots,t_{n-1})$$

because $\exp(0) = \exp(2\pi i\cdot p_L) = \mathrm{Id}_{V_{\mathbb{C}}}$; moreover,

$$\epsilon_n(L_1,\dots,L_n;\, t_1,\dots,t_i,t_i\dots,t_{n-1}) = \epsilon_{n-1}(L_1,\dots,L_i\oplus L_{i+1},\dots,L_n;\, t_1,\dots,t_{n-1})$$

for all $0 < i < n$, because $\exp(2\pi it \cdot p_{L_i}) \cdot \exp(2\pi it \cdot p_{L_{i+1}}) = \exp(2\pi it \cdot p_{L_i \oplus L_{i+1}})$; and finally

$$\epsilon_{n+1}(L_1, \ldots, L_i, 0, L_{i+1}, \ldots, L_n;\ t_1, \ldots, t_{n+1}) \\ = \epsilon_n(L_1, \ldots, L_n;\ t_1, \ldots, \widehat{t_{i+1}}, \ldots, t_{n+1})$$

for all $0 \leq i \leq n$. So the maps ϵ_n are compatible with the equivalence relation defining geometric realization, and they induce a continuous map

$$\epsilon(V) \ : \ |\mathbf{Gr}^{\mathbb{C}}_{\langle n \rangle}(V)| \ \longrightarrow \ U(V_{\mathbb{C}}) \ .$$

The map $\epsilon(V)$ is bijective because every unitary automorphism is diagonalizable with pairwise orthogonal eigenspaces and eigenvalues in $U(1)$. As a continuous bijection from a compact space to a Hausdorff space, $\epsilon(V)$ is a homeomorphism. The homeomorphisms $\epsilon(V)$ are compatible with linear isometric embeddings in V and the ultra-commutative multiplications on both sides, i.e., they define an isomorphism of ultra-commutative monoids $\epsilon : |\mathbf{Gr}^{\mathbb{C}}_{\langle \bullet \rangle}| \cong \mathbf{U}$.

Now we can conclude the proof. Unraveling all definitions shows that the composite

$$\mathbf{Gr}^{\mathbb{C}} \wedge S^1 \ \longrightarrow \ B(\mathbf{Gr}^{\mathbb{C}}) \ \xrightarrow{\zeta} \ |\mathbf{Gr}^{\mathbb{C}}_{\langle \bullet \rangle}| \ \xrightarrow{\epsilon} \ \mathbf{U}$$

is given at an inner product space V by the map

$$\begin{aligned} Gr^{\mathbb{C}}(V_{\mathbb{C}}) \wedge S^1 \ &\longrightarrow \quad \mathbf{U}(V_{\mathbb{C}}) \\ L \wedge x \ &\longmapsto \ \exp(2\pi i \cdot \log(c(x)) \cdot p_L) \ = \ c(x) \cdot p_L \ + \ p_{L^\perp} \ = \ \beta(V)(L)(x) \ . \end{aligned}$$

This means that the original morphism β is the composite

$$\mathbf{Gr}^{\mathbb{C}} \ \xrightarrow{\eta_{\mathbf{Gr}^{\mathbb{C}}}} \ \Omega B(\mathbf{Gr}^{\mathbb{C}}) \ \xrightarrow{\Omega(\zeta)} \ \Omega|\mathbf{Gr}^{\mathbb{C}}_{\bullet}(V)| \ \xrightarrow{\Omega(\epsilon)} \ \Omega\mathbf{U} \ ,$$

where $\eta_{\mathbf{Gr}^{\mathbb{C}}}$ was defined in (2.5.27). Thus the adjoint $\beta^\flat$ factors as $\epsilon \circ \zeta$, a global equivalence followed by an isomorphism. So $\beta^\flat$ is a global equivalence of ultra-commutative monoids.

We showed above that $\mathbf{Gr}^{\mathbb{C}}$ has a flat unit, so the adjunction unit $\eta_{\mathbf{Gr}^{\mathbb{C}}}$ is a global group completion by Corollary 2.5.31. Since $\beta^\flat$ is a global equivalence, so is $\Omega(\beta^\flat)$. Hence the composite $\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ is a global group completion. □

Theorem 2.5.33 and Theorem 2.5.40 show that the morphisms

$$i \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \mathbf{BUP} \qquad \text{and} \qquad \beta \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega\mathbf{U}$$

are both global group completions. The universal property of group completions already implies that **BUP** is isomorphic to $\Omega\mathbf{U}$ in the homotopy category of ultra-commutative monoids; the two can thus be linked by a chain of global equivalences of ultra-commutative monoids. In a sense we could stop here, and

call this 'complex global Bott periodicity'. However, we elaborate a bit more and exhibit an explicit chain of two global equivalences between **BUP** and Ω**U**, see Theorem 2.5.41 below.

We define a morphism of ultra-commutative monoids

$$\bar{\beta} \ : \ \mathbf{BUP} \ \longrightarrow \ \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U}) \ .$$

Here $\mathrm{sh}_{\otimes} = \mathrm{sh}_{\otimes}^{\mathbb{R}^2}$ is the multiplicative shift by $\mathbb{R}^2$ defined in Example 1.1.11. The orthogonal space **U** has a commutative multiplication by direct sum of unitary automorphisms; thus $\Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})$ inherits a commutative multiplication by pointwise multiplication of loops. The target of $\bar{\beta}$ is globally equivalent, as an ultra-commutative monoid, to $\Omega\mathbf{U}$, the objectwise loops of the unitary group monoid. The definition of the map

$$\bar{\beta}(V) \ : \ \mathbf{BUP}(V) \ \longrightarrow \ \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(V) \ = \ \mathrm{map}(S^1, \mathbf{U}(V_{\mathbb{C}}^2)) \ ,$$

for an inner product space V, is similar to, but slightly more elaborate than the definition of $\beta(V)$ in (2.5.38) above. An element of $\mathbf{BUP}(V)$ is a complex subspace L of $V_{\mathbb{C}}^2$; as before we denote by p_L and $p_{L^\perp}$ the orthogonal projections to L and to its orthogonal complement. We define the loop

$$\bar{\beta}(V)(L) \ : \ S^1 \ \longrightarrow \ U(V_{\mathbb{C}}^2)$$

by

$$\bar{\beta}(V)(L)(x) \ = \ ((c(x)\cdot p_L) \ + \ p_{L^\perp}) \circ ((c(-x)\cdot p_{V_{\mathbb{C}}\oplus 0}) + p_{0\oplus V_{\mathbb{C}}}) \ .$$

As before $c : S^1 \longrightarrow U(1)$ is the Cayley transform. The map $\bar{\beta}(V)$ is continuous in L.

For every inner product space V we have

$$\bar{\beta}(V)(V_{\mathbb{C}}\oplus 0)(x) \ = \ ((c(x)\cdot p_{V_{\mathbb{C}}\oplus 0}) \ + \ p_{0\oplus V_{\mathbb{C}}})\circ((c(-x)\cdot p_{V_{\mathbb{C}}\oplus 0})+p_{0\oplus V_{\mathbb{C}}}) \ = \ \mathrm{Id}_{V_{\mathbb{C}}} \ ;$$

so $\bar{\beta}(V)(V_{\mathbb{C}}\oplus 0)$ is the constant loop at the identity, which is the unit element of $\Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(V)$. Now we consider subspaces $L \in \mathbf{BUP}(V)$ and $L' \in \mathbf{BUP}(W)$. We recall that $\kappa^{V,W} : V_{\mathbb{C}}^2 \oplus W_{\mathbb{C}}^2 \cong (V \oplus W)_{\mathbb{C}}^2$ is the preferred natural isometry, which enters into the definition of the multiplication of **BUP**. The argument for the additivity relation

$$\bar{\beta}(V \oplus W)(L \oplus L')(x) \ = \ \kappa_*^{V,W}\left(\bar{\beta}(V)(L)(x) \oplus \bar{\beta}(W)(L')(x)\right)$$

is straightforward and similar to (but somewhat longer than) the argument for the map β in (2.5.39), and we omit it. Hence $\bar{\beta}$ is compatible with the multiplications on both sides. Since $\bar{\beta}$ respects multiplication and unit, it also respects the structure maps. The upshot is that $\bar{\beta}$ is a morphism of ultra-commutative

monoids. We also have

$$\begin{aligned}\det(\bar\beta(V)(L)(x)) &= \det((c(x)\cdot p_L) + p_{L^\perp})\cdot\det((c(-x)\cdot p_{V_{\mathbb{C}}\oplus 0}) + p_{0\oplus V_{\mathbb{C}}})\\ &= c(x)^{\dim(L)-\dim(V)}\ ,\end{aligned}$$

exploiting $c(-x) = \overline{c(x)} = c(x)^{-1}$. So the map $\bar\beta(V) : \mathbf{BUP}(V) \longrightarrow \Omega U(V^2_{\mathbb{C}})$ sends the subspace $\mathbf{BU}(V) = \mathbf{BUP}^{[0]}(V)$ to $\Omega(SU(V^2_{\mathbb{C}}))$. Hence the morphism $\bar\beta$ restricts to a morphism of ultra-commutative monoids

$$\bar\beta^{[0]} \ : \ \mathbf{BU} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{SU})\ .$$

Now we can properly state our global version of complex Bott periodicity. The embeddings $j : V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ as the first summand induce a morphism of ultra-commutative monoids $\mathbf{U}\circ j : \mathbf{U} \longrightarrow \mathrm{sh}_\otimes \mathbf{U}$.

Theorem 2.5.41 (Global Bott periodicity)**.** *The morphisms of ultra-commutative monoids*

$$\mathbf{BUP} \xrightarrow{\ \bar\beta\ } \Omega(\mathrm{sh}_\otimes \mathbf{U}) \xleftarrow{\ \Omega(\mathbf{U}\circ j)\ } \Omega\mathbf{U}$$

are global equivalences. The morphism $\bar\beta^{[0]} : \mathbf{BU} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{SU})$ *is a global equivalence.*

Proof The morphism $\mathbf{U}\circ j$ is a global equivalence by Theorem 1.1.10, hence so is $\Omega(\mathbf{U}\circ j)$. The following diagram of homomorphisms of ultra-commutative monoids commutes by direct inspection:

$$\begin{tikzcd} \mathbf{Gr}^{\mathbb{C}} \arrow[r, "\beta"] \arrow[d, "i"'] & \Omega\mathbf{U} \arrow[d, "\Omega(\mathbf{U}\circ j)"] \\ \mathbf{BUP} \arrow[r, "\bar\beta"'] & \Omega(\mathrm{sh}_\otimes \mathbf{U}) \end{tikzcd}$$

The morphism $\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ is a global group completion by Theorem 2.5.40. So the composite $\Omega(\mathbf{U}\circ j)\circ\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$ with a global equivalence is also a global group completion. The morphism $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ is a global group completion by Theorem 2.5.33. The universal property of group completions then shows that $\bar\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$ becomes an isomorphism in the homotopy category of ultra-commutative monoids. Since the global equivalences are part of a model structure, this implies that $\bar\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$ is a global equivalence. Since the morphism $\bar\beta^{[0]} : \mathbf{BU} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{SU})$ is a retract of the global equivalence $\bar\beta$, it is a global equivalence itself. □

Corollary 2.5.42. *For every compact Lie group G and every finite G-CW-complex A, the map*

$$[A,\beta]^G \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \Omega\mathbf{U}]^G$$

is a group completion of abelian monoids.

Proof We contemplate the following commutative square of abelian monoid homomorphisms:

$$\begin{tikzcd}
[A, \mathbf{Gr}^{\mathbb{C}}]^G \arrow[r, "{[A,\beta]^G}"] \arrow[d, "{[A,i]^G}"'] & [A, \Omega\mathbf{U}]^G \arrow[d, "{[A,\Omega(\mathbf{U}\circ j)]^G}", "\cong"'] \\
[A, \mathbf{BUP}]^G \arrow[r, "\cong", "{[A,\bar\beta]^G}"'] & [A, \Omega(\mathrm{sh}_\otimes \mathbf{U})]^G
\end{tikzcd}$$

Since $\bar\beta$ and $\Omega(\mathbf{U} \circ j)$ are global equivalences by Theorem 2.5.41, the two homomorphisms $[A, \bar\beta]^G$ and $[A, \Omega(\mathbf{U} \circ j)]^G$ are isomorphisms, by Proposition 1.5.3 (ii). The morphism $[A, i]^G$ is a group completion of abelian monoids by Proposition 2.4.5 (or rather its complex analog, which is proved analogously). So $[A, \beta]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \Omega\mathbf{U}]^G$ is also a group completion of abelian monoids. $\square$

3

Equivariant stable homotopy theory

In this chapter we give a largely self-contained exposition of many basics about equivariant stable homotopy theory for a fixed compact Lie group; our model is the category of orthogonal G-spectra. In Section 3.1 we review orthogonal spectra and orthogonal G-spectra; we define equivariant stable homotopy groups and prove their basic properties, such as the suspension isomorphism and long exact sequences of mapping cones and homotopy fibers, and the additivity of equivariant homotopy groups on sums and products. Section 3.2 discusses the Wirthmüller isomorphism that relates the equivariant homotopy groups of a spectrum over a subgroup to the equivariant homotopy groups of the induced spectrum; intimately related to the Wirthmüller isomorphism are various transfers that we also recall. In Section 3.3 we introduce and study geometric fixed-point homotopy groups. We establish the isotropy separation sequence that facilitates inductive arguments, and show that equivariant equivalences can also be detected by geometric fixed-points. We use geometric fixed-points to derive a functorial description of the 0th equivariant stable homotopy group of a G-space Y in terms of the path components of the fixed-point spaces Y^H. Section 3.4 gives a self-contained proof of the double coset formula for the composite of a transfer followed by a restriction to a closed subgroup. We also discuss various examples and end with a discussion of Mackey functors for finite groups. After inverting the group order, the category of G-Mackey functors splits as a product, indexed by conjugacy classes of subgroups, of module categories over the Weyl groups, see Theorem 3.4.22. We show that rationally and for finite groups, geometric fixed-point homotopy groups can be obtained from equivariant homotopy groups by dividing out transfers from proper subgroups. Section 3.5 is devoted to multiplicative aspects of equivariant stable homotopy theory. In our model, all multiplicative features can be phrased in terms of the smash product of orthogonal spectra (or orthogonal G-spectra), another example of a Day type convolution product. The smash product gives rise to pairings of equivariant homotopy groups; when specialized to equiv-

ariant ring spectra, these pairings turn the equivariant stable homotopy into graded rings.

We do not discuss model category structures for orthogonal G-spectra; the interested reader can find different ones in the memoir of Mandell and May [108], in the thesis of Stolz [163], the article by Brun, Dundas and Stolz [32] and (for finite groups) in the paper of Hill, Hopkins and Ravenel [77].

3.1 Equivariant orthogonal spectra

In this section we begin to develop some of the basic features of equivariant stable homotopy theory for compact Lie groups in the context of equivariant orthogonal spectra. After introducing orthogonal G-spectra and equivariant stable homotopy groups, we discuss shifts by a representation and show that they are equivariantly equivalent to smashing with the representation sphere (Proposition 3.1.25). We establish the loop and suspension isomorphisms (Proposition 3.1.30) and the long exact homotopy group sequences of homotopy fibers and mapping cones (Proposition 3.1.36). We prove that equivariant homotopy groups take wedges to sums and preserve finite products (Corollary 3.1.37). We end by showing that the equivariant homotopy group functor π_0^H, for a closed subgroup H of a compact Lie group G, is represented by the unreduced suspension spectrum of the homogeneous space G/H (Proposition 3.1.46).

We recall *orthogonal spectra*. These objects are used, at least implicitly, already in [112]; the term 'orthogonal spectrum' was introduced by Mandell, May, Shipley and the author in [107], where the (non-equivariant) stable model structure for orthogonal spectra was constructed. Orthogonal spectra are stable versions of orthogonal spaces, and before recalling the formal definition we try to motivate it – already with a view towards the global perspective. An orthogonal space Y assigns values to all finite-dimensional inner product spaces. The global homotopy type is encoded in the G-spaces $Y(\mathcal{U}_G)$, where $\mathcal{U}_G$ is a complete G-universe, which we can informally think of as 'the homotopy colimit of $Y(V)$ over all G-representations V'. So besides the values $Y(V)$, an orthogonal space uses the information about the $O(V)$-action (which is turned into a G-action when G acts on V) and the information about inclusions of inner product spaces (in order to be able to stabilize to the colimit $\mathcal{U}_G$). The information about the $O(V)$-actions and how to stabilize are conveniently encoded together as a continuous functor from the category $\mathbf{L}$ of linear isometric embeddings.

An orthogonal spectrum X is a stable analog of this: it assigns a based space $X(V)$ to every inner product space, and it keeps track of an $O(V)$-action on

$X(V)$ (to get G-homotopy types when G acts on V) and of a way to stabilize by suspensions (needed when exhausting a complete universe by its finite-dimensional subrepresentations). When doing this in a coordinate-free way, the stabilization data assigns to a linear isometric embedding $\varphi : V \longrightarrow W$ a continuous based map

$$\varphi_\star \ : \ S^{W-\varphi(V)} \wedge X(V) \ \longrightarrow \ X(W)$$

where $W-\varphi(V)$ is the orthogonal complement of the image of φ. This structure map should 'vary continuously with φ', but this phrase has no literal meaning because the source of $\varphi_\star$ depends on φ. The way to make the continuous dependence rigorous is to exploit the fact that the complements $W - \varphi(V)$ vary in a locally trivial way, i.e., they are the fibers of a distinguished vector bundle, the 'orthogonal complement bundle', over the space of $\mathbf{L}(V, W)$ of linear isometric embeddings. All the structure maps $\varphi_\star$ together define a map on the smash product of $X(V)$ with the Thom space of this complement bundle, and the continuity of the dependence on φ is formalized by requiring continuity of that map. All these Thom spaces together form the morphism spaces of a based topological category, and the data of an orthogonal spectrum can conveniently be packaged as a continuous based functor on this category.

Construction 3.1.1. We let V and W be inner product spaces. Over the space $\mathbf{L}(V, W)$ of linear isometric embeddings sits a certain 'orthogonal complement' vector bundle with total space

$$\xi(V, W) \ = \ \{\, (w, \varphi) \in W \times \mathbf{L}(V, W) \mid w \perp \varphi(V) \,\} \ .$$

The structure map $\xi(V, W) \longrightarrow \mathbf{L}(V, W)$ is the projection to the second factor. The vector bundle structure of $\xi(V, W)$ is as a vector subbundle of the trivial vector bundle $W \times \mathbf{L}(V, W)$, and the fiber over $\varphi : V \longrightarrow W$ is the orthogonal complement $W - \varphi(V)$ of the image of φ.

We let $\mathbf{O}(V, W)$ be the Thom space of the bundle $\xi(V, W)$, i.e., the one-point compactification of the total space of $\xi(V, W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V, W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V, W)$ is empty and $\mathbf{O}(V, W)$ consists of a single point at infinity. Otherwise we can choose a linear isometric embedding $\varphi : V \longrightarrow W$, and then the maps

$$\begin{aligned} O(W)/O(W - \varphi(V)) \quad &\longrightarrow \ \mathbf{L}(V, W) \ , \quad A \cdot O(W - \varphi(V)) \ \longmapsto \ A\varphi \quad \text{and} \\ O(W) \ltimes_{O(W-\varphi(V))} S^{W-\varphi(V)} \ &\longrightarrow \ \mathbf{O}(V, W) \ , \quad [A, w] \ \longmapsto \ (Aw, A\varphi) \end{aligned}$$

are homeomorphisms. Here, and in the following, we write

$$G \ltimes_H A \ = \ (G_+) \wedge_H A \ = \ (G_+ \wedge A)/ \sim$$

for a closed subgroup H of G and a based G-space A; the equivalence relation is $gh \wedge a \sim g \wedge ha$ for all $(g, h, a) \in G \times H \times A$. Put yet another way: if $\dim V = n$ and $\dim W = n + m$, then $\mathbf{L}(V, W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V, W)$ is homeomorphic to $O(n+m) \ltimes_{O(m)} S^m$. The vector bundle $\xi(V, W)$ becomes trivial upon product with the trivial bundle V, via the trivialization

$$\xi(V, W) \times V \;\cong\; W \times \mathbf{L}(V, W)\,, \quad ((w, \varphi), v) \;\longmapsto\; (w + \varphi(v), \varphi)\,.$$

When we pass to Thom spaces on both sides this becomes the *untwisting homeomorphism:*

$$\mathbf{O}(V, W) \wedge S^V \;\cong\; S^W \wedge \mathbf{L}(V, W)_+\,. \tag{3.1.2}$$

The Thom spaces $\mathbf{O}(V, W)$ are the morphism spaces of a based topological category. Given a third inner product space U, the bundle map

$$\xi(V, W) \times \xi(U, V) \;\longrightarrow\; \xi(U, W)\,, \quad ((w, \varphi),\, (v, \psi)) \;\longmapsto\; (w + \varphi(v),\, \varphi\psi)$$

covers the composition map $\mathbf{L}(V, W) \times \mathbf{L}(U, V) \longrightarrow \mathbf{L}(U, W)$. Passage to Thom spaces gives a based map

$$\circ \;:\; \mathbf{O}(V, W) \wedge \mathbf{O}(U, V) \;\longrightarrow\; \mathbf{O}(U, W)$$

which is clearly associative, and is the composition in the category $\mathbf{O}$. The identity of V is $(0, \mathrm{Id}_V)$ in $\mathbf{O}(V, V)$.

Definition 3.1.3. An *orthogonal spectrum* is a based continuous functor from $\mathbf{O}$ to the category $\mathbf{T}_*$ of based spaces. A *morphism* of orthogonal spectra is a natural transformation of functors. We denote the category of orthogonal spectra by $\mathcal{S}p$.

Given two inner product spaces V and W we define a continuous based map

$$i_V \;:\; S^V \;\longrightarrow\; \mathbf{O}(W, V \oplus W) \qquad \text{by} \qquad v \;\longmapsto\; ((v, 0), (0, -))\,,$$

where $(0, -) : W \longrightarrow V \oplus W$ is the embedding of the second summand. We define the *structure map* $\sigma_{V,W} : S^V \wedge X(W) \longrightarrow X(V \oplus W)$ of the orthogonal spectrum X as the composite

$$S^V \wedge X(W) \xrightarrow{i_V \wedge X(W)} \mathbf{O}(W, V \oplus W) \wedge X(W) \xrightarrow{X} X(V \oplus W)\,. \tag{3.1.4}$$

Often it will be convenient to use the *opposite structure map*

$$\sigma^{\mathrm{op}}_{V,W} \;:\; X(V) \wedge S^W \;\longrightarrow\; X(V \oplus W) \tag{3.1.5}$$

which we define as the following composite:

$$X(V) \wedge S^W \xrightarrow{\text{twist}} S^W \wedge X(V) \xrightarrow{\sigma_{W,V}} X(W \oplus V) \xrightarrow{X(\tau_{V,W})} X(V \oplus W)$$

Remark 3.1.6 (Coordinatized orthogonal spectra)**.** Every inner product space is isometrically isomorphic to $\mathbb{R}^n$ with standard inner product, for some $n \geq 0$. So the topological category $\mathbf{O}$ has a small skeleton, and the functor category of orthogonal spectra has 'small' morphism sets. This also leads to the following more explicit coordinatized description of orthogonal spectra in a way that resembles a presentation by generators and relations.

Up to isomorphism, an orthogonal spectrum X is determined by the values $X_n = X(\mathbb{R}^n)$ and the following additional data relating these spaces:

- a based continuous left $O(n)$-action on X_n for each $n \geq 0$,
- based maps $\sigma_n : S^1 \wedge X_n \longrightarrow X_{1+n}$ for $n \geq 0$.

This data is subject to the following condition: for all $m, n \geq 0$, the iterated structure map $S^m \wedge X_n \longrightarrow X_{m+n}$ defined as the composition

$$S^m \wedge X_n \xrightarrow{S^{m-1}\wedge\sigma_n} S^{m-1} \wedge X_{1+n} \xrightarrow{S^{m-2}\wedge\sigma_{1+n}} \quad \cdots \quad \xrightarrow{\sigma_{m-1+n}} X_{m+n}$$

is $(O(m) \times O(n))$-equivariant. Here the group $O(m) \times O(n)$ acts on the target by restriction, along orthogonal sum, of the $O(m+n)$-action. Indeed, the map

$$O(n)_+ \longrightarrow \mathbf{O}(\mathbb{R}^n, \mathbb{R}^n) , \quad A \longmapsto (0, A)$$

is a homeomorphism, so $O(n)$ 'is' the endomorphism monoid of $\mathbb{R}^n$ as an object of the category $\mathbf{O}$; via this map, $O(n)$ acts on the value at $\mathbb{R}^n$ of any functor on $\mathbf{O}$. The map $\sigma_n = \sigma_{\mathbb{R},\mathbb{R}^n}$ is just one of the structure maps (3.1.4).

Definition 3.1.7. Let G be a compact Lie group. An *orthogonal G-spectrum* is a based continuous functor from $\mathbf{O}$ to the category $G\mathbf{T}_*$ of based G-spaces. A *morphism* of orthogonal G-spectra is a natural transformation of functors. We write GSp for the category of orthogonal G-spectra and G-equivariant morphisms.

A continuous functor to based G-spaces is the same data as a G-object of continuous functors. So orthogonal G-spectra could equivalently be defined as orthogonal spectra equipped with a continuous G-action. An orthogonal G-spectrum X can be evaluated on a G-representation V, and then $X(V)$ is a $(G \times G)$-space by the 'external' G-action on X and the 'internal' G-action from the G-action on V and the $O(V)$-functoriality of X. We consider $X(V)$ as a G-space via the diagonal G-action. If V and W are G-representations, then the structure map (3.1.4) and the opposite structure map (3.1.5) are G-equivariant where the group G also acts on the representation spheres.

Remark 3.1.8. Our definition of orthogonal G-spectra is not the same as the one used by Mandell and May [108] and Hill, Hopkins and Ravenel [77], who define orthogonal G-spectra as G-functors on a G-enriched extension of the

category **O** that contains all G-representations as objects. However, our category of orthogonal G-spectra is equivalent to theirs by [108, V Thm. 1.5]. The substance of this equivalence is the fact that for every orthogonal G-spectrum in the sense of Mandell and May, the values at arbitrary G-representations are in fact determined by the values at trivial representations.

Next we recall the equivariant stable homotopy groups $\pi_*^G(X)$ (indexed by the complete G-universe) of an orthogonal G-spectrum X. We introduce a convenient piece of notation. If $\varphi : V \longrightarrow W$ is a linear isometric embedding and $f : S^V \longrightarrow X(V)$ a continuous based map, we define $\varphi_* f : S^W \longrightarrow X(W)$ as the composite

$$\begin{aligned} S^W \cong S^{W-\varphi(V)} \wedge S^V \xrightarrow{S^{W-\varphi(V)}\wedge f} & \ S^{W-\varphi(V)} \wedge X(V) \\ \xrightarrow{\sigma_{W-\varphi(V),V}} & \ X((W-\varphi(V))\oplus V) \xrightarrow{\cong} X(W) \end{aligned} \tag{3.1.9}$$

where two unnamed homeomorphisms use the linear isometry

$$(W-\varphi(V))\oplus V \cong W , \quad (w,v) \longmapsto w+\varphi(v) .$$

For example, if φ is bijective (i.e., an equivariant isometry), then $\varphi_* f$ becomes the φ-conjugate of f, i.e., the composite

$$S^W \xrightarrow{S^{\varphi^{-1}}} S^V \xrightarrow{f} X(V) \xrightarrow{X(\varphi)} X(W) .$$

The construction is continuous in both variables, i.e., the map

$$\mathbf{L}(V,W)\times \mathrm{map}_*(S^V, X(V)) \longrightarrow \mathrm{map}_*(S^W, X(W)) , \quad (\varphi, f) \longmapsto \varphi_* f$$

is continuous.

As before we let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite-dimensional G-subrepresentations of the chosen complete G-universe $\mathcal{U}_G$. We obtain a functor from $s(\mathcal{U}_G)$ to sets by sending $V \in s(\mathcal{U}_G)$ to

$$[S^V, X(V)]^G ,$$

the set of G-equivariant homotopy classes of based G-maps from S^V to $X(V)$. For $V \subseteq W$ in $s(\mathcal{U}_G)$ the inclusion $i : V \longrightarrow W$ is sent to the map

$$i_* : [S^V, X(V)]^G \longrightarrow [S^W, X(W)]^G , \quad [f] \longmapsto [i_* f] .$$

The *0th equivariant homotopy group* $\pi_0^G(X)$ is then defined as

$$\pi_0^G(X) = \mathrm{colim}_{V\in s(\mathcal{U}_G)} [S^V, X(V)]^G ,$$

the colimit of this functor over the poset $s(\mathcal{U}_G)$.

The sets $\pi_0^G(X)$ have a lot of extra structure; we start with the abelian group

structure. We consider a finite-dimensional G-subrepresentation V of the universe $\mathcal{U}_G$ with non-zero fixed-points. We choose a G-fixed unit vector $v_0 \in V$, and we let $V^\perp$ denote the orthogonal complement of v_0 in V. This induces a decomposition

$$\mathbb{R} \oplus V^\perp \ \cong\ V\ , \quad (t, v) \ \longmapsto\ tv_0 + v$$

that extends to a G-equivariant homeomorphism $S^1 \wedge S^{V^\perp} \cong S^V$ on one-point compactifications. From this we obtain a bijection

$$[S^V, X(V)]^G \ \cong\ [S^1, \mathrm{map}_*^G(S^{V^\perp}, X(V))]_* \ =\ \pi_1(\mathrm{map}_*^G(S^{V^\perp}, X(V)))\ , \quad (3.1.10)$$

natural in the orthogonal G-spectrum X. We use the bijection (3.1.10) to transfer the group structure on the fundamental group into a group structure on the set $[S^V, X(V)]^G$.

Now we suppose that the dimension of the fixed-point space V^G is at least 2. Then the space of G-fixed unit vectors in V is connected and similar arguments as for the commutativity of higher homotopy groups show:

- the group structure on the set $[S^V, X(V)]^G$ defined by the bijection (3.1.10) is commutative and independent of the choice of G-fixed unit vector;
- if W is another finite-dimensional G-subrepresentation of $\mathcal{U}_G$ containing V, then the map
$$i_* \ :\ [S^V, X(V)]^G \ \longrightarrow\ [S^W, X(W)]^G$$
is a group homomorphism.

The G-subrepresentations V of $\mathcal{U}_G$ with $\dim(V^G) \geq 2$ are cofinal in the poset $s(\mathcal{U}_G)$, so the two properties above show that the abelian group structures on $[S^V, X(V)]^G$ for $\dim(V^G) \geq 2$ assemble into a well-defined and natural abelian group structure on the colimit $\pi_0^G(X)$.

We generalize the definition of $\pi_0^G(X)$ to integer graded equivariant homotopy groups of an orthogonal G-spectrum X. If k is a positive integer, then we set

$$\begin{aligned} \pi_k^G(X) \ &=\ \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^{V \oplus \mathbb{R}^k}, X(V)]^G \qquad \text{and} \\ \pi_{-k}^G(X) \ &=\ \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^V, X(V \oplus \mathbb{R}^k)]^G\ . \end{aligned} \quad (3.1.11)$$

The colimits are taken over the analogous stabilization maps as for π_0^G, and they come with abelian group structures by the same reasoning as for $\pi_0^G(X)$.

Definition 3.1.12. A morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is a $\underline{\pi}_*$-*isomorphism* if the induced map $\pi_k^H(f) : \pi_k^H(X) \longrightarrow \pi_k^H(Y)$ is an isomorphism for all closed subgroups H of G and all integers k.

Construction 3.1.13. While the definition of $\pi_k^G(X)$ involves a case distinction in positive and negative dimensions k, every class in $\pi_k^G(X)$ can be represented by a G-map

$$f \;:\; S^{V\oplus\mathbb{R}^{n+k}} \;\longrightarrow\; X(V\oplus\mathbb{R}^n)$$

for suitable $n \in \mathbb{N}$ such that $n + k \geq 0$. Moreover, V can be any finite-dimensional G-representation, not necessarily a subrepresentation of the chosen complete G-universe. Since we will frequently use this way to represent elements of $\pi_k^G(X)$, we make the construction explicit here.

We start with the case $k \geq 0$. We choose a G-equivariant linear isometry $j : V\oplus\mathbb{R}^n \longrightarrow \bar{V}$ onto a G-subrepresentation $\bar{V}$ of $\mathcal{U}_G$. Then the composite

$$S^{\bar{V}\oplus\mathbb{R}^k} \xrightarrow[\cong]{(S^{j\oplus\mathbb{R}^k})^{-1}} S^{V\oplus\mathbb{R}^{n+k}} \xrightarrow{f} X(V\oplus\mathbb{R}^n) \xrightarrow[\cong]{X(j)} X(\bar{V})$$

represents a class $\langle f\rangle \in \pi_k^G(X)$. For $k \leq 0$, we choose a G-equivariant linear isometry $j : V\oplus\mathbb{R}^{n+k} \longrightarrow \bar{V}$ onto a G-subrepresentation $\bar{V}$ of $\mathcal{U}_G$. Then the composite

$$S^{\bar{V}} \xrightarrow[\cong]{(S^{j})^{-1}} S^{V\oplus\mathbb{R}^{n+k}} \xrightarrow{f} X(V\oplus\mathbb{R}^n) \xrightarrow[\cong]{X(j\oplus\mathbb{R}^{-k})} X(\bar{V}\oplus\mathbb{R}^{-k})$$

represents a class $\langle f\rangle \in \pi_k^G(X)$.

We also need a way to recognize that 'stabilization along a linear isometric embedding' does not change the class in $\pi_k^G(X)$. For this we let $\varphi : V \longrightarrow W$ be a G-equivariant linear isometric embedding and $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V\oplus\mathbb{R}^n)$ a continuous based G-map as above. We define $\varphi_* f : S^{W\oplus\mathbb{R}^{n+k}} \longrightarrow X(W\oplus\mathbb{R}^n)$ as the composite

$$\begin{aligned} S^{W\oplus\mathbb{R}^{n+k}} \;\cong\; S^{W-\varphi(V)}\wedge S^{V\oplus\mathbb{R}^{n+k}} &\xrightarrow{S^{W-\varphi(V)}\wedge f} S^{W-\varphi(V)}\wedge X(V\oplus\mathbb{R}^n) \\ &\xrightarrow{\sigma_{W-\varphi(V),V\oplus\mathbb{R}^n}} X((W-\varphi(V))\oplus V\oplus\mathbb{R}^n) \xrightarrow{\cong} X(W\oplus\mathbb{R}^n)\;; \end{aligned}$$

the two unnamed homeomorphisms use the linear isometry

$$(W-\varphi(V))\oplus V\oplus\mathbb{R}^m \;\cong\; W\oplus\mathbb{R}^m\;, \quad (w,v,x) \;\longmapsto\; (w+\varphi(v),x)$$

for $m = n + k$ and $m = n$, respectively. In the special case $n = k = 0$, this construction reduces to (3.1.9).

The same reasoning as in the unstable situation in Proposition 1.5.8 shows the following stable analog:

Proposition 3.1.14. *Let G be a compact Lie group and X an orthogonal G-spectrum. Let V be a G-representation and $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V\oplus\mathbb{R}^n)$ a based continuous G-map, where $n\in\mathbb{N}$ and $k\in\mathbb{Z}$ are such that $n + k \geq 0$.*

(i) *The class* $\langle f\rangle$ *in* $\pi_k^G(X)$ *is independent of the choice of linear isometry onto a subrepresentation of* $\mathcal{U}_G$.

(ii) *For every G-equivariant linear isometric embedding* $\varphi : V \longrightarrow W$ *the relation*

$$\langle\varphi_* f\rangle \;=\; \langle f\rangle \qquad \textit{holds in} \quad \pi_k^G(X).$$

Now we let K and G be two compact Lie groups. Every continuous based functor $F : G\mathbf{T}_* \longrightarrow K\mathbf{T}_*$ from based G-spaces to based K-spaces gives rise to a functor

$$F\circ - \;:\; G\mathcal{S}p \;\longrightarrow\; K\mathcal{S}p$$

from orthogonal G-spectra to orthogonal K-spectra by post-composition: if X is a G-orthogonal spectrum, then the composite

$$\mathbf{O} \xrightarrow{X} G\mathbf{T}_* \xrightarrow{F} K\mathbf{T}_* \,.$$

is an orthogonal K-spectrum. The next construction is an example of this.

Construction 3.1.15 (Restriction maps)**.** We let $\alpha : K \longrightarrow G$ be a continuous homomorphism between compact Lie groups. Given an orthogonal G-spectrum X, we apply restriction of scalars level-wise and obtain an orthogonal K-spectrum $\alpha^* X$. We define the *restriction homomorphism*

$$\alpha^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^K(\alpha^* X) \qquad \text{by} \qquad \alpha^*[f] \;=\; \langle\alpha^*(f)\rangle \,.$$

In other words, the class represented by a based G-map $f : S^V \longrightarrow X(V)$ is sent to the class represented by the K-map

$$\alpha^*(f) \;:\; S^{\alpha^*(V)} \;=\; \alpha^*(S^V) \longrightarrow \alpha^*(X(V)) \;=\; (\alpha^* X)(\alpha^* V)\,,$$

appealing to Construction 3.1.13. The restriction maps α^* are clearly transitive (contravariantly functorial) for composition of group homomorphisms.

For $g \in G$ the conjugation homomorphism is defined as

$$c_g \;:\; G \;\longrightarrow\; G\,, \quad c_g(h) \;=\; g^{-1} h g\,.$$

For every G-space A, left multiplication by g is then a G-equivariant homeomorphism $l_g^A : c_g^*(A) \longrightarrow A$. For an orthogonal G-spectrum X the maps $l_g^{X(V)} : (c_g^* X)(V) = c_g^*(X(V)) \longrightarrow X(V)$ assemble into an isomorphism of orthogonal G-spectra $l_g^X : c_g^* X \longrightarrow X$, as V runs over all inner product spaces (with trivial G-action).

Proposition 3.1.16. *Let G be a compact Lie group, X an orthogonal G-spectrum and* $g \in G$. *Then the two isomorphisms*

$$c_g^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^G(c_g^* X) \qquad \textit{and} \qquad (l_g^X)_* \;:\; \pi_0^G(c_g^* X) \;\longrightarrow\; \pi_0^G(X)$$

are inverse to each other.

Proof We let V be a G-representation, and we recall that the G-action on $X(V)$ is diagonally, from the external G-action on X and the internal G-action on V. Hence the map $l_g^{X(V)} : c_g^*(X(V)) \longrightarrow X(V)$ is the composite of the map $l_g^X(c_g^*V) : (c_g^*X)(c_g^*V) \longrightarrow X(c_g^*V)$ and the map $X(l_g^V) : X(c_g^*V) \longrightarrow X(V)$. Now we let $f : S^V \longrightarrow X(V)$ be a G-map representing a class in $\pi_0^G(X)$. The following diagram of G-spaces and G-maps commutes because f is G-equivariant:

$$\begin{tikzcd}
S^{c_g^*V} \arrow[r, "c_g^*f"] \arrow[d, "l_g^V"'] & c_g^*(X(V)) = (c_g^*X)(c_g^*V) \arrow[r, "l_g^X(c_g^*V)"] \arrow[dr, bend left, "l_g^{X(V)}"'] & X(c_g^*V) \arrow[d, "X(l_g^V)"] \\
S^V \arrow[rr, "f"'] & & X(V)
\end{tikzcd}$$

The upper horizontal composite represents the class $(l_g^X)_*(c_g^*[f])$. Since it differs from f by conjugation with an equivariant isometry, the upper composite represents the same class as f, by Proposition 3.1.14 (ii). Thus we conclude that $(l_g)_*(c_g^*[f]) = [f]$. □

Remark 3.1.17 (Weyl group action on equivariant homotopy groups)**.** We consider a closed subgroup H of a compact Lie group G and an orthogonal G-spectrum X. Every $g \in G$ gives rise to a conjugation homomorphism $c_g : H \longrightarrow H^g$ by $c_g(h) = g^{-1}hg$, where $H^g = \{g^{-1}hg \mid h \in H\}$ is the conjugate subgroup. One should beware that while $c_g^*(\mathrm{res}^G_{H^g}(X))$ and $\mathrm{res}^G_H(X)$ have the same underlying orthogonal spectrum, they come with *different* H-actions. However, left translation by g is an isomorphism of orthogonal H-spectra $l_g^X : c_g^*X \longrightarrow X$. So combining the restriction map along c_g with the effect of l_g^X gives an isomorphism

$$g_\star : \pi_0^{H^g}(X) \xrightarrow{(c_g)^*} \pi_0^H(c_g^*X) \xrightarrow{(l_g^X)_*} \pi_0^H(X) . \tag{3.1.18}$$

Moreover,

$$\begin{aligned} g_\star \circ g'_\star &= (l_g^X)_* \circ (c_g)^* \circ (l_{g'}^X)_* \circ (c_{g'})^* = (l_g^X)_* \circ ((c_g)^*(l_{g'}^X))_* \circ (c_g)^* \circ (c_{g'})^* \\ &= (l_g^X \circ (c_g)^*(l_{g'}^X))_* \circ (c_{g'} \circ c_g)^* = (l_{gg'}^X)_* \circ (c_{gg'})^* = (gg')_\star \end{aligned}$$

by naturality of $(c_g)^*$. If g normalizes H, then $g_\star$ is a self-map of the group $\pi_0^H(X)$. If moreover g belongs to H, then $g_\star$ is the identity by Proposition 3.1.16; so the maps $g_\star$ define an action of the Weyl group $W_GH = N_GH/H$ on the equivariant homotopy group $\pi_0^H(X)$.

If H has finite index in its normalizer, this is the end of the story concerning Weyl group actions on $\pi_0^H(X)$. In general, however the group H need not have

finite index in its normalizer $N_G H$, and the Weyl group $W_G H$ may have positive dimension, and hence a non-trivial identity path component $(W_G H)^\circ$. We will now show that the entire identity path component acts trivially on $\pi_0^H(X)$ for any orthogonal G-spectrum X. This is a consequence of the fact that every element of $(W_G H)^\circ$ has the form zH for an element z in $(C_G H)^\circ$, the identity component of the centralizer of H in G, compare Proposition A.26. But then $c_z : H \longrightarrow H$ is the identity because z centralizes H. On the other hand, any path from z and 1 in $C_G H$ induces a homotopy of morphisms of orthogonal H-spectra from $l_z : X \longrightarrow X$ to the identity of X. So

$$z_\star \;=\; (l_z^X)_* \circ (c_z)^* \;=\; \mathrm{Id}_{\pi_0^H(X)} \;.$$

This shows that the identity component of the Weyl group $W_G H$ acts trivially on $\pi_0^H(X)$. So the Weyl group action factors over an action of the discrete group

$$\pi_0(W_G H) \;=\; (W_G H)/(W_G H)^\circ \;.$$

Construction 3.1.19. If A is a pointed G-space, then smashing with A and taking based maps out of A are two continuous based endofunctors on the category of based G-spaces. So for every orthogonal G-spectrum X, we can define two new orthogonal G-spectra $X \wedge A$ and $\mathrm{map}_*(A, X)$ by smashing with A (and letting G act diagonally) or taking based maps from A level-wise (and letting G act by conjugation). More explicitly, we have

$$(X \wedge A)(V) \;=\; X(V) \wedge A \qquad \text{and} \qquad \mathrm{map}_*(A, X)(V) \;=\; \mathrm{map}_*(A, X(V))$$

for an inner product space V. The structure maps and actions of the orthogonal groups do not interact with A: the group $O(V)$ acts through its action on $X(V)$, and the structure maps are given by the composite

$$S^V \wedge (X \wedge A)(W) \;=\; S^V \wedge X(W) \wedge A \xrightarrow{\sigma_{V,W} \wedge A} X(V \oplus W) \wedge A \;=\; (X \wedge A)(V \oplus W)$$

and by the composite

$$S^V \wedge \mathrm{map}_*(A, X(W)) \longrightarrow \mathrm{map}_*(A, S^V \wedge X(W)) \xrightarrow{\mathrm{map}_*(A, \sigma_{V,W})} \mathrm{map}_*(A, X(V \oplus W))$$

where the first is an assembly map that sends $v \wedge f$ to the map sending $a \in A$ to $v \wedge f(a)$.

Just as the functors $- \wedge A$ and $\mathrm{map}_*(A, -)$ are adjoint on the level of based G-spaces, the two functors just introduced are an adjoint pair on the level of orthogonal G-spectra. The adjunction

$$G\mathcal{S}p(X, \mathrm{map}_*(A, Y)) \xrightarrow{\cong} G\mathcal{S}p(X \wedge A, Y) \tag{3.1.20}$$

takes a morphism $f : X \longrightarrow \mathrm{map}_*(A, Y)$ to the morphism $f^\flat : X \wedge A \longrightarrow Y$ whose Vth level $f^\flat(V) : X(V) \wedge A \longrightarrow Y(V)$ is $f^\flat(V)(x \wedge a) = f(V)(x)(a)$.

An important special case of this construction is when $A = S^W$ is a representation sphere, i.e., the one-point compactification of an orthogonal G-representation. The *Wth suspension* $X \wedge S^W$ is defined by

$$(X \wedge S^W)(V) \ = \ X(V) \wedge S^W \ ,$$

the smash product of the Vth level of X with the sphere S^W. The *Wth loop spectrum* $\Omega^W X = \mathrm{map}_*(S^W, X)$, defined by

$$(\Omega^W X)(V) \ = \ \Omega^W X(V) \ = \ \mathrm{map}_*(S^W, X(V)) \ ,$$

the based mapping space from S^W to the Vth level of X. We obtain an adjunction between $- \wedge S^W$ and Ω^W as the special case $A = S^W$ of (3.1.20).

Construction 3.1.21 (Shift of an orthogonal spectrum)**.** We introduce a spectrum analog of the additive shift of orthogonal spaces defined in Example 1.1.11. We let V be an inner product space and denote by

$$- \oplus V \ : \ \mathbf{O} \ \longrightarrow \ \mathbf{O}$$

the continuous functor given on objects by orthogonal direct sum with V, and on morphism spaces by

$$\mathbf{O}(U, W) \ \longrightarrow \ \mathbf{O}(U \oplus V, W \oplus V) \ , \quad (w, \varphi) \ \longmapsto \ ((w, 0), \varphi \oplus V) \ .$$

The *Vth shift* of an orthogonal spectrum X is the composite

$$\mathrm{sh}^V X \ = \ X \circ (- \oplus V) \ . \tag{3.1.22}$$

In other words, the value of $\mathrm{sh}^V X$ at an inner product space U is

$$(\mathrm{sh}^V X)(U) \ = \ X(U \oplus V) \ .$$

The orthogonal group $O(U)$ acts through the monomorphism $- \oplus V : O(U) \longrightarrow O(U \oplus V)$. The structure map $\sigma^{\mathrm{sh}^V X}_{U,W}$ of $\mathrm{sh}^V X$ is the structure map $\sigma^X_{U,W \oplus V}$ of X.

Since composition of functors is associative, the shift construction commutes on the nose with all constructions on orthogonal spectra that are given by post-composition with a continuous based functor as in Construction 3.1.19. This applies in particular to smashing with and taking mapping space from a based space A, i.e.,

$$(\mathrm{sh}^V X) \wedge A \ = \ \mathrm{sh}^V (X \wedge A) \quad \text{and} \quad \mathrm{map}_*(A, \mathrm{sh}^V X) \ = \ \mathrm{sh}^V(\mathrm{map}_*(A, X)) \ .$$

So we can – and will – omit the parentheses in expressions such as $\mathrm{sh}^V X \wedge A$.

The shift construction is also transitive in the following sense. The values of $\mathrm{sh}^V(\mathrm{sh}^W X)$ and $\mathrm{sh}^{V \oplus W} X$ at an inner product space U are given by

$$(\mathrm{sh}^V(\mathrm{sh}^W X))(U) \ = \ X((U \oplus V) \oplus W)$$

and

$$(\mathrm{sh}^{V\oplus W} X)(U) \;=\; X(U \oplus (V \oplus W))\,.$$

We use the effect of X on the associativity isomorphism

$$(U \oplus V) \oplus W \;\cong\; U \oplus (V \oplus W)\,, \quad ((u, v), w) \;\longmapsto\; (u, (v, w))$$

to identify these two spaces; then we abuse notation and write

$$\mathrm{sh}^V(\mathrm{sh}^W X) \;=\; \mathrm{sh}^{V\oplus W} X\,.$$

The suspension and the shift of an orthogonal spectrum X are related by a natural morphism

$$\lambda^V_X \;:\; X \wedge S^V \;\longrightarrow\; \mathrm{sh}^V X\,. \tag{3.1.23}$$

In level U, this is defined as $\lambda^V_X(U) = \sigma^{\mathrm{op}}_{U,V}$, the opposite structure map (3.1.5), i.e., the composite

$$X(U){\wedge}S^V \xrightarrow{\text{twist}} S^V{\wedge}X(U) \xrightarrow{\sigma_{V,U}} X(V{\oplus}U) \xrightarrow{X(\tau_{V,U})} X(U{\oplus}V) = (\mathrm{sh}^V X)(U)\,.$$

In the special case $V = \mathbb{R}$ we abbreviate $\lambda^{\mathbb{R}}_X$ to $\lambda_X : X{\wedge}S^1 \longrightarrow \mathrm{sh}\,X$. The λ-maps are transitive in the sense that for another inner product space W, the morphism $\lambda^{V\oplus W}_X$ coincides with the two composites in the commutative diagram:

$$\begin{tikzcd}
X \wedge S^{V\oplus W} \arrow[r, "\cong"] & X \wedge S^V \wedge S^W \arrow[rr, "\lambda^W_{X\wedge S^V}"] \arrow[d, "\lambda^V_X \wedge S^W"'] & & \mathrm{sh}^W X \wedge S^V \arrow[d, "\lambda^V_{\mathrm{sh}^W X}"] & \\
& \mathrm{sh}^V X \wedge S^W \arrow[rr, "\mathrm{sh}^V(\lambda^W_X)"'] & & \mathrm{sh}^V(\mathrm{sh}^W X) \arrow[r, equal] & \mathrm{sh}^{V\oplus W} X
\end{tikzcd}$$

Now we let G be a compact Lie group, V a G-representation and X an orthogonal G-spectrum. Then the orthogonal spectra $X \wedge S^V$ and $\mathrm{sh}^V X$ become orthogonal G-spectra by letting G act diagonally on X and V. With respect to these diagonal actions, the morphism $\lambda^V_X : X \wedge S^V \longrightarrow \mathrm{sh}^V X$ is a morphism of orthogonal G-spectra. Our next aim is to show that λ^V_X is in fact a $\underline{\pi}_*$-isomorphism. We define a homomorphism

$$\psi^V_X \;:\; \pi^G_k(\mathrm{sh}^V X) \;\longrightarrow\; \pi^G_k(X \wedge S^V) \tag{3.1.24}$$

by sending the class represented by a G-map

$$f \;:\; S^{U\oplus\mathbb{R}^{n+k}} \;\longrightarrow\; X(U \oplus \mathbb{R}^n \oplus V) \;=\; (\mathrm{sh}^V X)(U \oplus \mathbb{R}^n)$$

to the class represented by the composite

$$\begin{aligned}
S^{U\oplus V\oplus\mathbb{R}^{n+k}} &\xrightarrow{S^U\wedge\tau_{V,\mathbb{R}^{n+k}}} S^{U\oplus\mathbb{R}^{n+k}\oplus V} \xrightarrow{f\wedge S^V} X(U \oplus \mathbb{R}^n \oplus V) \wedge S^V \\
&\xrightarrow{X(U\oplus\tau_{\mathbb{R}^n,V})\wedge S^V} X(U \oplus V \oplus \mathbb{R}^n) \wedge S^V\,.
\end{aligned}$$

We omit the straightforward verification that this assignment is compatible with stabilization, and hence well-defined. The map ψ_X^V is natural for morphisms of orthogonal G-spectra in X. Finally, we define

$$\varepsilon_V \;:\; \pi_k^G(X \wedge S^V) \longrightarrow \pi_k^G(X \wedge S^V)$$

as the effect of the involution

$$X \wedge S^{-\mathrm{Id}_V} \;:\; X \wedge S^V \longrightarrow X \wedge S^V$$

induced by the 'negative' map of S^V.

Proposition 3.1.25. *Let G be a compact Lie group, X an orthogonal G-spectrum and V a G-representation.*

(i) *For every integer k, each of the three composites around the triangle*

$$\begin{tikzcd} \pi_k^G(X \wedge S^V) \arrow[rr, "(\lambda_X^V)_*"] & & \pi_k^G(\mathrm{sh}^V X) \arrow[dl, "\psi_X^V"] \\ & \pi_k^G(X \wedge S^V) \arrow[ul, "\varepsilon_V"] & \end{tikzcd}$$

is the respective identity.

(ii) *The morphism*

$$\lambda_X^V \;:\; X \wedge S^V \longrightarrow \mathrm{sh}^V X\,, \qquad \textit{its adjoint} \qquad \tilde{\lambda}_X^V \;:\; X \longrightarrow \Omega^V \mathrm{sh}^V X\,,$$

the adjunction unit $\eta_X^V : X \longrightarrow \Omega^V(X \wedge S^V)$ and the adjunction counit $\epsilon_X^V : (\Omega^V X) \wedge S^V \longrightarrow X$ are $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.

Proof We introduce an auxiliary functor $\pi^G(A; -)$ from orthogonal G-spectra to abelian groups that generalizes equivariant homotopy groups and depends on a based G-space A. We set

$$\pi^G(A; X) \;=\; \mathrm{colim}_{U \in s(\mathcal{U}_G)} \, [S^U \wedge A, X(U)]^G \,,$$

where the colimit is taken over the analogous stabilization maps as for π_0^G; the set $\pi^G(A; X)$ comes with a natural abelian group structure by the same reasoning as for $\pi_0^G(X)$. Then $\pi^G(S^k; X)$ is naturally isomorphic to $\pi_k^G(X)$; more generally, for a G-representation W, the adjunction bijections

$$[S^U \wedge S^{\mathbb{R}^k \oplus W}, X(U)]^G \;\cong\; [S^{U \oplus \mathbb{R}^k}, \Omega^W X(U)]^G$$

assemble into a natural isomorphism of abelian groups between $\pi^G(S^{\mathbb{R}^k \oplus W}; X)$ and $\pi_k^G(\Omega^W X)$.

The definition of the map ψ^V_X has a straightforward generalization to a natural homomorphism

$$\psi^V_X \;:\; \pi^G(A; \mathrm{sh}^V X) \;\longrightarrow\; \pi^G(A; X \wedge S^V)$$

by sending the class represented by a G-map

$$f \;:\; S^U \wedge A \;\longrightarrow\; X(U \oplus V) \;=\; (\mathrm{sh}^V X)(U)$$

to the class represented by the composite

$$S^{U\oplus V} \wedge A \xrightarrow{S^U \wedge \tau_{S^V,A}} S^U \wedge A \wedge S^V \xrightarrow{f \wedge S^V} X(U \oplus V) \wedge S^V .$$

We omit the straightforward verification that this assignment is compatible with stabilization, and hence well-defined. We claim that each of the three composites around the triangle

$$\begin{tikzcd}
\pi^G(A; X \wedge S^V) \arrow[rr, "(\lambda^V_X)_*"] & & \pi^G(A; \mathrm{sh}^V X) \arrow[dl, "\psi^V_X"] \\
& \pi^G(A; X \wedge S^V) \arrow[ul, "\varepsilon_V"] &
\end{tikzcd}$$

is the respective identity. We consider a based continuous G-map $f : S^U \wedge A \longrightarrow X(U) \wedge S^V$ that represents a class in $\pi^G(A; X \wedge S^V)$. Then the class $\varepsilon_V(\psi^V_X((\lambda^V_X)_*\langle f\rangle))$ is represented by the composite

$$S^{U\oplus V} \wedge A \;\cong\; S^U \wedge A \wedge S^V \xrightarrow{f \wedge S^V} X(U) \wedge S^V \wedge S^V \xrightarrow{\sigma^{\mathrm{op}}_{U,V} \wedge S^{-\mathrm{Id}_V}} X(U \oplus V) \wedge S^V .$$

The map $V \oplus (-\mathrm{Id}_V) : V \oplus V \longrightarrow V \oplus V$ is homotopic, through G-equivariant linear isometries, to the twist map $\tau_{V,V} : V \oplus V \longrightarrow V \oplus V$ that interchanges the two summands. So $\varepsilon_V(\psi^V_X((\lambda^V_X)_*\langle f\rangle))$ is also represented by the left vertical composite in the following diagram of based continuous G-maps:

$$\begin{tikzcd}
S^U \wedge S^V \wedge A \arrow[r, "\tau_{U,V} \wedge A"] \arrow[d, "S^U \wedge \tau_{A,S^V}"'] & S^V \wedge S^U \wedge A \arrow[ddd, "S^V \wedge f"] \\
S^U \wedge A \wedge S^V \arrow[d, "f \wedge S^V"'] & \\
X(U) \wedge S^V \wedge S^V \arrow[d, "X(U) \wedge \tau_{V,V}"'] \arrow[dr, "\tau_{X(U) \wedge S^V, S^V}"] & \\
X(U) \wedge S^V \wedge S^V \arrow[r, "\tau_{X(U), S^V \wedge S^V}"'] \arrow[d, "\sigma^{\mathrm{op}}_{U,V} \wedge S^V"'] & S^V \wedge X(U) \wedge S^V \arrow[d, "\sigma_{V,U} \wedge S^V"] \\
X(U \oplus V) \wedge S^V \arrow[r, "X(\tau_{U,V}) \wedge S^V"'] & X(V \oplus U) \wedge S^V
\end{tikzcd}$$

The right vertical composite $(\sigma_{V,U} \wedge S^V) \circ (S^V \wedge f)$ is the stabilization of f, so it represents the same class in $\pi^G(A; X \wedge S^V)$. Since the left and right vertical composites differ by conjugation with an equivariant isometry, they also represent the same class in $\pi^G(A; X \wedge S^V)$, by Proposition 3.1.14 (ii). Altogether this shows that the composite $\varepsilon_V \circ \psi^V_X \circ (\lambda^V_X)_*$ is the identity. Since ε^2_V is the identity, this also implies that the composite $\psi^V_X \circ (\lambda^V_X)_* \circ \varepsilon_V$ is the identity.

The remaining case is similar. We consider a based continuous G-map $g : S^U \wedge A \longrightarrow X(U \oplus V) = (\mathrm{sh}^V X)(U)$ that represents a class in $\pi^G(A; \mathrm{sh}^V X)$. Then the class $(\lambda^V_X)_*(\varepsilon_V(\psi^V_X\langle g\rangle))$ is represented by the composite

$$\begin{aligned} S^U \wedge S^V \wedge A &\xrightarrow{S^U \wedge \tau_{S^V,A}} S^U \wedge A \wedge S^V \xrightarrow{g \wedge S^V} X(U \oplus V) \wedge S^V \\ &\xrightarrow{X(U\oplus V)\wedge S^{-\mathrm{Id}_V}} X(U \oplus V) \wedge S^V \\ &\xrightarrow{\sigma^{\mathrm{op}}_{U\oplus V,V}} X(U \oplus V \oplus V) \ = \ (\mathrm{sh}^V X)(U \oplus V)\ . \end{aligned}$$

Since

$$\sigma^{\mathrm{op}}_{U\oplus V,V} \circ (X(U \oplus V) \wedge S^{-\mathrm{Id}}) \ = \ X(U \oplus V \oplus (-\mathrm{Id})) \circ \sigma^{\mathrm{op}}_{U\oplus V,V}$$

and $V \oplus (-\mathrm{Id}_V) : V \oplus V \longrightarrow V \oplus V$ is G-homotopic to the twist $\tau_{V,V}$, the class $(\lambda^V_X)_*(\varepsilon_V(\psi^V_X\langle g\rangle))$ is also represented by the left vertical composite in the following diagram:

$$\begin{tikzcd}
S^U \wedge S^V \wedge A \arrow[r, "\tau_{U,V}\wedge A"] \arrow[d, "S^U\wedge\tau_{S^V,A}"'] & S^V \wedge S^U \wedge A \arrow[ddd, "S^V\wedge g"] \\
S^U \wedge A \wedge S^V \arrow[d, "g\wedge S^V"'] & \\
X(U \oplus V) \wedge S^V \arrow[d, "\sigma^{\mathrm{op}}_{U\oplus V,V}"'] \arrow[dr, "\tau_{X(U)\wedge S^V,S^V}"] & \\
X(U \oplus V \oplus V) \arrow[d, "X(U\oplus\tau_{V,V})"'] \arrow[dr, "X(\tau_{U\oplus V,V})"] & S^V \wedge X(U \oplus V) \arrow[d, "\sigma_{V,U\oplus V}"] \\
X(U \oplus V \oplus V) \arrow[r, "X(\tau_{U,V}\oplus V)"'] \arrow[d, equal] & X(U \oplus V \oplus V) \arrow[d, equal] \\
(\mathrm{sh}^V X)(U \oplus V) \arrow[r, "(\mathrm{sh}^V X)(\tau_{U,V})"'] & (\mathrm{sh}^V X)(U \oplus V)
\end{tikzcd}$$

The right vertical composite $\sigma_{V,U\oplus V} \circ (S^V \wedge g)$ is the stabilization of g, so it represents the same class in $\pi^G(A; \mathrm{sh}^V X)$. Since the left and right vertical composites differ by conjugation with an equivariant isometry, they represent the same class, so the composite $(\lambda^V_X)_* \circ \varepsilon_V \circ \psi^V_X$ is the identity.

Now we prove claim (i) of the proposition. For $k \geq 0$, it is the special case $A = S^k$ of the discussion above. To deduce the claim for negative dimensional homotopy groups we use the isomorphism of orthogonal G-spectra

$$\tau_{k,V} \ : \ \mathrm{sh}^k(\mathrm{sh}^V X) \ \cong \ \mathrm{sh}^V(\mathrm{sh}^k X) \tag{3.1.26}$$

whose value at an inner product space U is the map

$$X(U \oplus \tau_{\mathbb{R}^k,V}) \ : \ X(U \oplus \mathbb{R}^k \oplus V) \ \cong \ X(U \oplus V \oplus \mathbb{R}^k) \ .$$

Then the following diagram commutes:

$$\begin{tikzcd}
\pi^G_{-k}(X \wedge S^V) \arrow[r, "(\lambda^V_X)_*"] \arrow[dd, equal] & \pi^G_{-k}(\mathrm{sh}^V X) \arrow[r, "\varepsilon_V \circ \psi^V_X"] \arrow[d, equal] & \pi^G_{-k}(X \wedge S^V) \arrow[dd, equal] \\
& \pi^G_0(\mathrm{sh}^k(\mathrm{sh}^V X)) \arrow[d, "(\tau_{k,V})_*"', "\cong"] & \\
\pi^G_0(\mathrm{sh}^k X \wedge S^V) \arrow[r, "(\lambda^V_{\mathrm{sh}^k X})_*"'] & \pi^G_0(\mathrm{sh}^V(\mathrm{sh}^k X)) \arrow[r, "\varepsilon_V \circ \psi^V_{\mathrm{sh}^k X}"'] & \pi^G_0(\mathrm{sh}^k X \wedge S^V)
\end{tikzcd}$$

So the claim in dimension $-k$ for the orthogonal G-spectrum X is a consequence of the previously established claim in dimension 0 for the orthogonal G-spectrum $\mathrm{sh}^k X$.

(ii) We start with the morphism $\tilde{\lambda}^V_X$, which can be treated fairly directly. We discuss the case $k \geq 0$ and leave the analogous argument for $k < 0$ to the reader. We define a map in the opposite direction

$$\kappa \ : \ \pi^G_k(\Omega^V \mathrm{sh}^V X) \ \longrightarrow \ \pi^G_k(X) \ .$$

We let $g : S^{U\oplus\mathbb{R}^k} \longrightarrow \Omega^V X(U \oplus V) = (\Omega^V \mathrm{sh}^V X)(U)$ represent a class of the left-hand side. The map κ sends $[g]$ to the class represented by the composite

$$S^{U\oplus V\oplus\mathbb{R}^k} \xrightarrow{S^U \wedge \tau_{V,\mathbb{R}^k}} S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{g^\flat} X(U \oplus V) \ ,$$

where $g^\flat$ is the adjoint of g. This is compatible with stabilization.

We claim that the map κ is injective. Indeed, if $g : S^{U\oplus\mathbb{R}^k} \longrightarrow \Omega^V X(U \oplus V)$ represents an element in the kernel of κ, then after increasing U, if necessary, the composite $g^\flat \circ (S^U \wedge \tau_{V,\mathbb{R}^k})$ is G-equivariantly null-homotopic. But then $g^\flat$, and hence also its adjoint g, are equivariantly null-homotopic. So κ is injective.

The composite $\kappa \circ (\tilde{\lambda}^V_X)_*$ sends the class of a G-map $f : S^{U\oplus\mathbb{R}^k} \longrightarrow X(U)$ to the class of the composite

$$S^{U\oplus V\oplus\mathbb{R}^k} \xrightarrow{S^U \wedge \tau_{V,\mathbb{R}^k}} S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{(\eta^V_X(U)\circ f)^\flat} X(U \oplus V) \ . \tag{3.1.27}$$

The adjoint $(\eta_X^V(U) \circ f)^\flat$ coincides with the composite

$$S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{f\wedge S^V} X(U)\wedge S^V \xrightarrow{\sigma_{U,V}^{\mathrm{op}}} X(U\oplus V)\ ,$$

so the composite (3.1.27) represents the same class as f. This proves that $\kappa \circ (\tilde\lambda_X^V)_*$ is the identity. Since κ is also injective, the map $(\tilde\lambda_X^V)_*$ is bijective.

If H is a closed subgroup of G we apply the previous argument to the underlying H-representation of V to conclude that $\tilde\lambda_X^V$ induces an isomorphism on π_*^H. So $(\tilde\lambda_X^V)_*$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.

Now we treat the morphism λ_X^V. Again we show a more general statement, namely that for every pair of G-representations V and W the morphism

$$\Omega^W(\lambda_X^V)\ :\ \Omega^W(X\wedge S^V)\ \longrightarrow\ \Omega^W(\mathrm{sh}^V X)$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. We start with the effect on G-equivariant homotopy groups. For $k \geq 0$ this follows by applying part (i) with $A = S^{\mathbb{R}^k\oplus W}$ and exploiting the natural isomorphism $\pi_k^G(\Omega^W Y) \cong \pi^G(S^{\mathbb{R}^k\oplus W}; Y)$. To get the same conclusion for negative dimensional homotopy groups we exploit the fact that $\pi_{-k}^G(Y) = \pi_0^G(\mathrm{sh}^k Y)$, by definition. Moreover, the following diagram commutes

$$\begin{tikzcd}
\pi_{-k}^G(\Omega^W(X\wedge S^V)) \arrow[r, equal] \arrow[dd, "\pi_{-k}^G(\Omega^W(\lambda_X^V))"'] & \pi_0^G(\Omega^W(\mathrm{sh}^k X\wedge S^V)) \arrow[d, "\pi_0^G(\Omega^W(\lambda_{\mathrm{sh}^k X}^V))"] \\
 & \pi_0^G(\Omega^W(\mathrm{sh}^V(\mathrm{sh}^k X))) \\
\pi_{-k}^G(\Omega^W(\mathrm{sh}^V X)) \arrow[r, equal] & \pi_0^G(\Omega^W(\mathrm{sh}^k(\mathrm{sh}^V X))) \arrow[u, "\cong"', "\pi_0^G(\Omega^W\tau_{k,V})"]
\end{tikzcd}$$

where the isomorphism $\tau_{k,V}$ was defined in (3.1.26). So the previous argument applied to the spectrum $\mathrm{sh}^k X$ shows that the morphism $\Omega^W(\lambda_X^V)$ also induces isomorphisms on G-equivariant homotopy groups in negative dimensions. If H is any closed subgroup of G, then we consider the underlying H-representations of V and W and conclude that the morphism $\Omega^W(\lambda_X^V)$ induces isomorphisms on π_*^H. This proves that $\Omega^W(\lambda_X^V)$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. The special case $W = 0$ proves that λ_X^V is a $\underline{\pi}_*$-isomorphism.

The morphism $\tilde\lambda_X^V$ factors as the composite

$$X \xrightarrow{\eta_X^V} \Omega^V(X\wedge S^V) \xrightarrow{\Omega^V(\lambda_X^V)} \Omega^V\,\mathrm{sh}^V X\ .$$

Since both $\tilde\lambda_X^V$ and $\Omega^V(\lambda_X^V)$ are $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, so is the adjunction unit η_X^V.

Finally, we treat the adjunction counit ϵ_X^V. The two homomorphisms of or-

thogonal G-spectra

$$\Omega^V(\tilde{\lambda}^V_X)\ ,\ \tilde{\lambda}^V_{\Omega^V X}\ :\ \Omega^V X \longrightarrow \Omega^V(\Omega^V \operatorname{sh}^V X)$$

are *not* the same; they differ by the involution on the target that interchanges the two V-loop coordinates. An equivariant homotopy of linear isometries from $\tau_V : V \oplus V \longrightarrow V \oplus V$ to $(-\operatorname{Id}) \oplus \operatorname{Id}$ thus induces an equivariant homotopy between the morphism $\Omega^V(\tilde{\lambda}^V_X)$ and the composite

$$\Omega^V X \xrightarrow{\tilde{\lambda}^V_{\Omega^V X}} \Omega^V(\Omega^V \operatorname{sh}^V X) \xrightarrow{\Omega^V \operatorname{map}_*(S^{-\operatorname{Id}}, \operatorname{sh}^V X)} \Omega^V(\Omega^V \operatorname{sh}^V X)\ .$$

Passing to adjoints shows that the square of morphisms of orthogonal G-spectra

$$\begin{tikzcd}
(\Omega^V X) \wedge S^V \arrow[rr, "\epsilon^V_X"] \arrow[d, "\lambda^V_{\Omega^V X}"'] & & X \arrow[d, "\tilde{\lambda}^V_X"] \\
\operatorname{sh}^V \Omega^V X \arrow[rr, "\cong", "\operatorname{sh}^V \operatorname{map}_*(S^{-\operatorname{Id}}, X)"'] & & \operatorname{sh}^V \Omega^V X
\end{tikzcd}$$

commutes up to G-equivariant homotopy. Since the two vertical morphisms are $\underline{\pi}_*$-isomorphisms, and the lower horizontal one is even an isomorphism, we conclude that ϵ^V_X is a $\underline{\pi}_*$-isomorphism. □

Now we recall some important properties of equivariant homotopy groups, such as stability under suspension and looping, and the long exact sequences associated with mapping cones and homotopy fibers. We define the *loop isomorphism*

$$\alpha\ :\ \pi^G_k(\Omega X) \longrightarrow \pi^G_{k+1}(X)\ . \tag{3.1.28}$$

We represent a given class in $\pi^G_k(\Omega X)$ by a based G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \Omega X(V\oplus\mathbb{R}^n)$ and let $f^\flat : S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V\oplus\mathbb{R}^n)$ denote the adjoint of f, which represents an element of $\pi^G_{k+1}(X)$. Then we set $\alpha[f] = [f^\flat]$.

Next we define the *suspension isomorphism*

$$-\wedge S^1\ :\ \pi^G_k(X) \longrightarrow \pi^G_{k+1}(X \wedge S^1)\ . \tag{3.1.29}$$

We represent a given class in $\pi^G_k(X)$ by a based G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V \oplus \mathbb{R}^n)$; then $f\wedge S^1 : S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V\oplus\mathbb{R}^n)\wedge S^1$ represents a class in $\pi^G_{k+1}(X\wedge S^1)$, and we set $[f] \wedge S^1 = [f \wedge S^1]$.

Proposition 3.1.30. *Let G be a compact Lie group, X an orthogonal G-spectrum and k an integer. Then the loop isomorphism* (3.1.28) *and the suspension isomorphism* (3.1.29) *are isomorphisms of abelian groups.*

Proof The inverse to the loop map (3.1.28) is given by sending the class

of a G-map $S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V\oplus\mathbb{R}^n)$ to the class of its adjoint $S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \Omega X(V\oplus\mathbb{R}^n)$. The suspension homomorphism is the composite of the two maps

$$\pi_k^G(X) \xrightarrow{(\eta_X)_*} \pi_k^G(\Omega(X\wedge S^1)) \underset{\cong}{\xrightarrow{\alpha}} \pi_{k+1}^G(X\wedge S^1)\ ;$$

since $(\eta_X)_*$ is an isomorphism by Proposition 3.1.25 (ii), so is the suspension isomorphism. □

A key feature that distinguishes stable from unstable equivariant homotopy theory – and at the same time an important calculational tool – is the fact that mapping cones give rise to long exact sequences of equivariant homotopy groups. Our next aim is to establish this long exact sequence, see Proposition 3.1.36.

Construction 3.1.31 (Mapping cone and homotopy fiber)**.** The *reduced mapping cone Cf* of a morphism of based spaces $f : A \longrightarrow B$ is defined by

$$Cf \;=\; (A\wedge[0,1])\cup_f B\ .$$

Here the unit interval $[0,1]$ is pointed by 0, so that $A\wedge[0,1]$ is the reduced cone of A. The mapping cone comes with an embedding $i : B \longrightarrow Cf$ and a projection $p : Cf \longrightarrow A\wedge S^1$; the projection sends B to the basepoint and is given on $A\wedge[0,1]$ by $p(a,x) = a\wedge t(x)$, where

$$t\;:\;[0,1]\;\longrightarrow\;S^1 \qquad \text{is defined as} \qquad t(x)\;=\;\frac{2x-1}{x(1-x)}\ . \tag{3.1.32}$$

What is relevant about the map t is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and $S^1 = \mathbb{R}\cup\{\infty\}$, and that it satisfies $t(1-x) = -t(x)$.

The *mapping cone Cf* of a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is now defined level-wise by

$$(Cf)(V)\;=\;C(f(V))\;=\;(X(V)\wedge[0,1])\cup_{f(V)} Y(V)\ ,$$

the reduced mapping cone of $f(V) : X(V) \longrightarrow Y(V)$. The groups G and $O(V)$ act on $(Cf)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. The embeddings $i(V) : Y(V) \longrightarrow C(f(V))$ and projections $p(V) : C(f(V)) \longrightarrow X(V)\wedge S^1$ assemble into morphisms of orthogonal G-spectra

$$i\;:\;Y\;\longrightarrow\;Cf \qquad\text{and}\qquad p\;:\;Cf\;\longrightarrow\;X\wedge S^1\ . \tag{3.1.33}$$

We define a *connecting homomorphism* $\partial : \pi_{k+1}^G(Cf) \longrightarrow \pi_k^G(X)$ as the composite

$$\pi_{k+1}^G(Cf) \xrightarrow{\pi_{k+1}^G(p)} \pi_{k+1}^G(X\wedge S^1) \xrightarrow{-\wedge S^{-1}} \pi_k^G(X)\ , \tag{3.1.34}$$

the effect of the projection $p : Cf \longrightarrow X \wedge S^1$ on homotopy groups, followed by the inverse of the suspension isomorphism (3.1.29).

The *homotopy fiber* is the construction ‘dual’ to the mapping cone. The homotopy fiber of a continuous map $f : A \longrightarrow B$ of based spaces is the fiber product

$$F(f) \;=\; * \times_B B^{[0,1]} \times_B A \;=\; \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\ \lambda(1) = f(a)\}\ ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \xrightarrow{\ i\ } F(f) \xrightarrow{\ p\ } A\ .$$

The map p is the projection to the second factor; the value of the map i on a based loop $\omega : S^1 \longrightarrow B$ is

$$i(\omega) = (\omega \circ t, *)\ ,$$

where $t : [0, 1] \longrightarrow S^1$ was defined in (3.1.32).

The homotopy fiber $F(f)$ of a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is the orthogonal G-spectrum defined by

$$F(f)(V) \;=\; F(f(V))\ ,$$

the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. The groups G and $O(V)$ act on $F(f)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. Put another way, the homotopy fiber is the pullback in the cartesian square of orthogonal G-spectra:

$$\begin{tikzcd} F(f) \arrow[r, "p"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\ Y^{[0,1]} \arrow[r, "\lambda \mapsto (\lambda(0),\lambda(1))"'] & Y \times Y \end{tikzcd}$$

The inclusions $i(V) : \Omega Y(V) \longrightarrow F(f(V))$ and projections $p(V) : F(f(V)) \longrightarrow X(V)$ assemble into morphisms of orthogonal G-spectra

$$i \;:\; \Omega Y \;\longrightarrow\; F(f) \qquad \text{and} \qquad p \;:\; F(f) \;\longrightarrow\; X\ .$$

We define a *connecting homomorphism* $\partial : \pi^G_{k+1}(Y) \longrightarrow \pi^G_k(F(f))$ as the composite

$$\pi^G_{k+1}(Y) \xrightarrow{\ \alpha^{-1}\ } \pi^G_k(\Omega Y) \xrightarrow{\ \pi^G_k(i)\ } \pi^G_k(F(f))\ ,$$

where the first map is the inverse of the loop isomorphism (3.1.28).

The proof of exactness for the mapping cone sequence will need some elementary homotopies that we spell out in the next proposition.

Proposition 3.1.35. *Let G be a topological group.*

(i) *For every continuous based map $f : A \longrightarrow B$ of based G-spaces the composites*

$$A \xrightarrow{f} B \xrightarrow{i} Cf \qquad \text{and} \qquad F(f) \xrightarrow{p} A \xrightarrow{f} B$$

are naturally based G-null-homotopic. Moreover, the diagram

$$\begin{tikzcd} & CA \cup_f CB \arrow[dl, "p_A \cup *"'] \arrow[dr, "* \cup p_B"] & \\ A \wedge S^1 \arrow[rr, "f \wedge \tau"'] & & B \wedge S^1 \end{tikzcd}$$

commutes up to natural, based G-homotopy, where τ is the sign involution of S^1 given by $x \mapsto -x$.

(ii) *For every based G-space Z the map $p_Z \cup * : CZ \cup_{Z \times 1} CZ \longrightarrow Z \wedge S^1$ which collapses the second cone is a based G-homotopy equivalence.*

(iii) *Let $f : A \longrightarrow B$ and $\beta : Z \longrightarrow B$ be morphisms of based G-spaces such that the composite $i\beta : Z \longrightarrow Cf$ is equivariantly null-homotopic. Then there exists a based G-map $h : Z \wedge S^1 \longrightarrow A \wedge S^1$ such that $(f \wedge S^1) \circ h : Z \wedge S^1 \longrightarrow B \wedge S^1$ is equivariantly homotopic to $\beta \wedge S^1$.*

Proof (i) We specify natural G-homotopies by explicit formulas. The map $if : A \longrightarrow Cf$ is null-homotopic by $A \times [0,1] \longrightarrow Cf$, $(a, s) \mapsto (a, s)$, i.e., the composite of the canonical maps $A \times [0,1] \longrightarrow A \wedge [0,1]$ and $A \wedge [0,1] \longrightarrow Cf$. The map $fp : F(f) \longrightarrow B$ is null-homotopic by $F(f) \times [0,1] \longrightarrow B$, $(\lambda, a, s) \mapsto \lambda(s)$.

The homotopy for the triangle will be glued together from two pieces. We define a based homotopy $H : CA \times [0,1] \longrightarrow B \wedge S^1$ by the formula

$$H(a, s, u) \ = \ f(a) \wedge t(2 - s - u)$$

which is to be interpreted as the basepoint if $2 - s - u \geq 1$. Another based homotopy $H' : CB \times [0,1] \longrightarrow B \wedge S^1$ is given by the formula

$$H'(b, s, u) \ = \ b \wedge t(s - u) \ ,$$

where $t : [0,1] \longrightarrow S^1$ was defined in (3.1.32). This formula is to be interpreted as the basepoint if $s \leq u$. The two homotopies are compatible in the sense that

$$H(a, 1, u) \ = \ f(a) \wedge t(1 - u) \ = \ H'(f(a), 1, u) \ ,$$

for all $a \in A$ and $u \in [0,1]$. So H and H' glue and yield a homotopy

$$\big(CA \cup_f CB\big) \times [0,1] \cong (CA \times [0,1]) \cup_{f \times [0,1]} (CB \times [0,1]) \xrightarrow{H \cup H'} B \wedge S^1 .$$

For $u = 0$ this homotopy starts with the map $* \cup p_B$, and it ends for $u = 1$ with the map $(f \wedge \tau) \circ (p_A \cup *)$.

(ii) Since the functor $Z \wedge -$ is a left adjoint and $Z \wedge \{0,1\} \cong Z \times 1$, the space $CZ \cup_{1 \times Z} CZ$ is homeomorphic to the smash product of Z and the pushout $[0,1] \cup_{\{0,1\}} [0,1]$. This identification

$$CZ \cup_{Z \times 1} CZ \ \cong \ Z \wedge ([0,1] \cup_{\{0,1\}} [0,1])$$

turns the map p_Z into the map

$$Z \wedge (t \cup *) \ : \ Z \wedge ([0,1] \cup_{\{0,1\}} [0,1]) \ \longrightarrow \ Z \wedge S^1 .$$

So the claim follows from the fact that the map $t \cup * : [0,1] \cup_{\{0,1\}} [0,1] \longrightarrow S^1$ is a based homotopy equivalence.

(iii) Let $H : CZ = Z \wedge [0,1] \longrightarrow Cf$ be a based, equivariant null-homotopy of the composite $i\beta : Z \longrightarrow Cf$, i.e., $H(z,1) = i(\beta(z))$ for all $z \in Z$. The composite $p_A H : CZ \longrightarrow A \wedge S^1$ then factors as $p_A H = h p_Z$ for a unique G-map $h : Z \wedge S^1 \longrightarrow A \wedge S^1$. We claim that h has the required property.

To prove the claim we need the G-homotopy equivalence $p_Z \cup * : CZ \cup_{Z \times 1} CZ \longrightarrow Z \wedge S^1$ which collapses the second cone. We obtain a sequence of equalities and G-homotopies

$$\begin{aligned}
(f \wedge S^1) \circ h \circ (p_Z \cup *) \ &= \ (f \wedge S^1) \circ (p_A \cup *) \circ (H \cup C(\beta)) \\
&= \ (B \wedge \tau) \circ (f \wedge \tau) \circ (p_A \cup *) \circ (H \cup C(\beta)) \\
&\simeq \ (B \wedge \tau) \circ (* \cup p_B) \circ (H \cup C(\beta)) \\
&= \ (B \wedge \tau) \circ (\beta \wedge S^1) \circ (* \cup p_Z) \\
&= \ (\beta \wedge S^1) \circ (Z \wedge \tau) \circ (* \cup p_Z) \ \simeq \ (\beta \wedge S^1) \circ (p_Z \cup *)
\end{aligned}$$

Here $H \cup C(\beta) : CZ \cup_{1 \times Z} CZ \longrightarrow Cf \cup_B CB \cong CA \cup_f CB$ and τ is the sign involution of S^1. The two homotopies result from part (i) applied to f and to the identity of Z, and we used the naturality of various constructions. Since the map $p_Z \cup *$ is a G-homotopy equivalence by part (ii), this proves that $(f \wedge S^1) \circ h$ is G-homotopic to $\beta \wedge S^1$. □

Now we are ready to prove the long exact homotopy group sequences for mapping cones and homotopy fibers.

Proposition 3.1.36. *For every compact Lie group G and every morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the long sequences of equivariant homotopy*

groups

$$\cdots \longrightarrow \pi_{k+1}^G(Cf) \xrightarrow{\ \partial\ } \pi_k^G(X) \xrightarrow{\pi_k^G(f)} \pi_k^G(Y) \xrightarrow{\pi_k^G(i)} \pi_k^G(Cf) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi_{k+1}^G(Y) \xrightarrow{\ \partial\ } \pi_k^G(F(f)) \xrightarrow{\pi_k^G(p)} \pi_k^G(X) \xrightarrow{\pi_k^G(f)} \pi_k^G(Y) \longrightarrow \cdots$$

are exact.

Proof We start with exactness of the first sequence at $\pi_k^G(Y)$. The composite of $f : X \longrightarrow Y$ and the inclusion $i : Y \longrightarrow Cf$ is equivariantly null-homotopic, so it induces the trivial map on π_k^G. It remains to show that every element in the kernel of $\pi_k^G(i) : \pi_k^G(Y) \longrightarrow \pi_k^G(Cf)$ is in the image of $\pi_k^G(f)$. We show this for $k \geq 0$; for $k < 0$ we can either use a similar argument or exploit the fact that $\pi_k^G(X) = \pi_0^G(\mathrm{sh}^{\mathbb{R}^{-k}} X)$ and shifting commutes with the formation of mapping cones. We let $\beta : S^{V\oplus\mathbb{R}^k} \longrightarrow Y(V)$ represent an element in the kernel of $\pi_k^G(i)$. By increasing V, if necessary, we can assume that the composite of β with the inclusion $i : Y(V) \longrightarrow (Cf)(V) = C(f(V))$ is equivariantly null-homotopic. Proposition 3.1.35 (iii) provides a G-map $h : S^{V\oplus\mathbb{R}^k} \wedge S^1 \longrightarrow X(V) \wedge S^1$ such that $(f(V) \wedge S^1) \circ h$ is G-homotopic to $\beta \wedge S^1$. The composite

$$S^{V\oplus\mathbb{R}^{k+1}} \xrightarrow{\ h\ } X(V) \wedge S^1 \xrightarrow{\sigma^{\mathrm{op}}_{V,\mathbb{R}}} X(V \oplus \mathbb{R})$$

represents an equivariant homotopy class in $\pi_k^G(X)$ and we have

$$\begin{aligned}\pi_k^G(f)\langle\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h\rangle \ &= \ \langle f(V \oplus \mathbb{R}) \circ \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h\rangle \\ &= \langle\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (f(V) \wedge S^1) \circ h\rangle \ = \langle\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (\beta \wedge S^1)\rangle \ = \ (-1)^k \cdot \langle\beta\rangle \ .\end{aligned}$$

To justify the last equation we let $\varphi : V \longrightarrow V \oplus \mathbb{R}$ denote the embedding of the first summand. Then the maps

$$\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (\beta \wedge S^1) \ : \ S^{V\oplus\mathbb{R}^k\oplus\mathbb{R}} \longrightarrow Y(V\oplus\mathbb{R}) \quad \text{and} \quad \varphi_*\beta \ : \ S^{V\oplus\mathbb{R}\oplus\mathbb{R}^k} \longrightarrow Y(V\oplus\mathbb{R})$$

differ by the permutation of the source that moves one sphere coordinate past k other sphere coordinates; this permutation has degree k, so Proposition 3.1.14 (ii) establishes the last equation. Altogether this shows that the class represented by β is the image under $\pi_k^G(f)$ of the class $(-1)^k \cdot \langle\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h\rangle$.

We now deduce the exactness at $\pi_k^G(Cf)$ and $\pi_{k-1}^G(X)$ by comparing the mapping cone sequence for $f : X \longrightarrow Y$ to the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$ (shifted to the left). We observe that the collapse map

$$* \cup p \ : \ Ci \cong CY \cup_f CX \ \longrightarrow \ X \wedge S^1$$

is an equivariant homotopy equivalence, and thus induces an isomorphism of

equivariant homotopy groups. Indeed, a homotopy inverse

$$r \;:\; X \wedge S^1 \longrightarrow CY \cup_f CX$$

is defined by the formula

$$r(x \wedge s) \;=\; \begin{cases} \quad (x, 2s) \quad\; \in CX & \text{for } 0 \le s \le 1/2, \text{ and} \\ (f(x), 2-2s) \in CY & \text{for } 1/2 \le s \le 1, \end{cases}$$

which is to be interpreted level-wise. We omit the G-homotopies $r(* \cup p) \simeq \mathrm{Id}$ and $(* \cup p)r \simeq \mathrm{Id}$; explicit formulas can be found in the proof of [180, Note (4.6.1)]. Now we consider the diagram

$$\begin{tikzcd} Cf \arrow[r, "i_i"] \arrow[dr, "p"'] & Ci \arrow[r, "p_i"] \arrow[d, "* \cup p"] & Y \wedge S^1 \\ & X \wedge S^1 \arrow[ur, "f \wedge \tau"'] & \end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$. The left triangle commutes on the nose and the right triangle commutes up to G-homotopy, by Proposition 3.1.35 (i). We get a diagram

$$\begin{tikzcd} \pi_k^G(Y) \arrow[r, "\pi_k^G(i)"] \arrow[d, equal] & \pi_k^G(Cf) \arrow[r, "\pi_k^G(i_i)"] \arrow[d, equal] & \pi_k^G(Ci) \arrow[r, "\partial"] \arrow[d, "(- \wedge S^{-1}) \circ (* \cup p)_*"', "\cong"] & \pi_{k-1}^G(Y) \arrow[d, equal] \\ \pi_k^G(Y) \arrow[r, "\pi_k^G(i)"'] & \pi_k^G(Cf) \arrow[r, "\partial"'] & \pi_{k-1}^G(X) \arrow[r, "\pi_{k-1}^G(f)"'] & \pi_{k-1}^G(Y) \end{tikzcd}$$

whose left and middle squares commutes. The right square commutes up to sign (the degree of the sign involution $\tau : S^1 \longrightarrow S^1$), using the naturality of the suspension isomorphism. By the previous paragraph, applied to $i : Y \longrightarrow Cf$ instead of f, the upper row is exact at $\pi_k^G(Cf)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_k^G(Cf)$. But the morphism f was arbitrary, so when applied to $i : Y \longrightarrow Cf$ instead of f, we obtain that the upper row is exact at $\pi_k^G(Ci)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_{k-1}^G(X)$. This finishes the proof of exactness of the first sequence.

Now we come to why the second sequence is exact. We show the claim for $k \ge 0$, the other case being similar. For every $V \in s(\mathcal{U}_G)$ the sequence $F(f)(V) = F(f(V)) \longrightarrow X(V) \longrightarrow Y(V)$ is an equivariant homotopy fiber sequence. So for every based G-CW-complex A, the long sequence of based

sets

$$\cdots \longrightarrow [A, \Omega Y(V)]^G \xrightarrow{[A,i(V)]} [A, F(f(V))]^G \xrightarrow{[A,p(V)]^G} [A, X(V)]^G \xrightarrow{[A,f(V)]^G} [A, Y(V)]^G$$

is exact. We take $A = S^{V\oplus\mathbb{R}^k}$ and form the colimit over the poset $s(\mathcal{U}_G)$. Since filtered colimits are exact, the resulting sequence of colimits is again exact, and that proves the second claim. □

Corollary 3.1.37. *Let G be a compact Lie group and k an integer.*

(i) *For every family of orthogonal G-spectra $\{X_i\}_{i\in I}$ the canonical map*

$$\bigoplus_{i\in I} \pi_k^G(X_i) \longrightarrow \pi_k^G\Big(\bigvee_{i\in I} X_i\Big)$$

is an isomorphism.

(ii) *For every finite index set I and every family $\{X_i\}_{i\in I}$ of orthogonal G-spectra the canonical map*

$$\pi_k^G\Big(\prod_{i\in I} X_i\Big) \longrightarrow \prod_{i\in I} \pi_k^G(X_i)$$

is an isomorphism.

(iii) *For every finite family of orthogonal G-spectra the canonical morphism from the wedge to the product induces isomorphisms of G-equivariant homotopy groups.*

Proof (i) We first show the special case of two summands. If X and Y are two orthogonal G-spectra, then the wedge inclusion $j : X \longrightarrow X\vee Y$ has a retraction. So the associated long exact homotopy group sequence of Proposition 3.1.36 splits into short exact sequences

$$0 \longrightarrow \pi_k^G(X) \xrightarrow{\pi_k^G(j)} \pi_k^G(X \vee Y) \xrightarrow{\pi_k^G(i)} \pi_k^G(Cj) \longrightarrow 0\,.$$

The mapping cone Cj is isomorphic to $(CX) \vee Y$ and thus G-homotopy equivalent to Y. So we can replace $\pi_k^G(Cj)$ by $\pi_k^G(Y)$ and conclude that $\pi_k^G(X \vee Y)$ splits as the sum of $\pi_k^G(X)$ and $\pi_k^G(Y)$, via the canonical map. The case of a finite index set I now follows by induction.

In the general case we consider the composite

$$\bigoplus_{i\in I} \pi_k^G(X_i) \longrightarrow \pi_k^G\Big(\bigvee_{i\in I} X_i\Big) \longrightarrow \prod_{i\in I} \pi_k^G(X_i)\,,$$

where the second map is induced by the projections to the wedge summands. This composite is the canonical map from a direct sum to a product of abelian groups, hence injective. So the first map is injective as well. For surjectivity we consider a G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \bigvee_{i\in I} X_i(V\oplus\mathbb{R}^n)$ that represents an element in the kth G-homotopy group of $\bigvee_{i\in I} X_i$. Since the source of f is compact, there

is a finite subset J of I such that f has image in $\bigvee_{j\in J} X_j(V \oplus \mathbb{R}^n)$, compare Proposition A.18. Then the given class is in the image of $\pi_k^G\left(\bigvee_{j\in J} X_j\right)$; since J is finite, the class is in the image of the canonical map, by the previous paragraph.

(ii) The functor $X \mapsto [S^{V\oplus\mathbb{R}^{n+k}}, X(V \oplus \mathbb{R}^n)]^G$ commutes with products. Finite products commute with filtered colimits, such as the one defining π_k^G, so passage to colimits gives the claim.

(iii) For a finite indexing set I and every integer k the composite map

$$\bigoplus_{i\in I} \pi_k^G(X_i) \longrightarrow \pi_k^G\left(\bigvee_{i\in I} X_i\right) \longrightarrow \pi_k^G\left(\prod_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} \pi_k^G(X_i)$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) and (ii), respectively, hence the middle map is an isomorphism. □

The equivariant homotopy group functor π_k^G does not in general commute with infinite products. The issue is that π_k^G involves a filtered colimit, and these do not always commute with infinite products. However, this defect is cured when we pass to the equivariant or global stable homotopy category, i.e., π_k^G takes 'derived' infinite products to products. We refer to Remark 4.4.6 below for more details.

We recall that a morphism $f : A \longrightarrow B$ of orthogonal G-spectra is an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism of orthogonal G-spectra $\varphi : B \longrightarrow X$ and a homotopy $H : A \wedge [0, 1]_+ \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : B \wedge [0, 1]_+ \longrightarrow X$ starting with φ such that $\bar{H} \circ (f \wedge [0, 1]_+) = H$. For every such h-cofibration $f : A \longrightarrow B$ the collapse map $c : Cf \longrightarrow B/A$ from the mapping cone to the cokernel of f is a G-homotopy equivalence. Indeed, since the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "-\wedge 1"'] & B \arrow[d] \\ A \wedge [0,1] \arrow[r, "g"'] & Cf \end{tikzcd}$$

is a pushout, the cobase change $g : CA = A \wedge [0, 1] \longrightarrow Cf$ also has the homotopy extension property. The cone CA is contractible, so the claim follows from the following more general statement: for every h-cofibration $i : C \longrightarrow D$ such that C is contractible, the quotient map $D \longrightarrow D/C$ is a based homotopy equivalence. The standard proof of this fact in the category of topological spaces (see for example [71, Prop. 0.17] or [180, Prop. 5.1.10] with $B = *$) only uses formal properties of homotopies, and carries over to the category of orthogonal G-spectra.

So for every h-cofibration $f : A \longrightarrow B$ of orthogonal G-spectra, the collapse map $c : Cf \longrightarrow B/A$ is an equivariant homotopy equivalence, hence it induces isomorphisms of all equivariant homotopy groups. We can thus define a modified connecting homomorphism $\partial : \pi_{k+1}^G(B/A) \longrightarrow \pi_k^G(A)$ as the composite

$$\pi_{k+1}^G(B/A) \xrightarrow{(\pi_{k+1}^G(c))^{-1}} \pi_{k+1}^G(Cf) \xrightarrow{\partial} \pi_k^G(A) .$$

We call a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra a *strong level fibration* if for every closed subgroup H of G and every H-representation V the map $f(V)^H : X(V)^H \longrightarrow Y(V)^H$ is a Serre fibration. For every such strong level fibration, the embedding $j : F \longrightarrow F(f)$ of the strict fiber into the homotopy fiber then induces isomorphisms on π_*^G. We can thus define a modified connecting homomorphism $\partial : \pi_{k+1}^G(Y) \longrightarrow \pi_k^G(F)$ as the composite

$$\pi_{k+1}^G(Y) \xrightarrow{\partial} \pi_k^G(F(f)) \xrightarrow{(\pi_k^G(j))^{-1}} \pi_k^G(F) .$$

So we deduce:

Corollary 3.1.38. *Let G be a compact Lie group.*

(i) *Let $f : A \longrightarrow B$ be an h-cofibration of orthogonal G-spectra. Then the long sequence of equivariant homotopy groups*

$$\cdots \longrightarrow \pi_{k+1}^G(B/A) \xrightarrow{\partial} \pi_k^G(A) \xrightarrow{\pi_k^G(f)} \pi_k^G(B) \xrightarrow{\pi_k^G(q)} \pi_k^G(B/A) \longrightarrow \cdots$$

is exact.

(ii) *Let $f : X \longrightarrow Y$ be a strong level fibration of orthogonal G-spectra and $j : F \longrightarrow X$ the inclusion of the point-set level fiber of f. Then the long sequence of equivariant homotopy groups*

$$\cdots \longrightarrow \pi_{k+1}^G(Y) \xrightarrow{\partial} \pi_k^G(F) \xrightarrow{\pi_{k-1}^G(j)} \pi_k^G(X) \xrightarrow{\pi_k^G(f)} \pi_k^G(Y) \longrightarrow \cdots$$

is exact.

Corollary 3.1.39. *Let G be a compact Lie group and*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

a commutative square of orthogonal G-spectra.

(i) *Suppose that the square is a pushout and the map $\pi_*^G(f) : \pi_*^G(A) \longrightarrow$*

$\pi_*^G(B)$ of G-equivariant homotopy groups is an isomorphism. If in addition f or g is an h-cofibration, then the map $\pi_*^G(k) : \pi_*^G(C) \longrightarrow \pi_*^G(D)$ is also an isomorphism.

(ii) *Suppose that the square is a pullback and the map $\pi_*^G(k) : \pi_*^G(C) \longrightarrow \pi_*^G(D)$ of G-equivariant homotopy groups is an isomorphism. If in addition k or h is a strong level fibration, then the map $\pi_*^G(f) : \pi_*^G(A) \longrightarrow \pi_*^G(B)$ is also an isomorphism.*

Proof (i) If f is an h-cofibration, then its long exact homotopy group sequence (Corollary 3.1.38) shows that all G-equivariant homotopy groups of the cokernel B/A are trivial. Since the square is a pushout, the induced morphism from B/A to any cokernel D/C of k is an isomorphism, so the groups $\pi_*^G(D/C)$ are all trivial. As a cobase change of the h-cofibration f, the morphism k is again an h-cofibration, so its long exact homotopy group sequence shows that $\pi_*^G(k)$ is an isomorphism.

If g is an h-cofibration, then so is its cobase change h. Moreover, any cokernel C/A of g maps by an isomorphism to any cokernel D/B of h, since the square is a pushout. The square induces compatible maps between the two long exact homotopy group sequences of g and h, and the five lemma then shows that $\pi_*^G(k)$ is an isomorphism. The argument for part (ii) is dual to (i). □

Since the group $\pi_k^G(\Omega^m X)$ is naturally isomorphic to $\pi_{k+m}^G(X)$, looping preserves equivariant stable equivalences. The next proposition generalizes this.

Proposition 3.1.40. *Let G be a compact Lie group and A a finite based G-CW-complex.*

(i) *Let $f : X \longrightarrow Y$ be a morphism of orthogonal G-spectra with the following property: for every closed subgroup H of G that fixes some non-basepoint of A, the map $\pi_*^H(f) : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism. Then the morphism $\mathrm{map}_*(A, f) : \mathrm{map}_*(A, X) \longrightarrow \mathrm{map}_*(A, Y)$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.*

(ii) *The functor $\mathrm{map}_*(A, -)$ preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.*

Proof (i) We start with a special case and let X be an orthogonal G-spectrum whose equivariant homotopy groups $\pi_*^H(X)$ vanish for all closed subgroups H of G that fix some non-basepoint of A. We show first that then the G-equivariant homotopy groups of the orthogonal G-spectrum $\mathrm{map}_*(A, X)$ vanish.

We argue by induction over the number of equivariant cells in a CW-structure of A. The induction starts when A consists only of the basepoint, in which case $\mathrm{map}_*(A, X)$ is a trivial spectrum and there is nothing to show. For the

inductive step we assume that the groups $\pi_*^G(\mathrm{map}_*(B,X))$ vanish and A is obtained from B by attaching an equivariant n-cell $G/H \times D^n$ along its boundary $G/H \times \partial D^n$, for some closed subgroup H of G. Then the restriction map $\mathrm{map}_*(A,X) \longrightarrow \mathrm{map}_*(B,X)$ is a strong level fibration of orthogonal G-spectra whose fiber is isomorphic to

$$\mathrm{map}_*(A/B,X) \cong \mathrm{map}_*(G/H_+ \wedge S^n,X) \cong \mathrm{map}_*(G/H_+,\Omega^n X)\ .$$

The G-equivariant stable homotopy groups of this spectrum are isomorphic to the H-equivariant homotopy groups of $\Omega^n X$, and hence to the shifted H-homotopy groups of X. Since H occurs as a stabilizer of a cell of A, the latter groups vanish by assumption. The long exact sequence of Corollary 3.1.38 (ii) and the inductive hypothesis for B then show that the groups $\pi_*^G(\mathrm{map}_*(A,X))$ vanish.

When H is a proper closed subgroup of G we exploit that the underlying H-space of A is H-equivariantly homotopy equivalent to a finite H-CW-complex, see [85, Cor. B]. We can thus apply the previous paragraph to the underlying H-spectrum of X and the underlying H-space of A and conclude that the H-equivariant homotopy groups of $\mathrm{map}_*(A,X)$ vanish. Altogether this proves the special case of the proposition.

The functor $\mathrm{map}_*(A,-)$ commutes with homotopy fibers; so two applications of the long exact homotopy group sequence of a homotopy fiber (Proposition 3.1.36) reduce the general case of the first claim to the special case. Part (ii) is a special case of (i). □

Proposition 3.1.41. *Let G be a compact Lie group.*

(i) *Let $e_n : X_n \longrightarrow X_{n+1}$ be morphisms of orthogonal G-spectra that are level-wise closed embeddings, for $n \geq 0$. Let X_∞ be a colimit of the sequence $\{e_n\}_{n\geq 0}$. Then for every integer k the canonical map*

$$\mathrm{colim}_{n\geq 0}\ \pi_k^G(X_n) \longrightarrow \pi_k^G(X_\infty)$$

is an isomorphism.

(ii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal G-spectra that are level-wise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a $\underline{\pi}_*$-isomorphism.*

(iii) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra that are level-wise closed embeddings, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to a colimit of the sequence $\{f_n\}_{n\geq 0}$ is a $\underline{\pi}_*$-isomorphism.*

Proof (i) We let V be a G-representation, and $m \geq 0$ such that $m+k \geq 0$. Since

the sphere $S^{V\oplus\mathbb{R}^{m+k}}$ is compact and $X_\infty(V\oplus\mathbb{R}^m)$ is a colimit of the sequence of closed embeddings $X_n(V\oplus\mathbb{R}^m) \longrightarrow X_{n+1}(V\oplus\mathbb{R}^m)$, any based G-map $f : S^{V\oplus\mathbb{R}^{m+k}} \longrightarrow X_\infty(V\oplus\mathbb{R}^m)$ factors through a continuous map

$$\bar{f} \ : \ S^{V\oplus\mathbb{R}^{m+k}} \ \longrightarrow \ X_n(V\oplus\mathbb{R}^m)$$

for some $n \geq 0$, see for example [80, Prop. 2.4.2] or Proposition A.15 (i). Since the canonical map $X_n(V\oplus\mathbb{R}^m) \longrightarrow X_\infty(V\oplus\mathbb{R}^m)$ is injective, $\bar{f}$ is again based and G-equivariant. The same applies to homotopies, so the canonical map

$$\operatorname{colim}_{n\geq 0}\,[S^{V\oplus\mathbb{R}^{m+k}}, X_n(V\oplus\mathbb{R}^m)]^G \ \longrightarrow \ [S^{V\oplus\mathbb{R}^{m+k}}, X_\infty(V\oplus\mathbb{R}^m)]^G$$

is bijective. Passing to colimits over m and over the poset $s(\mathcal{U}_G)$ proves the claim.

Part (ii) is a direct consequence of (i), applied to G and all its closed subgroups. Part (iii) is a special case of part (ii) where we set $X_n = Y_0$, $e_n = \mathrm{Id}_{Y_0}$ and $\psi_n = f_{n-1}\circ\cdots\circ f_0 : Y_0 \longrightarrow Y_n$. The morphism ψ_n is then a $\underline{\pi}_*$-isomorphism and Y_0 is a colimit of the constant first sequence. Since the morphism ψ_∞ induced on the colimits is the canonical morphism $Y_0 \longrightarrow Y_\infty$, part (ii) specializes to claim (iii). □

Example 3.1.42 (Equivariant suspension spectrum)**.** Every based G-space A gives rise to a *suspension spectrum* $\Sigma^\infty A$. This is the orthogonal G-spectrum with V-term

$$(\Sigma^\infty A)(V) \ = \ S^V \wedge A\ ,$$

with $O(V)$-action only on S^V, with G-action only on A, and with structure map $\sigma_{U,V}$ given by the canonical homeomorphism $S^U\wedge S^V\wedge A \cong S^{U\oplus V}\wedge A$. For an unbased G-space Y, we obtain an *unreduced suspension spectrum* $\Sigma^\infty_+ Y$ by first adding a disjoint basepoint to Y and then forming the suspension spectrum as above.

The suspension spectrum functor is homotopical on a large class of G-spaces; however, since the reduced suspension $S^1\wedge -$ does not preserve weak equivalences in complete generality, a non-degeneracy condition on the basepoint is often needed.

Definition 3.1.43. A based G-space is *well-pointed* if the basepoint inclusion has the homotopy extension property in the category of unbased G-spaces.

Equivalently, a based G-space (A, a_0) is well-pointed if and only if the inclusion of $A\times\{0\}\cup\{a_0\}\times[0,1]$ into $A\times[0,1]$ has a continuous G-equivariant retraction. Cofibrant based G-spaces are always well-pointed. Also, if we add a disjoint basepoint to any unbased G-space, the result is always well-pointed.

The reduced suspension of a well-pointed space is again well-pointed. Indeed, if (A, a_0) is well-pointed, then the subspace inclusion

$$A \times \{0, 1\} \cup \{a_0\} \times [0, 1] \longrightarrow A \times [0, 1]$$

is an h-cofibration, see for example [133, Satz 2] or [180, Prop. (5.1.6)]. Since h-cofibrations are stable under cobase change (see Corollary A.30 (ii)), the inclusion of the basepoint into the quotient space

$$A \times [0, 1]/(A \times \{0, 1\} \cup \{a_0\} \times [0, 1])$$

is an h-cofibration. Since $S^1 \wedge A$ is homeomorphic to this quotient space, this proves the claim.

We recall that a continuous map $f : X \longrightarrow Y$ is *m-connected*, for some $m \geq 0$, if the following condition holds: for $0 \leq k \leq m$ and all continuous maps $\alpha : \partial D^k \longrightarrow X$ and $\beta : D^k \longrightarrow Y$ such that $\beta|_{\partial D^k} = f \circ \alpha$, there is a continuous map $\lambda : D^k \longrightarrow X$ such that $\lambda|_{\partial D^k} = \alpha$ and such that $f \circ \lambda$ is homotopic, relative to ∂D^k, to β. An equivalent condition is that the map $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ is surjective, and for all $x \in X$ the map $\pi_k(f) : \pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$ is bijective for all $1 \leq k < m$ and surjective for $k = m$. The map f is a weak homotopy equivalence if and only if it is m-connected for every $m \geq 0$.

Proposition 3.1.44. *Let G be a compact Lie group.*

(i) *Let $f : X \longrightarrow Y$ be a continuous based G-map between well-pointed based G-spaces, and $m \geq 0$. Suppose that for every closed subgroup H of G the fixed-point map $f^H : X^H \longrightarrow Y^H$ is $(m - \dim(W_G H))$-connected. Then the induced map of G-equivariant stable homotopy groups*

$$\pi_k^G(\Sigma^\infty f) : \pi_k^G(\Sigma^\infty X) \longrightarrow \pi_k^G(\Sigma^\infty Y)$$

is bijective for every integer k with $k < m$ and surjective for $k = m$.

(ii) *For every well-pointed based G-space X and every negative integer k, the G-equivariant homotopy group $\pi_k^G(\Sigma^\infty X)$ is trivial.*

In particular, the suspension spectrum functor Σ^∞ takes G-weak equivalences between well-pointed G-spaces to $\underline{\pi}_*$-isomorphisms, and the unreduced suspension spectrum functor Σ^∞_+ takes G-weak equivalences between arbitrary unbased G-spaces to $\underline{\pi}_*$-isomorphisms.

Proof (i) We let V be a G-representation and we let $n \geq 0$ be such that $n + k \geq 0$. Reduced suspension increases the connectivity of continuous maps between well-pointed spaces, compare [180, Cor. 6.7.10], and it preserves well-pointedness. So the natural homeomorphism

$$(S^{V \oplus \mathbb{R}^n} \wedge X)^H \cong S^{V^H \oplus \mathbb{R}^n} \wedge X^H$$

shows that the map

$$(S^{V\oplus\mathbb{R}^n} \wedge f)^H \ : \ (S^{V\oplus\mathbb{R}^n} \wedge X)^H \ \longrightarrow \ (S^{V\oplus\mathbb{R}^n} \wedge Y)^H$$

is $(\dim(V^H)+m+n-\dim(W_G H))$-connected. On the other hand, we claim that the cellular dimension of $S^{V\oplus\mathbb{R}^{n+k}}$ at H, in the sense of [179, II.2, p. 106], is at most $\dim(V^H)+n+k-\dim(W_G H)$. Indeed, if any G-CW-structure for $S^{V\oplus\mathbb{R}^{n+k}}$ contains an equivariant cell of the form $G/H\times D^j$, then $(G/H)^H \times \mathring{D}^j$ embeds into $S^{V^H\oplus\mathbb{R}^{n+k}}$, and hence

$$\dim(W_G H) + j \ = \ \dim((G/H)^H) + j \ \leq \ \dim(V^H) + n + k \ .$$

The cellular dimension at H is the maximal j that occurs in this way, so this cellular dimension is at most $\dim(V^H)+n+k-\dim(W_G H)$. We conclude that for $k < m$ the cellular dimension of $S^{V\oplus\mathbb{R}^{n+k}}$ at H is smaller than the connectivity of $(S^{V\oplus\mathbb{R}^n} \wedge f)^H$, and for $k = m$ the former is less than or equal to the latter. So the induced map

$$[S^{V\oplus\mathbb{R}^{n+k}}, S^{V\oplus\mathbb{R}^n} \wedge f]^G \ : \ [S^{V\oplus\mathbb{R}^{n+k}}, S^{V\oplus\mathbb{R}^n} \wedge X]^G \ \longrightarrow \ [S^{V\oplus\mathbb{R}^{n+k}}, S^{V\oplus\mathbb{R}^n} \wedge X]^G$$

is bijective for $k < m$ and surjective for $k = m$, by [179, II Prop. 2.7]. Passage to the colimit over $V \in s(\mathcal{U}_G)$ and n then proves the claim.

(ii) The unique map $f : X \longrightarrow *$ to a one-point G-space has the property that f^H is 0-connected for every closed subgroup H of G. So for every negative integer k the induced map from $\pi_k^G(\Sigma^\infty X)$ to $\pi_k^G(\Sigma^\infty *)$ is an isomorphism by part (i). Since the latter group is trivial, this proves the claim. □

We end this section with a useful representability result for the functor π_0^H on the category of orthogonal G-spectra, where H is any closed subgroup of a compact Lie group G. We define a *tautological class*

$$e_H \ \in \ \pi_0^H(\Sigma_+^\infty G/H) \tag{3.1.45}$$

as the class represented by the H-fixed-point eH of G/H.

Proposition 3.1.46. *Let H be a closed subgroup of a compact Lie group G and $\Phi : G\mathcal{S}p \longrightarrow$ (sets) a set-valued functor on the category of orthogonal G-spectra that takes $\underline{\pi}_*$-isomorphisms to bijections. Then evaluation at the tautological class is a bijection*

$$\mathrm{Nat}^{G\mathcal{S}p}(\pi_0^H, \Phi) \ \longrightarrow \ \Phi(\Sigma_+^\infty G/H) \ , \quad \tau \ \longmapsto \ \tau(e_H) \ .$$

Proof To show that the evaluation map is injective we show that every natural transformation $\tau : \pi_0^H \longrightarrow \Phi$ is determined by the element $\tau(e_H)$. We let X be any orthogonal G-spectrum and $x \in \pi_0^H(X)$ an H-equivariant homotopy class. The class x is represented by a continuous based H-map

$$f \ : \ S^U \ \longrightarrow \ X(U)$$

for some H-representation U. By increasing U, if necessary, we can assume that U is underlying a G-representation. We can then view f as an H-fixed-point in $(\Omega^U \operatorname{sh}^U X)(0) = \Omega^U X(U)$. There is thus a unique morphism of orthogonal G-spectra

$$f^\sharp \ : \ \Sigma^\infty_+ G/H \longrightarrow \Omega^U \operatorname{sh}^U X$$

such that $f^\sharp(0) : G/H_+ \longrightarrow \Omega^U X(U)$ sends the distinguished coset eH to f. This morphism then satisfies

$$\pi_0^H(f^\sharp)(e_H) \ = \ \pi_0^H(\tilde\lambda_X^U)(x) \qquad \text{in } \pi_0^H(\Omega^U \operatorname{sh}^U X) ,$$

where $\tilde\lambda_X^U : X \longrightarrow \Omega^U \operatorname{sh}^U X$ is the $\underline{\pi}_*$-isomorphism discussed in Proposition 3.1.25 (ii). The diagram

$$\begin{tikzcd}
\pi_0^H(\Sigma^\infty_+ G/H) \arrow[r, "\pi_0^H(f^\sharp)"] \arrow[d, "\tau"'] & \pi_0^H(\Omega^U \operatorname{sh}^U X) \arrow[d, "\tau"'] & \pi_0^H(X) \arrow[l, "\pi_0^H(\tilde\lambda_X^U)"', "\cong"] \arrow[d, "\tau"] \\
\Phi(\Sigma^\infty_+ G/H) \arrow[r, "\Phi(f^\sharp)"'] & \Phi(\Omega^U \operatorname{sh}^U X) & \Phi(X) \arrow[l, "\cong", "\Phi(\tilde\lambda_X^U)"']
\end{tikzcd}$$

commutes and the two right horizontal maps are bijective. So

$$\Phi(f^\sharp)(\tau(e_H)) \ = \ \tau(\pi_0^H(f^\sharp)(e_H)) \ = \ \tau(\pi_0^H(\tilde\lambda_X^U)(x)) \ = \ \Phi(\tilde\lambda_X^U)(\tau(x)) .$$

Since $\Phi(\tilde\lambda_X^U)$ is bijective, this proves that τ is determined by its value on the tautological class e_H.

It remains to construct, for every element $y \in \Phi(\Sigma^\infty_+ G/H)$, a natural transformation $\tau : \pi_0^H \longrightarrow \Phi$ with $\tau(e_H) = y$. The previous paragraph dictates what to do: we represent a given class $x \in \pi_0^H(X)$ by a continuous based H-map $f : S^U \longrightarrow X(U)$, where U is underlying a G-representation. Then we set

$$\tau(x) \ = \ \Phi(\tilde\lambda_X^U)^{-1}(\Phi(f^\sharp)(y)) .$$

We verify that the element $\tau(x)$ only depends on the class x. To this end we need to show that $\tau(x)$ does not change if we either replace f by a homotopic H-map, or increase it by stabilization along a G-equivariant linear isometric embedding. If $\bar f : S^U \longrightarrow X(U)$ is H-equivariantly homotopic to f, then the morphism $\bar f^\sharp$ is homotopic to $f^\sharp$ via a homotopy of morphisms of orthogonal G-spectra

$$K \ : \ (\Sigma^\infty_+ G/H) \wedge [0,1]_+ \longrightarrow \Omega^U \operatorname{sh}^U X .$$

The morphism $q : (\Sigma^\infty_+ G/H) \wedge [0,1]_+ \longrightarrow \Sigma^\infty_+ G/H$ that maps $[0,1]$ to a single point is a homotopy equivalence, hence a $\underline{\pi}_*$-isomorphism, of orthogonal G-spectra. So $\Phi(q)$ is a bijection. The two embeddings $i_0, i_1 \ : \ (\Sigma^\infty_+ G/H) \longrightarrow$

$\Sigma^\infty_+ G/H \wedge [0,1]_+$ as the endpoints of the interval are right inverse to q, so $\Phi(q) \circ \Phi(i_0) = \Phi(q) \circ \Phi(i_1) = \mathrm{Id}$. Since $\Phi(q)$ is bijective, $\Phi(i_0) = \Phi(i_1)$. Hence

$$\Phi(\bar{f}^\sharp) \;=\; \Phi(K \circ i_0) \;=\; \Phi(K) \circ \Phi(i_0) \;=\; \Phi(K) \circ \Phi(i_1) \;=\; \Phi(K \circ i_1) \;=\; \Phi(f^\sharp)\,.$$

This shows that $\tau(x)$ does not change if we modify f by an H-homotopy.

Now we let V be another G-representation and $\varphi : U \longrightarrow V$ a G-equivariant linear isometric embedding. Then $\varphi_* f : S^V \longrightarrow X(V)$ is another representative of the class x. A morphism of orthogonal G-spectra

$$\varphi_\sharp \;:\; \Omega^U \operatorname{sh}^U X \;\longrightarrow\; \Omega^V \operatorname{sh}^V X$$

is defined at an inner product space W as the stabilization map

$$\begin{aligned} \varphi_* \;:\; (\Omega^U \operatorname{sh}^U X)(W) \;&=\; \operatorname{map}_*(S^U, X(W \oplus U)) \\ &\longrightarrow\; \operatorname{map}_*(S^V, X(W \oplus V)) \;=\; (\Omega^V \operatorname{sh}^V X)(W)\,. \end{aligned}$$

This morphism makes the following diagram commute:

$$\begin{tikzcd} \Sigma^\infty_+ G/H \arrow[r, "f^\sharp"] \arrow[dr, "(\varphi_* f)^\sharp"'] & \Omega^U \operatorname{sh}^U X \arrow[d, "\varphi_\sharp"] & X \arrow[l, "\tilde\lambda^U_X"'] \arrow[dl, "\tilde\lambda^V_X"] \\ & \Omega^V \operatorname{sh}^V X & \end{tikzcd}$$

So

$$\Phi(\tilde\lambda^U_X)^{-1} \circ \Phi(f^\sharp) \;=\; \Phi(\tilde\lambda^V_X)^{-1} \circ \Phi(\varphi_\sharp) \circ \Phi(f^\sharp) \;=\; \Phi(\tilde\lambda^V_X)^{-1} \circ \Phi((\varphi_* f)^\sharp)\,,$$

and hence the class $\tau(x)$ remains unchanged upon stabilization of f along φ.

Now we know that $\tau(x)$ is independent of the choice of representative for the class x, and it remains to show that τ is natural. But this is straightforward: if $\psi : X \longrightarrow Y$ is a morphism of orthogonal G-spectra and $f : S^U \longrightarrow X(U)$ a representative for $x \in \pi_0^H(X)$ as above, then $\psi(U) \circ f : S^U \longrightarrow Y(U)$ represents the class $\pi_0^H(\psi)(x)$. Moreover, the following diagram of orthogonal G-spectra commutes:

$$\begin{tikzcd} \Sigma^\infty_+ G/H \arrow[r, "f^\sharp"] \arrow[dr, "(\psi(U)\circ f)^\sharp"'] & \Omega^U \operatorname{sh}^U X \arrow[d, "\Omega^U \operatorname{sh}^U \psi"] & X \arrow[l, "\tilde\lambda^U_X"'] \arrow[d, "\psi"] \\ & \Omega^U \operatorname{sh}^U Y & Y \arrow[l, "\tilde\lambda^U_Y"] \end{tikzcd}$$

So naturality follows:

$$\begin{aligned} \tau(\pi_0^H(\psi)(x)) \;&=\; (\Phi(\tilde\lambda^U_Y)^{-1} \circ \Phi((\psi(U) \circ f)^\sharp))(y) \\ &=\; (\Phi(\tilde\lambda^U_Y)^{-1} \circ \Phi(\Omega^U \operatorname{sh}^U \psi) \circ \Phi(f^\sharp))(y) \\ &=\; (\Phi(\psi) \circ \Phi(\tilde\lambda^U_X)^{-1} \circ \Phi(f^\sharp))(y) \;=\; \Phi(\psi)(\tau(x))\,. \end{aligned}$$

Finally, the class e_H is represented by the H-map $- \wedge eH : S^0 \longrightarrow S^0 \wedge G/H_+$, which is adjoint to the identity of $\Sigma^\infty_+ G/H$. Hence $\tau(e_H) = \Phi(\mathrm{Id})(y) = y$. □

3.2 The Wirthmüller isomorphism and transfers

This section establishes the Wirthmüller isomorphism that relates the equivariant homotopy groups of a spectrum over a subgroup to the equivariant homotopy groups of the induced spectrum, see Theorem 3.2.15. Intimately related to the Wirthmüller isomorphism are various transfers that we discuss in Constructions 3.2.7 and 3.2.22. We show that transfers are transitive (Proposition 3.2.29) and commute with inflations (Proposition 3.2.32). Theorem 3.4.9 below establishes the 'double coset formula' that expresses the composite of a transfer followed by a restriction as a linear combination of restrictions followed by transfers.

We let H be a closed subgroup of a compact Lie group G. We write $i^* : G\mathbf{T}_* \longrightarrow H\mathbf{T}_*$ for the restriction functor from based G-spaces to based H-spaces. We write

$$G \ltimes_H - \;=\; (G_+) \wedge_H - \;:\; H\mathbf{T}_* \;\longrightarrow\; G\mathbf{T}_*$$

for the left adjoint induction functor. For a based H-space A and a based G-space B, the *shearing isomorphism* is the G-equivariant homeomorphism

$$B \wedge (G \ltimes_H A) \;\cong\; G \ltimes_H (i^* B \wedge A)\,, \quad b \wedge [g, a] \;\longmapsto\; [g, (g^{-1}b) \wedge a]\,.$$

Construction 3.2.1. As before we consider a closed subgroup H of a compact Lie group G. To define the Wirthmüller morphism, we need a specific H-equivariant map

$$l_A \;:\; G \ltimes_H A \;\longrightarrow\; A \wedge S^L$$

that is natural for continuous based H-maps in A, see (3.2.2) below. Here $L = T_{eH}(G/H)$ is the tangent H-representation, i.e., the tangent space at the preferred coset eH of the smooth manifold G/H. Since the left translation by H on G/H fixes eH, the tangent space L inherits an H-action, and S^L is its one-point compactification. When H has finite index in G, then L is trivial and the map $l_A : G \ltimes_H A \longrightarrow A$ is simply the projection onto the wedge summand indexed by the preferred H-coset eH. If the dimension of G is bigger than the dimension of H, then L is non-zero and the definition of the map l_A is substantially more involved.

We consider the left H^2-action on G given by

$$H^2 \times G \;\longrightarrow\; G\,, \quad (h', h) \cdot g \;=\; h' g h^{-1}\,.$$

There is another action of the group Σ_2 on G by inversion, i.e., the non-identity element of Σ_2 acts by $\tau \cdot g = g^{-1}$. These two actions combine into an action of the wreath product group $\Sigma_2 \wr H = \Sigma_2 \ltimes H^2$ on G. The subgroup H is the $(\Sigma_2 \ltimes H^2)$-orbit of $1 \in G$, whose stabilizer is the subgroup $\Sigma_2 \times \Delta$, where $\Delta = \{(h,h) \mid h \in H\}$ is the diagonal subgroup of H^2. The differential of the projection $G \longrightarrow G/H$ identifies

$$\nu \ = \ (T_1 G)/(T_1 H) \ ,$$

the normal space at $1 \in H$ of the inclusion $H \longrightarrow G$, with the tangent representation $L = T_{eH}(G/H)$. The representation ν is a representation of the stabilizer group $\Sigma_2 \times \Delta$ of 1, and if we identify H with Δ via $h \mapsto (h,h)$, then this identification takes the Δ-action on ν to the tangent H-action on L. Moreover, the differential at 1 of the inversion map $g \mapsto g^{-1}$ is multiplication by -1 on the tangent space; so the above identification of ν with L takes the Σ_2-action to the sign action on L.

We choose an H-invariant inner product on the vector space L. Since H is the H^2-orbit of 1 inside G, there is a slice, i.e., a $\Sigma_2 \times \Delta$-equivariant smooth embedding

$$s \ : \ D(L) \ \longrightarrow \ G$$

of the unit disc of L with $s(0) = 1$ and such that the differential at $0 \in D(L)$ of the composite

$$D(L) \ \xrightarrow{\ s\ } \ G \ \xrightarrow{\text{proj}} \ G/H$$

is the identity of L. The property that s is $(\Sigma_2 \times \Delta)$-equivariant means in more concrete terms that the relations

$$s(h \cdot l) \ = \ h \cdot s(l) \cdot h^{-1} \qquad \text{and} \qquad s(-l) \ = \ s(l)^{-1}$$

hold in G for all $(h,l) \in H \times D(L)$. After scaling the slice, if necessary, the map

$$D(L) \times H \ \longrightarrow \ G \ , \quad (l,h) \ \longmapsto \ s(l) \cdot h$$

is an embedding whose image is a tubular neighborhood of H in G. For a proof, see for example [26, II Thm. 5.4]. This embedding is equivariant for the action of H^2 on the source by

$$(h_1, h_2) \cdot (l,h) \ = \ (h_1 l, \ h_1 h h_2^{-1})$$

and for the action of Σ_2 on the source by

$$\tau \cdot (l,h) \ = \ (-h^{-1} l, \ h^{-1}) \ .$$

The map

$$l_H^G \ : \ G \ \longrightarrow \ S^L \wedge H_+$$

is then defined as the H^2-equivariant collapse map with respect to the above tubular neighborhood. So explicitly,

$$l_H^G(g) \;=\; \begin{cases} (l/(1-|l|)) \wedge h & \text{if } g = s(l)\cdot h \text{ with } (l,h) \in D(L)\times H, \text{ and} \\ \quad * & \text{if } g \text{ is not of this form.} \end{cases}$$

Given any based H-space A, we can now form $l \wedge_H A$, where $- \wedge_H -$ refers to the action of the second H-factor in H^2. We obtain an H-equivariant based map

$$l_A \;=\; l_H^G \wedge_H A \;:\; G \ltimes_H A \;\longrightarrow\; (S^L \wedge H_+) \wedge_H A \;\cong\; A \wedge S^L \tag{3.2.2}$$

that is natural in A. Here H acts by left translation on the source, and diagonally on the target. This map is thus given by

$$l_A[g,a] \;=\; \begin{cases} ha \wedge (l/(1-|l|)) & \text{if } g = s(l)\cdot h \text{ with } (l,h) \in D(L)\times H, \text{ and} \\ \quad * & \text{if } g \text{ is not of this form.} \end{cases}$$

If H has finite index in G, then $L = 0$ and the triangle of the following proposition commutes on the nose, by direct inspection. So the essential content of the next result is when the dimension of G exceeds the dimension of H.

Proposition 3.2.3. *Let H be a closed subgroup of a compact Lie group G, A a based H-space and B a based G-space. Then the following triangle commutes up to H-equivariant based homotopy:*

$$\begin{tikzcd} B \wedge (G \ltimes_H A) \arrow[d, "\text{shear}"', "\cong"] \arrow[dr, "B \wedge l_A"] & \\ G \ltimes_H (i^* B \wedge A) \arrow[r, "l_{i^*B\wedge A}"'] & B \wedge A \wedge S^L \end{tikzcd}$$

Proof We write down an explicit homotopy: we define the map

$$K \;:\; (B \wedge (G \ltimes_H A)) \times [0,1] \;\longrightarrow\; B \wedge A \wedge S^L$$

by

$$K(b \wedge [g,a], t) \;=\; \begin{cases} s(tl)^{-1} b \wedge ha \wedge \frac{l}{1-|l|} & \text{if } g = s(l)\cdot h \text{ for } (l,h) \in D(L)\times H, \\ \quad * & \text{if } g \text{ is not of this form.} \end{cases}$$

Then for all $(l,h) \in D(L) \times H$ with $g = s(l)\cdot h$ we have

$$K(b \wedge [g,a], 0) \;=\; b \wedge ha \wedge (l/(1-|l|)) \;=\; (B \wedge l_A)(b \wedge [g,a])$$

(because $s(0) = 1$), and

$$K(b \wedge [g,a], 1) \;=\; hg^{-1}b \wedge ha \wedge (l/(1-|l|) \;=\; l_{i^*B\wedge A}[g, g^{-1}b \wedge a])$$

(because $s(l)^{-1} = hg^{-1}$). So K is the desired H-equivariant homotopy. □

The restriction functor from orthogonal G-spectra to orthogonal H-spectra has a left and a right adjoint, and both are given by applying the space level adjoints $G \ltimes_H -$ and $\mathrm{map}^H(G, -)$ level-wise, respectively. We will mostly be concerned with the left adjoint, and so we spell out the construction in more detail.

Construction 3.2.4 (Induced spectrum)**.** We let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum. We denote by $G \ltimes_H Y$ the *induced G-spectrum* whose value at an inner product space V is $G \ltimes_H Y(V)$. When G-acts on V by linear isometries, then this value has the diagonal G-action, through the action on the external G and by functoriality in V. With this diagonal G-action, $(G \ltimes_H Y)(V)$ is equivariantly homeomorphic to $G \ltimes_H Y(i^*V)$ (where H acts diagonally on $Y(i^*V)$), via

$$G \ltimes_H Y(i^*V) \;\cong\; (G \ltimes_H Y)(V)\,, \quad [g, y] \;\longmapsto\; [g, Y(l_g)(y)]\,. \tag{3.2.5}$$

Under this isomorphism the structure map of the spectrum $G \ltimes_H Y$ becomes the combination of the shearing isomorphism and the structure map of the H-spectrum Y, i.e., for every G-representation W the following square commutes:

$$\begin{tikzcd}
S^V \wedge (G \ltimes_H Y(i^*W)) \arrow[r, "(3.2.5)", "\cong"'] \arrow[d, "\text{shear}"', "\cong"] & S^V \wedge (G \ltimes_H Y)(W) \arrow[dd, "\sigma^{G\ltimes_H Y}_{V,W}"] \\
G \ltimes_H (S^{i^*V} \wedge Y(i^*W)) \arrow[d, "G\ltimes_H(\sigma^Y_{V,W})"'] & \\
G \ltimes_H (Y(i^*V \oplus i^*W)) \arrow[r, "\cong", "(3.2.5)"'] & (G \ltimes_H Y)(V \oplus W)
\end{tikzcd}$$

If the group H has finite index in G, then the tangent representation $L = T_{eH}(G/H)$ is trivial and $l_A : G \ltimes_H A \longrightarrow A$ is the projection onto the wedge summand indexed by eH; the maps $l_{Y(V)} : G \ltimes_H Y(V) \longrightarrow Y(V)$ then form a morphism of orthogonal H-spectra $l_Y : G \ltimes_H Y \longrightarrow Y$. This morphism induces a map of H-equivariant homotopy groups. In general, however, the diagram

$$\begin{tikzcd}
S^V \wedge (G \ltimes_H Y(W)) \arrow[r, "S^V\wedge l_{Y(W)}"] \arrow[d, "\cong"', "\text{shear}"] \arrow[dd, bend right=60, "\sigma^{G\ltimes_H Y}_{V,W}"'] & S^V \wedge Y(W) \wedge S^L \arrow[dd, "\sigma^{Y\wedge S^L}_{V,W}"] \\
G \ltimes_H (S^V \wedge Y(W)) \arrow[ur, "l_{S^V\wedge Y(W)}"'] \arrow[d, "G\ltimes_H \sigma^Y_{V,W}"] & \\
G \ltimes_H Y(V \oplus W) \arrow[r, "l_{Y(V\oplus W)}"'] & Y(V \oplus W) \wedge S^L
\end{tikzcd}$$

does *not* commute on the nose (because the upper triangle does not commute);

so if the dimension of G is bigger than the dimension of H we do *not* obtain a morphism of orthogonal H-spectra in the strict sense. Still, Proposition 3.2.3 shows that the above diagram does commute up to based H-equivariant homotopy, and this is good enough to yield a well-defined homomorphism

$$(l_Y)_* \ : \ \pi_k^H(G \ltimes_H Y) \longrightarrow \pi_k^H(Y \wedge S^L) .$$

As we just explained, this is an abuse of notation, since $(l_Y)_*$ is in general *not* induced by a morphism of orthogonal H-spectra.

We can now consider the composite

$$\mathrm{Wirth}_H^G \ : \ \pi_k^G(G \ltimes_H Y) \xrightarrow{\mathrm{res}_H^G} \pi_k^H(G \ltimes_H Y) \xrightarrow{(l_Y)_*} \pi_k^H(Y \wedge S^L) , \qquad (3.2.6)$$

which we call the *Wirthmüller map*. We will show in Theorem 3.2.15 below that the Wirthmüller map is an isomorphism.

Construction 3.2.7 (Transfers)**.** We let H be a closed subgroup of a compact Lie group G, and we let Y be an orthogonal H-spectrum. The *external transfer*

$$G \ltimes_H - \ : \ \pi_0^H(Y \wedge S^L) \longrightarrow \pi_0^G(G \ltimes_H Y) \qquad (3.2.8)$$

is a map in the direction opposite to the Wirthmüller map. The construction involves another equivariant Thom-Pontryagin construction. We choose a G-representation V and a vector $v_0 \in V$ whose stabilizer group is H; this is possible, for example by [26, Ch. 0, Thm. 5.2], [28, III Thm. 4.6] or [131, Prop. 1.4.1]. This data determines a G-equivariant smooth embedding

$$i \ : \ G/H \longrightarrow V , \quad gH \longmapsto gv_0$$

whose image is the orbit Gv_0. We let

$$W \ = \ V - (di)_{eH}(L) \ = \ V - T_{v_0}(G \cdot v_0)$$

denote the orthogonal complement of the image of the tangent space at eH; this is an H-subrepresentation of V, and

$$L \oplus W \ \cong \ V , \quad (x, w) \longmapsto (di)_{eH}(x) \ + \ w \qquad (3.2.9)$$

is an isomorphism of H-representations.

By multiplying the original vector v_0 with a sufficiently large scalar, if necessary, we can assume that the embedding i is ‘wide’, in the sense that the exponential map

$$j \ : \ G \times_H D(W) \longrightarrow V , \quad [g, w] \longmapsto g \cdot (v_0 \ + \ w)$$

is an embedding, where $D(W)$ is the unit disc of the normal H-representation,

compare [26, Ch. II, Cor. 5.2]. So j defines an equivariant tubular neighborhood of the orbit Gv_0 inside V. The associated collapse map

$$c \; : \; S^V \longrightarrow G \ltimes_H S^W \tag{3.2.10}$$

then becomes the G-map defined by

$$c(v) \; = \; \begin{cases} \left[g, \frac{w}{1-|w|}\right] & \text{if } v = g \cdot (v_0 + w) \text{ for some } (g, w) \in G \times D(W), \text{ and} \\ \infty & \text{else.} \end{cases}$$

With the collapse map in place, we can now define the external transfer (3.2.8). We let U be an H-representation and $f : S^U \longrightarrow Y(U) \wedge S^L$ an H-equivariant based map that represents a class in $\pi_0^H(Y \wedge S^L)$. By enlarging U, if necessary, we can assume that it is underlying a G-representation, see for example [131, Prop. 1.4.2] or [28, III Thm. 4.5]. We stabilize f by W from the right to obtain the H-map

$$\begin{aligned} f \diamond W \; : \; S^U \wedge S^W & \xrightarrow{f \wedge S^W} Y(U) \wedge S^L \wedge S^W \\ & {}_{(3.2.9)} \cong \; Y(U) \wedge S^V \xrightarrow{\sigma^{\mathrm{op}}_{U,V}} Y(U \oplus V) \, . \end{aligned} \tag{3.2.11}$$

The composite G-map

$$\begin{aligned} S^{U \oplus V} \xrightarrow{S^U \wedge c} S^U \wedge (G \ltimes_H S^W) & \xrightarrow[\cong]{\text{shear}} G \ltimes_H (S^U \wedge S^W) \\ & \xrightarrow{G \ltimes_H (f \diamond W)} G \ltimes_H (Y(U \oplus V)) \cong_{(3.2.5)} (G \ltimes_H Y)(U \oplus V) \end{aligned}$$

then represents the external transfer

$$G \ltimes_H \langle f \rangle \qquad \text{in} \qquad \pi_0^G(G \ltimes_H Y) \, .$$

The next proposition provides the main geometric input for the Wirthmüller isomorphism.

Proposition 3.2.12. *Let H be a closed subgroup of a compact Lie group G.*

(i) *The composite*

$$S^V \xrightarrow{c} G \ltimes_H S^W \xrightarrow{l_H^G \wedge_H S^W} S^L \wedge S^W$$

is H-equivariantly homotopic to the map induced by the inverse of the isometry (3.2.9)*.*

(ii) *Let A be a based H-space and $f, f' : B \longrightarrow G \ltimes_H A$ two continuous based G-maps. If the composites $l_A \circ f, l_A \circ f' : B \longrightarrow A \wedge S^L$ are H-equivariantly homotopic, then the maps $f \wedge S^V, f' \wedge S^V : B \wedge S^V \longrightarrow (G \ltimes_H A) \wedge S^V$ are G-equivariantly homotopic.*

Proof (i) The composite $(l_H^G \wedge_H S^W) \circ c$ is the collapse map based on the composite closed embedding

$$\zeta \ : \ D(L) \times D(W) \xrightarrow{(l,w)\mapsto[s(l),w]} G \times_H D(W) \xrightarrow{[g,w]\mapsto g\cdot(v_0+w)} V \ .$$

Then $\zeta(0,0) = v_0$ and we let

$$D \ = \ (d\zeta)_{(0,0)} \ : \ L \times W \ \longrightarrow \ V$$

denote the differential of ζ at $(0,0)$. We observe that $D(0,w) = w$ because $\zeta(0,w) = v_0 + w$; on the other hand, $\zeta(l,0) = s(l) \cdot v_0$, so the restriction of D to L is the differential at 0 of the composite

$$D(L) \xrightarrow{s} G \xrightarrow{\text{proj}} G/H \xrightarrow{i} V \ .$$

Since the differential of the composite $\text{proj} \circ s$ is the identity, we obtain

$$D(l,0) \ = \ (di)_{eH}(l) \ .$$

Since the differential is additive we conclude that $D(l,w) = (di)_{eH}(l) + w$, i.e., D equals the isomorphism (3.2.9).

We consider the H-equivariant homotopy

$$\begin{aligned} K \ : \ D(L) \times D(W) \times [0,1] \ &\longrightarrow \quad V \\ K(l,w,t) \quad &= \begin{cases} \frac{\zeta(t\cdot(l,w))-v_0}{t} \ + \ t \cdot v_0 & \text{for } t > 0, \text{ and} \\ D(l,w) & \text{for } t = 0. \end{cases} \end{aligned}$$

That this assignment is continuous (in fact smooth) when t approaches 0 is the defining property of the differential. Moreover, for every $t \in [0,1]$ the map $K(-,-,t) : D(L) \times D(W) \longrightarrow V$ is a smooth equivariant embedding, so it gives rise to a collapse map $c_t : S^V \longrightarrow S^L \wedge S^W$ defined by

$$c_t(v) \ = \ \begin{cases} \left(\frac{l}{1-|l|}, \frac{w}{1-|w|}\right) & \text{if } v = K(l,w,t) \text{ for some } (l,w) \in D(L) \times D(W), \text{ and} \\ \infty & \text{else.} \end{cases}$$

The passage from the embedding to the collapse map is continuous, so the 1-parameter family c_t provides an H-equivariant based homotopy from the collapse map c_0 to the collapse map c_1 associated with the embedding $\zeta = K(-,-,1) \ : \ D(L) \times D(W) \ \longrightarrow \ V$, i.e., to the map $(l_H^G \wedge_H S^W) \circ c$. Since $D = (d\zeta)_{(0,0)}$ is the isometry (3.2.9), another scaling homotopy compares the collapse map c_0 to the one-point compactification of the inverse of (3.2.9).

(ii) We define a based continuous G-map

$$r \ : \ \text{map}^H(G, A \wedge S^L) \wedge S^V \ \longrightarrow \ G \ltimes_H (A \wedge S^L \wedge S^W)$$

as the composite

$$
\begin{aligned}
\mathrm{map}^H(G, A \wedge S^L) \wedge S^V &\xrightarrow{\mathrm{Id} \wedge c} \mathrm{map}^H(G, A \wedge S^L) \wedge (G \ltimes_H S^W) \\
&\xrightarrow{\text{shear}} G \ltimes_H \left(\mathrm{map}^H(G, A \wedge S^L) \wedge S^W\right) \\
&\xrightarrow{G \ltimes_H (\epsilon \wedge S^W)} G \ltimes_H (A \wedge S^L \wedge S^W)\ .
\end{aligned}
$$

Here $\epsilon : \mathrm{map}^H(G, A \wedge S^L) \longrightarrow A \wedge S^L$ is the adjunction counit, i.e., evaluation at $1 \in G$. We expand this definition. We let $\psi : G \longrightarrow A \wedge S^L$ be an H-equivariant based map and $v \in S^V$. If v is not in the image of the tubular neighborhood $j : G \times_H D(W) \longrightarrow V$, then $r(\psi \wedge v)$ is the basepoint. Otherwise, $v = j[g, w] = g \cdot (v_0 + w)$ for some $(g, w) \in G \times D(W)$, and then

$$
\begin{aligned}
r(\psi \wedge v) &= ((G \ltimes_H (\epsilon \wedge S^W)) \circ \text{shear})(\psi \wedge [g, w/(1 - |w|)]) \qquad (3.2.13) \\
&= [g, \epsilon(g^{-1} \cdot \psi) \wedge w/(1 - |w|)] = \left[g, \psi(g^{-1}) \wedge \frac{w}{1 - |w|}\right] .
\end{aligned}
$$

We denote by $l_A^\sharp : A \longrightarrow \mathrm{map}^H(G, A \wedge S^L)$ the H-map defined by $l_A^\sharp(a)(g) = l_A[g, a]$, and we let $\varphi : V \longrightarrow L \oplus W$ be the inverse of the isometry (3.2.9). Now we argue that the following square commutes up to H-equivariant based homotopy:

$$
\begin{tikzcd}
A \wedge S^V \arrow[r, "{[1,-]}"] \arrow[dd, "l_A^\sharp \wedge S^V"'] & G \ltimes_H (A \wedge S^V) \arrow[d, "G \ltimes_H S^\varphi", "\cong"'] \\
& G \ltimes_H (A \wedge S^L \wedge S^W) \arrow[d, "G \ltimes_H (S^{-\mathrm{Id}_L} \wedge S^W)", "\cong"'] \\
\mathrm{map}^H(G, A \wedge S^L) \wedge S^V \arrow[r, "r"'] & G \ltimes_H (A \wedge S^L \wedge S^W)
\end{tikzcd}
\qquad (3.2.14)
$$

To see this we define an H-homotopy

$$K : (A \wedge S^V) \times [0, 1] \longrightarrow G \ltimes_H (A \wedge S^L \wedge S^W)$$

as follows. We exploit the fact that the map

$$\zeta : D(L) \times D(W) \longrightarrow V , \quad \zeta(l, w) = s(l) \cdot (v_0 + w)$$

is a smooth embedding. This map already featured in the proof of part (i), because the collapse map based on ζ is the composite $(l_H^G \wedge_H S^W) \circ c$. If a vector is of the form $v = \zeta(l, w) = s(l) \cdot (v_0 + w)$ for some $(l, w) \in D(L) \times D(W)$ (necessarily unique), then we set

$$K(a \wedge v, t) = K(a \wedge (s(l) \cdot (v_0 + w)), t) = \left[s(t \cdot l), a \wedge \frac{-l}{1 - |l|} \wedge \frac{w}{1 - |w|}\right] .$$

For $|l| = 1$ or $|w| = 1$ this formula yields the basepoint, so we can extend this definition by sending all elements $a \wedge v$ to the basepoint whenever v is not in the image of the embedding ζ. We claim that for $t = 0$ we obtain $K(-, 1) = r \circ (l_A^\sharp \wedge S^V)$. Indeed, if $v = s(l) \cdot (v_0 + w)$ for $(l, w) \in D(L) \times D(W)$ (necessarily unique), then

$$\begin{aligned}(r \circ (l_A^\sharp \wedge S^V))(a \wedge v) \overset{(3.2.13)}{=} & \left[s(l), l_A^\sharp(a)(s(l)^{-1}) \wedge \frac{w}{1 - |w|} \right] \\ = & \left[s(l), l_A[s(-l), a] \wedge \frac{w}{1 - |w|} \right] \\ = & \left[s(l), a \wedge \frac{-l}{1 - |l|} \wedge \frac{w}{1 - |w|} \right] = K(a \wedge v, 1) .\end{aligned}$$

On the other hand, the map $K(-, 0)$ agrees with the composite

$$\begin{aligned} A \wedge S^V & \xrightarrow{A \wedge ((l_H^G \wedge_H S^W) \circ c)} A \wedge S^L \wedge S^W \\ & \xrightarrow{[1,-]} G \ltimes_H (A \wedge S^L \wedge S^W) \xrightarrow{G \ltimes_H (S^{-\mathrm{Id}_L} \wedge S^W)} G \ltimes_H (A \wedge S^L \wedge S^W) ;\end{aligned}$$

part (i) provides an H-homotopy between $(l_H^G \wedge_H S^W) \circ c$ and the homeomorphism S^φ; so this proves the claim.

We let $\Lambda : G \ltimes_H (A \wedge S^V) \longrightarrow \mathrm{map}^H(G, A \wedge S^L) \wedge S^V$ be the G-equivariant extension of the H-map $l_A^\sharp \wedge S^V$. Since the square (3.2.14) commutes up to H-equivariant homotopy, the composite

$$G \ltimes_H (A \wedge S^V) \xrightarrow{\Lambda} \mathrm{map}^H(G, A \wedge S^L) \wedge S^V \xrightarrow{r} G \ltimes_H (A \wedge S^L \wedge S^W)$$

is G-equivariantly homotopic to the G-homeomorphism $G \ltimes_H ((S^{-\mathrm{Id}_L} \wedge S^W) \circ S^\varphi)$, by adjointness. So the composite of $r\Lambda$ with the shearing homeomorphism $(G \ltimes_H A) \wedge S^V \cong G \ltimes_H (A \wedge S^V)$ is also G-homotopic to a homeomorphism. This composite equals the composite

$$(G \ltimes_H A) \wedge S^V \xrightarrow{\Psi_A \wedge S^V} \mathrm{map}^H(G, A \wedge S^L) \wedge S^V \xrightarrow{r} G \ltimes_H (A \wedge S^L \wedge S^W) ,$$

where $\Psi_A : G \ltimes_H A \longrightarrow \mathrm{map}^H(G, A \wedge S^L)$ is the adjoint of the H-map $l_A : G \ltimes_H A \longrightarrow A \wedge S^L$.

Now we consider based G-maps $f, f' : B \longrightarrow G \ltimes_H A$ such that $l_A \circ f$ and $l_A \circ f' : B \longrightarrow A \wedge S^L$ are H-equivariantly homotopic. Then the two composites

$$B \xrightarrow{f, f'} G \ltimes_H A \xrightarrow{\Psi_A} \mathrm{map}^H(G, A \wedge S^L)$$

are G-equivariantly homotopic, by adjointness. The composite $(\Psi_A \circ f) \wedge S^V = (\Psi_A \wedge S^V) \circ (f \wedge S^V)$ is then G-equivariantly homotopic to $(\Psi_A \circ f') \wedge S^V = (\Psi_A \wedge S^V) \circ (f' \wedge S^V)$. The map $\Psi_A \wedge S^V$ has a G-equivariant retraction, up

to G-homotopy, by the previous paragraph. So already the maps $f \wedge S^V$ and $f' \wedge S^V$ are G-homotopic. □

Now we can establish the Wirthmüller isomorphism, which first appeared in [191, Thm. 2.1]. Wirthmüller attributes parts of the ideas to tom Dieck, and his statement that G-spectra define a 'complete G-homology theory' amounts to Theorem 3.2.15. Our proof is essentially Wirthmüller's original argument, adapted to the context of equivariant orthogonal spectra. We recall that

$$\varepsilon_L \ : \ \pi_0^H(Y \wedge S^L) \ \longrightarrow \ \pi_0^H(Y \wedge S^L)$$

denotes the effect of the involution of $Y \wedge S^L$ induced by the linear isometry $-\operatorname{Id}_L : L \longrightarrow L$ given by multiplication by -1.

Theorem 3.2.15 (Wirthmüller isomorphism)**.** *Let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum. Let $L = T_{eH}(G/H)$ denote the tangent H-representation. Then the maps*

$$\mathrm{Wirth}_H^G \ : \ \pi_0^G(G \ltimes_H Y) \ \longrightarrow \ \pi_0^H(Y \wedge S^L)$$

and

$$(G \ltimes_H -) \circ \varepsilon_L \ : \ \pi_0^H(Y \wedge S^L) \ \longrightarrow \ \pi_0^G(G \ltimes_H Y)$$

are independent of the choices made in their definitions, and they are mutually inverse isomorphisms.

Proof We prove the various claims in a specific order. In a first step we show that the Wirthmüller map is left inverse to the map $(G \ltimes_H -) \circ \varepsilon_L$, independently of all the choices made in the definitions. We let U be a G-representation and $f : S^U \longrightarrow Y(U) \wedge S^L$ an H-equivariant based map that represents a class in $\pi_0^H(Y \wedge S^L)$. We also choose a wide G-equivariant embedding $i : G/H \longrightarrow V$ as in Construction 3.2.7. This provides a decomposition $L \oplus W \cong V$ of H-representations as in (3.2.9) and a G-equivariant Thom-Pontryagin collapse map $c : S^V \longrightarrow G \ltimes_H S^W$. We let $\varphi : V \cong L \oplus W$ denote the inverse of the linear

isomorphism (3.2.9). We contemplate the diagram of based H-maps:

$$\begin{tikzcd}
S^U \wedge S^V \arrow[d, "S^U\wedge c"'] \arrow[dr, bend left, "S^U\wedge S^\varphi"] \arrow[ddd, bend right=60] & \\
S^U \wedge (G \ltimes_H S^W) \arrow[r, "S^U\wedge(l^G_H\wedge_H S^W)"'] \arrow[d, "\text{shear}"', "\cong"] & S^U \wedge S^L \wedge S^W \arrow[d, "S^U\wedge\tau_{L,W}"] \\
G \ltimes_H (S^U \wedge S^W) \arrow[r, "l_{S^U\wedge S^W}"'] \arrow[d, "G\ltimes_H(f\diamond W)"] & S^U \wedge S^W \wedge S^L \arrow[d, "(f\diamond W)\wedge S^L"] \\
G \ltimes_H Y(U \oplus V) \arrow[r, "l_{Y(U\oplus V)}"'] & Y(U \oplus V) \wedge S^L
\end{tikzcd}$$

The left vertical composite represents the external transfer $G \ltimes_H \langle f\rangle$, so the composite around the lower left corner represents $(l_Y)_*(\mathrm{res}^G_H(G \ltimes_H \langle f\rangle))$. The upper triangle commutes up to H-equivariant homotopy by Proposition 3.2.12 (i). The middle square commutes up to H-homotopy by Proposition 3.2.3 and the fact that $l_A : G \ltimes_H A \longrightarrow A \wedge S^L$ is defined as the composite of $(l^G_H \wedge_H A)$ and the twist isomorphism $S^L \wedge A \cong A \wedge S^L$. The lower square commutes by naturality of the maps (3.2.2). Upon expanding the definition (3.2.11) of $f \diamond W$, the diagram shows that the class $(l_Y)_*(\mathrm{res}^G_H(G \ltimes_H \langle f\rangle))$ is also represented by the composite

$$S^{U\oplus V} \xrightarrow{f\wedge S^\varphi} Y(U) \wedge S^L \wedge S^L \wedge S^W \xrightarrow{Y(U)\wedge S^L\wedge\tau_{L,W}} Y(U) \wedge S^L \wedge S^W \wedge S^L$$
$$\xrightarrow{Y(U)\wedge S^{\varphi^{-1}}\wedge S^L} Y(U) \wedge S^V \wedge S^L \xrightarrow{\sigma^{\mathrm{op}}_{U,V}\wedge S^L} Y(U \oplus V) \wedge S^L .$$

The isometry

$$(\varphi^{-1} \oplus L) \circ (L \oplus \tau_{L,W}) \circ (L \oplus \varphi) \ : \ L \oplus V \ \longrightarrow \ V \oplus L$$

is *not* the twist isometry $\tau_{L,V}$; however $(\varphi^{-1} \oplus L) \circ (L \oplus \tau_{L,W}) \circ (L \oplus \varphi)$ is equivariantly homotopic to the composite

$$L \oplus V \xrightarrow{\tau_{L,V}} V \oplus L \xrightarrow{V\oplus(-\mathrm{Id}_L)} V \oplus L .$$

Hence the class $(l_Y)_*(\mathrm{res}^G_H(G \ltimes_H \langle f\rangle))$ is also represented by the composite

$$S^{U\oplus V} \xrightarrow{f\wedge S^V} Y(U)\wedge S^L\wedge S^V \xrightarrow{Y(U)\wedge\tau_{L,V}} Y(U)\wedge S^V\wedge S^L \xrightarrow{\sigma^{\mathrm{op}}_{U,V}\wedge S^{-\mathrm{Id}}} Y(U\oplus V)\wedge S^L .$$

So altogether this shows the desired relation

$$\mathrm{Wirth}^G_H(G \ltimes_H \langle f\rangle) \ = \ (l_Y)_*(\mathrm{res}^G_H(G \ltimes_H \langle f\rangle)) \ = \ \varepsilon_L\langle f\rangle .$$

In particular, the Wirthmüller map is surjective.

Now we show that the Wirthmüller map is injective. We let $f, f' : S^U \longrightarrow G \ltimes_H Y(U)$ be two G-maps that represent classes with the same image under the Wirthmüller map $\mathrm{Wirth}_H^G : \pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$. By increasing the G-representation U, if necessary, we can assume that the composites

$$S^U \xrightarrow{f, f'} G \ltimes_H Y(i^* U) \xrightarrow{l_{Y(i^*U)}} Y(i^* U) \wedge S^L$$

are H-equivariantly homotopic. Then by Proposition 3.2.12 (ii) the two maps

$$S^{U \oplus V} \xrightarrow{f \wedge S^V, f' \wedge S^V} (G \ltimes_H Y(i^* U)) \wedge S^V \cong (G \ltimes_H Y)(U) \wedge S^V$$

are G-equivariantly homotopic. The maps remain G-homotopic if we furthermore post-compose with the opposite structure map $\sigma_{U,V}^{\mathrm{op}} : (G \ltimes_H Y)(U) \wedge S^V \longrightarrow (G \ltimes_H Y)(U \oplus V)$. This shows that the stabilizations from the right of f and f' by V become G-homotopic. Since such a stabilization represents the same class in $\pi_0^G(G \ltimes_H Y)$ as the original map, this shows that $\langle f \rangle = \langle f' \rangle$, i.e., the Wirthmüller map is injective.

Now we know that the Wirthmüller map and the map $(G \ltimes_H -) \circ \varepsilon_L$ are inverse to each other, no matter which choices of slice $s : D(L) \longrightarrow G$, representation V and G-embedding $G/H \longrightarrow V$ we made. Since the choices for the Wirthmüller map and the choices for the external transfer are independent of each other, the two resulting maps are independent of all choices.

The last thing to show is that the Wirthmüller map and the external transfer are additive maps. If H has finite index in G, the Wirthmüller map is the composite of a restriction homomorphism and the effect of a morphism of orthogonal H-spectra, both of which are additive. In general, however, we need an additional argument, namely naturality. Indeed, for a fixed choice of slice $s : D(L) \longrightarrow G$, the Wirthmüller map $\pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$ is natural in the orthogonal H-spectrum Y. Since source and target are reduced additive functors from orthogonal H-spectra to abelian groups, any natural transformation is automatically additive, by Proposition 2.2.12. Since the Wirthmüller map is additive, so is its inverse, and hence also the external transfer. □

Theorem 3.2.15 states the Wirthmüller isomorphism only for 0-dimensional homotopy groups. We now extend it to homotopy groups in all integer dimensions. This extension is a rather formal consequence of the fact that the Wirthmüller maps commute with the loop and suspension isomorphisms

$$\alpha : \pi_k^G(\Omega X) \longrightarrow \pi_{k+1}^G(X) \quad \text{and} \quad - \wedge S^1 : \pi_k^G(X) \longrightarrow \pi_{k+1}^G(X \wedge S^1)$$

defined in (3.1.28) and (3.1.29), respectively.

Proposition 3.2.16. *Let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum.*

(i) *The following diagrams commute for all integers k:*

$$\begin{tikzcd}
\pi_k^G(G \ltimes_H (\Omega Y)) \arrow[r, "\mathrm{Wirth}_H^G"] \arrow[d, "\mathrm{assembly}"', "\cong"] & \pi_k^H((\Omega Y) \wedge S^L) \arrow[d, "\cong"', "\mathrm{assembly}"] \\
\pi_k^G(\Omega(G \ltimes_H Y)) \arrow[d, "\alpha"', "\cong"] & \pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^G(G \ltimes_H Y) \arrow[r, "\mathrm{Wirth}_H^G"'] & \pi_{k+1}^H(Y \wedge S^L)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_k^G(G \ltimes_H Y) \arrow[r, "\mathrm{Wirth}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^H(Y \wedge S^L) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^G((G \ltimes_H Y) \wedge S^1) \arrow[d, "b_*"', "\cong"] & \pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "\cong"', "(Y \wedge \tau_{S^L,S^1})_*"] \\
\pi_{k+1}^G(G \ltimes_H (Y \wedge S^1)) \arrow[r, "\mathrm{Wirth}_H^G"'] & \pi_{k+1}^H(Y \wedge S^1 \wedge S^L)
\end{tikzcd}$$

Here $\tau_{S^L,S^1} : S^L \wedge S^1 \longrightarrow S^1 \wedge S^L$ *is the twist homeomorphism, and the G-equivariant isomorphism*

$$b \ : \ (G \ltimes_H Y) \wedge S^1 \ \cong \ G \ltimes_H (Y \wedge S^1)$$

is given by $b([g, y] \wedge t) = [g, y \wedge t]$.

(ii) *The Wirthmüller map*

$$\mathrm{Wirth}_H^G \ : \ \pi_k^G(G \ltimes_H Y) \ \longrightarrow \ \pi_k^H(Y \wedge S^L)$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof (i) The loop and suspension isomorphisms commute with restriction from G to H, by direct inspection. So the proof comes down to checking that

the following diagrams commute:

$$\begin{tikzcd}
\pi_k^H(G \ltimes_H (\Omega Y)) \arrow[r, "(l_{\Omega Y})_*"] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^H((\Omega Y) \wedge S^L) \arrow[d, "\text{assembly}", "\cong"'] \\
\pi_k^H(\Omega(G \ltimes_H Y)) \arrow[d, "\alpha"', "\cong"] & \pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\alpha", "\cong"'] \\
\pi_{k+1}^H(G \ltimes_H Y) \arrow[r, "(l_Y)_*"'] & \pi_{k+1}^H(Y \wedge S^L)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_k^H(G \ltimes_H Y) \arrow[r, "(l_Y)_*"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^H(Y \wedge S^L) \arrow[d, "-\wedge S^1", "\cong"'] \\
\pi_{k+1}^H((G \ltimes_H Y) \wedge S^1) \arrow[d, "b_*"', "\cong"] & \pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "(Y \wedge \tau_{S^L,S^1})_*", "\cong"'] \\
\pi_{k+1}^H(G \ltimes_H (Y \wedge S^1)) \arrow[r, "(l_{Y \wedge S^1})_*"'] & \pi_{k+1}^H(Y \wedge S^1 \wedge S^L)
\end{tikzcd}$$

This in turn follows from the fact – again verified by direct inspection – that for every based H-space A the following two squares commute:

$$\begin{tikzcd}
G \ltimes_H (\Omega A) \arrow[r, "l_{\Omega A}"] \arrow[d, "\text{assembly}"'] & (\Omega A) \wedge S^L \arrow[d, "\text{assembly}"] \\
\Omega(G \ltimes_H A) \arrow[r, "\Omega l_A"'] & \Omega(A \wedge S^L)
\end{tikzcd}
\qquad
\begin{tikzcd}
(G \ltimes_H A) \wedge S^1 \arrow[r, "l_A \wedge S^1"] \arrow[d, "b"'] & A \wedge S^L \wedge S^1 \arrow[d, "A \wedge \tau_{S^L,S^1}"] \\
G \ltimes_H (A \wedge S^1) \arrow[r, "l_{A \wedge S^1}"'] & A \wedge S^1 \wedge S^L
\end{tikzcd}$$

(ii) We argue by induction over $|k|$, the absolute value of the integer k. The induction starts with $k = 0$, where Theorem 3.2.15 provides the desired conclusion. If k is positive, the compatibility of the Wirthmüller map with the loop isomorphism, established in part (i), provides the inductive step. If k is negative, the compatibility of the Wirthmüller map with the suspension isomorphism, also established in part (i), provides the inductive step. □

Construction 3.2.17 (External transfer in integer degrees)**.** So far we only defined the external transfer for 0-dimensional homotopy groups. Theorem 3.2.15 shows that in dimension 0 the external transfer is inverse to the map $\varepsilon_L \circ \mathrm{Wirth}_H^G : \pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$. We want the same property in all dimensions, and since the Wirthmüller map is an isomorphism by Proposition 3.2.16 we simply define the external transfer isomorphism

$$G \ltimes_H - \;:\; \pi_k^H(Y \wedge S^L) \;\longrightarrow\; \pi_k^G(G \ltimes_H Y) \qquad (3.2.18)$$

as the composite

$$\pi_k^H(Y \wedge S^L) \xrightarrow{\varepsilon_L} \pi_k^H(Y \wedge S^L) \xrightarrow{(\mathrm{Wirth}_H^G)^{-1}} \pi_k^G(G \ltimes_H Y) .$$

The compatibility of the Wirthmüller isomorphism with the loop and suspension isomorphisms then directly implies the analogous compatibility for the external transfer $G \ltimes_H -$, by reading the diagrams of Proposition 3.2.16 (i) backwards. For easier reference, we explicitly record this compatibility in Proposition 3.2.27 below.

In the next proposition we use the Wirthmüller isomorphism to show that smashing with a cofibrant G-space is homotopical. In [108, III Thm. 3.11], Mandell and May give a different proof of this fact which does not use the Wirthmüller isomorphism.

Proposition 3.2.19. *Let G be a compact Lie group and A a cofibrant based G-space.*

(i) *Let $f : X \longrightarrow Y$ be a morphism of orthogonal G-spectra with the following property: for every closed subgroup H of G that fixes some non-basepoint of A, the map $\pi_*^H(f) : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism. Then the morphism $f \wedge A : X \wedge A \longrightarrow Y \wedge A$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.*

(ii) *The functor $- \wedge A$ preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.*

Proof (i) Smashing with A commutes with mapping cones, so by the long exact homotopy group sequence of Proposition 3.1.36 it suffices to show the following special case. Let X be an orthogonal G-spectrum with the following property: for every closed subgroup H of G that occurs as the stabilizer group of a non-basepoint of A, the groups $\pi_*^H(X)$ vanish. Then for all closed subgroups K of G the equivariant homotopy groups $\pi_*^K(X \wedge A)$ vanish.

In a first step we show this when A is a finite-dimensional G-CW-complex. We argue by contradiction. If the claim were false, we could find a compact Lie group G of minimal dimension for which it fails. We let A be a G-CW-complex whose dimension n is minimal among all counterexamples. Then A can be obtained from an $(n-1)$-dimensional subcomplex B by attaching equivariant cells $G/H_i \times D^n$, for i in some indexing set I, where H_i is a closed subgroup of G. Then $X \wedge (A/B)$ is isomorphic to a wedge, over the set I, of orthogonal G-spectra $X \wedge (G/H_i)_+ \wedge S^n$. Since equivariant homotopy groups take wedges to sums, the suspension isomorphism and the Wirthmüller isomorphism allow

us to rewrite the equivariant homotopy groups of $X \wedge A/B$ as

$$\begin{aligned} \pi_*^G(X \wedge A/B) &\cong \bigoplus\nolimits_{i\in I} \pi_*^G(X \wedge (G/H_i)_+ \wedge S^n) \\ &\cong \bigoplus\nolimits_{i\in I} \pi_{*-n}^G(X \wedge (G/H_i)_+) \cong \bigoplus\nolimits_{i\in I} \pi_{*-n}^{H_i}(X \wedge S^{L_i}) , \end{aligned}$$

where L_i is the tangent representation of H_i in G. Since H_i is the stabilizer of a non-basepoint of A, the groups $\pi_*^K(X)$ vanish for all closed subgroups K of H_i, by hypothesis. If H_i has finite index in G, then $L_i = 0$ and the respective summand thus vanishes. The representation sphere S^{L_i} admits a finite H_i-CW-structure, so if H_i has strictly smaller dimension than G, then the respective summand vanishes by the minimality of G. So altogether we conclude that the groups $\pi_*^G(X \wedge A/B)$ vanish. The groups $\pi_*^G(X \wedge B)$ vanish by the minimality of A. The inclusion of B into A is an h-cofibration of based G-spaces, so the induced morphism $X\wedge B \longrightarrow X\wedge A$ is an h-cofibration of orthogonal G-spectra. Hence the groups $\pi_*^G(X \wedge A)$ vanish by the long exact sequence of Corollary 3.1.38 (i). Since A was supposed to be a counterexample to the proposition, we have reached the desired contradiction. Altogether this proves the claim when A admits the structure of a finite-dimensional G-CW-complex.

If A admits the structure of a G-CW-complex, possibly infinite dimensional, we choose a skeleton filtration by G-subspaces A_n. Then the G-homotopy groups of $X\wedge A_n$ vanish for all $n \geq 0$, and all the morphisms $X\wedge A_n \longrightarrow X\wedge A_{n+1}$ are h-cofibrations of orthogonal G-spectra. Since equivariant homotopy groups commute with such sequential colimits (see Proposition 3.1.41 (i)), the groups $\pi_*^G(X\wedge A)$ vanish as well. An arbitrary cofibrant based G-space is G-homotopy equivalent to a based G-CW-complex, so the groups $\pi_*^G(X \wedge A)$ vanish for all cofibrant A.

Now we let K be an arbitrary closed subgroup of G. The underlying K-space of A is again cofibrant by Proposition B.14 (i), so we can apply the previous reasoning to K instead of G and conclude that the groups $\pi_*^K(X\wedge A)$ vanish. □

Proposition 3.2.20. *Let G and K be compact Lie groups and A a cofibrant based $(G \times K)$-space such that the G-action is free away from the basepoint and the K-action is free away from the basepoint. Then the functor*

$$A \wedge_K - \;:\; K\mathcal{S}p \longrightarrow G\mathcal{S}p$$

takes $\underline{\pi}_*$-isomorphisms of orthogonal K-spectra to $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.

Proof The functor $A \wedge_K -$ preserves mapping cones, so by the long exact homotopy group sequence of Proposition 3.1.36 it suffices to show the following special case. Let C be an orthogonal K-spectrum all of whose equivariant homotopy groups vanish, for all closed subgroups of K. Then all equivariant

homotopy groups of the orthogonal G-spectrum $A \wedge_K C$ vanish, for all closed subgroups of G.

In a first step we show that the G-equivariant stable homotopy groups of $A \wedge_K C$ vanish. A cofibrant based $(G \times K)$-space is equivariantly homotopy equivalent to a based $(G \times K)$-CW-complex, so it is no loss of generality to assume an equivariant CW-structure with skeleton filtration

$$* = A_{-1} \subset A_0 \subset \ldots \subset A_n \subset \ldots .$$

We show first, by induction on n, that the orthogonal G-spectrum $(A_n) \wedge_K C$ has trivial G-equivariant homotopy groups. The induction starts with $n = -1$, where there is nothing to show. For $n \geq 0$ the quotient A_n/A_{n-1} is $(G \times K)$-equivariantly isomorphic to a wedge of summands of the form $((G \times K)/\Gamma_i)_+ \wedge S^n$, for certain closed subgroups Γ_i of $G \times K$. Since the K-action on A is free (away from the basepoint), each isotropy group Γ_i that occurs is the graph of a continuous homomorphism $\alpha_i : H_i \longrightarrow K$ defined on a closed subgroup H_i of G. Since the G-action on A is free (away from the basepoint), all the homomorphisms α_i that occur are injective.

Since equivariant homotopy groups take wedges to sums, the suspension isomorphism and the Wirthmüller isomorphism allow us to rewrite the equivariant homotopy groups of $(A_n/A_{n-1}) \wedge_K C$ as

$$\begin{aligned} \pi_*^G((A_n/A_{n-1}) \wedge_K C) &\cong \bigoplus \pi_*^G(((G \times K)/\Gamma_i)_+ \wedge_K C \wedge S^n) \\ &\cong \bigoplus \pi_{*-n}^G(G \ltimes_{H_i} \alpha_i^*(C)) \cong \bigoplus \pi_{*-n}^{H_i}(\alpha_i^*(C) \wedge S^{L_i}) , \end{aligned}$$

where L_i is the tangent representation of H_i in G. By hypothesis on C and because α_i is injective, the orthogonal H-spectrum $\alpha_i^*(C)$ is H_i-stably contractible. So $\alpha_i^*(C) \wedge S^{L_i}$ is H_i-stably contractible by Proposition 3.2.19 (ii). Altogether this shows that the orthogonal G-spectrum $(A_n/A_{n-1}) \wedge_K C$ has vanishing G-equivariant homotopy groups.

The inclusion $A_{n-1} \longrightarrow A_n$ is an h-cofibration of based $(G \times K)$-spaces, so the induced morphism $(A_{n-1}) \wedge_K C \longrightarrow (A_n) \wedge_K C$ is an h-cofibration of orthogonal G-spectra, giving rise to a long exact sequence of equivariant homotopy groups (Corollary 3.1.38). By the previous paragraph and the inductive hypothesis, the orthogonal G-spectrum $(A_n) \wedge_K C$ has vanishing G-equivariant homotopy groups. This completes the inductive step.

Since A is the sequential colimit, along h-cofibrations of based $(G \times K)$-spaces, of the skeleta A_n, the orthogonal G-spectrum $A \wedge_K C$ is the sequential colimit, along h-cofibrations of orthogonal G-spectra, of the sequence with terms $(A_n) \wedge_K C$. Equivariant homotopy groups commute with such sequential colimits (Proposition 3.1.41 (i)), so also $A \wedge_K C$ has vanishing G-equivariant homotopy groups.

Now we let H be any closed subgroup of G. The underlying $(H \times K)$-space of A is again cofibrant by Proposition B.14 (i), so we can apply the previous reasoning to H instead of G and conclude that the groups $\pi_*^H(A \wedge_K C)$ vanish. □

Corollary 3.2.21. *Let H be a closed subgroup of a compact Lie group G. Then the induction functor*

$$G \ltimes_H - \ : \ H\mathcal{S}p \longrightarrow G\mathcal{S}p$$

takes $\underline{\pi}_*$-isomorphisms of orthogonal H-spectra to $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.

Proof We let $G \times H$ act on G by left and right translation, i.e., via

$$(g, h) \cdot \gamma \ = \ g\gamma h^{-1} \ .$$

With this action, G is $(G \times H)$-equivariantly isomorphic to the homogeneous space $(G \times H)/\Delta$ for $\Delta = \{(h, h) \ : \ h \in H\}$. In particular, G is $(G \times H)$-cofibrant, and both partial actions are free. So Proposition 3.2.20 applies to the functor $(G_+) \wedge_H - = G \ltimes_H -$ and yields the desired conclusion. □

Now we discuss the transfer maps of equivariant homotopy groups.

Construction 3.2.22 (Transfers)**.** We let H be a closed subgroup of a compact Lie group G. As before we write $L = T_{eH}(G/H)$ for the tangent space of G/H at the coset eH, which inherits an H-action from the H-action on G/H. We let X be an orthogonal G-spectrum. We form the composite

$$\pi_k^H(X \wedge S^L) \xrightarrow[\cong]{G \ltimes_H -} \pi_k^G(G \ltimes_H X) \xrightarrow{\mathrm{act}_*} \pi_k^G(X)$$

of the external transfer (3.2.18) and the effect of the action map (i.e., the adjunction counit) $G \ltimes_H X \longrightarrow X$. We call this composite the *dimension shifting transfer* and denote it

$$\mathrm{Tr}_H^G \ : \ \pi_k^H(X \wedge S^L) \longrightarrow \pi_k^G(X) \ . \tag{3.2.23}$$

The (degree zero) transfer is then defined as the composite,

$$\mathrm{tr}_H^G \ : \ \pi_k^H(X) \xrightarrow{(X \wedge i)_*} \pi_k^H(X \wedge S^L) \xrightarrow{\mathrm{Tr}_H^G} \pi_k^G(X) \ , \tag{3.2.24}$$

where $i : S^0 \longrightarrow S^L$ is the 'inclusion of the origin', the based map sending 0 to 0. Both kinds of transfer are natural for morphisms of orthogonal G-spectra. For finite index inclusions, $L = 0$ and there is no difference between the dimension shifting transfer and the degree zero transfer. The external transfer is additive; since the dimension shifting and degree zero transfers are obtained

from there by applying morphisms of equivariant spectra, they are also additive.

The key properties of these transfer maps are:

- transfers are transitive (Proposition 3.2.29);
- transfers commute with inflation maps (Proposition 3.2.32); and
- the restriction of a degree zero transfer to a closed subgroup satisfies a double coset formula (Theorem 3.4.9).

Example 3.2.25 (Infinite Weyl group transfers)**.** If H has infinite index in its normalizer, then the degree zero transfer tr_H^G is trivial. Indeed, the inclusion of the normalizer $N_G H$ of H into G induces a smooth embedding

$$W_G H \ = \ (N_G H)/H \ \longrightarrow \ G/H$$

and thus a monomorphism of tangent H-representations

$$T_{eH}(W_G H) \ \longrightarrow \ T_{eH}(G/H) = L \ .$$

If $n \in N_G H$ is an element of the normalizer and $h \in H$, then

$$h \cdot nH \ = \ n \cdot (n^{-1}hn)H \ = \ nH \ ,$$

so $W_G H$ is H-fixed inside G/H. Consequently, the tangent space $T_{eH}(W_G H)$ is contained in the H-fixed space L^H. If H has infinite index in its normalizer, then the Weyl group $W_G H$ and the tangent space $T_{eH}(W_G H)$ have positive dimension. In particular, L has non-zero H-fixed-points. The point 0 in S^L can thus be moved through H-fixed-points to the basepoint at infinity. The first map in the composite (3.2.24) is thus the zero map, hence so is the transfer tr_H^G.

Remark 3.2.26. As the previous example indicates, the passage from the dimension shifting transfer Tr_H^G to the degree zero transfer tr_H^G loses information. An extreme case is when the subgroup H is normal in G. Then the action of the group H on G/H is trivial; hence also the H-action on the tangent space L is trivial. Upon choosing an isomorphism $L \cong \mathbb{R}^d$, the Wirthmüller isomorphism identifies $\pi_0^G(G \ltimes_H Y)$ with $\pi_0^H(Y \wedge S^d)$, where $d = \dim(G/H) = \dim(G) - \dim(H)$. The dimension shifting transfer then becomes a natural map

$$\pi_0^H(Y) \ \cong \ \pi_d^H(Y \wedge S^L) \ \xrightarrow{\mathrm{Tr}_H^G} \ \pi_d^G(Y) \ .$$

This transformation is generically non-trivial.

Now we recall some important properties of the transfer maps. We start with the compatibility with the loop and suspension isomorphisms.

Proposition 3.2.27. *Let H be a closed subgroup of a compact Lie group G.*

(i) *For every orthogonal H-spectrum Y and all $k \in \mathbb{Z}$, the following diagrams commute:*

$$\begin{tikzcd}
\pi_k^H((\Omega Y) \wedge S^L) \arrow[r, "G\ltimes_H -", "\cong"'] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^G(G \ltimes_H (\Omega Y)) \arrow[d, "\cong"', "\text{assembly}"] \\
\pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\alpha"', "\cong"] & \pi_k^G(\Omega(G \ltimes_H Y)) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^H(Y \wedge S^L) \arrow[r, "\cong", "G\ltimes_H -"'] & \pi_{k+1}^G(G \ltimes_H Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_k^H(Y \wedge S^L) \arrow[r, "G\ltimes_H -", "\cong"'] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(G \ltimes_H Y) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "(Y\wedge\tau_{S^L,S^1})_*"', "\cong"] & \pi_{k+1}^G((G \ltimes_H Y) \wedge S^1) \arrow[d, "\cong"', "b_*"] \\
\pi_{k+1}^H(Y \wedge S^1 \wedge S^L) \arrow[r, "\cong", "G\ltimes_H -"'] & \pi_{k+1}^G(G \ltimes_H (Y \wedge S^1))
\end{tikzcd}$$

Here $\tau_{S^L,S^1} : S^L \wedge S^1 \longrightarrow S^1 \wedge S^L$ is the twist isomorphism, and the G-equivariant isomorphism

$$b \;:\; (G \ltimes_H Y) \wedge S^1 \;\cong\; G \ltimes_H (Y \wedge S^1)$$

is given by $b([g,y] \wedge t) = [g, y \wedge t]$.

(ii) *For every orthogonal G-spectrum X and all $k \in \mathbb{Z}$, the following diagrams commute:*

$$\begin{tikzcd}
\pi_k^H((\Omega X) \wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^G(\Omega X) \arrow[dd, "\cong"', "\alpha"] \\
\pi_k^H(\Omega(X \wedge S^L)) \arrow[d, "\alpha"', "\cong"] & \\
\pi_{k+1}^H(X \wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"'] & \pi_{k+1}^G(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(X \wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(X) \arrow[dd, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(X \wedge S^L \wedge S^1) \arrow[d, "(X\wedge\tau_{S^L,S^1})_*"', "\cong"] & \\
\pi_{k+1}^H(X \wedge S^1 \wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"'] & \pi_{k+1}^G(X \wedge S^1)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_k^H(\Omega X) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "\alpha"', "\cong"] & \pi_k^G(\Omega X) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^H(X) \arrow[r, "\mathrm{tr}_H^G"'] & \pi_{k+1}^G(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(X) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(X) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(X \wedge S^1) \arrow[r, "\mathrm{tr}_H^G"'] & \pi_{k+1}^G(X \wedge S^1)
\end{tikzcd}$$

Proof The first two diagrams commute because we can read the diagrams of Proposition 3.2.16 (i) backwards. The commutativity of the other diagrams then follows by naturality. □

Now we establish the transitivity property for a nested triple of compact Lie groups $K \leq H \leq G$. We continue to denote by $L = T_{eH}(G/H)$ the tangent H-representation in G, and we write $\bar{L} = T_{eK}(H/K)$ for the tangent K-representation in H. We choose a slice

$$s \; : \; D(L) \longrightarrow G$$

as in the construction of the map $l_H^G : G \longrightarrow S^L \wedge H_+$ in Construction 3.2.1; so s is a wide smooth embedding of the unit disc of L satisfying

$$s(0) \;=\; 1 \qquad \text{and} \qquad s(h \cdot l) \;=\; h \cdot s(l) \cdot h^{-1}$$

for all $(h, l) \in H \times D(L)$, and the differential at $0 \in D(L)$ of the composite $\mathrm{proj}_H \circ s : D(L) \longrightarrow G/H$ is the identity of L. The differential at $0 \in D(L)$ of the composite

$$D(L) \xrightarrow{\;s\;} G \xrightarrow{\;\mathrm{proj}_K\;} G/K$$

is then a K-equivariant linear monomorphism

$$d(\mathrm{proj}_K \circ s)_0 \; : \; L \longrightarrow T_{eK}(G/K) \;=\; L(K, G)$$

that splits the differential at eK of the projection $q : G/K \longrightarrow G/H$. So the combined map

$$\begin{aligned} (d(\mathrm{proj}_K \circ s)_0,\, (dq)_{eH}) \; : \; L \oplus \bar{L} \;&=\; T_{eH}(G/H) \oplus T_{eK}(H/K) \\ &\longrightarrow\; T_{eK}(G/K) \;=\; L(K, G) \end{aligned} \tag{3.2.28}$$

is an isomorphism of K-representations. Upon one-point compactification this isomorphism induces a homeomorphism of K-spaces

$$S^L \wedge S^{\bar{L}} \;\cong\; S^{L(K,G)} \,.$$

Any two slices are H-equivariantly isotopic (compare [26, VI Thm. 2.6]), so the K-equivariant homotopy class of the latter isomorphism is independent of the choice of slice.

Proposition 3.2.29 (Transitivity of transfers)**.** *Let G be a compact Lie group, $K \leq H \leq G$ nested closed subgroups and X an orthogonal G-spectrum. Then the composite*

$$\pi_k^K(X \wedge S^{L(K,G)}) \cong_{(3.2.28)} \pi_k^K(X \wedge S^L \wedge S^{\bar{L}}) \xrightarrow{\;\mathrm{Tr}_K^H\;} \pi_k^H(X \wedge S^L) \xrightarrow{\;\mathrm{Tr}_H^G\;} \pi_k^G(X)$$

agrees with the transfer Tr_K^G. *Moreover, the degree zero transfers satisfy*

$$\mathrm{tr}_H^G \circ \mathrm{tr}_K^H \;=\; \mathrm{tr}_K^G \;:\; \pi_k^K(X) \;\longrightarrow\; \pi_k^G(X)\ .$$

Proof We start by establishing transitivity of the Wirthmüller maps. We choose a slice for the inclusion of K into H, i.e., a wide smooth embedding $\bar{s} : D(\bar{L}) \longrightarrow H$ satisfying

$$\bar{s}(0) \;=\; 1 \qquad \text{and} \qquad \bar{s}(k \cdot \bar{l}) \;=\; k \cdot \bar{s}(\bar{l}) \cdot k^{-1}$$

for all $(k, \bar{l}) \in K \times D(\bar{L})$, and such that the differential at $0 \in D(\bar{L})$ lifts the identity of $\bar{L}$. We combine the two slices into a slice for K inside G: the K-equivariant map

$$D(L \oplus \bar{L}) \;\longrightarrow\; G\ , \quad (l, \bar{l}) \;\longmapsto\; s(l) \cdot \bar{s}(\bar{l})$$

sends $(0,0)$ to 1, and its differential at $(0,0)$ is exactly the identification (3.2.28). So we can – and will – define the map $l_K^G : G \longrightarrow S^{L(K,G)} \wedge K_+$ from the slice

$$s' \;:\; D(L(K,G)) \xrightarrow[(3.2.28)^{-1}]{\cong} D(L \oplus \bar{L}) \xrightarrow{(l,\bar{l})\mapsto s(l)\cdot\bar{s}(\bar{l})} G\ .$$

The maps $l_H^G : G \longrightarrow S^L \wedge H_+$ and $l_K^H : H \longrightarrow S^{\bar{L}} \wedge K_+$ are the Thom-Pontryagin collapses based on the H^2-equivariant smooth embedding

$$\tilde{s} \;:\; D(L) \times H \;\longrightarrow\; G\ , \quad (l, h) \;\longmapsto\; s(l) \cdot h$$

and the K^2-equivariant smooth embedding

$$\hat{s} \;:\; D(\bar{L}) \times K \;\longrightarrow\; H\ , \quad (\bar{l}, k) \;\longmapsto\; \bar{s}(\bar{l}) \cdot k\ .$$

The composite $(S^L \wedge l_K^H) \circ l_H^G$ is thus K^2-equivariantly homotopic to the Thom-Pontryagin collapse based on the K^2-equivariant smooth embedding

$$D(L) \times D(\bar{L}) \times K \;\longrightarrow\; G\ , \quad (l, \bar{l}, k) \;\longmapsto\; s(l) \cdot \bar{s}(\bar{l}) \cdot k\ .$$

The following diagram of K^2-equivariant smooth embeddings then commutes by construction:

$$\begin{tikzcd}
D(L \oplus \bar{L}) \times K \arrow[r, "(3.2.28)", "\cong"'] \arrow[d, "\mathrm{incl}"'] & D(L(K,G)) \times K \arrow[dd, "(l,k)\mapsto s'(l)\cdot k"] \\
D(L) \times D(\bar{L}) \times K \arrow[d, "(l,\bar{l},k)\mapsto(l,\bar{s}(\bar{l})\cdot k)"'] & \\
D(L) \times H \arrow[r, "(l,h)\mapsto s(l)\cdot h"'] & G
\end{tikzcd}$$

So the associated diagram of K^2-equivariant collapse maps also commutes:

$$\begin{tikzcd} G_+ \arrow[r, "l^G_H"] \arrow[dd, "l^G_K"'] & S^L \wedge H_+ \arrow[d, "S^L \wedge l^H_K"] \\ & S^L \wedge S^{\bar{L}} \wedge K_+ \arrow[d, "\Psi"] \\ S^{L(K,G)} \wedge K_+ \arrow[r, "(3.2.28)^{-1}", "\cong"'] & S^{L\oplus\bar{L}} \wedge K_+ \end{tikzcd}$$

Here $\Psi : S^L \wedge S^{\bar{L}} \longrightarrow S^{L\oplus\bar{L}}$ is the collapse map for the inclusion $D(L \oplus \bar{L}) \longrightarrow D(L) \times D(\bar{L})$. A rescaling homotopy connects Ψ to the canonical homeomorphism $S^L \wedge S^{\bar{L}} \cong S^{L\oplus\bar{L}}$, so the following square commutes up to K^2-equivariant based homotopy:

$$\begin{tikzcd} G_+ \arrow[r, "l^G_H"] \arrow[d, "l^G_K"'] & S^L \wedge H_+ \arrow[d, "S^L \wedge l^H_K"] \\ S^{L(K,G)} \wedge K_+ \arrow[r, "(3.2.28)^{-1}", "\cong"'] & S^{L\oplus\bar{L}} \wedge K_+ \end{tikzcd}$$

We can thus conclude that the map l^G_K/K is K-equivariantly homotopic to the composite

$$\begin{aligned} G/K_+ \ &\cong\ G \ltimes_H (H/K_+) \xrightarrow{l^G_H \wedge_H (H/K_+)} (H/K_+) \wedge S^L \\ &\cong\ H \ltimes_K S^L \xrightarrow{l^H_K \wedge_K S^L} S^L \wedge S^{\bar{L}} \ \cong_{(3.2.28)}\ S^{L(K,G)} . \end{aligned}$$

Naturality and transitivity of restriction maps then show that the following

diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X \wedge G/K_+) \arrow[rr, "\mathrm{res}^G_K"] \arrow[d, "\cong"] & & \pi_k^K(X \wedge G/K_+) \arrow[ddd, "l^G_K/K"] \\
\pi_k^G(X \wedge G \ltimes_H (H/K_+)) \arrow[d, "\mathrm{res}^G_H"] \arrow[dd, bend right, "\mathrm{Wirth}^G_H"'] & & \\
\pi_k^H(X \wedge G \ltimes_H (H/K_+)) \arrow[d, "l^G_H \wedge_H (H/K_+)"] & & \\
\pi_k^H(X \wedge (H/K_+) \wedge S^L) \arrow[d, "\cong"] & & \pi_k^K(X \wedge S^{L(K,G)}) \arrow[d, "\cong"] \\
\pi_k^H(X \wedge H \ltimes_K S^L)) \arrow[r, "\mathrm{res}^H_K"] \arrow[rr, bend right, "\mathrm{Wirth}^H_K"'] & \pi_k^K(X \wedge H \ltimes_K S^L) \arrow[r, "l^H_K \wedge_K S^L"] & \pi_k^K(X \wedge S^L \wedge S^{\bar{L}})
\end{tikzcd}$$

By Theorem 3.2.15 the Wirthmüller map is inverse to the composite

$$\pi_k^K(X \wedge S^{L(K,G)}) \xrightarrow{\varepsilon_{L(K,G)}} \pi_k^K(X \wedge S^{L(K,G)}) \xrightarrow{G \ltimes_K -} \pi_k^G(G \ltimes_K X)\ .$$

The map $\varepsilon_{L(K,G)}$ is induced by the 'negative' map of $S^{L(K,G)}$. Under the homeomorphism between $S^{L(K,G)}$ and $S^L \wedge S^{\bar{L}}$, this becomes the smash product of the 'negative' maps of L and $\bar{L}$. So reading the diagram backwards gives a commutative diagram of external transfers

$$\begin{tikzcd}
\pi_k^K(X \wedge S^L \wedge S^{\bar{L}}) \arrow[r, "H \ltimes_K -"] \arrow[dd, "\cong"] & \pi_k^H(X \wedge H \ltimes_K S^L) \arrow[r, "\cong"] & \pi_k^H(X \wedge H/K_+ \wedge S^L) \arrow[d, "G \ltimes_H -"] \\
& & \pi_k^G(X \wedge G \ltimes_H H/K_+) \arrow[d, "\cong"] \\
\pi_k^K(X \wedge S^{L(K,G)}) \arrow[rr, "G \ltimes_K -"'] & & \pi_k^G(X \wedge G/K_+)
\end{tikzcd}$$

(3.2.30)

Post-Composing with the effect of the projection $G/K \longrightarrow *$ and exploiting naturality gives the claim about the dimension shifting transfer. The second claim follows by pre-composing with the inclusion of the origin of $L(K, G)$. □

Example 3.2.31. We let $K \leq H \leq G$ be nested closed subgroups and X an orthogonal G-spectrum. We let $p : G/K \longrightarrow G/H$ denote the projection. For later reference we show that under the external transfer isomorphisms the effect of the morphism $X \wedge p_+ : X \wedge G/K_+ \longrightarrow X \wedge G/H_+$ corresponds to the transfer

from K to H. For simplicity we restrict to the case where $\dim(K) = \dim(H)$, i.e., when K has finite index in H; the general case only differs by more complicated notation. If K has finite index in H, then the differential of the projection $G/K \longrightarrow G/H$ is an isomorphism from the K-representation $T_{eK}(G/K)$ to the underlying K-representation of $L = T_{eH}(G/H)$. We identify these two representations via this isomorphism. We claim that then the following square commutes:

$$\begin{tikzcd}
\pi_*^K(X \wedge S^L) \arrow[r, "\mathrm{tr}_K^H"] \arrow[d, "G\ltimes_K -"', "\cong"] & \pi_*^H(X \wedge S^L) \arrow[d, "\cong"', "G\ltimes_H -"] \\
\pi_*^G(X \wedge G/K_+) \arrow[r, "(X\wedge p_+)_*"'] & \pi_*^G(X \wedge G/H_+)
\end{tikzcd}$$

To see this we compose the commutative diagram (3.2.30) in the proof of Proposition 3.2.29 that encodes the transitivity of external transfers with the map $\pi_*^G(X \wedge p_+)$ and arrive at another commutative diagram:

$$\begin{tikzcd}
\pi_k^K(X \wedge S^L) \arrow[rr, equal] \arrow[d, "H\ltimes_K -"] \arrow[ddd, bend right=60, "\mathrm{tr}_K^H"'] & & \pi_k^K(X \wedge S^L) \arrow[dd, "G\ltimes_K -"] \\
\pi_k^H(X \wedge H \ltimes_K S^L) \arrow[d, "\cong"] & & \\
\pi_k^H(X \wedge H/K_+ \wedge S^L) \arrow[r, "G\ltimes_H -"] \arrow[d] & \pi_k^G(X \wedge G \ltimes_H H/K_+) \arrow[r, "\cong"] & \pi_k^G(X \wedge G/K_+) \arrow[d, "(X\wedge p_+)_*"] \\
\pi_k^H(X \wedge S^L) \arrow[rr, "G\ltimes_H -"'] & & \pi_k^G(X \wedge G/H_+)
\end{tikzcd}$$

The lower left vertical map is induced by the projection $H/K \longrightarrow *$ and the bottom part of the diagram commutes by naturality of the external transfer.

Now we prove the compatibility of transfers with inflations, i.e., restriction along continuous epimorphisms $\alpha : K \longrightarrow G$. For every closed subgroup H of G, the map

$$\bar{\alpha} \ : \ K/J \ \longrightarrow \ G/H \ , \quad kJ \ \longmapsto \ \alpha(k)H$$

is a diffeomorphism, where $J = \alpha^{-1}(H)$. The differential at the coset eJ is an isomorphism

$$(d\bar{\alpha})_{eJ} \ : \ \bar{L} = T_{eJ}(K/J) \ \longrightarrow \ (\alpha|_J)^*(T_{eH}(G/H)) \ = \ (\alpha|_J)^*(L)$$

of J-representations. In the statement and proof of the following proposition

a couple of unnamed isomorphisms occur. One of them is the natural isomorphism of K-spaces

$$K \ltimes_J (\alpha|_J)^*(A) \;\cong\; \alpha^*(G \ltimes_H A)\,, \quad [k,a] \;\longmapsto\; [\alpha(k),a]\,.$$

Proposition 3.2.32. *Let K and G be compact Lie groups and $\alpha : K \longrightarrow G$ a continuous epimorphism. Let H be a closed subgroup of G, set $J = \alpha^{-1}(H)$, and let $\alpha|_J : J \longrightarrow H$ denote the restriction of α.*

(i) *For every orthogonal H-spectrum Y the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^H(Y \wedge S^L) \arrow[r, "(\alpha|_J)^*"] \arrow[d, "G\ltimes_H -"'] & \pi_k^J((\alpha|_J)^*(Y \wedge S^L)) \arrow[r, "\cong"] & \pi_k^J((\alpha|_J)^*(Y) \wedge S^{\bar L}) \arrow[d, "K\ltimes_J -"] \\
\pi_k^G(G \ltimes_H Y) \arrow[r, "\alpha^*"'] & \pi_k^K(\alpha^*(G \ltimes_H Y)) \arrow[r, "\cong"'] & \pi_k^K(K \ltimes_J (\alpha|_J)^*(Y))
\end{tikzcd}$$

(ii) *For every orthogonal G-spectrum X the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^H(X \wedge S^L) \arrow[r, "(\alpha|_J)^*"] \arrow[d, "\mathrm{Tr}_H^G"'] & \pi_k^J((\alpha|_J)^*(X \wedge S^L)) \arrow[r, "\cong"] & \pi_k^J((\alpha|_J)^*(X) \wedge S^{\bar L}) \arrow[d, "\mathrm{Tr}_J^K"] \\
\pi_k^G(X) \arrow[rr, "\alpha^*"'] & & \pi_k^K(\alpha^*(X))
\end{tikzcd}$$

Moreover, the degree zero transfers satisfy the relation

$$\alpha^* \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_J^K \circ (\alpha|_J)^*$$

as maps $\pi_k^H(X) \longrightarrow \pi_k^K(\alpha^*(X))$.

(iii) *For every orthogonal G-spectrum X, every $g \in G$ and all closed subgroups $K \le H$ of G the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^{K^g}(X) \arrow[r, "g_\star"] \arrow[d, "\mathrm{tr}_{K^g}^{H^g}"'] & \pi_k^K(X) \arrow[d, "\mathrm{tr}_K^H"] \\
\pi_k^{H^g}(X) \arrow[r, "g_\star"'] & \pi_k^H(X)
\end{tikzcd}$$

Proof (i) The restriction maps commute with the loop and suspension isomorphisms, and so do the external transfer maps (by Proposition 3.2.27). So it suffices to prove the claim in dimension $k = 0$. We choose a wide G-equivariant embedding $i : G/H \longrightarrow V$ into some G-representation; from this input data we form the collapse map

$$c_{G/H} \;:\; S^V \;\longrightarrow\; G \ltimes_H S^W$$

and the external transfer $G \ltimes_H -$. Then the composite

$$\alpha^*(i) \circ \bar{\alpha} \ : \ K/J \longrightarrow \alpha^*(G/H) \longrightarrow \alpha^*(V) \tag{3.2.33}$$

is a wide K-equivariant embedding, and we can (and will) base the external transfer $K \ltimes_J -$ on this embedding. Since $i : G/H \longrightarrow V$ and $\alpha^*(i) \circ \bar{\alpha}$ have the same image, they define the same decomposition of V into tangent and normal subspaces, i.e.,

$$\bar{W} \ = \ \alpha^*(V) - d(\alpha^*(i) \circ \bar{\alpha})_{eJ}(\bar{L}) \ = \ (\alpha|_J)^*(W) \ .$$

Moreover, the composite

$$S^{\alpha^*(V)} \ = \ \alpha^*(S^V) \xrightarrow{\alpha^*(c_{G/H})} \alpha^*(G \ltimes_H S^W) \ \cong \ K \ltimes_J (\alpha|_J)^*(S^W) \ = \ K \ltimes_J S^{\bar{W}}$$

is precisely the collapse map based on the wide embedding (3.2.33). From here the commutativity of the square is straightforward from the definitions.

(ii) The dimension shifting transfer Tr_H^G is defined as the composite of the external transfer $G \ltimes_H -$ and the effect of the action map $G \ltimes_H X \longrightarrow X$, and similarly for Tr_J^K. The action map for the orthogonal K-spectrum $K \ltimes_J \alpha^*(X)$ coincides with the composite

$$K \ltimes_J \alpha^*(X) \xrightarrow{\cong} \alpha^*(G \ltimes_H X) \xrightarrow{\alpha^*(\mathrm{act})} \alpha^*(X) \ .$$

So the following diagram commutes by naturality of the restriction map:

$$\begin{tikzcd}
\pi_k^G(G \ltimes_H X) \arrow[r, "\alpha^*"] \arrow[d, "\pi_k^G(\mathrm{act})"'] & \pi_k^K(\alpha^*(G \ltimes_H X)) \arrow[r, "\cong"] \arrow[dr, "\pi_k^K(\alpha^*(\mathrm{act}))"'] & \pi_k^K(K \ltimes_J \alpha^*(X)) \arrow[d, "\pi_k^K(\mathrm{act})"] \\
\pi_k^G(X) \arrow[rr, "\alpha^*"'] & & \pi_k^K(\alpha^*(X))
\end{tikzcd}$$

Part (ii) then follows by stacking this commutative diagram to the one of part (i). The second claim follows by pre-composing with the inclusion of the origin of L.

Part (iii) follows from part (ii) for the epimorphism $c_g : H \longrightarrow H^g$ and the closed subgroup K of H, and naturality of the transfer:

$$\begin{aligned} g_\star \circ \mathrm{tr}_{K^g}^{H^g} \ &= \ (l_g^X)_* \circ (c_g)^* \circ \mathrm{tr}_{K^g}^{H^g} \\ \overset{\text{(ii)}}{=} \ & (l_g^X)_* \circ \mathrm{tr}_K^H \circ (c_g)^* \ = \ \mathrm{tr}_K^H \circ (l_g^X)_* \circ (c_g)^* \ = \ \mathrm{tr}_K^H \circ g_\star \end{aligned} \qquad \square$$

Now we know how transfers compose and interact with inflations. The remaining compatibility issue is to rewrite the composite of a transfer map followed by a restriction map. The answer is given by the *double coset formula* that we will prove in Theorem 3.4.9 below.

3.3 Geometric fixed-points

In this section we study the *geometric fixed-point homotopy groups* $\Phi_*^G(X)$ of an orthogonal G-spectrum X. We establish the isotropy separation sequence (3.3.9) that is often useful for inductive arguments, and we prove that equivariant equivalences can also be detected by geometric fixed-points, see Proposition 3.3.10. In Proposition 3.3.11 we show that geometric fixed-points annihilate transfers from proper subgroups. Theorem 3.3.15 provides a functorial description of the 0th equivariant stable homotopy group of a G-space Y in terms of the path components of the H-fixed-points spaces Y^H for closed subgroups H of G with finite Weyl group.

We define the geometric fixed-point homotopy groups of an orthogonal G-spectrum X. As before we let $s(\mathcal{U}_G)$ denote the set of finite-dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$, considered as a poset under inclusion. We obtain a functor from $s(\mathcal{U}_G)$ to sets by

$$V \longmapsto [S^{V^G}, X(V)^G] ,$$

the set of (non-equivariant) homotopy classes of based maps from the fixed-point sphere S^{V^G} to the fixed-point space $X(V)^G$. An inclusion $V \subseteq W$ in $s(\mathcal{U}_G)$ is sent to the map

$$[S^{V^G}, X(V)^G] \longrightarrow [S^{W^G}, X(W)^G]$$

that takes the homotopy class of $f : S^{V^G} \longrightarrow X(V)^G$ to the homotopy class of the composite

$$\begin{aligned} S^{W^G} \cong S^{(V^\perp)^G} \wedge S^{V^G} &\xrightarrow{\mathrm{Id}\wedge f} S^{(V^\perp)^G} \wedge X(V)^G \\ &= (S^{V^\perp} \wedge X(V))^G \xrightarrow{(\sigma_{V^\perp,V})^G} X(V^\perp \oplus V)^G = X(W)^G . \end{aligned}$$

Definition 3.3.1. Let G be a compact Lie group and X an orthogonal G-spectrum. The 0th *geometric fixed-point homotopy group* is defined as

$$\Phi_0^G(X) = \mathrm{colim}_{V\in s(\mathcal{U}_G)} [S^{V^G}, X(V)^G] .$$

If k is an arbitrary integer, we define the kth geometric fixed-point homotopy group $\Phi_k^G(X)$ by replacing S^V by $S^{V\oplus\mathbb{R}^k}$ (for $k > 0$) or replacing $X(V)$ by $X(V\oplus \mathbb{R}^{-k})$ (for $k < 0$), analogous to the definition of $\pi_k^G(X)$ in (3.1.11).

The construction comes with a *geometric fixed-point map*

$$\begin{aligned} \Phi^G : \pi_0^G(X) &\longrightarrow \Phi_0^G(X) \\ [f : S^V \longrightarrow X(V)] &\longmapsto [f^G : S^{V^G} \longrightarrow X(V)^G] \end{aligned} \tag{3.3.2}$$

from the G-equivariant homotopy group to the geometric fixed-point homotopy group. For a trivial group, equivariant and geometric fixed-point groups coincide and the geometric fixed-point map $\Phi^e : \pi_0^e(X) \longrightarrow \Phi_0^e(X)$ is the identity.

Example 3.3.3 (Geometric fixed-points of suspension spectra)**.** If A is any based G-space, then the geometric fixed-points $\Phi_*^G(\Sigma^\infty A)$ of the suspension spectrum are given by

$$\Phi_k^G(\Sigma^\infty A) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\, [S^{V^G\oplus\mathbb{R}^k}, S^{V^G}\wedge A^G]\ .$$

As V ranges over $s(\mathcal{U}_G)$ the dimension of the fixed-points grows to infinity. So the composite

$$\pi_k^e(\Sigma^\infty A^G) \xrightarrow{p_G^*} \pi_k^G(\Sigma^\infty A^G) \xrightarrow{\mathrm{incl}_*} \pi_k^G(\Sigma^\infty A) \xrightarrow{\Phi^G} \Phi_k^G(\Sigma^\infty A)$$

is an isomorphism, where p_G^* is inflation along the unique group homomorphism $p_G : G \longrightarrow e$. We will sometimes refer to this isomorphism by saying that 'geometric fixed-points commute with suspension spectra'.

Construction 3.3.4. We let X be a G-orthogonal spectrum and $\alpha : K \longrightarrow G$ a continuous epimorphism. We define *inflation maps*

$$\alpha^* \ : \ \Phi_0^G(X) \ \longrightarrow \ \Phi_0^K(\alpha^* X)$$

on geometric fixed-point homotopy groups. We choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ of the restriction along α of the complete G-universe into the complete K-universe. We let $f : S^{V^G} \longrightarrow X(V)^G$ be a based map representing an element of $\Phi_0^G(X)$, for some $V \in s(\mathcal{U}_G)$. Since α is surjective, $V^G = (\alpha^* V)^K$ and $X(V)^G = (\alpha^*(X(V)))^K = ((\alpha^* X)(\alpha^* V))^K$. We use ψ to identify $\alpha^* V$ with $\psi(V)$ as K-representations, and hence also $(\alpha^* V)^K$ with $\psi(V)^K$. This turns f into a based map

$$S^{\psi(V)^K} \ \cong \ S^{(\alpha^* V)^K} \ = \ S^{V^G} \xrightarrow{\ f\ } X(V)^G \ = \ ((\alpha^* X)(\alpha^* V))^K \ \cong \ ((\alpha^* X)(\psi(V)))^K\ .$$

Any two equivariant embeddings of $\alpha^*(\mathcal{U}_G)$ into $\mathcal{U}_K$ are homotopic through K-equivariant linear isometric embeddings, so the restriction map is independent of the choice of ψ. This latter map represents the element $\alpha^*[f]$ in $\Phi_0^K(\alpha^* X)$. The element $\alpha^*[f]$ depends only on the class of f in $\Phi_0^G(X)$, so the inflation map α^* is a well-defined homomorphism.

The surjectivity of α is essential to obtain an inflation map α^* on geometric fixed-point homotopy groups, and geometric fixed-points do not have natural restriction maps to subgroups. These inflation maps between the geometric fixed-point homotopy groups are clearly natural in the orthogonal G-spectrum. The next proposition lists the other naturality properties.

Proposition 3.3.5. *Let G be a compact Lie group and X an orthogonal G-spectrum.*

(i) *For every pair of composable continuous epimorphisms $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$ we have*

$$\beta^* \circ \alpha^* \ = \ (\alpha\beta)^* \ : \ \Phi_0^G(X) \ \longrightarrow \ \Phi_0^L((\alpha\beta)^* X)\ .$$

(ii) *For every element $g \in G$ the composite*

$$\Phi_0^G(X) \ \xrightarrow{(c_g)^*} \Phi_0^G(c_g^* X) \ \xrightarrow{(l_g^X)_*} \ \Phi_0^G(X)$$

is the identity.

(iii) *For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the following square commutes:*

$$\begin{tikzcd} \pi_0^G(X) \arrow[r, "\Phi^G"] \arrow[d, "\alpha^*"'] & \Phi_0^G(X) \arrow[d, "\alpha^*"] \\ \pi_0^K(\alpha^* X) \arrow[r, "\Phi^K"'] & \Phi_0^K(\alpha^* X) \end{tikzcd}$$

Proof (i) We choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ and an L-equivariant linear isometric embedding $\varphi : \beta^*(\mathcal{U}_K) \longrightarrow \mathcal{U}_L$. If we then use the L-equivariant linear isometric embedding

$$\varphi \circ \beta^*(\psi) \ : \ (\alpha \circ \beta)^*(\mathcal{U}_G) \ \longrightarrow \ \mathcal{U}_L$$

for the calculation of $(\alpha \circ \beta)^*$, the desired equality even holds on the level of representatives.

(ii) We let V be a finite-dimensional G-subrepresentation of $\mathcal{U}_G$ and $f : S^{V^G} \longrightarrow X(V)^G$ a based map representing an element of $\Phi_0^G(X)$. We use the G-equivariant linear isometry $l_g : c_g^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_G$ given by left multiplication by g. Then $c_g^*(V)$ and $l_g(V)$ have the same underlying sets and the restriction of l_g to $(c_g^*(V))^G$ is the identity onto $(l_g(V))^G$.

The class $c_g^*[f]$ is represented by the composite

$$S^{(l_g(V))^G} = S^{(c_g^* V)^G} = S^{V^G} \ \xrightarrow{f} \ X(V)^G = ((c_g^* X)(c_g^* V))^G \\ \xrightarrow{((c_g^* X)(l_g^V))^G} \ ((c_g^* X)(l_g(V)))^G\ .$$

Consequently, $(l_g^X)_*(c_g^*[f])$ is represented by the composite

$$S^{(l_g(V))^G} \ = \ S^{V^G} \ \xrightarrow{f} \ X(V)^G \ = \ ((c_g^* X)(c_g^* V))^G \\ \xrightarrow{((c_g^* X)(l_g^V))^G} \ ((c_g^* X)(l_g(V)))^G \ \xrightarrow{((l_g^X)(l_g(V)))^G} \ (X(l_g(V)))^G\ .$$

The G-action on $X(V)$ is diagonally, from the external G-action on X and the internal G-action on V. Hence the map $l_g^{X(V)} : c_g^*(X(V)) \longrightarrow X(V)$ is the composite of $(c_g^*X)(l_g^V) : (c_g^*X)(c_g^*V) \longrightarrow (c_g^*X)(V)$ and $(l_g^X)(V) : (c_g^*X)(V) \longrightarrow X(V)$. Since the restriction of $l_g^{X(V)}$ to the G-fixed-points is the identity, the composite of $((c_g^*X)(l_g^V))^G$ and $((l_g^X)(V))^G$ is the identity. This shows that $(l_g^X)_*(c_g^*[f])$ is again represented by f, and hence $(l_g^X)_* \circ (c_g)^* = \mathrm{Id}$.

(iii) We consider a based continuous G-map $f : S^V \longrightarrow X(V)$; then $\alpha^*[f]$ is represented by the K-map

$$\alpha^*(f) \ : \ S^{\alpha^*(V)} \ = \ \alpha^*(S^V) \ \longrightarrow \ \alpha^*(X(V)) \ = \ (\alpha^*X)(\alpha^*(V)) ,$$

and so $\Phi^K(\alpha^*[f])$ is represented by

$$(\alpha^*(f))^K \ : \ (S^{\alpha^*(V)})^K \ = \ (\alpha^*(S^V))^K \ \longrightarrow \ (\alpha^*(X(V)))^K \ = \ ((\alpha^*X)(\alpha^*(V)))^K .$$

Since α is surjective, this is the same map as

$$f^G \ : \ (S^V)^G \ \longrightarrow \ X(V)^G .$$

To calculate $\alpha^*(\Phi^G([f]))$ we choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ and conjugate f^G by the isometry

$$(\psi|_{\alpha^*(V)})^K \ : \ (\alpha^*(V))^K \ \cong \ \psi(V)^K .$$

But conjugation by an isometry does not change the stable homotopy class, by Proposition 3.1.14 (ii) for the trivial group. So

$$\Phi^K(\alpha^*[f]) \ = \ [\alpha(f)^K] \ = \ [f^G] \ = \ \alpha^*(\Phi^G[f]) . \qquad \square$$

Remark 3.3.6 (Weyl group action on geometric fixed-points)**.** We let H be a closed subgroup of a compact Lie group G, and X an orthogonal G-spectrum. Every $g \in G$ gives rise to a conjugation homomorphism $c_g : H \longrightarrow H^g$ by $c_g(h) = g^{-1}hg$. Moreover, left translation by g is a homomorphism of orthogonal H-spectra $l_g^X : c_g^*(X) \longrightarrow X$. So combining inflation along c_g with the effect of l_g^X gives a homomorphism

$$g_\star \ : \ \Phi_0^{H^g}(X) \ \xrightarrow{(c_g)^*} \ \Phi_0^H(c_g^*(X)) \ \xrightarrow{(l_g^X)_*} \ \Phi_0^H(X) .$$

In the special case when g normalizes H, this is a self-map of the geometric fixed-point group $\Phi_0^H(X)$. If moreover g belongs to H, then $g_\star$ is the identity by Proposition 3.3.5 (ii). So the maps $g_\star$ define an action of the Weyl group $W_GH = N_GH/H$ on the geometric fixed-point homotopy group $\Phi_0^H(X)$. By the same arguments as for equivariant homotopy groups in Remark 3.1.17, the identity path component of the Weyl group acts trivially, so the action factors over an action of the discrete group $\pi_0(W_GH) = W_GH/(W_GH)^\circ$. Since the geometric fixed-point map $\Phi : \pi_0^H(X) \longrightarrow \Phi_0^H(X)$ is compatible with inflation and natural in X, this map is $\pi_0(W_GH)$-equivariant.

Now we recall the interpretation of geometric fixed-point homotopy groups as the equivariant homotopy groups of the smash product of X with a certain universal G-space. We denote by $\mathcal{P}_G$ the family of proper closed subgroups of G, and by $E\mathcal{P}_G$ a universal space for the family $\mathcal{P}_G$. So $E\mathcal{P}_G$ is a cofibrant G-space without G-fixed-points and such that the fixed-point space $(E\mathcal{P}_G)^H$ is contractible for every closed proper subgroup H of G. These properties determine $E\mathcal{P}_G$ uniquely up to G-homotopy equivalence, see Proposition B.11.

We denote by $\tilde{E}\mathcal{P}_G$ the reduced mapping cone of the based G-map $(E\mathcal{P}_G)_+ \longrightarrow S^0$ that sends $E\mathcal{P}_G$ to the non-basepoint of S^0. So $\tilde{E}\mathcal{P}_G$ is the unreduced suspension of the universal space $E\mathcal{P}_G$. The G-fixed-points of $E\mathcal{P}_G$ are empty and fixed-points commute with mapping cones, so the map $S^0 \longrightarrow (\tilde{E}\mathcal{P}_G)^G$ is an isomorphism. For all proper subgroups H of G the map $(E\mathcal{P}_G)_+^H \longrightarrow (S^0)^H = S^0$ is a weak equivalence, so $(\tilde{E}\mathcal{P}_G)^H$ is contractible.

Example 3.3.7. We let $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$ be the orthogonal complement of the G-fixed-points in the complete G-universe $\mathcal{U}_G$. We claim that the unit sphere $S(\mathcal{U}_G^\perp)$ of this complement is a universal space $E\mathcal{P}_G$. Indeed, the unit sphere $S(\mathcal{U}_G^\perp)$ is G-equivariantly homeomorphic to the space $\mathbf{L}(\mathbb{R}, \mathcal{U}_G^\perp)$, so it is cofibrant as a G-space by Proposition 1.1.19 (ii). Since $S(\mathcal{U}_G^\perp)$ has no G-fixed-points, any stabilizer group is a proper subgroup of G, i.e., in the family $\mathcal{P}_G$. On the other hand, for every proper subgroup H of G there is a G-representation V with $V^G = 0$ but $V^H \neq 0$, see for example [28, III Prop. 4.2]. Since $\mathcal{U}_G^\perp$ contains infinitely many isomorphic copies of V, the H-fixed-points

$$(S(\mathcal{U}_G^\perp))^H = S((\mathcal{U}_G^\perp)^H)$$

form an infinite-dimensional sphere, and hence are contractible. So $S(\mathcal{U}_G^\perp)$ is a universal G-space for the family of proper subgroups.

Since $\tilde{E}\mathcal{P}_G$ is an unreduced suspension of $E\mathcal{P}_G$, it is equivariantly homeomorphic to

$$S(\mathbb{R} \oplus \mathcal{U}_G^\perp),$$

the unit sphere in $\mathbb{R} \oplus \mathcal{U}_G^\perp$. So $S(\mathbb{R} \oplus \mathcal{U}_G^\perp)$ is a model for $\tilde{E}\mathcal{P}_G$.

The inclusion $i : S^0 \longrightarrow \tilde{E}\mathcal{P}_G$ induces an isomorphism of G-fixed-points $S^0 \cong (\tilde{E}\mathcal{P}_G)^G$. So for every based G-space A the map $A \wedge i : A \longrightarrow A \wedge \tilde{E}\mathcal{P}_G$ induces an isomorphism of G-fixed-points. Hence also for every orthogonal G-spectrum the induced map of geometric fixed-point homotopy groups

$$\Phi_k^G(X \wedge i) \ : \ \Phi_k^G(X) \ \cong \ \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$$

is an isomorphism. If we compose the inverse with the geometric fixed-point homomorphism (3.3.2), we arrive at a homomorphism $\Phi : \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \longrightarrow \Phi_k^G(X)$.

Proposition 3.3.8. *For every orthogonal G-spectrum X and every integer k, the geometric fixed-point map*

$$\Phi \ : \ \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \ \longrightarrow \ \Phi_k^G(X)$$

is an isomorphism.

Proof We claim that for every finite based G-CW-complex A and every based G-space Y the map

$$(-)^G \ : \ \mathrm{map}_*^G(A, Y \wedge \tilde{E}\mathcal{P}_G) \ \longrightarrow \ \mathrm{map}_*(A^G, Y^G)$$

that takes a G-map $f : A \longrightarrow Y \wedge \tilde{E}\mathcal{P}_G$ to the induced map on G-fixed-points

$$f^G \ : \ A^G \ \longrightarrow \ (Y \wedge \tilde{E}\mathcal{P}_G)^G = Y^G \wedge (\tilde{E}\mathcal{P}_G)^G \cong Y^G$$

is a weak equivalence and Serre fibration.

Indeed, since A is a G-CW-complex, the inclusion of fixed-points $A^G \longrightarrow A$ is a G-cofibration and induces a Serre fibration of equivariant mapping spaces

$$\mathrm{map}_*^G(A, Y \wedge \tilde{E}\mathcal{P}_G) \ \longrightarrow \ \mathrm{map}_*^G(A^G, Y \wedge \tilde{E}\mathcal{P}_G) \ .$$

Since every G-map from A^G lands in the G-fixed-points of $Y \wedge \tilde{E}\mathcal{P}_G$ and because $(Y \wedge \tilde{E}\mathcal{P}_G)^G = Y^G$, the target space is the non-equivariant mapping space $\mathrm{map}_*(A^G, Y^G)$. The G-space A is built from its fixed-points by attaching G-cells $G/H \times D^n$ whose isotropy H is a proper subgroup. Since the H-fixed-points of $Y \wedge \tilde{E}\mathcal{P}_G$ are contractible for all proper subgroups H of G, the fibration is also a weak equivalence.

Now we consider a finite-dimensional G-representation V. When applied to $A = S^V$ and $Y = X(V)$, the previous claim implies that the fixed-point map

$$[S^V, X(V) \wedge \tilde{E}\mathcal{P}_G]^G \ \longrightarrow \ [S^{V^G}, X(V)^G]$$

is bijective for every G-representation V. Passing to colimits over the poset $s(\mathcal{U}_G)$ proves the result for $k = 0$. The argument in the other dimensions is similar, and we leave it to the reader. □

A consequence of the previous proposition is the following *isotropy separation sequence*. The mapping cone sequence of based G-spaces

$$(E\mathcal{P}_G)_+ \ \longrightarrow \ S^0 \ \longrightarrow \ \tilde{E}\mathcal{P}_G$$

becomes a mapping cone sequence of G-spectra

$$X \wedge (E\mathcal{P}_G)_+ \ \longrightarrow \ X \ \longrightarrow \ X \wedge \tilde{E}\mathcal{P}_G$$

after smashing with any given orthogonal G-spectrum X. So taking equivariant

homotopy groups gives a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \pi_k^G(X \wedge (E\mathcal{P}_G)_+) \longrightarrow \pi_k^G(X) \xrightarrow{\Phi} \\ \Phi_k^G(X) \longrightarrow \pi_{k-1}^G(X \wedge (E\mathcal{P}_G)_+) \longrightarrow \cdots \end{aligned} \tag{3.3.9}$$

where we exploited the identification of Proposition 3.3.8.

Proposition 3.3.10. *Let G be a compact Lie group. For a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the following are equivalent:*

(i) *The morphism f is a $\underline{\pi}_*$-isomorphism.*
(ii) *For every closed subgroup H of G and every integer k the map $\Phi_k^H(f)$ of geometric H-fixed-point homotopy groups is an isomorphism.*

Proof (i)$\Longrightarrow$(ii) If f is is an equivalence of orthogonal G-spectra, then so is $f \wedge \tilde{E}\mathcal{P}_G$ by Proposition 3.2.19 (ii). Proposition 3.3.8 then implies that $\Phi_k^H(f) : \Phi_k^H(X) \longrightarrow \Phi_k^H(Y)$ is an isomorphism for all k.

(ii)$\Longrightarrow$(i) We argue by induction on the size of the group G (i.e., of the dimension of G and the order of $\pi_0 G$). If G is the trivial group, then the geometric fixed-point map $\Phi : \pi_k^e(X) \longrightarrow \Phi_k^e(X)$ does not do anything, and is an isomorphism. Since $\Phi_*^e(f)$ is an isomorphism, so is $\pi_*^e(f)$.

If G is a non-trivial group we know by induction hypothesis that f is an equivalence of orthogonal H-spectra for every proper closed subgroup H of G. Since $E\mathcal{P}_G$ is a cofibrant G-space without G-fixed-points, Proposition 3.2.19 (i) lets us conclude that $f \wedge (E\mathcal{P}_G)_+$ is an equivalence of orthogonal G-spectra. Since $\Phi_*^G(f) : \Phi_*^G(X) \longrightarrow \Phi_*^G(Y)$ is also an isomorphism, the isotropy separation sequence and the five lemma let us conclude that $\pi_*^G(f) : \pi_*^G(X) \longrightarrow \pi_*^G(Y)$ is an isomorphism. □

The next proposition shows that 'geometric fixed-points vanish on transfers'. In fact, it is often a helpful slogan to think of geometric fixed-points as 'dividing out transfers from proper subgroups' – despite the fact that the kernel of the geometric fixed-point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ is in general larger than the subgroup generated by proper transfers. For finite groups G, the slogan is in fact true *up to torsion*, i.e., the geometric fixed-point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ is rationally surjective and its kernel is rationally generated by transfers from proper subgroups, compare Proposition 3.4.26 below. A more general version of part (iii) below will appear in Proposition 3.4.2 (ii).

Proposition 3.3.11. *Let K be a closed subgroup of a compact Lie group G and X an orthogonal G-spectrum.*

(i) *Let H be a closed subgroup of G such that K is not subconjugate to H.*

Then the composite

$$\pi_0^G(X) \xrightarrow{\mathrm{res}^G_K} \pi_0^K(X) \xrightarrow{\Phi^K} \Phi_0^K(X)$$

annihilates the images of the dimension shifting transfer $\mathrm{Tr}_H^G : \pi_0^H(X \wedge S^L) \longrightarrow \pi_0^G(X)$ *and of the degree zero transfer* $\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$.

(ii) *The geometric fixed-point map* $\Phi^G : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ *annihilates the images of the dimension shifting transfer and of the degree zero transfer from all proper closed subgroups of* G.

(iii) *If the Weyl group* $W_G K$ *is finite, then the relation*

$$\Phi^K \circ \mathrm{res}^G_K \circ \mathrm{tr}^G_K \ = \ \sum_{gK \in W_G K} \Phi^K \circ g_\star$$

holds as natural transformations from $\pi_0^K(X)$ *to* $\Phi_0^K(X)$.

Proof (i) Let V be any G-representation. The G-space $(G \ltimes_H X)(V)$ is isomorphic to $G \ltimes_H X(i^* V)$. If K is not subconjugate to H, then both G-spaces have only one K-fixed-point, the base point. So the geometric fixed-point homotopy group $\Phi_0^K(G \ltimes_H X)$ vanishes. The dimension shifting transfer is defined as the composite

$$\pi_0^H(X \wedge S^L) \xrightarrow{G \ltimes_H -} \pi_0^G(G \ltimes_H X) \xrightarrow{\mathrm{act}_*} \pi_0^G(X) \ .$$

The geometric fixed-point map is natural for G-maps, so the composite $\Phi^K \circ \mathrm{res}^G_K \circ \mathrm{act}_* : \pi_0^G(G \ltimes_H X) \longrightarrow \Phi_0^K(X)$ factors through the trivial group $\Phi_0^K(G \ltimes_H X)$. Thus the dimension shifting transfer vanishes. The degree 0 transfer factors through the dimension shifting transfer, so it vanishes as well. Part (ii) is the special case of (i) for $K = G$.

(iii) Since the functor π_0^K is represented by the suspension spectrum of G/K (in the sense of Proposition 3.1.46), is suffices to check the relation for the orthogonal G-spectrum $\Sigma^\infty_+ G/K$ and the tautological class e_K defined in (3.1.45).

For every $g \in N_G K$ and every class $x \in \pi_0^K(X)$ we have

$$g_\star(\Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(x)))) \ = \ \Phi^K(\mathrm{res}^G_K(g_\star(\mathrm{tr}^G_K(x)))) \ = \ \Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(x)))$$

because $g_\star$ is the identity on π_0^G. So all classes of the form $\Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(x)))$ are invariant under the action of the Weyl group $W_G K$ specified in Remark 3.3.6. In the universal case this class lives in the group $\Phi_0^K(\Sigma^\infty_+ G/K)$ which is $W_G K$-equivariantly isomorphic to $\pi_0^e(\Sigma^\infty_+ (G/K)^K) = \pi_0^e(\Sigma^\infty_+ W_G K)$ and hence a free module of rank 1 over the integral group ring of the Weyl group $W_G K$, generated by the class $\Phi^K(e_K)$. So the class $\Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(e_K)))$ is an integer multiple of the norm element, i.e.,

$$\Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(e_K))) \ = \ \lambda \cdot \sum_{gK \in W_G K} g_\star(\Phi^K(e_K))$$

for some $\lambda \in \mathbb{Z}$.

It remains to show that $\lambda = 1$. We let $1 \in \pi_0^K(\mathbb{S})$ be the class represented by the identity of S^0. Inspection of Construction 3.2.22 reveals that the transfer $\mathrm{tr}_K^G(1)$ in $\pi_0^G(\mathbb{S})$ is represented by the G-map

$$S^V \xrightarrow{c} G \ltimes_K S^W \xrightarrow{a} S^V$$

where c is the collapse map based on any wide embedding of $i : G/K \longrightarrow V$ into a G-representation, W is the orthogonal complement of the image of $T_{eK}(G/K)$ under the differential of i, and $a[g, w] = gw$. The class $\Phi_0^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(1)))$ is then represented by the restriction to K-fixed-points of the above composite, i.e., by the map

$$S^{V^K} \xrightarrow{c^K} (G \ltimes_K S^W)^K \xrightarrow{a^K} S^{V^K} .$$

Every K-fixed-point of $G \ltimes_K S^W$ is of the form $[g, w]$ with $g \in N_G K$ and $w \in S^{W^K}$, i.e., the map

$$(W_G K)_+ \wedge S^{W^K} \longrightarrow (G \ltimes_K S^W)^K , \quad gK \wedge w \longmapsto [g, w]$$

is a homeomorphism.

Since the Weyl group $W_G K$ is finite we have $(T_{eK}(G/K))^K = 0$. Indeed, the translation action of K on the homogeneous space G/K is smooth, so by the differentiable slice theorem, the K-fixed-point eK has an open K-invariant neighborhood inside G/K that is K-equivariantly diffeomorphic to the tangent space $T_{eK}(G/K)$, compare [131, Thm. 1.6.5] or [26, VI Cor. 2.4]. Since $W_G K = (G/K)^K$ is finite, eK is an isolated K-fixed-point in G/K, and hence 0 is an isolated K-fixed-point in $T_{eK}(G/K)$, i.e., $(T_{eK}(G/K))^K = 0$.

Since $(T_{eK}(G/K))^K = 0$ we have $W^K = V^K$. Under these identifications, the map c^K becomes a pinch map

$$S^{V^K} \longrightarrow (W_G K)_+ \wedge S^{V^K} \cong \bigvee_{gK \in W_G K} S^{V^K} ,$$

i.e., the projection to each wedge summand has degree 1. On the other hand, the map a^K becomes the fold map

$$(W_G K)_+ \wedge S^{V^K} \longrightarrow S^{V^K} .$$

So the degree of the composite $a^K \circ c^K$ is the order of the Weyl group $W_G K$. We have thus shown that

$$\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(1))) = |W_G K| \cdot \Phi^K(1)$$

in the group $\Phi_0^K(\mathbb{S})$. On the other hand, the class 1 is invariant under the action

of the Weyl group, and hence

$$\Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(1))) \;=\; \lambda \cdot \sum_{gK\in W_GK} g_\star(\Phi^K(1)) \;=\; \lambda\cdot |W_GK|\cdot \Phi^K(1)\,.$$

Since the abelian group $\Phi^K_0(\mathbb{S})$ is freely generated by $\Phi^K(1)$, we can compare coefficients in the last two expressions and deduce that $\lambda = 1$. □

The 0th equivariant homotopy groups of equivariant *spectra* have two extra pieces of structure, compared to equivariant *spaces*: an abelian group structure and transfers. Theorem 3.3.15 and Proposition 4.1.11 make precise, first for suspension spectra of G-spaces and then for suspension spectra of orthogonal spaces, that at the level of 0th equivariant homotopy sets, the suspension spectrum 'freely builds in' the extra structure that is available stably.

We introduce specific stabilization maps that relate unstable homotopy sets to stable homotopy groups. We let H be a compact Lie group and Y an H-space. We define a map

$$\sigma^H \;:\; \pi_0(Y^H) \longrightarrow \pi_0^H(\Sigma^\infty_+ Y) \tag{3.3.12}$$

by sending the path component of an H-fixed-point $y\in Y^H$ to the equivariant stable homotopy class $\sigma^H[y]$ represented by the H-map

$$S^0 \xrightarrow{-\wedge y} S^0\wedge Y_+ \;=\; (\Sigma^\infty_+ Y)(0)\,.$$

By direct inspection, the map σ^H can be factored as the composition

$$\pi_0(Y^H) \xrightarrow{\sigma^e} \pi_0^e(\Sigma^\infty_+ Y^H) \xrightarrow{p_H^*} \pi_0^H(\Sigma^\infty_+ Y^H) \xrightarrow{\mathrm{incl}_*} \pi_0^H(\Sigma^\infty_+ Y)\,,$$

where $p_H : H \longrightarrow e$ is the unique group homomorphism.

We recall that for every space Z the non-equivariant stable homotopy group $\pi_0^e(\Sigma^\infty_+ Z)$ is free abelian generated by the classes $\sigma^e(y)$ for all $y\in\pi_0(Z)$, i.e,

$$\pi_0^e(\Sigma^\infty_+ Z) \;\cong\; \mathbb{Z}\{\pi_0(Z)\}\,. \tag{3.3.13}$$

Indeed, for all $n\geq 2$ the group $\pi_n(S^n\wedge Z_+,*)$ is free abelian, with basis the classes of the maps $-\wedge y : S^n \longrightarrow S^n\wedge Z_+$ as y runs over the path components of Z, see for example [180, Prop. 7.1.7]. Passing to the colimit over n proves the claim.

If H is a closed subgroup of a compact Lie group G, and Y is the underlying H-space of a G-space, then the normalizer N_GH leaves Y^H invariant, and the action of N_GH factors over an action of the Weyl group $W_GH = N_GH/H$ on Y^H. This, in turn, induces an action of the component group $\pi_0(W_GH)$ on the set $\pi_0(Y^H)$. For all $g\in G$, the following square commutes, again by direct

inspection:

$$\begin{tikzcd} \pi_0(Y^{H^g}) \arrow[r, "\sigma^{H^g}"] \arrow[d, "\pi_0(l_g)"'] & \pi_0^{H^g}(\Sigma^\infty_+ Y) \arrow[d, "g_\star"] \\ \pi_0(Y^H) \arrow[r, "\sigma^H"'] & \pi_0^H(\Sigma^\infty_+ Y) \end{tikzcd}$$

Here $l_g : Y^{H^g} \longrightarrow Y^H$ is left multiplication by g. In particular, the map σ^H is equivariant for the action of the group $\pi_0(W_G H)$.

After stabilizing along the map $\sigma^H : \pi_0(Y^H) \longrightarrow \pi_0^H(\Sigma^\infty_+ Y)$, we can then transfer from H to G. For an element $g \in N_G H$ and a class $x \in \pi_0(Y^H)$ we have

$$\begin{aligned} \operatorname{tr}_H^G(\sigma^H(\pi_0(l_g)(x))) &= \operatorname{tr}_H^G(g_\star(\sigma^H(x))) \\ &= g_\star(\operatorname{tr}_H^G(\sigma^H(x))) = \operatorname{tr}_H^G(\sigma^H(x)) , \end{aligned} \tag{3.3.14}$$

because transfer commutes with conjugation, and inner automorphisms act as the identity. So the composite $\operatorname{tr}_H^G \circ \sigma^H$ coequalizes the $\pi_0(W_G H)$-action on $\pi_0(Y^H)$.

Our proof of the following theorem is based on an inductive argument with the isotropy separation sequence. A different proof, based on the tom Dieck splitting, can be found in [100, V Cor. 9.3].

Theorem 3.3.15. *Let G be a compact Lie group and Y a G-space.*

(i) *The equivariant homotopy group $\pi_0^G(\Sigma^\infty_+ Y)$ is a free abelian group with a basis given by the elements*

$$\operatorname{tr}_H^G(\sigma^H(x)) ,$$

where H runs through all conjugacy classes of closed subgroups of G with finite Weyl group and x runs through a set of representatives of the $W_G H$-orbits of the set $\pi_0(Y^H)$.

(ii) *Let $z \in \pi_0^G(\Sigma^\infty_+ Y)$ be a class such that for every closed subgroup K of G with finite Weyl group the geometric fixed-point class*

$$\Phi^K(\operatorname{res}^G_K(z)) \in \Phi_0^K(\Sigma^\infty_+ Y)$$

is trivial. Then $z = 0$.

Proof (i) In (3.3.13) we recalled property (i) when G is a trivial group. For the trivial group the geometric fixed-point map $\Phi^e : \pi_0^e(\Sigma^\infty_+ Y) \longrightarrow \Phi_0^e(\Sigma^\infty_+ Y)$ is the identity, so part (ii) is tautologically true.

Now we let G be any compact Lie group. We let H be a closed subgroup of G with finite Weyl group. By (3.3.13) we know that the group $\pi_0^e(\Sigma^\infty_+ Y^H)$ is free abelian on the set of path components of Y^H; moreover, the Weyl group

$W_G H$ permutes the basis elements, i.e., $\pi_0^e(\Sigma_+^\infty Y^H)$ is an integral permutation representation of the group $W_G H$. For any representation M of a finite group W the norm map

$$N \ : \ M \ \longrightarrow \ M\ , \quad x \ \longmapsto \ \sum_{w\in W} w\cdot x$$

factors over the group of coinvariants

$$M_W \ = \ M/\langle x - wx \,|\, x\in M, w\in W\rangle\ .$$

For the integral permutation representation $M = \mathbb{Z}[S]$ of a W-set S, a special feature is that the induced map $\bar{N} : M_W \longrightarrow M$ is injective.

So part (i) is equivalent to the claim that the map

$$T \ : \ \bigoplus_{(H)} (\pi_0^e(\Sigma_+^\infty Y^H))_{W_G H} \ \longrightarrow \ \pi_0^G(\Sigma_+^\infty Y)$$

is an isomorphism, where the sum is indexed by representatives of the conjugacy classes of closed subgroups with finite Weyl group, and the restriction of T to the H-summand is induced by the composite

$$\pi_0^e(\Sigma_+^\infty Y^H) \ \xrightarrow{p_H^*}\ \pi_0^H(\Sigma_+^\infty Y^H) \ \xrightarrow{\text{incl}_*}\ \pi_0^H(\Sigma_+^\infty Y) \ \xrightarrow{\text{tr}_H^G}\ \pi_0^G(\Sigma_+^\infty Y)\ .$$

We consider the total geometric fixed-point homomorphism

$$\Phi^{\text{total}} \ : \ \pi_0^G(\Sigma_+^\infty Y) \ \longrightarrow \ \prod_{K:\, W_G K \text{ finite}} \Phi_0^K(\Sigma_+^\infty Y)\ , \quad y \ \longmapsto \ (\Phi^K(\text{res}_K^G(y)))_K\ .$$

Property (ii) is equivalent to the claim that Φ^{total} is injective.

We show now that the composite $\Phi^{\text{total}} \circ T$ is injective. We argue by contradiction and suppose that

$$z \ = \ (z_H)_H \ \in \ \bigoplus_{(H)} (\pi_0^e(\Sigma_+^\infty Y^H))_{W_G H}$$

is a non-zero element in the source of T such that $\Phi^{\text{total}}(T(z))$ vanishes. As an element in a direct sum, z has only finitely many non-zero components z_H. We let K be of maximal dimension and with maximal number of path components among all indexing subgroups such that $z_K \neq 0$. Then

$$T(z) \ = \ \text{tr}_K^G(\text{incl}_*(p_K^*(y))) \ + \ \sum\nolimits_{i=1}^m \ \text{tr}_{H_i}^G(y_i)$$

with y an element of $\pi_0^e(\Sigma_+^\infty Y^K)$ with non-zero image in the $W_G K$-coinvariants, and with certain closed subgroups H_i of G that are not conjugate to K and 'no larger' in the sense that either $\dim(H_i) < \dim(K)$, or $\dim(H_i) = \dim(K)$ and $|\pi_0(H_i)| \leq |\pi_0(K)|$. This means in particular that K is not subconjugate to any of the groups $H_1, \ldots, H_m$. Thus

$$\Phi^K(\text{res}_K^G(\text{tr}_{H_i}^G(y_i))) \ = \ 0$$

for all $i = 1, \dots, m$, by Proposition 3.3.11 (i). Hence

$$\begin{aligned} 0 \;=\; \Phi^K(T(z)) \;&=\; \Phi^K(\mathrm{res}^G_K(\mathrm{tr}^G_K(\mathrm{incl}_*(p_K^*(y))))) \\ &=\; \sum_{gK\in W_GK} \Phi^K(g_\star(\mathrm{incl}_*(p_K^*(y)))) \\ &=\; \sum_{gK\in W_GK} \Phi^K(\mathrm{incl}_*(g_\star(p_K^*(y)))) \\ &=\; \Phi^K(\mathrm{incl}_*(p_K^*(\textstyle\sum_{gK\in W_GK}(l_g)_*(y)))) \,. \end{aligned}$$

The third equation is Proposition 3.3.11 (iii). Since the composite

$$\pi_0^e(\Sigma^\infty_+ Y^K) \xrightarrow{\ p_K^*\ } \pi_0^K(\Sigma^\infty_+ Y^K) \xrightarrow{\ \mathrm{incl}_*\ } \pi_0^K(\Sigma^\infty_+ Y) \xrightarrow{\ \Phi^K\ } \Phi_0^K(\Sigma^\infty_+ Y)$$

is an isomorphism, we conclude that the norm of the element $y \in \pi_0^e(\Sigma^\infty_+ Y^K)$ is zero. But since $\pi_0^e(\Sigma^\infty_+ Y^K)$ is an integral permutation representation of the Weyl group, this only happens if y maps to 0 in the coinvariants, which contradicts our assumption.

Now we show that the classes $\mathrm{tr}_H^G(\sigma^H(x))$ in the statement of (i) generate the group $\pi_0^G(\Sigma^\infty_+ Y)$ (i.e., the homomorphism T is surjective). We argue by induction on the size of G, i.e., by a double induction over the dimension and number of path components of G. The induction starts when G is the trivial group, which we dealt with above. Now we let G be a non-trivial compact Lie group. We start with the special case $Y = G/K_+$ for a proper closed subgroup K of G. The composite

$$G \ltimes_K \mathbb{S} \xrightarrow{\ G\ltimes_K (eK)_*\ } G \ltimes_K (\Sigma^\infty_+ G/K) \xrightarrow{\ \mathrm{act}\ } \Sigma^\infty_+ G/K$$

is an isomorphism of orthogonal G-spectra. Hence the composite

$$\begin{aligned} \pi_0^K(\Sigma^\infty S^V) \xrightarrow[\cong]{\ G\ltimes_K -\ } \pi_0^G(G \ltimes \mathbb{S}) &\xrightarrow{\ \pi_0^G(G\ltimes_K(eK)_*)\ } \\ &\pi_0^G(G \ltimes_K (\Sigma^\infty_+ G/K)) \xrightarrow{\ \pi_0^G(\mathrm{act})\ } \pi_0^G(\Sigma^\infty_+ G/K) \end{aligned}$$

is an isomorphism of abelian groups, where $V = T_{eK}(G/K)$ is the tangent representation and the first map is the external transfer (an isomorphism by Theorem 3.2.15).

The inclusion $S^0 \longrightarrow S^V$ is an equivariant h-cofibration and its quotient S^V/S^0 is G-homeomorphic to the unreduced suspension of the unit sphere $S(V)$ (with respect to any K-invariant scalar product on V). So the group

$$\pi_0^G(\Sigma^\infty(S^V/S^0)) \;\cong\; \pi_0^G(\Sigma^\infty(S(V)_+ \wedge S^1)) \;\cong\; \pi_{-1}^G(\Sigma^\infty_+ S(V))$$

vanishes by the suspension isomorphism and Proposition 3.1.44 (ii). The long

exact sequence of Corollary 3.1.38 (i) then shows that the map

$$\mathrm{incl}_* \;:\; \pi_0^K(\Sigma_+^\infty\{0\}) \;\longrightarrow\; \pi_0^K(\Sigma^\infty S^V)$$

is surjective. Since K is a proper closed subgroup of G, it either has smaller dimension or fewer path components, so we know by the inductive hypothesis that the group $\pi_0^K(\Sigma_+^\infty\{0\})$ is generated by the elements $\mathrm{tr}_L^K(\sigma^L[0])$ where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group.

Putting this all together lets us conclude that the group $\pi_0^G(\Sigma_+^\infty G/K)$ is generated by the classes

$$\left(\pi_0^G(\mathrm{act}\circ(G\ltimes_K (eK)_*))\circ(G\ltimes_K -)\circ\mathrm{tr}_L^K\circ\sigma^L\right)[0]$$

for all K-conjugacy classes of closed subgroups $L \leq K$ that have finite Weyl group in K. However, this long expression in fact defines a familiar class, as we shall now see. Indeed, the following diagram commutes by the various naturality properties:

$$\begin{tikzcd}
\pi_0(\{0\}) \arrow[r, "\pi_0((eK)_*)"] \arrow[d, "p_L^*\circ\sigma^e"] \arrow[ddd, bend right, "\sigma^L"'] & \pi_0((G/K)^L) \arrow[d, "p_L^*\circ\sigma^e"'] \arrow[ddr, bend left, "\sigma^L"] & \\
\pi_0^L(\Sigma^\infty S^0) \arrow[r, "\pi_0^L((eK)_*)"] \arrow[dd, "\mathrm{incl}_*"] & \pi_0^L(\Sigma_+^\infty (G/K)^L) \arrow[dr, "\mathrm{incl}_*"] & \\
 & & \pi_0^L(\Sigma_+^\infty G/K) \arrow[dl, "\mathrm{incl}_*"'] \arrow[d, "\mathrm{tr}_L^K"] \\
\pi_0^L(\Sigma_+^\infty S^V) \arrow[r, "\pi_0^L((eK)_*\wedge S^V)"] \arrow[d, "\mathrm{tr}_L^K"'] & \pi_0^L(\Sigma_+^\infty G/K\wedge S^V) \arrow[d, "\mathrm{tr}_L^K"'] & \pi_0^K(\Sigma_+^\infty G/K) \arrow[dl, "\mathrm{incl}_*"'] \arrow[dd, "\mathrm{tr}_K^G"] \\
\pi_0^K(\Sigma_+^\infty S^V) \arrow[r, "\pi_0^K((eK)_*\wedge S^V)"] \arrow[d, "G\ltimes_K -"', "\cong"] & \pi_0^K(\Sigma_+^\infty G/K\wedge S^V) \arrow[d, "G\ltimes_K -"'] & \\
\pi_0^G(G\ltimes_K \mathbb{S}) \arrow[r, "\pi_0^G(G\ltimes_K (eK)_*)"'] & \pi_0^G(G\ltimes_K \Sigma_+^\infty G/K) \arrow[r, "\pi_0^G(\mathrm{act})"'] & \pi_0^G(\Sigma_+^\infty G/K)
\end{tikzcd}$$

So

$$\begin{aligned}&\left(\pi_0^G(\mathrm{act}\circ(G\ltimes_K (eK)_*))\circ(G\ltimes_K -)\circ\mathrm{tr}_L^K\circ\sigma^L\right)[0]\\ &\qquad = \;\mathrm{tr}_K^G(\mathrm{tr}_L^K(\sigma^L((eK)_*[0]))) \;=\; \mathrm{tr}_L^G(\sigma^L[eK])\,.\end{aligned}$$

So the group $\pi_0^G(\Sigma_+^\infty G/K)$ is generated by the classes $\mathrm{tr}_L^G(\sigma^L[eK])$ for all K-conjugacy classes of closed subgroups $L \leq K$ that have finite Weyl group in K. If the Weyl group of L in the ambient group G happens to be infinite, then

$\mathrm{tr}_L^G = 0$ and the generator is redundant. Otherwise eK is an L-fixed-point of G/K, so the generator is one of the classes mentioned in the statement of (i). This shows the generating property for the G-space G/K.

Next we observe that whenever the claim is true for a family of G-spaces, then it is also true for their disjoint union; this follows from the fact that both fixed-points and π_0 commute with disjoint unions, and that equivariant homotopy groups take wedges to direct sums. In particular, the claim holds when Y is a disjoint union of homogeneous space G/H for varying proper closed subgroups H of G.

Now we prove the claim when Y is any G-space without G-fixed-points. For every proper closed subgroup H of G we choose representatives for the path components of the H-fixed-point space Y^H. These choices determine a continuous G-map

$$f \;:\; Z \;=\; \coprod_H \coprod_{[x]\in\pi_0(Y^H)} G/H \;\longrightarrow\; Y$$

by sending gH in the summand indexed by $x \in Y^H$ to gx. Because Y has no G-fixed-points, the induced map $\pi_0(f^H) : \pi_0(Z^H) \longrightarrow \pi_0(Y^H)$ is then surjective for every closed subgroup H of G, by construction. In the commutative square

$$\begin{tikzcd}
\bigoplus_{(H)} (\pi_0^e(\Sigma_+^\infty Z^H))_{W_GH} \arrow[r, "T"] \arrow[d, "\bigoplus(\pi_0^e(\Sigma_+^\infty f^H))_{W_GH}"'] & \pi_0^G(\Sigma_+^\infty Z) \arrow[d, "\pi_0^G(\Sigma_+^\infty f)"] \\
\bigoplus_{(H)} (\pi_0^e(\Sigma_+^\infty Y^H))_{W_GH} \arrow[r, "T"'] & \pi_0^G(\Sigma_+^\infty Y)
\end{tikzcd}$$

the right vertical map is then surjective by Proposition 3.1.44 (i) for $m = 0$, and the upper horizontal map is surjective by the above. So the lower horizontal map is surjective as well.

Now we let Y be any G-space, possibly with G-fixed-points. The composite

$$\pi_*^e(\Sigma_+^\infty Y^G) \xrightarrow{p_G^*} \pi_*^G(\Sigma_+^\infty Y^G) \xrightarrow{\mathrm{incl}_*} \pi_*^G(\Sigma_+^\infty Y) \xrightarrow{\Phi^G} \Phi_*^G(\Sigma_+^\infty Y)$$

is an isomorphism, i.e., the map $\mathrm{incl}_* \circ p_G^*$ splits the geometric fixed-point homomorphism. The long exact isotropy separation sequence (3.3.9) thus decomposes into short exact sequences and the map

$$(q_*, \mathrm{incl}_* \circ p_G^*) \;:\; \pi_*^G(\Sigma_+^\infty(Y \times E\mathcal{P}_G)) \oplus \pi_*^e(\Sigma_+^\infty Y^G) \;\longrightarrow\; \pi_*^G(\Sigma_+^\infty Y)$$

is an isomorphism, where $q : Y \times E\mathcal{P}_G \longrightarrow Y$ is the projection. The G-space $Y \times E\mathcal{P}_G$ has no G-fixed-points. So by the previous part, the group $\pi_0^G(\Sigma_+^\infty(Y \times E\mathcal{P}_G))$ is generated by the classes $\mathrm{tr}_H^G(\sigma^H(x))$ for H with finite Weyl group as above and $x \in \pi_0((Y \times E\mathcal{P}_G)^H)$. If H is a proper subgroup, then $(E\mathcal{P}_G)^H$ is contractible and so the projection $q : Y \times E\mathcal{P}_G \longrightarrow Y$ induces a bijection on

$\pi_0((-)^H)$. So the generators coming from $Y \times E\mathcal{P}_G$ map to the desired basis elements for Y that are indexed by proper subgroups of G. On the other hand, the group $\pi_*^e(\Sigma_+^\infty Y^G)$ is free with basis the classes $\sigma^e(x)$ for all components $x \in \pi_0(Y^G)$, by (3.3.13). Because

$$\mathrm{incl}_*(p_G^*(\sigma^e(x))) \;=\; \sigma^G(x)\;,$$

the basis of the second summand precisely maps to the desired basis elements for Y that are indexed by the group G itself. This completes the argument that the map T is surjective.

Now we can complete the proofs of part (i) and (ii). Since the composite $\Phi^{\mathrm{total}} \circ T$ is injective, the homomorphism T is injective; since T is also surjective, it is bijective, which shows (i) for the group G. Since T is bijective and $\Phi^{\mathrm{total}} \circ T$ is injective, the homomorphism Φ^{total} is injective, which shows (ii) for the group G. □

Example 3.3.16 (Equivariant 0-stems and Burnside rings)**.** We recall that for a finite group G, the *Burnside ring* $A(G)$ is the Grothendieck group of the abelian monoid, under disjoint union, of isomorphism classes of finite G-sets. Every finite G-set is the disjoint union of transitive G-sets, so $A(G)$ is a free abelian group generated by the classes of the G-sets G/H, as H runs over all conjugacy classes of subgroups of G.

The sphere spectrum $\mathbb{S}$ is also the unreduced suspension spectrum of the one-point G-space, $\mathbb{S} \cong \Sigma_+^\infty\{0\}$. So Theorem 3.3.15 (i) says that the map

$$\psi_G \;:\; A(G) \;\longrightarrow\; \pi_0^G(\mathbb{S})\;, \quad [G/H] \;\longmapsto\; \mathrm{tr}_H^G(1)$$

is an isomorphism, a result that is originally due to Segal [152]. The Burnside rings for different groups are related by restriction homomorphisms α^* : $A(G) \longrightarrow A(K)$ along homomorphisms $\alpha : K \longrightarrow G$, induced by restriction of the action along α. The Burnside rings also enjoy transfer maps

$$\mathrm{tr}_H^G \;:\; A(H) \;\longrightarrow\; A(G)$$

induced by sending an H-set S to the induced G-set $G \times_H S$.

The compatibility of transfers with inflations (Proposition 3.2.32 (iii)) implies that for every surjective homomorphism α the relation

$$\begin{aligned}\alpha^*(\psi_G[G/H]) \;&=\; \alpha^*(\mathrm{tr}_H^G(1)) \;=\; \mathrm{tr}_L^K((\alpha|_L)^*(1))\\ &=\; \mathrm{tr}_L^K(1) \;=\; \psi_K([K/L]) \;=\; \psi_K(\alpha^*[G/H])\end{aligned}$$

holds. In other words, the isomorphisms ψ_G are compatible with inflation. The double coset formula (see Example 3.4.11 below), and the double coset formula in the Burnside rings imply that the isomorphisms ψ_G are also compatible with transfers.

If G is a compact Lie group of positive dimension, there is no direct interpretation of $\pi_0^G(\mathbb{S})$ in terms of finite G-sets; in that situation, some authors *define* the Burnside ring of G as the group $\pi_0^G(\mathbb{S})$. In [178, Prop. 5.5.1], tom Dieck gives an interpretation of this Burnside ring in terms of certain equivalence classes of compact G-ENRs. Theorem 3.3.15 (i) identifies $\pi_0^G(\mathbb{S})$ as a free abelian group with basis the classes $\mathrm{tr}_H^G(1)$ for all conjugacy classes of closed subgroups H with finite Weyl group.

Theorem 3.3.15 (ii) provides a convenient detection criterion for elements in the equivariant 0-stems; as we explain now, this can be rephrased as a degree function. For every compact Lie group K the geometric fixed-point group $\Phi_0^K(\mathbb{S})$ is the 0th non-equivariant stable stem, and hence free abelian of rank 1, generated by the class $\Phi^K(1)$. Moreover, if $z \in \pi_0^K(\mathbb{S})$ is represented by the K-map $f : S^V \longrightarrow S^V$ for some K-representation V, then

$$\Phi^K(z) \ = \ [f^K \ : \ S^{V^K} \longrightarrow S^{V^K}] \ = \ \deg(f^K) \cdot \Phi^K(1) \ .$$

So in terms of the basis $\Phi^K(1)$, the geometric fixed-point homomorphism $\Phi^K : \pi_0^K(\mathbb{S}) \longrightarrow \Phi_0^K(\mathbb{S})$ is extracting the degree of the restriction of any representative to K-fixed-points. We let $C(G)$ denote the set of integer valued, conjugation-invariant functions from the set of closed subgroup of G with finite Weyl group. We define a homomorphism

$$\deg \ : \ \pi_0^G(\mathbb{S}) \ \longrightarrow \ C(G) \tag{3.3.17}$$

by

$$\deg[f : S^V \longrightarrow S^V](K) \ = \ \deg(f^K \ : \ S^{V^K} \longrightarrow S^{V^K}) \ .$$

Theorem 3.3.15 (ii) then says that this degree homomorphism is injective. Whenever the group G is non-trivial, the degree homomorphism is *not* surjective. In [178, Prop. 5.8.5], tom Dieck exhibits an explicit set of congruences that together with a certain continuity condition characterize the image of the degree homomorphism (3.3.17). When G is finite, these congruences – combined with the isomorphism ψ_G to the Burnside ring $A(G)$ – specialize to the congruences between the cardinalities of the various fixed-point sets of a finite G-set, specified for example in [178, Prop. 1.3.5].

We will revisit the equivariant 0-stems from a global perspective in Example 4.2.7 and with a view towards products and multiplicative power operations in Example 5.3.1.

Theorem 3.3.15 gives a functorial description of the 0th equivariant stable homotopy group of an unbased G-space. We will now deduce a similar result for reduced suspension spectra of *based* G-spaces. This version, however, needs a non-degeneracy hypothesis, i.e., we must restrict to well-pointed G-spaces.

Theorem 3.3.18. *Let G be a compact Lie group and Y a well-pointed G-space. Then the group $\pi_0^G(\Sigma^\infty Y)$ is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_H^G(\sigma^H(x)) ,$$

where H runs through all conjugacy classes of closed subgroups of G with finite Weyl group and x runs through a set of representatives of the $W_G H$-orbits of the non-basepoint components of the set $\pi_0(Y^H)$.

Proof If the inclusion of the basepoint $i : \{y_0\} \longrightarrow Y$ is an unbased h-cofibration, then the based map $i_+ : \{y_0\}_+ \longrightarrow Y_+$ is a based h-cofibration. The induced morphism of suspension spectra $\Sigma^\infty_+ i : \Sigma^\infty_+ \{y_0\} \longrightarrow \Sigma^\infty_+ Y$ is then an h-cofibration of orthogonal G-spectra, so it gives rise to a long exact sequence of homotopy groups as in Corollary 3.1.38 (i). The cokernel of $i_+ : \{y_0\}_+ \longrightarrow Y_+$ is G-homeomorphic to Y (with the original basepoint). Also, the map i has a section, so the long exact sequence degenerates into a short exact sequence:

$$0 \longrightarrow \pi_0^G(\Sigma^\infty_+ \{y_0\}) \xrightarrow{(\Sigma^\infty_+ i)_*} \pi_0^G(\Sigma^\infty_+ Y) \xrightarrow{\mathrm{proj}_*} \pi_0^G(\Sigma^\infty Y) \longrightarrow 0$$

Theorem 3.3.15 (i), applied to the unbased G-spaces $\{y_0\}$ and Y, provides bases of the left and middle group; by naturality the map $(\Sigma^\infty_+ i)_*$ hits precisely the subgroup generated by the basis elements coming from basepoint components of the space Y^H. So the cokernel of $(\Sigma^\infty_+ i)_*$, and hence the group $\pi_0^G(\Sigma^\infty Y)$, has a basis of the desired form. □

3.4 The double coset formula

The main aim of this section is to establish the double coset formula for the composite of a transfer followed by a restriction to a closed subgroup, see Theorem 3.4.9 below. We also discuss various examples and special cases in Examples 3.4.10 through 3.4.13. For finite groups (or more generally for transfers along finite index inclusions), the statement and proof of the double coset formula are significantly simpler, see Example 3.4.11, as this special case does not need any equivariant differential topology. We end the section with a discussion of Mackey functors for finite groups, and prove that rationally and for finite groups, geometric fixed-point homotopy groups can be obtained from equivariant homotopy groups by dividing out transfers from proper subgroups (Proposition 3.4.26).

For use in the proof of the double coset formula, and as an interesting result on its own, we calculate the geometric fixed-points of the restriction of any

transfer. In fact, when $K = e$ is the trivial subgroup of G, then $\Phi_0^e(X) = \pi_0^e(X)$, the map Φ^e is the identity and the following proposition reduces to a special case of the double coset formula. For the statement we use the fact that the action of a closed subgroup K on a homogeneous space G/H by left translation is smooth, and hence the fixed-points $(G/H)^K$ form a disjoint union of closed smooth submanifolds, possibly of varying dimensions.

We let K be a compact Lie group and B a closed smooth K-manifold. The Mostow-Palais embedding theorem [124, 130] provides a smooth K-equivariant embedding $i : B \longrightarrow V$, for some K-representation V. We can assume without loss of generality that V is a subrepresentation of the chosen complete K-universe $\mathcal{U}_K$. We use the inner product on V to define the normal bundle ν of the embedding at $b \in B$ by

$$\nu_b \;=\; V - (di)(T_bB) \,,$$

the orthogonal complement of the image of the tangent space T_bB in V. By multiplying with a suitably large scalar, if necessary, we can assume that the embedding is *wide* in the sense that the exponential map

$$D(\nu) \;\longrightarrow\; V \,, \quad (b, v) \;\longmapsto\; i(b) \,+\, v$$

is injective on the unit disc bundle of the normal bundle, and hence a closed K-equivariant embedding. The image of this map is a tubular neighborhood of radius 1 around $i(B)$, and it determines a K-equivariant Thom-Pontryagin collapse map

$$c_B \;:\; S^V \;\longrightarrow\; S^V \wedge B_+ \tag{3.4.1}$$

as follows: every point outside of the tubular neighborhood is sent to the basepoint, and a point $i(b) + v$, for $(b, v) \in D(\nu)$, is sent to

$$c_B(i(b) + v) \;=\; \left(\frac{v}{1 - |v|}\right) \wedge b \,.$$

The homotopy class of the map c_B is then an element in the equivariant homotopy group $\pi_0^K(\Sigma_+^\infty B)$. The next result determines the image of the class $[c_B]$ under the geometric fixed-point map.

In (3.3.12) we defined the map $\sigma^K : \pi_0(B^K) \longrightarrow \pi_0^K(\Sigma_+^\infty B)$ that produces equivariant stable homotopy classes from fixed-point information.

Proposition 3.4.2. *Let K be a compact Lie group.*

(i) *For every closed smooth K-manifold B the relation*

$$\Phi^K[c_B] \;=\; \sum_{M \in \pi_0(B^K)} \chi(M) \cdot \Phi^K(\sigma^K[M])$$

holds in the group $\Phi_0^K(\Sigma_+^\infty B)$, where the sum runs over the connected components of the fixed-point manifold B^K.

(ii) *Let G be a compact Lie group containing K and let H be another closed subgroup of G. Then*

$$\Phi^K \circ \mathrm{res}_K^G \circ \mathrm{tr}_H^G \;=\; \sum_{M\in\pi_0((G/H)^K)} \chi(M)\cdot \Phi^K \circ g_\star \circ \mathrm{res}^H_{K^g}\ ,$$

where the sum runs over the connected components of the fixed-point space $(G/H)^K$ and $g \in G$ is such that $gH \in M$.

Proof (i) Since the composite

$$\pi_0^e(\Sigma_+^\infty B^K) \xrightarrow{p_K^*} \pi_0^K(\Sigma_+^\infty B^K) \xrightarrow{\mathrm{incl}_*} \pi_0^K(\Sigma_+^\infty B) \xrightarrow{\Phi^K} \Phi_0^K(\Sigma_+^\infty B)$$

is an isomorphism and the source is free abelian on the path components of the space B^K, the group $\Phi_0^K(\Sigma_+^\infty B)$ is free abelian with basis given by the classes

$$\Phi^K(\sigma^K[M]) \;=\; \Phi^K(\mathrm{incl}_*(p_K^*(\sigma^e[M])))$$

for $M \in \pi_0(B^K)$. We let $i : B \longrightarrow V$ be a wide smooth K-equivariant embedding into a K-representation. Then the class $\Phi^K[c_B]$ is represented by the non-equivariant map

$$S^{V^K} \xrightarrow{(c_B)^K} S^{V^K} \wedge B_+^K\ .$$

Since K acts smoothly on B, the fixed-points B^K are a disjoint union of finitely many closed smooth submanifolds. For each component M of B^K we let $p_M : B_+^K \longrightarrow M_+$ denote the projection, i.e., p_M is the identity on M and sends all other path components of B^K to the basepoint. Then the composite

$$S^{V^K} \xrightarrow{(c_B)^K} S^{V^K} \wedge B_+^K \xrightarrow{S^{V^K}\wedge p_M} S^{V^K} \wedge M_+$$

coincides with

$$S^{V^K} \xrightarrow{c_M} S^{V^K} \wedge M_+\ ,$$

the collapse map (3.4.1) based on the non-equivariant wide smooth embedding $(i^K)|_M : M \longrightarrow V^K$. Since M is path connected, the group of based homotopy classes of maps $[S^{V^K}, S^{V^K}\wedge M_+]$ is isomorphic to $\mathbb{Z}$, by (3.3.13), and an element is determined by the degree of the composite with the projection $S^{V^K}\wedge M_+ \longrightarrow S^{V^K}$. It is a classical fact that the degree of the composite

$$S^{V^K} \xrightarrow{c_M} S^{V^K} \wedge M_+ \xrightarrow{\mathrm{proj}} S^{V^K}$$

is the Euler characteristic of the manifold M, see for example [6, Thm. 2.4]. So the summand indexed by M precisely contributes the term $\chi(M)\cdot\Phi^K(\sigma^K[M])$, which proves the desired relation.

(ii) Both sides of the equation are natural transformations on the category of orthogonal G-spectra from the functor π_0^H to the functor Φ_0^K. Since the functor π_0^H is represented by the suspension spectrum of G/H (in the sense of Proposition 3.1.46), is suffices to check the relation for the orthogonal G-spectrum $\Sigma_+^\infty G/H$ and the tautological class e_H defined in (3.1.45). Inspection of the definition in Construction 3.2.22 reveals that the transfer of the tautological class $\mathrm{tr}_H^G(e_H)$ in $\pi_0^G(\Sigma_+^\infty G/H)$ is represented by the G-map

$$S^V \xrightarrow{c} G \ltimes_H S^W \xrightarrow{a} S^V \wedge G/H_+$$

where c is the collapse map based on any wide embedding of $i : G/H \longrightarrow V$ into a G-representation, and $a[g, w] = gw \wedge gH$. So the class $\mathrm{res}_K^G(\mathrm{tr}_H^G(e_H))$ is represented by the underlying K-map of the above composite, which is precisely the map $c_{G/H}$ for the underlying K-manifold of G/H. Part (i) thus provides the relation

$$\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_H^G(e_H))) \;=\; \Phi^K[c_{G/H}] \;=\; \sum_{M\in\pi_0((G/H)^K)} \chi(M)\cdot\Phi^K(\sigma^K[M])$$

where the sum runs over the connected components of $(G/H)^K$. On the other hand, if $g \in G$ is such that $gH \in M \subset (G/H)^K$, then $K^g \leq H$ and

$$\sigma^K[M] \;=\; g_\star(\sigma^{K^g}\langle eH\rangle) \;=\; g_\star(\mathrm{res}^H_{K^g}(e_H)) \;;$$

this proves the desired relation for the universal class e_H. □

Now we can proceed towards the double coset formula for a transfer followed by a restriction. The double coset formula was first proved by Feshbach for Borel cohomology theories [53, Thm. II.11] and later generalized to equivariant cohomology theories by Lewis and May [100, IV §6]. I am not aware of a published account of the double coset formula in the context of orthogonal G-spectra. Our method of proof for the double coset formula is different from the approach of Feshbach and Lewis-May: we verify the double coset formula after passage to geometric fixed-points for all closed subgroups. The detection criterion of Theorem 3.3.15 (ii) then lets us deduce the double coset formula for the universal example, the tautological class e_H in $\pi_0^H(\Sigma_+^\infty G/H)$.

Before we can state the double coset formula we have to recall some additional concepts, such as orbit type submanifolds and the internal Euler characteristic.

Definition 3.4.3. Let L be a closed subgroup of a compact Lie group K and B a K-space. The *orbit type space* associated with the conjugacy class of L is the K-invariant subspace

$$B_{(L)} \;=\; \{x \in B \mid \text{the stabilizer group of } x \text{ is conjugate to } L\}\;.$$

The points in the space $B_{(L)}$ are said to have 'orbit type' the conjugacy class (L). We mostly care about orbit type subspaces for equivariant smooth manifolds. Two sources that collect general information about orbit type manifolds are Chapters IV and VI of Bredon's book [26] and Section I.5 of tom Dieck's book [179]. If B is a smooth K-manifold, then the subspace $B_{(L)}$ is a smooth submanifold of B, and locally closed in B, compare [179, I Prop. 5.12] or [26, VI Cor. 2.5]. Thus $B_{(L)}$ is called the *orbit type manifold*. If the smooth K-manifold is compact, then it only has finitely many orbit types – this was first shown by Yang [193]; other references are [26, IV Prop. 1.2], [131, Thm. 1.7.25], or [179, I Thm. 5.11]. The orbit type manifolds $B_{(L)}$ need not be closed inside B; but if one orbit type manifold $B_{(L)}$ lies in the closure of another one $B_{(L')}$, then L' is subconjugate to L in K, compare [26, IV Thm. 3.3]. For every conjugacy class (L), the quotient map $B_{(L)} \longrightarrow K\backslash B_{(L)}$ is a locally trivial smooth fiber bundle with fiber K/L and structure group the normalizer $N_K L$, compare [26, VI Cor. 2.5]. Thus every path component of every orbit space $K\backslash B_{(L)}$ of an orbit type manifold is again a smooth manifold, in such a way that the quotient maps are smooth submersions. In particular, if the action happens to have only one orbit type (i.e., all stabilizer groups are conjugate), then the quotient space $K\backslash B$ is a manifold and inherits a smooth structure from B.

The terms in the double coset formula will be indexed by path components of the orbit type orbit manifolds $K\backslash B_{(L)}$. These manifolds need not be connected, and different components may have varying dimensions, but each $K\backslash B_{(L)}$ has only finitely many path components. Indeed, by [182, Cor. 3.7 and 3.8] the orbit space $K\backslash B$ admits a triangulation such that the orbit type is constant on every open simplex. So if a path component M of $K\backslash B_{(L)}$ intersects a particular open simplex, then that simplex is entirely contained in M, i.e., M is a union of open simplices in such a triangulation. Since $K\backslash B$ is compact, any triangulation has only finitely many simplices.

Construction 3.4.4 (Internal Euler characteristic)**.** We continue to consider a compact Lie group K acting smoothly on a closed smooth manifold B. We let L be a closed subgroup of K and $M \subset K\backslash B_{(L)}$ a connected component of the orbit type manifold for the conjugacy class (L). We let $\bar{M}$ denote the closure of M inside $K\backslash B$ and $\delta M = \bar{M} - M$ the complement of M inside its closure. Since $K\backslash B$ is compact, so is $\bar{M}$. Since $K\backslash B_{(L)}$ is locally closed in $K\backslash B$, the set M is open inside its closure $\bar{M}$. So the complement δM is closed in $\bar{M}$, hence compact. One should beware that while M is a topological manifold (without boundary, and typically not compact), $\bar{M}$ need not be a topological manifold with boundary.

By [182, Cor. 3.7 and 3.8] the orbit space $K\backslash B$ admits a triangulation (necessarily finite) such that the orbit type is constant on every open simplex. So

if the orbit type component M intersects a particular open simplex, then that simplex is entirely contained in M, i.e., M is a union of open simplices in such a triangulation. This means that $\bar{M}$ is obtained from M by adding some simplices of smaller dimension. Hence M is a simplicial subcomplex of any such triangulation, and δM is a subcomplex of M. In particular, $\bar{M}$ and δM admit the structure of finite simplicial complexes of dimension less than or equal to the dimension of B. Thus the integral singular homology groups of $\bar{M}$ and δM are finitely generated in every degree, and they vanish above the dimension of B. So the Euler characteristics of $\bar{M}$ and δM are well-defined integers. The *internal Euler characteristic* of M is then defined as

$$\chi^\sharp(M) \ = \ \chi(\bar{M}) \ - \ \chi(\delta M) \, .$$

The internal Euler characteristic $\chi^\sharp(M)$ also has an intrinsic interpretation that does not refer to the ambient space $K\backslash B$. Indeed since the integral homology groups of $\bar{M}$ and δM are finitely generated and vanish for almost all degrees, the same is true for the relative singular homology groups $H_*(\bar{M}, \delta M; \mathbb{Z})$, and the internal Euler characteristic satisfies

$$\chi^\sharp(M) \ = \ \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H_n(\bar{M}, \delta M; \mathbb{Z})) \ = \ \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H^n(\bar{M}, \delta M; \mathbb{Z})) \, .$$

The second equality uses that by the universal coefficient theorem, we can use *co*-homology instead of homology to calculate the Euler characteristic. Since δM is a simplicial subcomplex of $\bar{M}$, it is a neighborhood deformation retract inside $\bar{M}$. So the relative cohomology group $H^n(\bar{M}, \delta M; \mathbb{Z})$ is isomorphic to the reduced cohomology group of the quotient space $\bar{M}/\delta M$, by excision. This quotient space is homeomorphic to $\hat{M}$, the one-point compactification of $M = \bar{M} - \delta M$. Since M is a topological manifold, it is locally compact, and so the cohomology groups of $\hat{M}$ are isomorphic to the compactly supported cohomology groups of M. Altogether, this provides isomorphisms

$$H^n(\bar{M}, \delta M; \mathbb{Z}) \ \cong \ H^n_c(M; \mathbb{Z}) \, .$$

So we conclude that the internal Euler characteristic $\chi^\sharp(M)$ can also be defined as the compactly supported Euler characteristic of M, i.e.,

$$\chi^\sharp(M) \ = \ \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H^n_c(M; \mathbb{Z})) \, .$$

Now we let K and H be two closed subgroups of a compact Lie group G. Then the homogeneous space G/H is a smooth manifold and the K-action on G/H by left translations is smooth. The *double coset space* $K\backslash G/H$ is the quotient space of G/H by this K-action, i.e., the quotient of G by the left K- and right H-action by translation. In contrast to a homogeneous space G/H, the double coset space is in general not a smooth manifold. However, the orbit

type decomposition expresses the double coset space as the union of certain subspaces that are manifolds of varying dimensions. Indeed, G/H decomposes into orbit type manifolds with respect to K:

$$K\backslash G/H \;=\; \bigcup_{(L)} K\backslash (G/H)_{(L)} \;,$$

where the set-theoretic union is indexed by conjugacy classes of closed subgroups of L; all except finitely many of the subspaces $K\backslash(G/H)_{(L)}$ are empty. The double coset formula expresses the composite $\mathrm{res}^G_K \circ \mathrm{tr}^G_H$ as a sum of terms, indexed by all connected components M of orbit type orbit manifolds $K\backslash(G/H)_{(L)}$. The coefficient of the contribution of M is the internal Euler characteristic $\chi^\sharp(M)$.

Our next result is a technical proposition that contains the essential input to the double coset formula from equivariant differential topology.

Proposition 3.4.5. *Let K be a compact Lie group, B a closed smooth K-manifold and $L \leq J \leq K$ nested closed subgroups. Let N be a connected component of the fixed-point space B^L, and M a connected component of the orbit space $K\backslash B_{(J)}$. Let $b \in B$ be any point with stabilizer group J and with $Kb \in M$. Let $W[M,N]$ be the preimage of $N \cap B_{(J)}$ under the map*

$$(K/J)^L \;\longrightarrow\; (B_{(J)})^L \;, \quad kJ \;\longmapsto\; kb \;.$$

Then

$$\chi^\sharp(N \cap B_{(J)}) \;=\; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(W[M,N]) \;.$$

Proof The projection

$$B_{(J)} \xrightarrow{\;p\;} K\backslash B_{(J)}$$

is a smooth fiber bundle with fiber K/J and structure group $N_K J$, compare [131, Thm. 1.7.35] or [26, VI Cor. 2.5]. Since L is contained in J the composite

$$(B_{(J)})^L \xrightarrow{\;\mathrm{incl}\;} B_{(J)} \xrightarrow{\;p\;} K\backslash B_{(J)}$$

is surjective and the restriction of p to $(B_{(J)})^L$ is another smooth fiber bundle

$$(B_{(J)})^L \;\longrightarrow\; K\backslash B_{(J)} \;,$$

now with fiber $(K/J)^L$, and the map

$$\Psi_M \;:\; (K/J)^L \;\longrightarrow\; (B_{(J)})^L \;, \quad kJ \;\longmapsto\; kb$$

is the inclusion of the fiber over $Kb \in M$.

For every connected component M of the base $K\backslash B_{(J)}$ the inverse image $p^{-1}(M)$ is open and closed in $(B_{(J)})^L$. For every connected component N of B^L

the intersection $N \cap B_{(J)}$ is open and closed in $(B_{(J)})^L$. So the subset $p^{-1}(M) \cap N$ is open and closed in $(B_{(J)})^L$. As M varies, the subsets $p^{-1}(M) \cap N$ cover all of $N \cap B_{(J)}$. Moreover, the restriction of the projection to a map

$$p \;:\; p^{-1}(M) \cap N \;\longrightarrow\; M$$

is a smooth fiber bundle with connected base and with fiber $W[M, N]$, by definition of the latter. The internal Euler characteristic is multiplicative on smooth fiber bundles with closed fiber, so

$$\chi^\sharp(p^{-1}(M) \cap N) \;=\; \chi^\sharp(M) \cdot \chi(W[M, N]) \;.$$

The internal Euler characteristic is additive on disjoint unions, so

$$\chi^\sharp(N \cap B_{(J)}) \;=\; \sum_{M \in \pi_0(K \backslash B_{(J)})} \chi^\sharp(N \cap p^{-1}(M)) \;=\; \sum_{M \in \pi_0(K \backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(W[M, N]) \;.$$

□

Example 3.4.6. For later reference we point out a direct consequence of Proposition 3.4.5 for $L = e$, the trivial subgroup of K. We let J be any closed subgroup of K, and M a connected component of $K \backslash B_{(J)}$. The space K/J is the disjoint union of its subspaces $W[M, N]$, as N varies over the connected components of B. So summing the formula of Proposition 3.4.5 over all components of B yields

$$\chi^\sharp(B_{(J)}) \;=\; \sum_{M \in \pi_0(K \backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(K/J) \;,$$

by additivity of Euler characteristics for disjoint unions. Because $\chi(B)$ is the sum of the internal Euler characteristics $\chi^\sharp(B_{(J)})$ over all conjugacy classes of closed subgroups of K, summing up over conjugacy classes gives the formula

$$\chi(B) \;=\; \sum_{(J) \leq K} \; \sum_{M \in \pi_0(K \backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(K/J) \;. \tag{3.4.7}$$

We let K be a compact Lie group and B a closed smooth K-manifold. The Mostow-Palais embedding theorem [124, 130] provides a wide smooth K-equivariant embedding $i : B \longrightarrow V$, for some K-representation V. The associated collapse map

$$c_B \;:\; S^V \;\longrightarrow\; S^V \wedge B_+$$

was defined (3.4.1). The homotopy class of the map c_B is then an element in the equivariant homotopy group $\pi_0^K(\Sigma_+^\infty B)$. Theorem 3.3.15 (i) exhibits a basis of the group $\pi_0^K(\Sigma_+^\infty B)$, and the next theorem expands the class of c_B in that basis. As we shall explain in the proof of Theorem 3.4.9 below, the double coset formula for $\mathrm{res}^G_K \circ \mathrm{tr}^G_H$ is essentially the special case $B = G/H$ of the following

theorem. A compact smooth K-manifold only has finitely many orbit types, so the sum occurring in the following theorem is finite.

Theorem 3.4.8. *Let K be a compact Lie group and B a closed smooth K-manifold. Then the relation*

$$[c_B] \;=\; \sum_{(J)\leq K}\ \sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot \mathrm{tr}_J^K(\sigma^J\langle b_M\rangle)$$

holds in the group $\pi_0^K(\Sigma^\infty_+ B)$. Here the sum runs over all connected components M of all orbit type orbit manifolds $K\backslash B_{(J)}$, and the element $b_M\in B$ that occurs is such that $Kb_M\in M$ and with stabilizer group J, and $\langle b_M\rangle$ is the path component of b_M in the space B^J.

Proof Because the desired relation lies in the 0th equivariant homotopy group of a suspension spectrum, Theorem 3.3.15 (ii) applies and shows that we only need to verify the formula after taking geometric fixed-points to any closed subgroup L of K. We let J be another closed subgroup of K containing L. We calculate the contribution of the conjugacy class (J) to the effect of Φ^L on the right-hand side of the formula. For every connected component M of $K\backslash B_{(J)}$ we choose an element $b_M\in B$ with stabilizer group J and $Kb_M\in M$. Then

$$\begin{aligned}
&\sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot\Phi^L(\mathrm{res}^K_L(\mathrm{tr}^K_J(\sigma^J\langle b_M\rangle)))\\
&= \sum_{M\in\pi_0(K\backslash B_{(J)})}\ \sum_{W\in\pi_0((K/J)^L)} \chi^\sharp(M)\cdot\chi(W)\cdot\Phi^L(k_\star(\mathrm{res}^J_{L^k}(\sigma^J\langle b_M\rangle)))\\
&= \sum_{M\in\pi_0(K\backslash B_{(J)})}\ \sum_{W\in\pi_0((K/J)^L)} \chi^\sharp(M)\cdot\chi(W)\cdot\Phi^L(\sigma^L(k_\star(\mathrm{res}^J_{L^k}\langle b_M\rangle)))\\
&= \sum_{M\in\pi_0(K\backslash B_{(J)})}\ \sum_{N\in\pi_0(B^L)} \chi^\sharp(M)\cdot\chi(W[M,N])\cdot\Phi^L(\sigma^L[N])\\
&= \sum_{N\in\pi_0(B^L)} \chi^\sharp(N\cap B_{(J)})\cdot\Phi^L(\sigma^L[N])\,.
\end{aligned}$$

The first equation is Proposition 3.4.2 (ii), applied to the closed subgroups L and J of K. The element $k\in K$ is chosen so that $kJ\in W\subset(K/J)^L$; in particular this forces the relation $L\leq {}^kJ\leq {}^k\mathrm{stab}(b_M)$, and thus $kb_M\in B^L$. The third equation uses that for every M,

$$(K/J)^L \;=\; \coprod_{N\in\pi_0(B^L)} W[M,N]\,,$$

that the Euler characteristic is additive on disjoint unions, and that for all $kJ\in W[M,N]$ the element kb_M lies in the path component N of B^L, by the very definition of $W[M,N]$. The fourth equation is Proposition 3.4.5.

Now we sum up the contributions from the different conjugacy classes of

subgroups of K. The smooth K-manifold B is stratified by the relatively closed smooth submanifolds $B_{(J)}$, indexed over the poset of conjugacy classes of subgroup of K, where only finitely many orbit types occur. Intersecting the strata with a connected component N of the fixed-point submanifold B^L gives a stratification of N by relatively closed submanifolds, still indexed over the conjugacy classes of subgroup of K. The internal Euler characteristic is additive for such stratifications, i.e.,

$$\chi(N) \;=\; \sum_{(J)\leq K} \chi^\sharp(N \cap B_{(J)}) \ .$$

An application of Proposition 3.4.2 (i) gives

$$\begin{aligned}
\Phi^L(\mathrm{res}^K_L[c_B]) \;&=\; \sum_{N\in\pi_0(B^L)} \chi(N)\cdot \Phi^L(\sigma^L[N]) \\
&=\; \sum_{(J)\leq K}\sum_{N\in\pi_0(B^L)} \chi^\sharp(N\cap B_{(J)})\cdot \Phi^L(\sigma^L[N]) \\
&=\; \sum_{(J)\leq K}\sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot \Phi^L(\mathrm{res}^K_L(\mathrm{tr}^K_J(\sigma^J\langle b_M\rangle)))\\
&=\; \Phi^L\left(\mathrm{res}^K_L\left(\sum_{(J)\leq K}\sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot \mathrm{tr}^K_J(\sigma^J\langle b_M\rangle)\right)\right) \ .
\end{aligned}$$

This proves the desired formula after composition with the geometric fixed-point map to any closed subgroup of K. $\square$

Now we have all ingredients for the statement and proof of the double coset formula in place, which gives an expression for the operation $\mathrm{res}^G_K\circ\mathrm{tr}^G_H$ on the equivariant homotopy groups of an orthogonal G-spectrum X, as a sum indexed by the path components M of the orbit type orbit manifolds $K\backslash(G/H)_{(L)}$. The operation associated with M involves the transfer $\mathrm{tr}^K_{K\cap{}^gH}$, where $g\in G$ is an element such that $KgH\in M$; since $K\cap{}^gH$ is the stabilizer of the coset gH, the group $K\cap{}^gH$ lies in the conjugacy class (L). The group $K\cap{}^gH$ may have infinite index in its normalizer in K, in which case $\mathrm{tr}^K_{K\cap{}^gH}=0$, and then the corresponding summand in the double coset formula vanishes.

Theorem 3.4.9 (Double coset formula)**.** *Let H and K be closed subgroups of a compact Lie group G. Then for every orthogonal G-spectrum X the relation*

$$\mathrm{res}^G_K\circ\mathrm{tr}^G_H \;=\; \sum_M \chi^\sharp(M)\cdot \mathrm{tr}^K_{K\cap{}^gH}\circ g_\star\circ\mathrm{res}^H_{K^g\cap H}$$

holds as homomorphisms $\pi_0^H(X)\longrightarrow\pi_0^K(X)$. Here the sum runs over all connected components M of all orbit type orbit manifolds $K\backslash(G/H)_{(L)}$, and $g\in G$ is an element such that $KgH\in M$.

Proof Both sides of the double coset formula are natural transformations on the category of orthogonal G-spectra from the functor π_0^H to the functor π_0^K. Since the functor π_0^H is represented by the suspension spectrum of G/H (in the sense of Proposition 3.1.46), is suffices to check the relation for the orthogonal G-spectrum $\Sigma_+^\infty G/H$ and the tautological class e_H defined in (3.1.45).

Inspection of the definition in Construction 3.2.22 reveals that the transfer of the tautological class $\mathrm{tr}_H^G(e_H)$ in $\pi_0^G(\Sigma_+^\infty G/H)$ is represented by the G-map

$$S^V \xrightarrow{\ c\ } G \ltimes_H S^W \xrightarrow{\ a\ } S^V \wedge G/H_+$$

where c is the collapse map based on any wide embedding $i : G/H \longrightarrow V$ into a G-representation, and $a[g,w] = gw \wedge gH$. So the class $\mathrm{res}_K^G(\mathrm{tr}_H^G(e_H))$ is represented by the underlying K-map of the above composite, which is precisely the map $c_{G/H}$ for the underlying K-manifold of G/H.

Theorem 3.4.8 thus yields the formula

$$\mathrm{res}_K^G(\mathrm{tr}_H^G(e_H)) \ = \ [c_{G/H}] \ = \sum_{(J)\leq K} \ \sum_{M\in\pi_0(K\backslash(G/H)_{(J)})} \chi^\sharp(M)\cdot \mathrm{tr}_J^K(\sigma^J\langle g_M H\rangle)$$

in the group $\pi_0^K(\Sigma_+^\infty G/H)$. Here $g_M \in G$ is such that $K g_M H \in M$ and $K \cap {}^{g_M}H = J$. On the other hand,

$$\begin{aligned} \sigma^J\langle g_M H\rangle \ &= \ (g_M)_\star(\sigma^{J^{g_M}}\langle eH\rangle) \\ &= \ (g_M)_\star(\mathrm{res}^H_{J^{g_M}}(\sigma^H\langle eH\rangle)) \ = \ (g_M)_\star(\mathrm{res}^H_{K^{g_M}\cap H}(e_H)) \ , \end{aligned}$$

so this proves the double coset formula for the universal class e_H. □

Example 3.4.10. A special case of the double coset formula is when $K = e$, i.e., when we restrict a transfer all the way to the trivial subgroup of G. In this case the orbit type component manifolds are the path components of the coset space G/H, and the sum in the double coset formula is indexed by these. Since all path components of G/H are homeomorphic, they have the same Euler characteristic $\chi(G/H)/|\pi_0(G/H)|$, so the double coset formula specializes to

$$\begin{aligned} \mathrm{res}_e^G \circ \mathrm{tr}_H^G \ &= \sum_{M\in\pi_0(G/H)} \chi(M)\cdot g_\star \circ \mathrm{res}_e^H \\ &= \ \chi(G/H)/|\pi_0(G/H)| \cdot \sum_{[gH]\in\pi_0(G/H)} g_\star \circ \mathrm{res}_e^H \ . \end{aligned}$$

In the special case where the conjugation action of $\pi_0(G)$ on $\pi_0^e(X)$ happens to be trivial (for example for all global stable homotopy types), all the maps $g_\star$ are the identity and the formula simplifies to

$$\mathrm{res}_e^G \circ \mathrm{tr}_H^G \ = \ \chi(G/H)\cdot \mathrm{res}_e^H \ .$$

Example 3.4.11 (Double coset formula for finite index transfers)**.** If H has finite index in G, then the double coset formula simplifies. In this situation the homogeneous space G/H, and hence also the double coset space $K\backslash G/H$, is discrete, all orbit type manifold components are points, and all internal Euler characteristics that occur in the double coset formula are 1. Since the intersection $K \cap {}^gH$ also has finite index in K, only finite index transfers show up in the double coset formula, which specializes to the relation

$$\mathrm{res}^G_K \circ \mathrm{tr}^G_H \ = \ \sum_{[g]\in K\backslash G/H} \mathrm{tr}^K_{K\cap {}^gH} \circ g_\star \circ \mathrm{res}^H_{K^g\cap H} \ .$$

Example 3.4.12. We calculate the double coset formula for the maximal torus

$$H \ = \ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \ : \ \lambda, \mu \in U(1) \right\}$$

of $G = U(2)$. We take $K = N_{U(2)}H$, the normalizer of the maximal torus; this is Example VI.2 in [53]. Then K is isomorphic to $\Sigma_2 \ltimes H$, the semidirect product with Σ_2 permuting the two diagonal entries of matrices in H; the group K is generated by H and the involution $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We calculate the double coset space $K\backslash U(2)/H$ by identifying the space $H\backslash U(2)/H$ and the residual action of the symmetric group Σ_2 on this space. A homeomorphism from $H\backslash U(2)/H$ to the unit interval $[0,1]$ is induced by

$$h \ : \ U(2) \ \longrightarrow \ [0,1] \ , \quad A \ \longmapsto \ |a_{11}|^2 \ ,$$

the square of the length of the upper left entry a_{11} of A. Lengths are non-negative and every column of a unitary matrix is a unit vector, so the map really lands in the interval $[0,1]$. The number $h(A)$ only depends on the double coset of the matrix A, so the map h factors over the double coset space $H\backslash U(2)/H$. For every $x \in [0,1]$ the vector $(\sqrt{x}, \sqrt{1-x}) \in \mathbb{C}^2$ has length 1, so it can be complemented to an orthonormal basis, and so it occurs as the first column of a unitary matrix; the map h is thus surjective. On the other hand, if $h(A) = h(B)$ for two unitary matrices A and B, then left multiplication by an element in H makes the first row of B equal to the first row of A. Right multiplication by an element of $1 \times U(1)$ then makes the matrices equal. So A and B represent the same element in the double coset space. The induced map

$$\bar h \ : \ H\backslash U(2)/H \ \longrightarrow \ [0,1]$$

is thus bijective, hence a homeomorphism. The action of Σ_2 on the orbit space $H\backslash U(2)/H$ permutes the two rows of a matrix; under the homeomorphism $\bar h$, this action thus corresponds to the involution of $[0,1]$ sending x to $1-x$. Altogether this specifies a homeomorphism from the double coset space to the

interval $[0,1/2]$ that sends a double coset $K\cdot A\cdot H$ to the minimum of $|a_{11}|^2$ and $|a_{21}|^2$. The inverse homeomorphism is

$$g \;:\; [0,1/2] \;\cong\; K\backslash U(2)/H\ , \quad g(t) \;=\; K\cdot\begin{pmatrix}\sqrt{1-t} & \sqrt{t}\\ -\sqrt{t} & \sqrt{1-t}\end{pmatrix}\cdot H\ .$$

The orbit type decomposition is as

$$\{0\}\cup(0,1/2)\cup\{1/2\}\ .$$

So as representatives of the orbit types we can choose

$$g(0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\quad g(1/5)=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\quad\text{and}\quad g(1/2)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\ .$$

For the intersections (which are representatives of the conjugacy classes of those subgroups of K with non-empty orbit type manifolds) we get

$$\begin{aligned} K\cap{}^{g(0)}H &= H\ ,\\ K\cap{}^{g(1/5)}H &= \Delta\ , \text{ and}\\ K\cap{}^{g(1/2)}H &= \Sigma_2\times\Delta\ . \end{aligned}$$

Here $\Delta = \left\{\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)\ :\ \lambda\in U(1)\right\}$ is the diagonal copy of $U(1)$. The group Δ is normal in K, so it has infinite index in its normalizer, and the corresponding transfer does not contribute to the double coset formula. The internal Euler characteristic of a point is 1, so the double coset formula has two non-trivial summands, and it specializes to

$$\mathrm{res}^{U(2)}_K\ \circ\ \mathrm{tr}^{U(2)}_H\ =\ \mathrm{tr}^K_H\ +\ \mathrm{tr}^K_{\Sigma_2\times\Delta}\circ\gamma_\star\circ\mathrm{res}^H_{(\Sigma_2\times\Delta)^\gamma}\ ,$$

where $\gamma = g(1/2) = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right)$.

Example 3.4.13. For later use in the 'explicit Brauer induction' (see Remark 5.3.19 below), we work out another double coset formula for the unitary group $G=U(n)$. We write $U(k,n-k)$ for the subgroup of those elements of $U(n)$ that leave the subspaces $\mathbb{C}^k\oplus 0$ and $0\oplus\mathbb{C}^{n-k}$ invariant. We also use the analogous notation for more than two factors. As subgroups we take

$$H\ =\ U(1,n-1)\qquad\text{and}\qquad K\ =\ U(k,n-k)\ ,$$

where $1\le k\le n-1$. Again the relevant double coset space 'is' an interval. Indeed, the continuous map

$$U(n)\ \longrightarrow\ [0,1]\ ,\quad A\ \longmapsto\ |a_{11}|^2+\cdots+|a_{k1}|^2\ ,$$

the partial length of the entries in the first column of A, is invariant under right multiplication by elements of $U(1,n-1)$, and under left multiplication by elements of $U(k,n-k)$. Every column of a unitary matrix is a unit vector, so

the map really lands in the interval $[0,1]$. The map factors over a continuous surjective map

$$h \;:\; U(k,n-k)\backslash U(n)/U(1,n-1) \;\longrightarrow\; [0,1]\,.$$

On the other hand, if $h[A] = h[B]$ for two unitary matrices A and B, then left multiplication by an element in $U(k,n-k)$ makes the first columns of A and B equal. Right multiplication by an element of $U(1,n-1)$ then makes the matrices equal. So A and B represent the same element in the double coset space. This shows that the map h is bijective, hence a homeomorphism. The inverse homeomorphism is

$$[0,1] \;\cong\; U(k,n-k)\backslash U(n)/U(1,n-1)\,, \quad t \;\mapsto\; U(k,n-k)\cdot g(t)\cdot U(1,n-1)$$

with

$$g(t) \;=\; \begin{pmatrix} \sqrt{t} & 0 & \cdots & 0 & \sqrt{1-t} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\sqrt{1-t} & 0 & \cdots & 0 & \sqrt{t} \end{pmatrix}.$$

The orbit type decomposition is as $\{0\} \cup (0,1) \cup \{1\}$. As representatives of the orbit types we can choose

$$g(0) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad g(1/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the identity $g(1)$. For the intersections (which are representatives of the conjugacy classes of those subgroups of $U(k,n-k)$ with non-empty orbit type manifolds) we get

$$\begin{aligned} U(k,n-k) \cap {}^{g(0)}U(1,n-1) &= U(k,n-k-1,1) \\ U(k,n-k) \cap {}^{g(1/2)}U(1,n-1) &= \Delta \\ U(k,n-k) \cap {}^{g(1)}U(1,n-1) &= U(1,k-1,n-k)\,. \end{aligned}$$

Here Δ is the subgroup of $U(n)$ consisting of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

for $(\lambda, A, B) \in U(1) \times U(k-1) \times U(n-k-1)$. The subgroup Δ is normalized by $U(1,\, k-1,\, n-k-1,\, 1)$, so it has infinite index in its normalizer, and the corresponding transfer does not contribute to the double coset formula. The double coset formula thus has two non-trivial summands, and comes out as

$$\begin{aligned} \mathrm{res}^{U(n)}_{U(k,n-k)} \circ \mathrm{tr}^{U(n)}_{U(1,n-1)} \ = \ & \\ \mathrm{tr}^{U(k,n-k)}_{U(1,k-1,n-k)} \circ \mathrm{res}^{U(1,n-1)}_{U(1,k-1,n-k)} \ + \ & \mathrm{tr}^{U(k,n-k)}_{U(k,n-k-1,1)} \circ g(0)_\star \circ \mathrm{res}^{U(1,n-1)}_{U(1,k,n-k-1)} \ . \end{aligned} \tag{3.4.14}$$

For the rest of this section we restrict our attention to finite groups. For easier reference we record that the equivariant homotopy groups of an orthogonal G-spectrum form a Mackey functor. We first recall one of several equivalent definitions of this concept.

Definition 3.4.15. Let G be a finite group. A *G-Mackey functor M* consists of the following data:

- an abelian group $M(H)$ for every subgroup H of G,
- conjugation homomorphisms $g_\star : M(H^g) \longrightarrow M(H)$ for all $H \leq G$ and $g \in G$,
- restriction homomorphisms $\mathrm{res}^H_K : M(H) \longrightarrow M(K)$ for all $K \leq H \leq G$, and
- transfer homomorphisms $\mathrm{tr}^H_K : M(K) \longrightarrow M(H)$ for all $K \leq H \leq G$.

This data has to satisfy the following conditions:

(i) (Unit conditions)

$$\mathrm{res}^H_H \ = \ \mathrm{tr}^H_H \ = \ \mathrm{Id}_{M(H)}$$

for all subgroups H, and $h_\star = \mathrm{Id}_{M(H)}$ for all $h \in H$.

(ii) (Transitivity conditions)

$$\mathrm{res}^K_L \circ \mathrm{res}^H_K \ = \ \mathrm{res}^H_L \quad \text{and} \quad \mathrm{tr}^H_K \circ \mathrm{tr}^K_L \ = \ \mathrm{tr}^H_L$$

for all $L \leq K \leq H \leq G$.

(iii) (Interaction conditions)

$$g_\star \circ g'_\star \ = \ (gg')_\star\, , \quad \mathrm{tr}^H_K \circ g_\star \ = \ g_\star \circ \mathrm{tr}^{H^g}_{K^g} \quad \text{and} \quad \mathrm{res}^H_K \circ g_\star \ = \ g_\star \circ \mathrm{res}^{H^g}_{K^g}$$

for all $g, g' \in G$ and $K \leq H \leq G$.

(iv) (Double coset formula) for every pair of subgroups K, L of H the relation

$$\mathrm{res}^H_L \circ \mathrm{tr}^H_K \ = \ \sum_{[h] \in L\backslash H/K} \mathrm{tr}^L_{L \cap {}^h K} \circ h_\star \circ \mathrm{res}^K_{L^h \cap K}$$

holds as maps $M(K) \longrightarrow M(L)$; here $[h]$ runs over a set of representatives for the L-K-double cosets.

A *morphism* of G-Mackey functors $f : M \longrightarrow N$ is a collection of group homomorphisms $f(H) : M(H) \longrightarrow N(H)$, for all subgroups H of G, such that

$$\mathrm{res}^H_K \circ f(H) \ = \ f(K) \circ \mathrm{res}^H_K \quad \text{and} \quad \mathrm{tr}^H_K \circ f(K) \ = \ f(H) \circ \mathrm{tr}^H_K$$

for all $K \leq H \leq G$, and

$$g_\star \circ f(H^g) \ = \ f(H) \circ g_\star$$

for all $g \in G$.

We denote the category of G-Mackey functors by G-$\mathcal{M}$ack. To my knowledge, the concept of a Mackey functor goes back to Dress [44] and Green [65]. Green in fact defined what is nowadays called a *Green functor* (which he called *G-functor* in [65, Def. 1.3]), which amounts to G-Mackey functor whose values underlie commutative rings, where restriction and conjugation maps are ring homomorphisms, and where the transfer maps satisfy Frobenius reciprocity. Definition 3.4.15 is the 'down to earth' definition of Mackey functor, and the one that is most useful for concrete calculations. There are two alternative (and equivalent) definitions that are often used:

- As a pair (M_*, M^*) of additive functors on the category of finite G-sets, where M_* is covariant and M^* is contravariant; the two functors must agree on objects and for every pullback diagram of finite G-sets

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

  the relation

$$M_*(f) \circ M^*(g) \ = \ M^*(h) \circ M_*(k) \ : \ M(C) \ \longrightarrow \ M(B)$$

  holds. This is (a special case of) the definition introduced by Dress [44, §6].
- As an additive functor on the category of spans of finite G-sets; the equivalence with Dress' definition is due to Lindner [103, Thm. 4].

Our main source of examples of G-Mackey functors comes from orthogonal G-spectra: as we verified in Sections 3.1 and 3.2 and Theorem 3.4.9, the restriction, conjugation and transfer maps make the homotopy groups $\pi^H_k(X)$ for varying subgroups H a G-Mackey functor.

Proposition 3.3.11 above shows that the geometric fixed-point map (3.3.2) factors over the quotient of $\pi^G_0(X)$ by the subgroup generated by proper transfers. This should serve as motivation for the following algebraic interlude about

Mackey functors for finite groups, where we study the process of 'dividing out transfers' systematically.

Construction 3.4.16. We let G be a finite group and M a G-Mackey functor. For a subgroup H of G we let $t_H M$ be the subgroup of $M(H)$ generated by transfers from proper subgroups of H, and we define

$$\tau_H M \;=\; M(H)/t_H M \;.$$

For $g \in G$, the conjugation isomorphism $g_\star : M(H^g) \longrightarrow M(H)$ descends to a homomorphism

$$g_\star \;:\; \tau_{H^g} M \;\longrightarrow\; \tau_H M \;. \tag{3.4.17}$$

Moreover, the transitivity relation $g_\star \circ g'_\star = (gg')_\star$ still holds. In particular, the action of the Weyl group $W_G H$ on $M(H)$ descends to a $W_G H$-action on the quotient group $\tau_H M$.

Now we recall how a G-Mackey functor M can rationally be recovered from the $W_G H$-modules $\tau_H M$ for all subgroups H of G. We let

$$\bar{\psi}^M_G \;:\; M(G) \;\longrightarrow\; \prod\nolimits_{H \leq G} \tau_H M$$

be the homomorphism whose H-component is the composite

$$M(G) \xrightarrow{\mathrm{res}^G_H} M(H) \xrightarrow{\mathrm{proj}} \tau_H M \;;$$

here the product is indexed over all subgroups H of G. The group G acts on the product via the maps (3.4.17), permuting the factors within conjugacy classes. Since inner automorphisms induce the identity we have $g_\star \circ \mathrm{res}^G_{H^g} = \mathrm{res}^G_H \circ g_\star = \mathrm{res}^G_H$, so the map $\bar{\psi}^M_G$ lands in the subgroup of G-invariant tuples. We let

$$\psi^M_G \;:\; M(G) \;\longrightarrow\; \Big(\prod\nolimits_{H \leq G} \tau_H M\Big)^G$$

denote the same map as $\bar{\psi}^M_G$, but now with image the subgroups of G-invariants. While the above description of the target of ψ^M_G is the most natural one, it is somewhat redundant. Indeed, for every $H \leq G$, the image of the restriction map $\mathrm{res}^G_H : M(G) \longrightarrow M(H)$ lands in the subgroup $M(H)^{W_G H}$ of invariants under the Weyl group $W_G H$. Moreover, if we choose representatives of the conjugacy classes of subgroups, then projection from the full product (over all subgroups of G) to the product indexed by the representatives restricts to an isomorphism

$$\Big(\prod\nolimits_{H \leq G} \tau_H M\Big)^G \xrightarrow{\;\cong\;} \prod\nolimits_{(H)} (\tau_H M)^{W_G H} \;.$$

For explicit calculations, the second description of the target of ψ^M_G is often more convenient.

For a finite group G we set

$$d_G \;=\; \prod\nolimits_{(H)} |W_G H| \;,$$

the product, over all conjugacy classes of subgroups of G, of the orders of the respective Weyl groups. Since the order of $W_G H$ divides the order of G, inverting $|G|$ also inverts the number d_G. The following result is well known; the earliest reference that I am aware of is [172, Cor. 4.4].

Proposition 3.4.18. *For every finite group G and every G-Mackey functor M, the kernel and cokernel of the homomorphism ψ^M_G are annihilated by d_G. In particular, ψ^M_G becomes an isomorphism after inverting the order of G.*

Proof We reproduce the proof given in [172]. We choose representatives for the conjugacy classes of subgroups and number them by non-decreasing order

$$e = H_1\;, H_2, \ldots, H_n = G\;.$$

Then

- for all $i < j$ and $g \in G$ the group $H_j \cap {}^g H_i$ is a proper subgroup of H_j, and
- every proper subgroup of H_i is conjugate to one of the groups occurring before H_i.

We set

$$K_j \;=\; \ker(\psi^M_G) \;\cap\; \sum\nolimits_{i=1}^{j} \operatorname{tr}^G_{H_i}(M(H_i)) \;\subseteq\; M(G)\;;$$

this defines a nested sequence of subgroups

$$0 \;=\; K_0 \;\subseteq\; K_1 \;\subseteq\; K_2 \;\subseteq\; \ldots\; K_{n-1} \;\subseteq\; K_n \;=\; \ker(\psi^M_G)\;.$$

We show that

$$|W_G H_j| \cdot K_j \;\subseteq\; K_{j-1} \tag{3.4.19}$$

for all $j = 1, \ldots, n$. Altogether this means that

$$d_G \cdot K_n \;=\; \prod\nolimits_{i=1}^{n} |W_G H_i| \cdot K_n \;\subseteq\; K_0 \;=\; 0\;.$$

In the course of proving (3.4.19) we call an element of $M(H)$ *degenerate* if it maps to zero in $\tau_H M$, i.e., if it is a sum of transfers from proper subgroups of H. So the kernel of ψ^M_G is precisely the subgroup of those elements of $M(G)$ that restrict to degenerate elements on all subgroups.

We write any given element $x \in K_j$ as

$$x \;=\; \operatorname{tr}^G_{H_j}(y) \;+\; \bar{x}$$

for suitable $y \in M(H_j)$ and with $\bar{x}$ a sum of transfers from the groups $H_1, \ldots, H_{j-1}$. For $i = 1, \ldots, j-1$ the double coset formula for $\operatorname{res}^G_{H_j} \circ \operatorname{tr}^G_{H_i}$ lets us write $\operatorname{res}^G_{H_j}(\bar{x})$

as a sum of transfers from proper subgroups of H_j; this uses the hypothesis on the enumeration of the subgroups. So the class $\mathrm{res}^G_{H_j}(\mathrm{tr}^G_{H_i}(\bar{x}))$ is degenerate. Since x is in the kernel of ψ^M_G, the class $\mathrm{res}^G_{H_j}(x)$ is also degenerate. So the class

$$\mathrm{res}^G_{H_j}(\mathrm{tr}^G_{H_j}(y)) \ = \ \mathrm{res}^G_{H_j}(x) \ - \mathrm{res}^G_{H_j}(\bar{x})$$

is degenerate. The double coset formula expresses $\mathrm{res}^G_{H_j}(\mathrm{tr}^G_{H_j}(y))$ as the sum of the $W_G H_j$-conjugates of y, plus transfers from proper subgroups of H_j. So the element

$$\sum_{gH_j \in W_G H_j} g_\star(y) \ \in \ M(H_j)$$

is degenerate and hence the element

$$|W_G H_j| \cdot \mathrm{tr}^G_{H_j}(y) \ = \ \mathrm{tr}^G_{H_j}\left(\sum_{gH_j \in W_G H_j} g_\star(y) \right) \ \in \ M(G)$$

is a sum of transfers from proper subgroups of H_j. Every proper subgroup of H_j is conjugate to one of the group $H_1, \dots, H_{j-1}$, so

$$|W_G H_j| \cdot x \ = \ |W_G H_j| \cdot \mathrm{tr}^G_{H_j}(y) \ + \ |W_G H_j| \cdot \bar{x}$$

is both in the kernel of ψ^M_G and a sum of transfers from the groups $H_1, \dots, H_{j-1}$. This proves the claim that $|W_G H_j| \cdot x$ belongs to K_{j-1}. Altogether this finishes the proof that the kernel of ψ^M_G is annihilated by the number d_G.

Now we show that the cokernel of ψ^M_G is annihilated by d_G. We let I_j denote the subgroup of $\prod_{i=1}^n (\tau_{H_i} M)^{W_G H_i}$ consisting of those tuples

$$x \ = \ (x_i)_{1 \leq i \leq n}$$

such that $x_{j+1} = x_{j+2} = \cdots = x_n = 0$. This defines a nested sequence

$$0 \ = \ I_0 \ \subseteq \ I_1 \ \subseteq \ I_2 \ \subseteq \ \dots \ I_{n-1} \ \subseteq \ I_n \ = \ \prod_{i=1,\dots,n} (\tau_{H_i} M)^{W_G H_i} \ .$$

We show that

$$|W_G H_j| \cdot I_j \ \subseteq \ \mathrm{Im}(\psi^M_G) \ + \ I_{j-1} \qquad (3.4.20)$$

for all $j = 1, \dots, n$. Altogether this means that

$$d_G \cdot I_n \ \subseteq \ \mathrm{Im}(\psi^M_G) \ ,$$

i.e., the cokernel of ψ^M_G is annihilated by d_G.

To prove (3.4.20) we consider a tuple $x = (x_n)$ in I_j and choose a representative $y \in M(H_j)$ for the 'last' non-zero component, i.e., such that y maps to x_j in $\tau_{H_j} M$. Since x_j is invariant under the action of the Weyl group $W_G H_j$, the

element y is at least $W_G H_j$-invariant modulo transfers from proper subgroups of H_j. The double coset formula for $\mathrm{res}^G_{H_j} \circ \mathrm{tr}^G_{H_j}$ thus gives that

$$\mathrm{res}^G_{H_j}(\mathrm{tr}^G_{H_j}(y)) \equiv \sum_{gH_j \in W_G H_j} g_\star(y) \equiv |W_G H_j| \cdot y ,$$

with both congruences modulo transfers from proper subgroups of H_j. So the composite

$$M(G) \xrightarrow{\mathrm{res}^G_{H_j}} M(H_j) \xrightarrow{\mathrm{proj}} \tau_{H_j} M$$

takes $\mathrm{tr}^G_{H_j}(y)$ to $|W_G H_j| \cdot x_j$. For $i > j$ the double coset formula shows that $\mathrm{res}^G_{H_i}(\mathrm{tr}^G_{H_j}(y))$ is a sum of transfers from proper subgroups of H_i. So the element $\mathrm{res}^G_{H_i}(\mathrm{tr}^G_{H_j}(y))$ maps to 0 in $\tau_{H_i} M$ for all $i = j+1, \dots, n$. In other words, the tuple $\psi^M_G(\mathrm{tr}^G_{H_j}(y))$ belongs to I_j. Since $\psi^M_G(\mathrm{tr}^G_{H_j}(y))$ and $|W_G H_j| \cdot x$ both belong to I_j and agree at the component of H_j, we can thus conclude that

$$\psi^M_G(\mathrm{tr}^G_{H_j}(y)) - |W_G H_j| \cdot x \in I_{j-1} .$$

This proves (3.4.20) and finishes the proof that the cokernel of ψ^M_G is annihilated by the number d_G. □

The previous Proposition 3.4.18 shows that for every G-Mackey functor M the group $\mathbb{Z}[1/|G|] \otimes (\tau_G M)$ is a direct summand of the group $\mathbb{Z}[1/|G|] \otimes M(G)$, and the splitting is natural for morphisms of G-Mackey functors. So the functor

$$G\text{-}\mathcal{M}\mathrm{ack} \longrightarrow \mathcal{A}b , \quad M \longmapsto \mathbb{Z}[1/|G|] \otimes (\tau_G M)$$

is a natural direct summand of an exact functor. So we can conclude:

Corollary 3.4.21. *Let G be a finite group. The functor that assigns to a G-Mackey functor M the abelian group $\mathbb{Z}[1/|G|] \otimes (\tau_G M)$ is exact.*

Now we will show that after inverting the order of the finite group G, the category of G-Mackey functors splits as a product of categories, indexed by the conjugacy classes of subgroups of G; the factor corresponding to $H \leq G$ is the category of $W_G H$-modules, i.e., abelian groups with an action of Weyl group of G (also known as modules over the group ring $\mathbb{Z}[W_G H]$). The argument is not particularly difficult, but somewhat lengthy because the details involve a substantial amount of book-keeping. The following result is folklore, but I do not know a reference that states it in precisely this form. The discussion in Appendix A of [67] is closely related, but the exposition of Greenlees and May differs from ours in that it emphasizes idempotents in the rationalized Burnside ring, as opposed to dividing out transfers. One could deduce part (ii) of the following theorem from the results in [67, App. A] by using the action of

the Burnside ring $A(H)$ on the value $M(H)$ of any G-Mackey functor M, and showing that after inverting the group order, the functor $\tau_H : G\text{-}\mathcal{M}\text{ack} \longrightarrow W_GH\text{-mod}$ becomes isomorphic to $e_H \cdot M(H)$, where $e_H \in A(H)[1/|G|]$ is the idempotent 'supported at H'.

The special case $Q = \mathbb{Q}$ of the following theorem in particular shows that rational G-Mackey functors form a semisimple abelian category, i.e., every object is both projective and injective.

Theorem 3.4.22. *Let G be a finite group.*

(i) *The functor*

$$\tau \ = \ (\tau_H)_{(H)} \ : \ G\text{-}\mathcal{M}\text{ack} \ \longrightarrow \ \prod_{(H)} W_GH\text{-mod}$$

has a right adjoint $\rho : \prod_{(H)} W_GH\text{-mod} \longrightarrow G\text{-}\mathcal{M}\text{ack}$.

(ii) *Let Q be a subring of the ring of rational numbers in which the order of G is invertible. Then the adjoint functors (τ, ρ) restrict to inverse equivalences between the category of Q-local G-Mackey functors and the product of the categories of Q-local W_GH-modules.*

Proof (i) We start by exhibiting a right adjoint $\rho_H : W_GH\text{-mod} \longrightarrow G\text{-}\mathcal{M}\text{ack}$ to the functor τ_H. We recall that the Weyl group W_GH acts on the H-fixed-points of every G-set S by

$$W_GH \times S^H \ \longrightarrow \ S^H \ , \quad (gH, s) \ \longmapsto \ gs \ .$$

We let N be a W_GH-module and K another subgroup of G. We set

$$(\rho_H N)(K) \ = \ \mathrm{map}^{W_GH}((G/K)^H, N) \ ,$$

the abelian group of W_GH-equivariant functions from $(G/K)^H$ to N, under pointwise addition of functions. The conjugation map $\gamma_\star : (\rho_H N)(K^\gamma) \longrightarrow (\rho_H N)(K)$, for $\gamma \in G$, is pre-composition with the W_GH-map

$$(G/K)^H \ \longrightarrow \ (G/K^\gamma)^H \ , \quad g \cdot K \ \longmapsto \ g\gamma \cdot K^\gamma \ .$$

For $L \leq K$, the restriction map $\mathrm{res}^K_L : (\rho_H N)(K) \longrightarrow (\rho_H N)(L)$ is pre-composition with the W_GH-map

$$(G/L)^H \ \longrightarrow \ (G/K)^H \ , \quad gL \ \longmapsto \ gK \ .$$

The transfer map $\mathrm{tr}^K_L : (\rho_H N)(L) \longrightarrow (\rho_H N)(K)$ is given by 'summation over preimages', i.e.,

$$\mathrm{tr}^K_L(f)(gK) \ = \sum_{\gamma L \in (G/L)^H \,:\, \gamma K = gK} f(\gamma L) \ ,$$

where $f : (G/L)^H \longrightarrow N$ is W_GH-equivariant and $gK \in (G/K)^H$. We omit the verification that $\mathrm{tr}_L^K(f)$ is again W_GH-equivariant.

With these definitions, the verification of properties (i), (ii) and (iii) of Definition 3.4.15 is straightforward, and we also omit it. Checking the double coset formula is more involved, so we spell out the argument. We consider a subgroup K of G and subgroups J and L of K. We let $g \in G$ be such that $gJ \in (G/J)^H$, i.e., $H^g \leq J$. We let R be a set of representatives of the J-L-double cosets in K. For every $k \in R$ we let S_k be a set of representatives j of those $(J \cap {}^kL)$-cosets $j(J \cap {}^kL) \in (G/(J \cap {}^kL))^H$ such that $jJ = gJ$. Then the sets $S_k \cdot k$ are pairwise disjoint for $k \in R$, and their union

$$\bigcup\nolimits_{k\in R} S_k \cdot k$$

is a set of representatives of those H-fixed L-coset $\gamma L \in (G/L)^H$ such that $\gamma K = gK$. Hence for every W_GH-map $f : (G/L)^H \longrightarrow N$ all $gJ \in (G/J)^H$ we obtain the relation

$$\begin{aligned}(\mathrm{res}_J^K(\mathrm{tr}_L^K(f)))(gJ) &= \sum_{\gamma L \in (G/L)^H \,:\, \gamma K = gK} f(\gamma L) \\ &= \sum_{k\in R}\sum_{j\in S_k} f(jkL) \\ &= \sum_{k\in R}\sum_{j\in S_k} k_\star(\mathrm{res}^L_{J^k\cap L}(f))(j(J\cap {}^kL)) \\ &= \sum_{k\in R} \mathrm{tr}^J_{J\cap^k L}(k_\star(\mathrm{res}^L_{J^k\cap L}(f)))(gJ)\ .\end{aligned}$$

For varying gJ and f, this proves the double coset relation

$$\mathrm{res}_J^K \circ \mathrm{tr}_L^K \ = \ \sum\nolimits_{k\in R} \mathrm{tr}^J_{J\cap^k L} \circ k_\star \circ \mathrm{res}^L_{J^k\cap L} \ : \ (\rho_H N)(L) \longrightarrow (\rho_H N)(J)\ .$$

Now we have shown that $\rho_H N$ is a G-Mackey functor, so this completes the definition of the functor ρ_H on objects; on morphisms, ρ_H is simply given by post-composition with a W_GH-module homomorphism.

The action of W_GH on $(G/H)^H$ is free and transitive, so the evaluation map

$$(\rho_H N)(H) \ = \ \mathrm{map}^{W_GH}((G/H)^H, N) \ \longrightarrow \ N\ , \quad f \ \longmapsto \ f(eH) \tag{3.4.23}$$

is an isomorphism. Moreover, if H is not subconjugate to K, then the set $(G/K)^H$ is empty, and so $(\rho_H N)(K) = 0$. This holds in particular for all proper subgroups of H, so in the G-Mackey functor $\rho_H N$, all transfers from proper subgroups to H are trivial. Hence the isomorphism (3.4.23) passes to a natural isomorphism of W_GH-modules

$$\epsilon_H^N \ : \ \tau_H(\rho_H N) \ \cong \ N\ , \quad \epsilon_H^N[f] \ = \ f(eH)\ .$$

Here $[y]$ denotes the class in $\tau_H M$ of an element $y \in M(H)$. Given a G-Mackey functor M, we also define a homomorphism

$$\eta_H^M \ : \ M \ \longrightarrow \ \rho_H(\tau_H M)$$

of G-Mackey functors as follows. For a subgroup K of G we define

$$\eta_H^M(K) \ : \ M(K) \ \longrightarrow \ \mathrm{map}^{W_G H}((G/K)^H, \tau_H M) \ = \ \rho_H(\tau_H M)(K)$$

by

$$\eta_H^M(K)(x)(gK) \ = \ [g_\star(\mathrm{res}^K_{H^g}(x))] \ .$$

We show that these additive maps indeed define a morphism of G-Mackey functors. For $L \leq K \leq G$, $x \in M(K)$ and $gL \in (G/L)^H$ we have

$$\begin{aligned} \mathrm{res}^K_L(\eta_H^M(K)(x))(gL) \ &= \ \eta_H^M(K)(x)(gK) \ = \ [g_\star(\mathrm{res}^K_{H^g}(x))] \\ &= \ [g_\star(\mathrm{res}^L_{H^g}(\mathrm{res}^K_L(x)))] \ = \ \eta_H^M(L)(\mathrm{res}^K_L(x))(gL) \ . \end{aligned}$$

So $\mathrm{res}^K_L \circ \eta_H^M(K) = \eta_H^M(L) \circ \mathrm{res}^K_L$, i.e., the η_H^M-maps are compatible with restrictions. Similarly, for $\gamma \in G$ and $x \in M(K^\gamma)$ we have

$$\begin{aligned} \gamma_\star(\eta_H^M(K^\gamma)(x))(g \cdot K) \ &= \ \eta_H^M(K^\gamma)(x)(g\gamma \cdot K^\gamma) \ = \ [(g\gamma)_\star(\mathrm{res}^{K^\gamma}_{H^{g\gamma}}(x))] \\ &= \ [g_\star(\mathrm{res}^K_{H^g}(\gamma_\star(x)))] \ = \ \eta_H^M(K)(\gamma_\star(x))(g \cdot K) \ . \end{aligned}$$

So $\gamma_\star \circ \eta_H^M(K^\gamma) = \eta_H^M(K) \circ \gamma_\star$, i.e., the η_H^M-maps are compatible with conjugations.

For compatibility with transfers we consider $gK \in (G/K)^H$ (i.e., $H^g \leq K$), and use the double coset formula

$$\begin{aligned} g_\star \circ \mathrm{res}^K_{H^g} \circ \mathrm{tr}^K_L \ &= \ \sum_{(H^g)kL \in H^g \backslash K/L} g_\star \circ \mathrm{tr}^{H^g}_{H^g \cap {}^kL} \circ k_\star \circ \mathrm{res}^L_{H^{gk} \cap L} \\ &\equiv \ \sum_{(H^g)kL \,:\, H^{gk} \leq L} (gk)_\star \circ \mathrm{res}^L_{H^{gk}} \\ &= \ \sum_{\gamma L \in (G/L)^H \,:\, \gamma K = gK} \gamma_\star \circ \mathrm{res}^L_{H^{gk}} \end{aligned} \tag{3.4.24}$$

as maps $M(L) \longrightarrow \tau_{H^g}(M)$. The second relation is modulo transfers from proper subgroups of H^g, i.e., we have dropped all summands where H^g is not contained in ${}^kL$. The third relation exploits that sending $(H^g)kL$ to gkL is a bijection between the set of those H^g-L-double cosets with $H^{gk} \leq L$ and the set of those $\gamma L \in (G/L)^H$ that satisfy $\gamma K = gK$. So for all $x \in M(L)$ we obtain the relation

$$\begin{aligned} \mathrm{tr}^K_L(\eta_H^M(L)(x))(gK) \ &= \ \sum \eta_H^M(L)(x)(\gamma L) \ = \ \sum [\gamma_\star(\mathrm{res}^L_{H^\gamma}(x))] \\ &\overset{(3.4.24)}{=} \ [g_\star(\mathrm{res}^K_{H^g}(\mathrm{tr}^K_L(x)))] \ = \ \eta_H^M(K)(\mathrm{tr}^K_L(x))(gK) \ . \end{aligned}$$

The sums run over those H-fixed L-cosets $\gamma L \in (G/L)^H$ such that $\gamma K = gK$. So $\mathrm{tr}_L^K \circ \eta_H^M(L) = \eta_H^M(K) \circ \mathrm{tr}_L^K$, i.e., the η_H^M-maps are compatible with transfers.

It is now straightforward to check that the composite maps

$$\rho_H N \xrightarrow{\eta_H^{\rho_H N}} \rho_H(\tau_H(\rho_H N)) \xrightarrow[\cong]{\rho_H(\epsilon_H^N)} \rho_H N \qquad \text{and}$$
$$\tau_H M \xrightarrow{\tau_H(\eta_H^M)} \tau_H(\rho_H(\tau_H M)) \xrightarrow[\cong]{\epsilon_H^{\tau_H M}} \tau_H M$$

are the respective identity morphisms. Indeed,

$$\begin{aligned} \rho_H(\epsilon_H^N)(\eta_H^{\rho_H N}(K)(f))(gK) &= \epsilon_H^N(\eta_H^{\rho_H N}(K)(f)(gK)) = \epsilon_H^N[g_\star(\mathrm{res}^K_{H^g}(f))] \\ &= g_\star(\mathrm{res}^K_{H^g}(f))(eH) = \mathrm{res}^K_{H^g}(f)(gH) = f(gK) \end{aligned}$$

for all $W_G H$-equivariant $f : (G/K)^H \longrightarrow N$ and all $gK \in (G/K)^H$. Moreover,

$$\epsilon_H^{\tau_H M}(\tau_H(\eta_H^M)[x]) = \epsilon_H^{\tau_H M}[\eta_H^M(H)(x)] = \eta_H^M(H)(x)(eH) = [x]$$

for all $x \in M(H)$. This shows that η_H and ϵ_H are the unit and counit of an adjunction between τ_H and ρ_H.

Now we can easily prove part (i). Since (τ_H, ρ_H) is an adjoint pair for every subgroup H of G, a right adjoint to the product functor τ is given by

$$\rho : \prod\nolimits_{(H)} W_G H\text{-mod} \longrightarrow G\text{-}\mathcal{M}\mathrm{ack}, \quad \rho((N_H)_{(H)}) = \prod\nolimits_{(H)} \rho_H(N_H),$$

the product of the G-Mackey functors $\rho_H(N_H)$. The universal property of the product of G-Mackey functors and the previous adjunctions combine into a natural isomorphism

$$\begin{aligned} G\text{-}\mathcal{M}\mathrm{ack}(M, \rho((N_H)_{(H)})) &\xrightarrow{\cong} \prod\nolimits_{(H)} G\text{-}\mathcal{M}\mathrm{ack}(M, \rho_H(N_H)) \\ &\cong \prod\nolimits_{(H)} W_G H\text{-mod}(\tau_H M, N_H). \end{aligned}$$

This completes the construction of the adjunction between τ and ρ.

(ii) We show first that for every G-Mackey functor M, the adjunction unit

$$\eta^M = (\eta_H^M) : M \longrightarrow \rho(\tau M) = \prod\nolimits_{(H)} \rho_H(\tau_H M)$$

becomes an isomorphism after inverting the order of G. When evaluated at G, this is the content of Proposition 3.4.18. At a general subgroup of G, we reduce the claim to Proposition 3.4.18 as follows. We let H and K be two subgroups of G. We let $R_{H,K}$ be a set of representatives of the K-conjugacy classes of subgroups $L \leq K$ that are G-conjugate to H. For each $L \in R_{H,K}$ we choose an element $g_L \in G$ such that $H^{g_L} = L$. Then $g_L K \in (G/K)^H$, and the map

$$\coprod\nolimits_{L \in R_{H,K}} (W_G H)/(W_{g_L K} H) \longrightarrow (G/K)^H, \quad [nH] \mapsto n g_L K$$

is an isomorphism of $W_G H$-sets. So taking $W_G H$-equivariant maps into $\tau_H M$ provides an isomorphism of abelian groups

$$\begin{aligned} \rho_H(\tau_H M)(K) \ = \ \mathrm{map}^{W_G H}((G/K)^H, \tau_H M) \ &\longrightarrow \ \prod\nolimits_{L \in R_{H,K}} (\tau_H M)^{W_{g_L K} H} \\ f \qquad &\longmapsto \quad (f(g_L K))_L \ . \end{aligned}$$

Because $H^{g_L} = L$, the map $(g_L)_\star : \tau_L M \longrightarrow \tau_H M$ defined in (3.4.17) is an isomorphism of abelian groups, equivariant for the isomorphism $W_K L = W_K H^{g_L} \cong W_{g_L K} H$ induced by conjugation by g_L. So the map

$$\zeta_H \ : \ \rho_H(\tau_H M)(K) \ \longrightarrow \ \prod\nolimits_{L \in R_{H,K}} (\tau_L M)^{W_K L} \ , \quad f \ \longmapsto \ ((g_L)_\star^{-1} f(g_L K))_L$$

is an isomorphism.

Now the following diagram commutes by direct inspection, where the right vertical map is the product of the maps ζ_H over all conjugacy classes of subgroups of G:

$$\begin{tikzcd} M(K) \arrow[r, "\eta_M(K)"] \arrow[dr, bend right, "\psi_K^M"'] & \rho(\tau M)(K) \arrow[r, equal] & \prod_{(H)} \rho_H(\tau_H M)(K) \arrow[d, "\cong"', "\prod \zeta_H"] \\ & \prod_{(L) \leq K} (\tau_L M)^{W_K L} \arrow[r, equal] & \prod_{(H) \leq G} \prod_{L \in R_{H,K}} (\tau_L M)^{W_K L} \end{tikzcd}$$

The map ψ_K^M becomes an isomorphism after inverting the order of K, by Proposition 3.4.18, applied to the underlying K-Mackey functor of M. Since the right vertical map is an isomorphism and the order of K divides the order of G, we conclude that the map $\eta_M(K)$ becomes an isomorphism after inverting the order of G. This completes the proof that $\eta_M : M \longrightarrow \rho(\tau M)$ becomes an isomorphism after inverting $|G|$.

Now $|G|$ is invertible in Q, so when restricted to Q-local objects, the adjunction unit $\eta : \mathrm{Id} \longrightarrow \rho \circ \tau$ is a natural isomorphism. Hence the restriction of the functor τ to the full subcategory of Q-local G-Mackey functors is fully faithful. We claim that additionally, every family of Q-local $W_G H$-modules is the image of a Q-local G-Mackey functor. Since the functor τ is additive, it commutes with finite products, so it suffices to show that for every individual subgroup H of G and every Q-local $W_G H$-module N, there is a Q-local G-Mackey functor M such that τM is isomorphic to N. We let $e : \tau(\rho_H N) \longrightarrow \tau(\rho_H N)$ be the idempotent endomorphism whose H-component is the identity of $\tau_H(\rho_H N)$, and such that $e_K = 0$ for all K that are not conjugate to H. Since $\rho_H N$ is Q-local and τ is fully faithful on Q-local G-Mackey functors, there exists an idempotent endomorphism $\tilde{e} : \rho_H N \longrightarrow \rho_H N$ such that $e = \tau(\tilde{e})$. Then $\rho_H N$ is isomorphic to the direct sum of the image of $\tilde{e}$ and the image of $\mathrm{Id} - \tilde{e}$. Since the functor τ is additive, $\tau(\tilde{e} \cdot \rho_H N)$ is isomorphic to $e \cdot \tau(\rho_H N) = \tau_H(\rho_H N)$. As

we showed in the proof of part (i), the $W_G H$-module $\tau_H(\rho_H N)$ is isomorphic to N, so this proves that N is isomorphic to $\tau(\tilde{e} \cdot \rho_H N)$. Altogether this shows that the functor τ is essentially surjective on Q-local objects. On this subcategory, τ is also fully faithful, and hence an equivalence of categories. □

Now we return to equivariant spectra. For every orthogonal G-spectrum X, Proposition 3.3.11 (ii) shows that the geometric fixed-point map (3.3.2) factors over the quotient of $\pi_0^G(X)$ by the subgroup generated by proper transfers. We denote by

$$\bar{\Phi} \ : \ \pi_k^G(X)/t_G(\underline{\pi}_k(X)) \ = \ \tau_G(\underline{\pi}_k(X)) \ \longrightarrow \ \Phi_k^G(X) \qquad (3.4.25)$$

the induced map on the factor group, and call it the *reduced geometric fixed-point homomorphism*. Our next result shows that for finite groups G, the 'corrected' (i.e., reduced) geometric fixed-point map becomes an isomorphism after inverting the order of G. The following proposition is well known, and closely related statements appear in Appendix A of [67]; however, I am not aware of a reference for the following statement in this form.

Proposition 3.4.26. *For every finite group G, every orthogonal G-spectrum X and every integer k the reduced geometric fixed-point map*

$$\bar{\Phi} \ : \ \tau_G(\underline{\pi}_k(X)) \ \longrightarrow \ \Phi_k^G(X)$$

becomes an isomorphism after inverting the order of G.

Proof We start by showing that for every orthogonal G-spectrum X and every subgroup H of G the transfer map

$$\mathrm{tr}_H^G \ : \ \pi_k^H(X \wedge G/H_+) \ \longrightarrow \ \pi_k^G(X \wedge G/H_+)$$

for the spectrum $X \wedge G/H_+$ is surjective. Indeed, the transfer is defined as the composite

$$\pi_k^H(X \wedge G/H_+) \xrightarrow[\cong]{G \ltimes_H -} \pi_k^G(G \ltimes_H (X \wedge G/H_+)) \xrightarrow{\mathrm{act}_*} \pi_k^G(X \wedge G/H_+) \ .$$

The first map – the external transfer – is an isomorphism by Theorem 3.2.15. The second map is surjective because the action morphism has a G-equivariant section

$$X \wedge G/H_+ \ \longrightarrow \ G \ltimes_H (X \wedge G/H_+) \ , \quad x \wedge gH \ \longmapsto \ [g, g^{-1}x \wedge eH] \ .$$

Now we let A be a G-CW-complex without G-fixed-points. We claim that for every orthogonal G-spectrum X, the entire group $\pi_k^G(X \wedge A_+)$ is generated by transfers from proper subgroups after inverting $|G|$. In a first step we show this when A is a finite-dimensional G-CW-complex. We argue by induction over the dimension of A. The induction starts when A is empty, in which case

$X \wedge A_+$ is the trivial spectrum and there is nothing to show. Now we consider $n \geq 0$ and assume the claim for all $(n-1)$-dimensional G-CW-complexes without G-fixed-points. We suppose that A is obtained from such an $(n-1)$-dimensional G-CW-complex B by attaching equivariant n-cells. After choosing characteristic maps for the n-cells we can identify the quotient A/B as a wedge

$$A/B \cong \bigvee_{i \in I} S^n \wedge (G/H_i)_+$$

for some index set I and certain subgroups H_i of G. Equivariant homotopy groups commute with wedges, so we can identify the homotopy group Mackey functor of $X \wedge (A/B)$ as

$$\underline{\pi}_k(X \wedge A/B) \cong \bigoplus_{i \in I} \underline{\pi}_k(X \wedge S^n \wedge (G/H_i)_+) .$$

Since A has no G-fixed-points, the groups H_i are all proper subgroups of G, so the group $\pi_k^G(X \wedge S^n \wedge (G/H_i)_+)$ is generated by transfers from the proper subgroup H_i, by the previous paragraph. So altogether the group $\pi_k^G(X \wedge A/B)$ is generated by transfers from proper subgroups (even before any localization).

The inclusion $B \longrightarrow A$ is a G-equivariant h-cofibration, so it induces an h-cofibration of orthogonal G-spectra $X \wedge B_+ \longrightarrow X \wedge A_+$ that results in a long exact sequence of homotopy group Mackey functors as in Corollary 3.1.38 (i). The functor sending a G-Mackey functor M to $\mathbb{Z}[1/|G|] \otimes (\tau_G M)$ is exact (Corollary 3.4.21), the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge B_+))$ vanish by induction, and the groups $\tau_G(\underline{\pi}_*(X \wedge A/B))$ vanish by the previous paragraph. So the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge A_+))$ vanish by exactness, and this finishes the inductive step.

Now we suppose that A is an arbitrary G-CW-complex without G-fixed-points, possibly infinite-dimensional. We choose a skeleton filtration by G-subspaces A^n. Then the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge A^n_+))$ vanish for all $n \geq 0$, by the previous paragraph. All the morphisms $X \wedge A^n_+ \longrightarrow X \wedge A^{n+1}_+$ are h-cofibrations of orthogonal G-spectra, hence level-wise closed embeddings. Since equivariant homotopy groups and the functor $\mathbb{Z}[1/|G|] \otimes \tau_G(-)$ both commute with such sequential colimits, this shows that the groups $\tau_G(\underline{\pi}_k(X \wedge A_+))$ vanish after inverting $|G|$.

Now we can prove the proposition. The inclusion $i : S^0 \longrightarrow \tilde{E}\mathcal{P}_G$ gives rise to a commutative square:

$$\begin{tikzcd}
\tau_G(\underline{\pi}_k(X)) \arrow[r, "\tau_G(\underline{\pi}_k(X \wedge i))"] \arrow[d, "\bar{\Phi}"'] & \tau_G(\underline{\pi}_k(X \wedge \tilde{E}\mathcal{P}_G)) \arrow[d, "\bar{\Phi}"] \\
\Phi_k^G(X) \arrow[r, "\Phi_k^G(X \wedge i)"'] & \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G)
\end{tikzcd} \tag{3.4.27}$$

The lower horizontal map is an isomorphism because i identifies S^0 with the

fixed-points $(\tilde{E}\mathcal{P}_G)^G$. Proposition 3.3.8 shows that the geometric fixed-point map $\Phi : \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \longrightarrow \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$ is an isomorphism *before dividing out transfers*. So all transfers from proper subgroups are in fact trivial in $\pi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$, and the right vertical map in the square (3.4.27) is an isomorphism (even before any localization). By the previous paragraph, the group $\pi_k^G(X \wedge (E\mathcal{P}_G)_+)$ is generated by transfers from proper subgroups after inverting $|G|$, so

$$\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_k(X \wedge (E\mathcal{P}_G)_+) \ = \ 0 \ .$$

The functor sending a G-Mackey functor M to the group $\mathbb{Z}[1/|G|] \otimes (\tau_G M)$ is exact (Corollary 3.4.21), so the isotropy separation sequence (3.3.9) shows that the upper horizontal map in the square (3.4.27) becomes an isomorphism after inverting $|G|$. So the left vertical map in the square (3.4.27) becomes an isomorphism after inverting $|G|$. □

When X is an orthogonal G-spectrum, we can apply the earlier algebraic Proposition 3.4.18 to the G-Mackey functor $\underline{\pi}_k(X)$; this allows us – after inverting $|G|$ – to reconstruct $\pi_k^G(X)$ from the groups $\tau_H(\underline{\pi}_k(X))$ with their Weyl group action. Moreover, after inverting the group order we can use the reduced geometric fixed-point map (3.4.25), for the underlying H-spectrum of X, to identify the group $\tau_H(\underline{\pi}_k(X))$ with the group $\Phi_k^H(X)$. Under this identification, the map $\bar{\psi}_G^M$ becomes the product of the maps

$$\pi_k^G(X) \xrightarrow{\mathrm{res}_H^G} \pi_k^H(X) \xrightarrow{\Phi^H} \Phi_k^H(X) \ .$$

So Propositions 3.4.18 and 3.4.26 together prove:

Corollary 3.4.28. *For every finite group G, every orthogonal G-spectrum X and every integer k the map*

$$(\Phi^H \circ \mathrm{res}_H^G)_H \ : \ \pi_k^G(X) \ \longrightarrow \ \prod\nolimits_{(H)} \left(\Phi_k^H(X)\right)^{W_G H}$$

becomes an isomorphism after inverting the order of G.

3.5 Products

In this section we recall the smash product of orthogonal spectra and orthogonal G-spectra and study its formal and homotopical properties. Like the box product of orthogonal spaces, the smash product of orthogonal spectra is a special case of Day's convolution product on categories of enriched functors, compare Appendix C. The main homotopical result about the smash product is the 'flatness theorem' (Theorem 3.5.10), showing that smashing with

G-flat orthogonal G-spectra (in the sense of Definition 3.5.7) preserves $\underline{\pi}_*$-isomorphisms.

The smash product is intimately related to pairings of equivariant stable homotopy groups that we recall in Construction 3.5.12; the main properties of these pairings are summarized in Theorem 3.5.14. When specialized to equivariant ring spectra, these pairings turn the equivariant stable homotopy groups into a graded ring, compare Corollary 3.5.17.

The smash product of orthogonal spectra is characterized by a universal property that we recall now. The indexing category $\mathbf{O}$ for orthogonal spectra has a symmetric monoidal product by direct sum as follows. We denote by $\mathbf{O} \wedge \mathbf{O}$ the category enriched in based spaces whose objects are pairs of inner product spaces and whose morphism spaces are smash products of morphism spaces in $\mathbf{O}$. A based continuous functor

$$\oplus \ : \ \mathbf{O} \wedge \mathbf{O} \ \longrightarrow \ \mathbf{O}$$

is defined on objects by orthogonal direct sum, and on morphism spaces by

$$\begin{aligned} \mathbf{O}(V, W) \wedge \mathbf{O}(V', W') \ &\longrightarrow \ \mathbf{O}(V \oplus W, V' \oplus W') \\ (w, \varphi) \wedge (w', \varphi') \quad &\longmapsto \quad ((w, w'), \varphi \oplus \varphi') \ . \end{aligned}$$

A *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spectra (X, Y) to an orthogonal spectrum Z is a natural transformation

$$b \ : \ X \bar{\wedge} Y \ \longrightarrow \ Z \circ \oplus$$

of continuous functors $\mathbf{O} \wedge \mathbf{O} \longrightarrow \mathbf{T}_*$; here $X \bar{\wedge} Y$ is the 'external smash product' defined by $(X \bar{\wedge} Y)(V, W) = X(V) \wedge Y(W)$. A bimorphism thus consists of based continuous maps

$$b_{V,W} \ : \ X(V) \wedge Y(W) \ \longrightarrow \ Z(V \oplus W)$$

for all inner product spaces V and W that form morphisms of orthogonal spectra in each variable separately. A smash product of two orthogonal spectra is now a universal example of a bimorphism from (X, Y).

Definition 3.5.1. A *smash product* of two orthogonal spectra X and Y is a pair $(X \wedge Y, i)$ consisting of an orthogonal spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every orthogonal spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \ f \mapsto fi = \{f(V \oplus W) \circ i_{V,W}\}_{V,W} \quad (3.5.2)$$

is bijective.

Since the index category $\mathbf{O}$ is skeletally small and the base category $\mathbf{T}_*$ is cocomplete, every pair of orthogonal spectra has a smash product by Proposition C.5. Often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (3.5.2) as the *universal property* of the smash product of orthogonal spectra.

While the smash products for pairs of orthogonal spectra are choices, there is a preferred way to extend any chosen smash products to a functor in two variables (Construction C.8); this functor has a preferred symmetric monoidal structure, i.e., distinguished natural associativity and symmetry isomorphisms

$$\alpha_{X,Y,Z} \;:\; (X \wedge Y) \wedge Z \;\longrightarrow\; X \wedge (Y \wedge Z) \quad \text{and} \quad \tau_{X,Y} \;:\; X \wedge Y \;\longrightarrow\; Y \wedge X$$

(see Construction C.9). Together with strict unit isomorphisms $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$, these satisfy the coherence conditions of a symmetric monoidal category, compare Day's Theorem C.10. The smash product of orthogonal spectra is *closed* symmetric monoidal in the sense that the smash product is adjoint to an internal Hom spectrum. We won't use the internal function spectrum, so we don't go into any details.

When a compact Lie group G acts on the orthogonal spectra X and Y, then $X \wedge Y$ becomes an orthogonal G-spectrum via the diagonal action. So the smash product lifts to a symmetric monoidal closed structure

$$\wedge \;:\; G\mathcal{S}p \times G\mathcal{S}p \;\longrightarrow\; G\mathcal{S}p$$

on the category of orthogonal G-spectra.

Example 3.5.3 (Smash product with a free orthogonal spectrum)**.** We give a 'formula' for the smash product with a free orthogonal spectrum, i.e., the represented functor $\mathbf{O}(W, -)$ for some inner product space W. A general feature of Day type convolution products is that the convolution product of represented functors is represented, see Remark C.11. In the case at hand this specializes to a preferred isomorphism

$$\mathbf{O}(V, -) \wedge \mathbf{O}(W, -) \;\cong\; \mathbf{O}(V \oplus W, -)$$

specified, via the universal property (3.5.2), by the bimorphism with (U, U')-component

$$\oplus \;:\; \mathbf{O}(V, U) \wedge \mathbf{O}(W, U') \;\longrightarrow\; \mathbf{O}(V \oplus W, U \oplus U') \,. \tag{3.5.4}$$

Now we let X be any orthogonal spectrum. We give a formula for the value at $V \oplus W$ of the smash product $X \wedge \mathbf{O}(W, -)$. The composite

$$X(V) \xrightarrow{-\wedge \mathrm{Id}_W} X(V) \wedge \mathbf{O}(W, W) \xrightarrow{i_{V,W}} (X \wedge \mathbf{O}(W, -))(V \oplus W)$$

is $O(V)$-equivariant, where $O(V)$ acts on the target by restriction along the homomorphism $- \oplus W : O(V) \longrightarrow O(V \oplus W)$. So the map extends to an $O(V \oplus W)$-equivariant based map

$$O(V \oplus W) \ltimes_{O(V)} X(V) \longrightarrow (X \wedge \mathbf{O}(W, -))(V \oplus W) . \tag{3.5.5}$$

We claim that this map is a homeomorphism. To see that we first check the special case where $X = \mathbf{O}(U, -)$ is itself a represented functor. The isomorphism (3.5.4) turns (3.5.5) into the map

$$\begin{aligned} O(V \oplus W) \ltimes_{O(V)} \mathbf{O}(U, V) &\longrightarrow \quad \mathbf{O}(U \oplus W, V \oplus W) \\ [A, (v, \varphi)] \quad &\longmapsto (A \cdot (v, 0), A \circ (\varphi \oplus W)) \end{aligned}$$

which is a homeomorphism by inspection. Source and target of the map (3.5.5) commute with smash products with based spaces and colimits in the variable X. Since every orthogonal spectrum is a coend of represented orthogonal spectra smashed with based spaces, the map (3.5.5) is an isomorphism in general.

Our next topic is a flatness result, proving that smashing with certain orthogonal G-spectra preserves $\underline{\pi}_*$-isomorphisms. To define the relevant class, the 'G-flat orthogonal G-spectra', we introduce the *skeleton filtration*, a functorial way to write an orthogonal spectrum as a sequential colimit of spectra which are made from the information below a fixed level. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal spectrum need not be injective. Since an orthogonal G-spectrum is the same data as an orthogonal spectrum with continuous G-action, and since skeleta are functorial, the skeleta of an orthogonal G-spectrum automatically come as orthogonal G-spectra.

Construction 3.5.6 (Skeleton filtration of orthogonal spectra)**.** As in the unstable situation in Section 1.2, the skeleton filtration is a special case of the general skeleton filtration on certain enriched functor categories that we discuss in Appendix C. Indeed, if we specialize the base category to $\mathcal{V} = \mathbf{T}_*$, the category of based spaces under smash product, and the index category to $\mathcal{D} = \mathbf{O}$, then the functor category $\mathcal{D}^*$ becomes the category $\mathcal{S}p$ of orthogonal spectra. The dimension function needed in the construction and analysis of skeleta is the vector space dimension. For every orthogonal spectrum X and every $m \geq 0$, the general theory provides an *m-skeleton*,

$$\mathrm{sk}^m X \;=\; l_m(X^{\leq m}) \,,$$

the extension of the restriction of X to $\mathbf{O}_{\leq m}$, and a natural morphism i_m : $\mathrm{sk}^m X \longrightarrow X$, the counit of the adjunction $(l_m, (-)^{\leq m})$. The value

$$i_m(V) \;:\; (\mathrm{sk}^m X)(V) \;\longrightarrow\; X(V)$$

is an isomorphism for all inner product spaces V of dimension at most m. The *mth latching space* of X is the based $O(m)$-space

$$L_m X \ = \ (\mathrm{sk}^{m-1} X)(\mathbb{R}^m) \ ;$$

it comes with a natural based $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) \ : \ L_m X \ \longrightarrow \ X(\mathbb{R}^m) \ ,$$

the *mth latching map*. We also agree to set $\mathrm{sk}^{-1} X = *$, the trivial orthogonal spectrum, and $L_0 X = *$, a one-point space.

The different skeleta are related by natural morphisms $j_m : \mathrm{sk}^{m-1} X \longrightarrow \mathrm{sk}^m X$, for all $m \geq 0$, such that $i_m \circ j_m = i_{m-1}$. The sequence of skeleta stabilizes to X in a very strong sense. For every inner product space V, the maps $j_m(V)$ and $i_m(V)$ are isomorphisms as soon as $m > \dim(V)$. In particular, $X(V)$ is a colimit, with respect to the maps $i_m(V)$, of the sequence of maps $j_m(V)$. Since colimits in the category of orthogonal spectra are created objectwise, we deduce that the orthogonal spectrum X is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Now we can define the class of G-flat orthogonal G-spectra for which smash product is homotopical. As we remarked earlier, the skeleta of (the underlying orthogonal spectrum of) an orthogonal G-spectrum inherit a continuous G-action by functoriality. In other words, the skeleta and the various morphisms between them lift to endofunctors and natural transformations on the category of orthogonal G-spectra. If X is an orthogonal G-spectrum, then the $O(m)$-space $L_m X$ comes with a commuting action by G, again by functoriality of the latching space. Moreover, the latching morphism $\nu_m : L_m X \longrightarrow X(\mathbb{R}^m)$ is $(G \times O(m))$-equivariant.

Definition 3.5.7. Let G be a compact Lie group. An orthogonal G-spectrum X is *G-flat* if for every $m \geq 0$ the latching map $\nu_m : L_m X \longrightarrow X(\mathbb{R}^m)$ is a $(G \times O(m))$-cofibration.

When G is the trivial group, we simply speak of *flat* orthogonal spectra. These flat orthogonal spectra play a special role as the cofibrant objects in the global model structure on the category of orthogonal spectra, to be established in Theorem 4.3.18 below.

Remark 3.5.8. The G-flat orthogonal spectra are the cofibrant objects in a certain model structure on the category of orthogonal G-spectra, the $\mathbb{S}$-model structure of Stolz [163, Thm. 2.3.27]. This model structure has more cofibrant objects than the one of Mandell-May [108, III Thm. 4.2], whose 'q-cofibrant' orthogonal G-spectra are precisely the ones for which the latching map ν_m :

$L_m X \longrightarrow X(\mathbb{R}^m)$ is a $(G \times O(m))$-cofibration and in addition the group $O(m)$ acts freely on the complement of the image of ν_m.

A convenient feature of the class of flat equivariant spectra is that they are closed under various change-of-group functors. For example, if $\alpha : K \longrightarrow G$ is a continuous homomorphism and X an orthogonal G-spectrum, then the orthogonal K-spectrum $\alpha^* X$ satisfies

$$L_m(\alpha^* X) \;=\; (\alpha \times O(m))^*(L_m X) \quad \text{and} \quad (\alpha^* X)(\mathbb{R}^m) \;=\; (\alpha \times O(m))^*(X(\mathbb{R}^m))$$

as $(K \times O(m))$-spaces. Since restriction along $\alpha \times O(m)$ preserves cofibrations (see Proposition B.14 (i)), the restriction functor $\alpha^* : G\mathcal{S}p \longrightarrow K\mathcal{S}p$ takes G-flat orthogonal G-spectra to K-flat orthogonal K-spectra. Of particular relevance for global considerations is the special case of the unique homomorphism $G \longrightarrow e$ to the trivial group. In this case we obtain that a flat orthogonal spectrum is in particular G-flat when we give it the trivial G-action.

A similar argument applies to induction of equivariant spectra. Indeed, if H is a closed subgroup of G and Y an orthogonal H-spectrum, then

$$\begin{aligned} L_m(G \ltimes_H Y) \;&\cong\; (G \times O(m)) \ltimes_{H \times O(m)} (L_m Y) \quad \text{and} \\ (G \ltimes_H Y)(\mathbb{R}^m) \;&\cong\; (G \times O(m)) \ltimes_{H \times O(m)} Y(\mathbb{R}^m)\,. \end{aligned}$$

Since induction from $H \times O(m)$ to $G \times O(m)$ preserves cofibrations (see Proposition B.14 (ii)), the induction functor $G \ltimes_H - : H\mathcal{S}p \longrightarrow G\mathcal{S}p$ takes H-flat orthogonal H-spectra to G-flat orthogonal G-spectra.

The main purpose of the latching object and latching morphism are to record how one skeleton of an orthogonal G-spectrum is obtained from the previous one. We denote by

$$G_m \;:\; O(m)\mathbf{T}_* \;\longrightarrow\; \mathcal{S}p$$

the left adjoint to the evaluation functor $X \mapsto X(\mathbb{R}^m)$. Explicitly, this functor sends a based $O(m)$-space A to the orthogonal spectrum

$$G_m A \;=\; \mathbf{O}(\mathbb{R}^m, -) \wedge_{O(m)} A\,.$$

Since G_m is a continuous functor, it takes $(G \times O(m))$-spaces to G-orthogonal spectra, by functoriality. Proposition C.17 and the fact that pushouts of orthogonal G-spectra are created on underlying orthogonal spectra yield:

Proposition 3.5.9. *For every orthogonal G-spectrum X and every $m \geq 0$ the commutative square*

$$\begin{tikzcd} G_m L_m X \arrow[r, "G_m \nu_m"] \arrow[d] & G_m X(\mathbb{R}^m) \arrow[d] \\ \mathrm{sk}^{m-1} X \arrow[r, "j_m"'] & \mathrm{sk}^m X \end{tikzcd}$$

is a pushout of orthogonal G-spectra. The two vertical morphisms are adjoint to the identity of $L_m X$ and $X(\mathbb{R}^m)$, respectively.

The next theorem shows that smashing with a G-flat orthogonal G-spectrum preserves $\underline{\pi}_*$-isomorphisms. This result is due to Stolz [163, Prop. 2.3.29], because the G-flat orthogonal G-spectra are precisely the $\mathbb{S}$-cofibrant objects in the sense of [163, Def. 2.3.4]; a proof can also be found in [32, Prop. 2.10.1]. The theorem is stronger than the earlier result of Mandell and May [108, III Prop. 7.3] because the class of G-flat of orthogonal G-spectra is strictly larger than the cofibrant G-spectra in the sense of [108, Ch. III].

Since Stolz' thesis [163] is not published, the notation and level of generality in [32] is different from ours, and the characterization of flat objects in terms of latching maps is not explicitly mentioned in [32] nor [163], we spell out the argument.

Theorem 3.5.10 (Flatness theorem)**.** *Let G be a compact Lie group and X a G-flat orthogonal G-spectrum. Then the functor $- \wedge X$ preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.*

Proof We go through a sequence of six steps, proving successively more general cases of the theorem. We call an orthogonal G-spectrum *G-stably contractible* if all of its equivariant homotopy groups vanish, for all closed subgroups of G. In Step 1 through 5, we let C be a G-stably contractible orthogonal G-spectrum, and we show for successively more general X that $C \wedge X$ is again G-stably contractible.

Step 1: We let K be another compact Lie group, W a K-representation and A a based $(G \times K)$-space. We define an orthogonal G-spectrum

$$A \triangleright_{K,W} C \;=\; A \wedge_K (\mathrm{sh}^W C)$$

at an inner product space U as

$$(A \triangleright_{K,W} C)(U) \;=\; A \wedge_K C(U \oplus W) \;;$$

the K-action on $C(U \oplus W)$ is through the K-action on W; the G-action on the smash product is diagonally, through the G-actions on A and on C.

Now we assume in addition that A is a finite based $(G\times K)$-CW-complex and

the K-action is free (away from the basepoint). We claim that then $A \triangleright_{K,W} C$ is also G-stably contractible. We argue by the number of equivariant cells, starting with $A = *$, in which case there is nothing to show. Now we assume that the claim holds for A; we let B be obtained from A by attaching an equivariant cell $(G \times K)/\Gamma \times D^k$, where Γ is a closed subgroup of $G \times K$. The inclusion $A \longrightarrow B$ is an h-cofibration of $(G \times K)$-spaces, hence the induced morphism $A \triangleright_{K,W} C \longrightarrow B \triangleright_{K,W} C$ is an h-cofibration of orthogonal G-spectra. By the long exact homotopy group sequence (Corollary 3.1.38 (i)) it suffices to show the claim for the quotient:

$$(B \triangleright_{K,W} C)/(A \triangleright_{K,W} C) \;\cong\; (A/B) \triangleright_{K,W} C \;\cong\; S^k \wedge ((G \times K)/\Gamma)_+ \triangleright_{K,W} C$$

By the suspension isomorphism we may show that $((G \times K)/\Gamma)_+ \triangleright_{K,W} C$ is G-stably contractible. Since the K-action on B is free, the stabilizer group Γ is the graph of a continuous homomorphism $\alpha : H \longrightarrow K$ from some closed subgroup H of G (namely the projection of Γ to G). So we can rewrite

$$((G \times K)/\Gamma)_+ \triangleright_{K,W} C \;\cong\; G \ltimes_H (\mathrm{sh}^{\alpha^*(W)} C) .$$

Since C is G-stably contractible, it is also H-stably contractible. The orthogonal H-spectrum $C \wedge S^{\alpha^*(W)}$ is then H-stably contractible by Proposition 3.2.19 (ii). Since $\mathrm{sh}^{\alpha^*(W)} C$ is $\underline{\pi}_*$-isomorphic to $C \wedge S^{\alpha^*(W)}$ (by Proposition 3.1.25 (ii)), it is H-stably contractible. Since the induction functor $G \ltimes_H -$ preserves $\underline{\pi}_*$-isomorphisms (by Corollary 3.2.21), we conclude that $((G \times K)/\Gamma)_+ \triangleright_{K,W} C$ is G-stably contractible. This finishes the inductive argument, and hence the proof that the orthogonal G-spectrum $A \triangleright_{K,W} C$ is G-stably contractible.

Step 2: We let G be a compact Lie group, K a closed subgroup of $O(m)$ and $\alpha : G \longrightarrow W_{O(m)}K$ a continuous homomorphism to the Weyl group of K in $O(m)$. The represented orthogonal spectrum $\mathbf{O}(\mathbb{R}^m, -)$ has a right $O(m)$-action through the tautological action on $\mathbb{R}^m$. The semifree orthogonal spectrum

$$G_m(O(m)/K_+) \;=\; \mathbf{O}(\mathbb{R}^m, -)/K$$

has a residual action of the Weyl group $W_{O(m)}K$, and we restrict scalars along α to obtain the orthogonal G-spectrum

$$\alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K) .$$

We claim that then the orthogonal G-spectrum $C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K)$ is G-stably contractible. We show that $\pi_k^H(C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K)) = 0$ for $k \geq 0$ and every closed subgroup H of G; for $k < 0$ we exploit the fact that $\pi_k^H(C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K))$ is isomorphic to $\pi_0^H((C \wedge S^{-k}) \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K))$, so applying the argument to $C \wedge S^{-k}$ instead of C shows the vanishing in negative dimensions. Since the input data is stable under passage to closed subgroups

of G (just restrict α to such a subgroup), it is no loss of generality to assume $H = G$.

We can represent every class of $\pi_k^G(C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K))$ by a based G-map

$$f \ : \ S^{V \oplus \mathbb{R}^{m+k}} \ \longrightarrow \ (C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K))(V \oplus \mathbb{R}^m)$$

for some G-representation V. In (3.5.5) we exhibited an isomorphism

$$O(V \oplus \mathbb{R}^m) \ltimes_{O(V)} C(V) \ \xrightarrow{\cong} \ (C \wedge \mathbf{O}(\mathbb{R}^m, -))(V \oplus \mathbb{R}^m) \ .$$

Under this isomorphism, the K-action on $\mathbf{O}(\mathbb{R}^m, -)$ becomes the right translation K-action on $O(V \oplus \mathbb{R}^m)$, so passage to K-orbits yields an isomorphism

$$O(V \oplus \mathbb{R}^m)/K_+ \wedge_{O(V)} C(V) \ \cong \ (C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m, -)/K))(V \oplus \mathbb{R}^m) \ .$$

The G-action on the right-hand side is diagonally from three ingredients: the G-action on C, the G-action on V, and the G-action on $\mathbf{O}(\mathbb{R}^m, -)/K$ through α. So G acts on the left-hand side diagonally, also through the actions on V (by left translation on $O(V \oplus \mathbb{R}^m)$), on $\mathbb{R}^m$ through α, and on C.

The target of f is thus G-equivariantly homeomorphic to

$$O(V \oplus \mathbb{R}^m)/K_+ \wedge_{O(V)} C(V) \ = \ \left(O(V \oplus \mathbb{R}^m)/K_+ \triangleright_{O(V),V} C\right)(0) \ .$$

Via this isomorphism, we view f as representing a G-equivariant homotopy class in

$$\pi_{m+k}^G \left(\Omega^V (O(V \oplus \mathbb{R}^m)/K_+ \triangleright_{O(V),V} C)\right) \ . \tag{3.5.11}$$

The homogeneous space $O(V \oplus \mathbb{R}^m)/K$ has a left G-action by translation through the composite

$$G \ \longrightarrow \ O(V) \ \xrightarrow{-\oplus \mathbb{R}^m} \ O(V \oplus \mathbb{R}^m)$$

and another G-action by restriction of the residual $W_{O(m)}(K)$-action along α; altogether, the group G acts diagonally. This space also has a commuting free right action of $O(V)$ by right translation. Since $O(V \oplus \mathbb{R}^m)/K$ is a smooth manifold and the $(G \times O(V))$-action is smooth, Illman's theorem [84, Cor. 7.2] provides a finite $(G \times O(V))$-CW-structure on $O(V \oplus \mathbb{R}^m)/K$; so $O(V \oplus \mathbb{R}^m)/K_+$ is cofibrant as a based $(G \times O(V))$-space, with free right $O(V)$-action. Since C is G-stably contractible, so is $O(V \oplus \mathbb{R}^m)/K_+ \triangleright_{O(V),V} C$, by Step 1. The orthogonal G-spectrum $\Omega^V (O(V \oplus \mathbb{R}^m)/K_+ \triangleright_{O(V),V} C)$ is then G-stably contractible by Proposition 3.1.40 (ii). In particular, the group (3.5.11) is trivial, so there is a G-representation U such that the composite

$$\begin{aligned} S^{U \oplus V \oplus \mathbb{R}^{m+k}} \ &\xrightarrow{S^U \wedge f} \ S^U \wedge (O(V \oplus \mathbb{R}^m)/K_+) \wedge_{O(V)} C(V) \\ &\xrightarrow{\sigma_{U,V}} \ O(V \oplus \mathbb{R}^m)/K_+ \wedge_{O(V)} C(U \oplus V) \end{aligned}$$

is G-equivariantly null-homotopic. The structure map

$$\begin{aligned}\sigma_{U,V\oplus\mathbb{R}^m} \ : \ S^U \wedge (C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K))(V\oplus\mathbb{R}^m) \\ \longrightarrow \ (C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K))(U\oplus V\oplus\mathbb{R}^m)\end{aligned}$$

of the G-spectrum $C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K)$ factors as the composite

$$\begin{aligned}& S^U \wedge O(V\oplus\mathbb{R}^m)/K_+ \wedge_{O(V)} C(V) \xrightarrow{O(V\oplus\mathbb{R}^m)/K_+\wedge_{O(V)}\sigma_{U,V}} \\ & O(V\oplus\mathbb{R}^m)/K_+ \wedge_{O(V)} C(U\oplus V) \ \longrightarrow \ O(U\oplus V\oplus\mathbb{R}^m)/K_+ \wedge_{O(U\oplus V)} C(U\oplus V)\ .\end{aligned}$$

So also the stabilization $\sigma_{U,V} \circ (S^U \wedge f)$ in the orthogonal G-spectrum $C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K)$ is equivariantly null-homotopic. Since the stabilization represents the same element as f, this shows that $\pi_k^G(C \wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K)) = 0$.

Step 3: We let Γ be a closed subgroup of $G\times O(m)$. We claim that smashing with the orthogonal G-spectrum

$$G_m\,((G\times O(m))/\Gamma_+) \ = \ \mathbf{O}(\mathbb{R}^m,-) \wedge_{O(m)} (G\times O(m))/\Gamma_+$$

preserves G-stably contractible orthogonal G-spectra. To see that we interpret Γ as a 'generalized graph'. We let $H\leq G$ be the projection of Γ onto the first factor. We let

$$K \ = \ \{A\in O(m)\ :\ (1,A)\in\Gamma\}$$

be the trace of Γ in $O(m)$. Then Γ is the graph of the continuous homomorphism

$$\alpha \ : \ H \ \longrightarrow \ W_{O(m)}K\ , \quad \alpha(h) \ = \ \{A\in O(m)\ :\ (h,A)\in\Gamma\}\ ,$$

in the sense that

$$\Gamma \ = \ \bigcup\nolimits_{h\in H} \{h\}\times\alpha(h)\ .$$

Moreover, the orthogonal G-spectrum $G_m\,((G\times O(m))/\Gamma_+)$ is isomorphic to

$$G\ltimes_H (\alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K))\ .$$

So for every orthogonal G-spectrum C, the shearing isomorphism becomes an isomorphism of orthogonal G-spectra

$$\begin{aligned}C\wedge (G_m((G\times O(m))/\Gamma_+)) \ &\cong \ C\wedge (G\ltimes_H (\alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K))) \\ &\cong \ G\ltimes_H (\mathrm{res}^G_H(C)\wedge\alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K))\ .\end{aligned}$$

If C is G-stably contractible, then it is also H-stably contractible; so $\mathrm{res}^G_H(C)\wedge \alpha^*(\mathbf{O}(\mathbb{R}^m,-)/K)$ is H-stably contractible by Step 2. Since the induction functor $G\ltimes_H -$ takes $\underline{\pi}_*$-isomorphisms of orthogonal H-spectra to $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra (Corollary 3.2.21), this shows that $C\wedge G_m((G\times O(m))/\Gamma_+)$ is G-stably contractible.

Step 4: We let A be a cofibrant based $(G\times O(m))$-space. We claim that smashing with the orthogonal G-spectrum $G_m A$ preserves G-stably contractible orthogonal G-spectra. A cofibrant based $(G \times O(m))$-space is equivariantly homotopy equivalent to a based $(G \times O(m))$-CW-complex, so it is no loss of generality to assume an equivariant CW-structure with skeleton filtration

$$* \;=\; A_{-1} \;\subset\; A_0 \;\subset\; \ldots \;\subset\; A_n \;\subset\; \ldots \;.$$

We show first, by induction on n, that smashing with the orthogonal G-spectrum $G_m A_n$ preserves G-stably contractible orthogonal G-spectra. The induction starts with $n = -1$, where there is nothing to show. For $n \geq 0$ the inclusion $A_{n-1} \longrightarrow A_n$ is an h-cofibration of based $(G \times O(m))$-spaces, so the induced morphism $C \wedge G_m A_{n-1} \longrightarrow C \wedge G_m A_n$ is an h-cofibration of orthogonal G-spectra. By induction and the long exact sequence of equivariant homotopy groups (Corollary 3.1.38) it suffices to show that the quotient

$$(C \wedge G_m A_n)/(C \wedge G_m A_{n-1}) \;\cong\; C \wedge G_m(A_n/A_{n-1})$$

is G-stably contractible. Now A_n/A_{n-1} is $(G \times O(m))$-equivariantly isomorphic to a wedge of summands of the form $S^n \wedge (G \times O(m))/\Gamma_+$, for various closed subgroups Γ of $G \times O(m)$. So the quotient spectrum is isomorphic to the n-fold suspension of a wedge of orthogonal G-spectra of the form $C \wedge G_m((G \times O(m))/\Gamma_+)$. These wedge summands are G-stably contractible by Step 3. Since equivariant homotopy groups take wedges to direct sums and suspension shifts equivariant homotopy groups, this proves that $C \wedge G_m(A_n/A_{n-1})$ is G-stably contractible. This completes the inductive step, and the proof for all finite-dimensional based $(G \times O(m))$-CW-complexes.

Since A is the sequential colimit, along h-cofibrations of based $(G \times O(m))$-spaces, of the skeleta A_n, the orthogonal G-spectrum $C \wedge G_m A$ is the sequential colimit, along h-cofibrations of orthogonal G-spectra, of the sequence with terms $C \wedge G_m A_n$. Since h-cofibrations are in particular level-wise closed embeddings, equivariant homotopy groups commute with such sequential colimits (Proposition 3.1.41 (i)), so also $C \wedge G_m A$ is G-stably trivial.

Step 5: We let X be a G-flat orthogonal G-spectrum. We show first, by induction on m, that $C \wedge (\mathrm{sk}^m X)$ is G-stably contractible, where $\mathrm{sk}^m X$ is the m-skeleton in the sense of Construction 3.5.6. The induction starts with $m = -1$, where there is nothing to show. For $m \geq 0$ the morphism $j_m : \mathrm{sk}^{m-1} X \longrightarrow \mathrm{sk}^m X$ is an h-cofibration of orthogonal G-spectra, hence so is the induced morphism $C \wedge j_m : C \wedge (\mathrm{sk}^{m-1} X) \longrightarrow C \wedge (\mathrm{sk}^m X)$. By induction and the long exact sequence of equivariant homotopy groups (Corollary 3.1.38) we may show that the quotient

$$(C \wedge \mathrm{sk}^m X)/(C \wedge \mathrm{sk}^{m-1} X) \;\cong\; C \wedge (\mathrm{sk}^m X/\,\mathrm{sk}^{m-1} X) \;\cong\; C \wedge G_m(X(\mathbb{R}^m)/L_m X)$$

is G-stably contractible. Since X is G-flat, $X(\mathbb{R}^m)/L_m X$ is a cofibrant based $(G\times O(m))$-space, and the G-equivariant homotopy groups of $C \wedge G_m(X(\mathbb{R}^m)/L_m X)$ vanish by Step 4. This finishes the inductive proof that $C \wedge (\mathrm{sk}^m X)$ is G-stably contractible. The spectrum $C\wedge X$ is the sequential colimit, along h-cofibrations, of the orthogonal G-spectra $C\wedge(\mathrm{sk}^m X)$; equivariant homotopy groups commute with such colimits, so $C \wedge X$ is G-stably contractible.

Step 6: We let X be a G-flat orthogonal G-spectrum and $f : A \longrightarrow B$ a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. Then the mapping cone $C(f)$ is G-stably contractible by the long exact homotopy group sequence (Proposition 3.1.36). So the smash product $C(f) \wedge X$ is G-stably contractible by Step 5. Since $C(f) \wedge X$ is isomorphic to the mapping cone of $f \wedge X : A \wedge X \longrightarrow B \wedge X$, the morphism $f \wedge X$ induces isomorphisms on G-equivariant stable homotopy groups, again by the long exact homotopy group sequence. □

Now we turn to products on equivariant homotopy groups.

Construction 3.5.12. Given a compact Lie group G and two orthogonal G-spectra X and Y, we endow the equivariant homotopy groups with a pairing

$$\times \; : \; \pi_k^G(X) \times \pi_l^G(Y) \longrightarrow \pi_{k+l}^G(X \wedge Y) , \tag{3.5.13}$$

where k and l are integers. We let

$$f \; : \; S^{U\oplus\mathbb{R}^{m+k}} \longrightarrow X(U \oplus \mathbb{R}^m) \quad \text{and} \quad g \; : \; S^{V\oplus\mathbb{R}^{n+l}} \longrightarrow Y(V \oplus \mathbb{R}^n)$$

represent classes in $\pi_k^G(X)$ and $\pi_l^G(Y)$, respectively, for suitable G-representations U and V. The class $[f]\times[g]$ in $\pi_{k+l}^G(X\wedge Y)$ is then represented by the composite

$$\begin{aligned} S^{U\oplus V\oplus\mathbb{R}^{m+n+k+l}} \cong S^{U\oplus\mathbb{R}^{m+k}} \wedge S^{V\oplus\mathbb{R}^{n+l}} &\xrightarrow{f\wedge g} X(U \oplus \mathbb{R}^m) \wedge Y(V \oplus \mathbb{R}^n) \\ &\xrightarrow{i_{U\oplus\mathbb{R}^m, V\oplus\mathbb{R}^n}} (X \wedge Y)(U \oplus \mathbb{R}^m \oplus V \oplus \mathbb{R}^n) \\ &\xrightarrow{(X\wedge Y)(U\oplus\tau_{\mathbb{R}^m,V}\oplus\mathbb{R}^n)} (X \wedge Y)(U \oplus V \oplus \mathbb{R}^{m+n}) . \end{aligned}$$

The first homeomorphism shuffles the sphere coordinates. We omit the proof that the class of the composite only depends on the classes of f and g.

The pairing of equivariant homotopy groups has several expected properties that we summarize in the next theorem.

Theorem 3.5.14. *Let G be a compact Lie group and X, Y and Z orthogonal G-spectra.*

(i) (Biadditivity) *The product $\times : \pi_k^G(X)\times\pi_l^G(Y) \longrightarrow \pi_{k+l}^G(X\wedge Y)$ is biadditive.*
(ii) (Unitality) *The class $1 \in \pi_0^G(\mathbb{S})$ represented by the identity of S^0 is a two-sided unit for the pairing $\times$.*

(iii) (Associativity) *For all* $x \in \pi_k^G(X)$, $y \in \pi_l^G(Y)$ *and* $z \in \pi_m^G(Z)$ *the relation*

$$x \times (y \times z) \;=\; (x \times y) \times z$$

holds in $\pi_{k+l+m}^G(X \wedge Y \wedge Z)$.

(iv) (Commutativity) *For all* $x \in \pi_k^G(X)$ *and* $y \in \pi_l^G(Y)$ *the relation*

$$\tau_*^{X,Y}(x \times y) \;=\; (-1)^{kl} \cdot (y \times x)$$

holds in $\pi_{k+l}^G(Y \wedge X)$, *where* $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ *is the symmetry isomorphism of the smash product.*

(v) (Restriction) *For all* $x \in \pi_k^G(X)$ *and* $y \in \pi_l^G(Y)$ *and all continuous homomorphisms* $\alpha : K \longrightarrow G$ *the relation*

$$\alpha^*(x) \times \alpha^*(y) \;=\; \alpha^*(x \times y)$$

holds in $\pi_{k+l}^K(\alpha^*(X \wedge Y))$.

(vi) (Transfer) *Let* H *be a closed subgroup of* G. *For all* $x \in \pi_k^G(X)$ *and* $z \in \pi_l^H(Y \wedge S^L)$, *the relation*

$$x \times \mathrm{Tr}_H^G(z) \;=\; \mathrm{Tr}_H^G(\mathrm{res}_H^G(x) \times z)$$

holds in $\pi_{k+l}^G(X \wedge Y)$, *where* $L = T_{eH}(G/H)$ *is the tangent* H*-representation. For all* $y \in \pi_l^H(Y)$, *the relation*

$$x \times \mathrm{tr}_H^G(y) \;=\; \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \times y)$$

holds in $\pi_{k+l}^G(X \wedge Y)$.

Proof (i) We deduce the additivity in the first variable from the general additivity statement in Proposition 2.2.12. We consider the two reduced additive functors

$$X \;\longmapsto\; \pi_k^G(X) \qquad \text{and} \qquad X \longmapsto \pi_{k+l}^G(X \wedge Y)$$

from the category of orthogonal G-spectra to the category of abelian groups. Proposition 2.2.12 shows that for every $y \in \pi_l^G(Y)$ the natural transformation

$$- \times y \;:\; \pi_k^G(-) \;\longrightarrow\; \pi_{k+l}^G(- \wedge Y)$$

is additive. Additivity in the second variable is proved in the same way. Properties (ii), (iii) and (v) are straightforward consequences of the definitions. The sign $(-1)^{kl}$ in the commutativity relation (iv) is the degree of the map that interchanges a k-sphere with an l-sphere. Indeed, if

$$f \;:\; S^{U \oplus \mathbb{R}^{m+k}} \;\longrightarrow\; X(U \oplus \mathbb{R}^m) \quad \text{and} \quad g \;:\; S^{V \oplus \mathbb{R}^{n+l}} \;\longrightarrow\; Y(V \oplus \mathbb{R}^n)$$

represent classes in $\pi_k^G(X)$ respectively $\pi_l^G(Y)$, then the left and right vertical composites in the diagram

$$\begin{tikzcd}
S^{U\oplus V\oplus\mathbb{R}^{m+n+k+l}} \arrow[rr, "\tau_{U,V}\wedge\tau_{m,n}\wedge S^{k+l}"] \arrow[d, "f\times g"'] & & S^{V\oplus U\oplus\mathbb{R}^{n+m+k+l}} \arrow[d, "S^{V\oplus U\oplus\mathbb{R}^{n+m}}\wedge\tau_{k,l}"] \\
(X\wedge Y)(U\oplus V\oplus\mathbb{R}^{m+n}) \arrow[d, "\tau^{X,Y}(U\oplus V\oplus\mathbb{R}^{m+n})"'] & & S^{V\oplus U\oplus\mathbb{R}^{n+m+l+k}} \arrow[d, "g\times f"] \\
(Y\wedge X)(U\oplus V\oplus\mathbb{R}^{m+n}) \arrow[rr, "(Y\wedge X)(\tau_{U,V}\oplus\tau_{m,n})"'] & & (Y\wedge X)(V\oplus U\oplus\mathbb{R}^{n+m})
\end{tikzcd}$$

represent $\tau_*^{X,Y}([f]\times[g])$ and $(-1)^{kl}\cdot([g]\times[f])$. Since the two composites differ by conjugation by a G-equivariant linear isometry, they represent the same class by Proposition 3.1.14 (ii).

(vi) The following diagram of abelian groups commutes by naturality of the pairing (3.5.13), and because restriction from G to H is multiplicative:

$$\begin{tikzcd}
\pi_k^G(X)\times\pi_l^G(Y\wedge G/H_+) \arrow[r, "\times"] \arrow[d, "\mathrm{Id}\times\mathrm{res}^G_H"] \arrow[dd, bend right, "\mathrm{Id}\times\mathrm{Wirth}^G_H"'] & \pi^G_{k+l}(X\wedge Y\wedge G/H_+) \arrow[d, "\mathrm{res}^G_H"'] \arrow[dddr, bend left, "\mathrm{Wirth}^G_H"] & \\
\pi_k^G(X)\times\pi_l^H(Y\wedge G/H_+) \arrow[d, "\mathrm{Id}\times(Y\wedge l)_*"] \arrow[dr, "\mathrm{res}^G_H\times\mathrm{Id}"] & \pi^H_{k+l}(X\wedge Y\wedge G/H_+) \arrow[ddr, bend left, "(X\wedge Y\wedge l)_*"] & \\
\pi_k^G(X)\times\pi_l^H(Y\wedge S^L) \arrow[dr, "\mathrm{res}^G_H\times\mathrm{Id}"'] & \pi_k^H(X)\times\pi_l^H(Y\wedge G/H_+) \arrow[u, "\times"'] \arrow[d, "\mathrm{Id}\times(Y\wedge l)_*"] & \\
 & \pi_k^H(X)\times\pi_l^H(Y\wedge S^L) \arrow[r, "\times"'] & \pi^H_{k+l}(X\wedge Y\wedge S^L)
\end{tikzcd}$$

Here $l : G/H_+ \longrightarrow S^L$ is the H-equivariant collapse map (3.2.2). The two vertical composites are the respective Wirthmüller isomorphisms (Theorem 3.2.15). Since the external transfer is inverse to the Wirthmüller isomorphism (up to the effect of the involution $S^{-\mathrm{Id}} : S^L \longrightarrow S^L$), we can read the diagram backwards and conclude that the upper part of the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X)\times\pi_l^H(Y\wedge S^L) \arrow[r, "\mathrm{res}^G_H\times\mathrm{Id}"] \arrow[dd, "\cong"', "\mathrm{Id}\times(G\ltimes_H-)"] \arrow[ddd, bend right, "\mathrm{Id}\times\mathrm{Tr}^G_H"'] & \pi_k^H(X)\times\pi_l^H(Y\wedge S^L) \arrow[d, "\times"] \\
 & \pi^H_{k+l}(X\wedge Y\wedge S^L) \arrow[d, "G\ltimes_H-"', "\cong"] \arrow[dd, bend left, "\mathrm{Tr}^G_H"] \\
\pi_k^G(X)\times\pi_l^G(Y\wedge G/H_+) \arrow[r, "\times"'] \arrow[d, "\mathrm{Id}\times(Y\wedge p)_*"] & \pi^G_{k+l}(X\wedge Y\wedge G/H_+) \arrow[d, "(X\wedge Y\wedge p)_*"'] \\
\pi_k^G(X)\times\pi_l^G(Y) \arrow[r, "\times"'] & \pi^G_{k+l}(X\wedge Y)
\end{tikzcd}$$

Here $p : G/H_+ \longrightarrow S^0$ is the projection. The lower part of the diagram commutes by naturality of the pairing. This proves the first claim about dimension shifting transfers. The formula for the degree zero transfers follows by naturality for the inclusion of the origin $S^0 \longrightarrow S^L$. □

Definition 3.5.15. An *orthogonal ring spectrum* is a monoid in the category of orthogonal spectra with respect to the smash product. For a compact Lie group G, an *orthogonal G-ring spectrum* is a monoid in the category of orthogonal G-spectra with respect to the smash product.

An orthogonal ring spectrum is thus an orthogonal spectrum R equipped with a multiplication morphism $\mu : R \wedge R \longrightarrow R$ and a unit morphism $\eta : \mathbb{S} \longrightarrow R$ such that the associativity and unit diagrams commute (compare [105, VII.3]). A *morphism* of orthogonal ring spectra is a morphism $f : R \longrightarrow S$ of orthogonal spectra that satisfies $f \circ \mu^R = \mu^S \circ (f \wedge f)$ and $f \circ \eta^R = \eta^S$. Via the universal property of the smash product the data contained in the multiplication morphism can be made more explicit: $\mu : R \wedge R \longrightarrow R$ corresponds to a collection of based continuous maps $\mu_{V,W} : R(V) \wedge R(W) \longrightarrow R(V \oplus W)$ that together form a bimorphism. The associativity and unit conditions can also be rephrased in more explicit forms, and then we are requiring that the multiplication and unit maps make $R : \mathbf{O} \longrightarrow \mathbf{T}_*$ a lax monoidal functor. Most of the time we will specify the data of an orthogonal ring spectrum in the explicit bimorphism form.

An orthogonal ring spectrum R (or an orthogonal G-ring spectrum) is *commutative* if the multiplication morphism satisfies $\mu \circ \tau_{R,R} = \mu$. In the explicit form this is equivalent to the commutativity of the square

$$\begin{tikzcd} R(V) \wedge R(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\tau_{R(V),R(W)}"'] & R(V \oplus W) \arrow[d, "R(\tau_{V,W})"] \\ R(W) \wedge R(V) \arrow[r, "\mu_{W,V}"'] & R(W \oplus V) \end{tikzcd}$$

for all inner product spaces V and W. Equivalently, the multiplication and unit maps make $R : \mathbf{O} \longrightarrow \mathbf{T}_*$ a lax *symmetric* monoidal functor. Commutative orthogonal ring spectra already appear, with an extra point-set topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [112, IV Def. 2.1].

Since the smash product of orthogonal G-spectra is just the smash product of the underlying orthogonal spectra, endowed with the diagonal G-action, an orthogonal G-ring spectrum is nothing but an orthogonal ring spectrum equipped with a continuous G-action through homomorphisms of orthogonal ring spectra. Via the universal property of the smash product, yet another way to pack-

age the data in an orthogonal G-ring spectrum is as a continuous lax monoidal functor from the category $\mathbf{O}$ to the category of based G-spaces.

Given a compact Lie group G and an orthogonal G-ring spectrum R, we define an internal pairing

$$\cdot \;:\; \pi_k^G(R) \times \pi_l^G(R) \longrightarrow \pi_{k+l}^G(R) \tag{3.5.16}$$

on the equivariant homotopy groups of R as the composite

$$\pi_k^G(R) \times \pi_l^G(R) \xrightarrow{\times} \pi_{k+l}^G(R \wedge R) \xrightarrow{\mu_*} \pi_{k+l}^G(R) .$$

Theorem 3.5.14 then immediately implies:

Corollary 3.5.17. *Let G be a compact Lie group, R an orthogonal G-ring spectrum and H a closed subgroup of G.*

(i) *The products* (3.5.16) *make the abelian groups $\{\pi_k^G(R)\}_{k\in\mathbb{Z}}$ a graded ring. The multiplicative unit is the class of the unit map $S^0 \longrightarrow R(0)$.*
(ii) *If the multiplication of R is commutative, then the relation*

$$x \cdot y \;=\; (-1)^{kl} \cdot y \cdot x$$

holds for all classes $x \in \pi_k^G(R)$ and $y \in \pi_l^G(R)$.
(iii) (Restriction) *The restriction maps* $\mathrm{res}_H^G : \pi_*^G(R) \longrightarrow \pi_*^H(R)$ *form a homomorphism of graded rings.*
(iv) (Conjugation) *For every $g \in G$ the conjugation maps $g_\star : \pi_*^{H^g}(R) \longrightarrow \pi_*^H(R)$ form a homomorphism of graded rings.*
(v) (Reciprocity) *For all $x \in \pi_k^G(R)$ and all $y \in \pi_l^H(R)$, the relation*

$$x \cdot \mathrm{tr}_H^G(y) \;=\; \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \cdot y)$$

holds in $\pi_{k+l}^G(R)$.

Remark 3.5.18 (Norm maps)**.** A lot more is happening for commutative orthogonal G-ring spectra: strict commutativity of the multiplication not only makes the homotopy pairings graded-commutative, but it also gives rise to new operations, usually called *norm maps* $N_H^G : \pi_0^H(R) \longrightarrow \pi_0^G(R)$ for all closed subgroups H of finite index in G. To my knowledge, these norm maps were first defined by Greenlees and May [68, Def. 7.10], and they were later studied for example in [31, 77].

We don't discuss norms maps for commutative orthogonal G-ring spectra here, but norms maps and power operations will be a major topic in the context of ultra-commutative ring spectra. In Section 5.1 we define power operations on the 0th equivariant homotopy groups of an ultra-commutative ring spectrum, and we recall in Remark 5.1.7 how to turn the power operations into norm maps.

4
Global stable homotopy theory

In this chapter we embark on the investigation of global stable homotopy theory. In Section 4.1 we specialize the equivariant theory of the previous chapter to global stable homotopy types, which we model by orthogonal spectra (with no additional action of any groups). Section 4.2 introduces the category of *global functors*, the natural home of the collection of equivariant homotopy groups of a global stable homotopy type (i.e., an orthogonal spectrum). Global functors play the same role for global homotopy theory as the category of abelian groups in ordinary homotopy theory, or the category of G-Mackey functors for G-equivariant homotopy theory. Global functors are defined as additive functors on the *global Burnside category*; an explicit calculation of the global Burnside category provides the link to other notions of global Mackey functors. In the global context, the pairings on equivariant homotopy groups also give rise to a symmetric monoidal structure on the global Burnside category, and to a symmetric monoidal 'box product' of global functors.

Section 4.3 establishes the global model structure on the category of orthogonal spectra; more generally, we consider a global family $\mathcal{F}$ and define the $\mathcal{F}$-global model structure, where weak equivalences are tested on equivariant homotopy groups for all Lie groups in $\mathcal{F}$. The $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra, provided that $\mathcal{F}$ is closed under products. Section 4.4 collects aspects of global stable homotopy theory that refer to the triangulated structure of the global stable homotopy category. Specific topics are compact generators, Brown representability, a t-structure whose heart is the category of global functors, global Postnikov sections and Eilenberg-Mac Lane spectra of global functors.

The final Section 4.5 is a systematic study of the effects of changing the global family. We show that the 'forgetful' functor between the global stable homotopy categories of two nested global families has fully faithful left and right adjoints, which are part of a recollement. We provide characterizations of the global homotopy types in the image of the two adjoints; for example,

the right adjoint all the way from the non-equivariant to the global stable homotopy category models Borel cohomology theories. We also relate the global homotopy category to the G-equivariant stable homotopy category for a fixed compact Lie group G; here the forgetful functor also has both adjoints, but these are no longer fully faithful if the group is non-trivial.

Also in Section 4.5 we establish an algebraic model for *rational $\mathcal{F}in$-global* stable homotopy theory, i.e., for rational global stable homotopy theory based on the global family of finite groups. Indeed, spectral Morita theory provides a chain of Quillen equivalences to the category of chain complexes of rational global functors on finite groups. On the algebraic side, the abelian category of rational $\mathcal{F}in$-global functors is equivalent to an even simpler category, namely functors from finite groups and conjugacy classes of epimorphisms to $\mathbb{Q}$-vector spaces. Under the two equivalences, homology groups of chain complexes correspond to equivariant stable homotopy groups, and geometric fixed-point homotopy groups, of spectra.

4.1 Orthogonal spectra as global homotopy types

In this section we specialize the equivariant stable homotopy theory of Chapter 3 to global stable homotopy types, which we model by orthogonal spectra (with no additional group action). Given an orthogonal spectrum X and a compact Lie group G, we obtain an orthogonal G-spectrum by letting G act trivially on the values of X. We call this the *underlying orthogonal G-spectrum* of X and denote it X_G. To simplify notation we omit the subscript '$_G$' when we refer to equivariant homotopy groups, i.e., we simply write $\pi_k^G(X)$ instead of $\pi_k^G(X_G)$.

For global homotopy types (i.e., orthogonal spectra), the notation related to restriction maps simplifies, and some special features happen. Indeed, if X is an orthogonal spectrum, then for every continuous homomorphism $\alpha : K \longrightarrow G$ we have $\alpha^*(X_G) = X_K$, because both K and G act trivially. So for global homotopy types the restriction maps become homomorphisms

$$\alpha^* \ : \ \pi_*^G(X) \ \longrightarrow \ \pi_*^K(X) \ .$$

In particular, we have these restriction maps when α is an epimorphism; in that case we refer to α^* as an *inflation map*.

If H is a closed subgroup of G and $g \in G$, the conjugation homomorphism $c_g : H \longrightarrow H^g$ is given by $c_g(h) = g^{-1}hg$. We recall that for an orthogonal G-spectrum Y the conjugation morphism $g_\star : \pi_0^{H^g}(Y) \longrightarrow \pi_0^H(Y)$ is defined as $g_\star = (l_g^Y)_* \circ (c_g)^*$, where $l_g^Y : c_g^*(Y) \longrightarrow Y$ is left multiplication by g, compare (3.1.18). If $Y = X_G$ for an orthogonal spectrum X, then l_g^X is the identity; so for

global homotopy types we have

$$g_\star = (c_g)^* : \pi_0^{H^g}(X) \longrightarrow \pi_0^H(X) . \qquad (4.1.1)$$

A different way to express the significance of the relation (4.1.1) is that whenever $Y = X_G$ underlies a global homotopy type, then the action of the group $W_G H = (N_G H)/H$ on $\pi_0^H(X)$ discussed in Construction 3.1.17 factors through an action of the quotient group

$$(N_G H)/(H \cdot C_G H) .$$

The difference is illustrated most drastically for $H = e$, the trivial subgroup of G. For a general orthogonal G-spectrum the action of $G = W_G e$ on $\pi_0^e(Y)$ is typically non-trivial. If $Y = X_G$ arises from an orthogonal spectrum, then this G-action is trivial.

Remark 4.1.2 (Global homotopy types are split G-spectra)**.** Obviously, only very special orthogonal G-spectra Y are part of a 'global family', i.e., arise as X_G for an orthogonal spectrum X. However, it is not a priori clear what the homotopical significance of the point-set level condition that G must act trivially on the values of Y (at trivial representations) is. Now we formulate obstructions to 'being global' in terms of the Mackey functor homotopy groups of an orthogonal G-spectrum.

The equivariant homotopy groups $\pi_0^G(X) = \pi_0^G(X_G)$ of an orthogonal spectrum X come equipped with restriction maps along arbitrary continuous group homomorphisms, not necessarily injective. This is in contrast to the situation for a fixed compact Lie group, where one can only restrict to subgroups, or along conjugation maps by elements of the ambient group, but where there are no inflation maps. One obstruction to Y being part of a 'global family' is that the G-Mackey functor structure can be extended to a 'global functor' (in the sense of Definition 4.2.2 below). In particular, the G-Mackey functor homotopy groups can be complemented by restriction maps along arbitrary group homomorphisms between the subgroups of G. So every global functor takes isomorphic values on a pair of isomorphic subgroups of G; for G-Mackey functors this is true when the subgroups are conjugate in G, but not in general when they are merely abstractly isomorphic.

As an extreme case the global structure includes an inflation map $p_G^* : \pi_0^e(X) \longrightarrow \pi_0^G(X)$ associated with the unique homomorphism $p_G : G \longrightarrow e$, splitting the restriction map $\pi_0^G(X) \longrightarrow \pi_0^e(X)$. So one obstruction to being global is that this restriction map from $\pi_0^G(X)$ to $\pi_0^e(X)$ needs to be a split epimorphism. This is the algebraic shadow of the fact that the G-equivariant homotopy types that underlie global homotopy types (i.e., are represented by orthogonal spectra) are 'split' in the sense that there is a morphism from the

underlying non-equivariant spectrum to the genuine G-fixed-point spectrum that splits the restriction map.

Definition 4.1.3. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all compact Lie groups G and all integers k.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. The global equivalences are the weak equivalences of the *global model structure* on the category of orthogonal spectra, see Theorem 4.3.18 below. So the methods of homotopical algebra are available for studying global equivalences and the associated global homotopy category. In later sections we will also consider a relative notion of global equivalence, the '$\mathcal{F}$-equivalences', based on a global family $\mathcal{F}$ of compact Lie groups. There we require that the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all compact Lie groups G that belong to the global family $\mathcal{F}$.

When we specialize Proposition 3.1.25 (ii), Corollary 3.1.37, Proposition 3.1.40, Proposition 3.2.19 (ii) and Proposition 3.1.41 to the underlying orthogonal G-spectra of orthogonal spectra, we obtain the following consequences:

Proposition 4.1.4. (i) *For every orthogonal spectrum X the morphism*

$$\lambda_X \ : \ X \wedge S^1 \ \longrightarrow \ \mathrm{sh}\, X\,, \quad \textit{its adjoint} \quad \tilde{\lambda}_X \ : \ X \ \longrightarrow \ \Omega\, \mathrm{sh}\, X\,,$$

the adjunction unit $\eta_X : X \longrightarrow \Omega(X \wedge S^1)$ and the adjunction counit $\epsilon_X : (\Omega X) \wedge S^1 \longrightarrow X$ are global equivalences.

(ii) *For every finite family of orthogonal spectra the canonical morphism from the wedge to the product is a global equivalence.*

(iii) *For every finite based CW-complex A, the functor* $\mathrm{map}_*(A, -)$ *preserves global equivalences of orthogonal spectra.*

(iv) *For every cofibrant based space A, the functor $- \wedge A$ preserves global equivalences of orthogonal spectra.*

(v) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spectra that are level-wise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be global equivalences that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a global equivalence.*

(vi) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be global equivalences of orthogonal spectra that are level-wise closed embeddings, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to a colimit of the sequence $\{f_n\}_{n \geq 0}$ is a global equivalence.*

For later use we record another closure property of global equivalences.

Corollary 4.1.5. *The class of h-cofibrations of orthogonal spectra that are simultaneously global equivalences is closed under cobase change, coproducts and sequential compositions.*

Proof The class of h-cofibrations is closed under coproducts, cobase change and composition (finite or sequential), compare Corollary A.30 (i). The class of global equivalences is closed under coproducts because equivariant homotopy groups take wedges to direct sums (Corollary 3.1.37 (i)). The cobase change of an h-cofibration that is also a global equivalence is another global equivalence by Corollary 3.1.39 (i). Every h-cofibration of orthogonal spectra is in particular level-wise a closed embedding. So the class of h-cofibrations that are also global equivalences is closed under sequential composition by Proposition 4.1.4 (vi). □

For an orthogonal spectrum X the 0th equivariant homotopy groups $\pi_0^G(X)$ and the restriction maps between them coincide with the homotopy Rep-functor, in the sense of (1.5.6), of a certain orthogonal space $\Omega^\bullet X$ that we now recall.

Construction 4.1.6. We introduce the functor

$$\Omega^\bullet \;:\; \mathcal{S}p \;\longrightarrow\; spc$$

from orthogonal spectra to orthogonal spaces. Given an orthogonal spectrum X, the value of $\Omega^\bullet X$ at an inner product space V is

$$(\Omega^\bullet X)(V) \;=\; \mathrm{map}_*(S^V, X(V)) \;.$$

If $\varphi : V \longrightarrow W$ is a linear isometric embedding, the induced map

$$\varphi_* \;:\; (\Omega^\bullet X)(V) \;=\; \mathrm{map}_*(S^V, X(V)) \;\longrightarrow\; \mathrm{map}_*(S^W, X(W)) \;=\; (\Omega^\bullet X)(W)$$

was defined in (3.1.9). In particular, the orthogonal group $O(V)$ acts on $(\Omega^\bullet X)(V) = \mathrm{map}_*(S^V, X(V))$ by conjugation. If $\psi : U \longrightarrow V$ is another isometric embedding, then we have $\varphi_*(\psi_* f) = (\varphi\psi)_* f$. The assignment $(\varphi, f) \mapsto \varphi_* f$ is continuous in both variables, so we have really defined an orthogonal space $\Omega^\bullet X$. The construction is clearly functorial in the orthogonal spectrum X; moreover, $\Omega^\bullet$ has a left adjoint 'unreduced suspension spectrum' functor Σ^∞_+ that we discuss in Construction 4.1.7 below.

If G acts on V by linear isometries, then the G-fixed subspace of $(\Omega^\bullet X)(V)$ is the space of G-equivariant based maps from S^V to $X(V)$:

$$((\Omega^\bullet X)(V))^G \;=\; \mathrm{map}_*^G(S^V, X(V)) \;.$$

The path components of this space are precisely the equivariant homotopy classes of based G-maps, i.e.,

$$\pi_0\left(((\Omega^\bullet X)(V))^G\right) \;=\; \pi_0\left(\mathrm{map}_*^G(S^V, X(V))\right) \;=\; [S^V, X(V)]^G \;.$$

Passing to colimits over the poset $s(\mathcal{U}_G)$ gives

$$\pi_0^G(\Omega^\bullet X) \;=\; \pi_0^G(X)\,,$$

i.e., the G-equivariant homotopy group of the orthogonal spectrum X equals the G-equivariant homotopy set of the orthogonal space $\Omega^\bullet X$. A direct inspection shows that the spectrum level restriction maps defined in Construction 3.1.15 coincide with the restriction maps for orthogonal spaces introduced in (1.5.9).

Construction 4.1.7 (Suspension spectra of orthogonal spaces)**.** To every orthogonal space Y we can associate an unreduced *suspension spectrum* $\Sigma^\infty_+ Y$ whose value on an inner product space

$$(\Sigma^\infty_+ Y)(V) \;=\; S^V \wedge Y(V)_+\;;$$

here the orthogonal group $O(V)$ acts diagonally and the structure map

$$\sigma_{V,W}\;:\; S^V \wedge (S^W \wedge Y(W)_+) \;\longrightarrow\; S^{V\oplus W} \wedge Y(V\oplus W)_+$$

is the combination of the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$ and the map $Y(i_W) : Y(W) \longrightarrow Y(V\oplus W)$. If Y is the constant orthogonal space with value a topological space A, then $\Sigma^\infty_+ Y = \Sigma^\infty_+ A$ specializes to the suspension spectrum of A with a disjoint basepoint added. The functor

$$\Sigma^\infty_+\;:\; spc \;\longrightarrow\; \mathcal{S}p$$

is left adjoint to the functor $\Omega^\bullet$ of Construction 4.1.6.

Let G be a compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of finite-dimensional G-subrepresentations of the complete universe $\mathcal{U}_G$. Given an orthogonal space Y, we denote by $\mathrm{tel}_i\, Y(V_i)$ the mapping telescope of the sequence of G-spaces

$$Y(V_1) \;\longrightarrow\; Y(V_2) \;\longrightarrow\; \cdots \;\longrightarrow\; Y(V_i) \;\longrightarrow\; \cdots\;;$$

the maps in the sequence are induced by the inclusions, so they are G-equivariant, and the telescope inherits a natural G-action. The canonical maps $Y(V_j) \longrightarrow \mathrm{tel}_i\, Y(V_i)$ induce maps of equivariant homotopy classes

$$\begin{aligned}[S^{V_j\oplus\mathbb{R}^k}, (\Sigma^\infty_+ Y)(V_j)]^G \;&=\; [S^{V_j\oplus\mathbb{R}^k}, S^{V_j} \wedge Y(V_j)_+]^G\\ &\longrightarrow [S^{V_j\oplus\mathbb{R}^k}, S^{V_j} \wedge \mathrm{tel}_i\, Y(V_i)_+]^G \;\longrightarrow\; \pi_k^G(\Sigma^\infty_+\, \mathrm{tel}_i\, Y(V_i))\,,\end{aligned}$$

where the second map is the canonical one to the colimit. These maps are compatible with stabilization in the source when we increase j. Since the exhaustive sequence is cofinal in the poset $s(\mathcal{U}_G)$, the colimit over j calculates

the kth equivariant homotopy group of $\Sigma^\infty_+ Y$. So altogether, the maps assemble into a natural group homomorphism

$$\pi_k^G(\Sigma^\infty_+ Y) \longrightarrow \pi_k^G(\Sigma^\infty_+ \operatorname{tel}_i Y(V_i)) ,$$

for $k \geq 0$. For negative values of k, we obtain a similar map by inserting $\mathbb{R}^{-k}$ into the second variable of the above sets of equivariant homotopy classes.

Proposition 4.1.8. *Let G be a compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-representations. Then for every orthogonal space Y and every integer k the map*

$$\pi_k^G(\Sigma^\infty_+ Y) \longrightarrow \pi_k^G(\Sigma^\infty_+ \operatorname{tel}_i Y(V_i))$$

is an isomorphism.

Proof The mapping telescope is the colimit, along h-cofibrations, of the truncated mapping telescopes $\operatorname{tel}_{[0,n]} Y(V_i)$. So the space $S^{V_j} \wedge \operatorname{tel}_i Y(V_i)_+$ is the colimit, along a sequence of closed embeddings, of the spaces $S^{V_j} \wedge \operatorname{tel}_{[0,n]} Y(V_i)_+$. For every compact based G-space A the canonical map

$$\operatorname{colim}_{n\geq 1} [A, S^{V_j} \wedge \operatorname{tel}_{[0,n]} Y(V_i)_+]^G \longrightarrow [A, S^{V_j} \wedge \operatorname{tel}_i Y(V_i)_+]^G$$

is thus bijective. Indeed, every continuous map from the compact spaces A and $A \wedge [0,1]_+$ to $S^{V_j} \wedge \operatorname{tel}_i Y(V_i)_+$ factors through $S^{V_j} \wedge \operatorname{tel}_{[0,n]} Y(V_i)$ some n, for example by Proposition A.15. The canonical map $Y(V_n) \longrightarrow \operatorname{tel}_i Y(V_i)$ factors through $Y(V_n) \longrightarrow \operatorname{tel}_{[0,n]} Y(V_i)$, and this factorization is an equivariant homotopy equivalence. So we can replace $\operatorname{tel}_{[0,n]} Y(V_i)$ by $Y(V_n)$ and conclude that the map

$$\operatorname{colim}_{n\geq 1} [A, S^{V_j} \wedge Y(V_n)_+]^G \longrightarrow [A, S^{V_j} \wedge \operatorname{tel}_i Y(V_i)_+]^G$$

is bijective. We specialize to $A = S^{V_j \oplus \mathbb{R}^k}$ and pass to the colimit over j by stabilization in source and target. The result is a bijection

$$\begin{aligned} &\operatorname{colim}_{j\geq 1} \operatorname{colim}_{n\geq 1} [S^{V_j\oplus\mathbb{R}^k}, S^{V_j} \wedge Y(V_n)_+]^G \\ &\qquad\qquad \longrightarrow \operatorname{colim}_{j\geq 1} [S^{V_j\oplus\mathbb{R}^k}, S^{V_j} \wedge \operatorname{tel}_i Y(V_i)_+]^G . \end{aligned}$$

The source is isomorphic to the diagonal colimit

$$\operatorname{colim}_{j\geq 1} [S^{V_j\oplus\mathbb{R}^k}, S^{V_j} \wedge Y(V_j)_+]^G$$

by cofinality. Since the exhaustive sequence is cofinal in the poset $s(\mathcal{U}_G)$, the two colimits over j also calculate the colimits over $s(\mathcal{U}_G)$, and hence the kth equivariant homotopy group of $\Sigma^\infty_+ Y$ and of $\Sigma^\infty_+ \operatorname{tel}_i Y(V_i)$. This shows the claim for $k \geq 0$. For negative values of k, the argument is similar: we insert $\mathbb{R}^{-k}$ into the second variable of the sets of equivariant homotopy classes. □

Corollary 4.1.9. *The unreduced suspension spectrum functor Σ^∞_+ takes global equivalences of orthogonal spaces to global equivalences of orthogonal spectra.*

Proof Let $f : X \longrightarrow Y$ be a global equivalence of orthogonal spaces and G a compact Lie group. We choose an exhaustive sequence of G-representations $\{V_i\}_{i\geq 1}$. Then the G-map

$$\mathrm{tel}_i\, f(V_i) \ : \ \mathrm{tel}_i\, X(V_i) \ \longrightarrow \ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence by Proposition 1.1.7 (iii). So the map of suspension spectra $\Sigma^\infty_+ \mathrm{tel}_i\, f(V_i)$ induces isomorphisms on all G-equivariant stable homotopy groups, by Proposition 3.1.44. Since the group $\pi_k^G(\Sigma^\infty_+ \mathrm{tel}_i\, X(V_i))$ is naturally isomorphic to the group $\pi_k^G(\Sigma^\infty_+ X)$ (by Proposition 4.1.8), the morphism of orthogonal suspension spectra $\Sigma^\infty_+ f : \Sigma^\infty_+ X \longrightarrow \Sigma^\infty_+ Y$ induces an isomorphism on π_*^G. □

Now we let Y be an orthogonal space and K a compact Lie group. We define a map

$$\sigma^K \ : \ \pi_0^K(Y) \ \longrightarrow \ \pi_0^K(\Sigma^\infty_+ Y) \tag{4.1.10}$$

as the effect of the adjunction unit $Y \longrightarrow \Omega^\bullet(\Sigma^\infty_+ Y)$ on the K-equivariant homotopy set π_0^K, using that $\pi_0^K(\Omega^\bullet(\Sigma^\infty_+ Y)) = \pi_0^K(\Sigma^\infty_+ Y)$. If we unravel the definitions, this comes out as follows: if V is a finite-dimensional K-subrepresentation of the complete K-universe $\mathcal{U}_K$ and $y \in Y(V)^K$ a K-fixed-point, then $\sigma^K[y]$ is represented by the K-map

$$S^V \ \xrightarrow{-\wedge y} \ S^V \wedge Y(V)_+ \ = \ (\Sigma^\infty_+ Y)(V) \ .$$

As K varies, the maps σ^K are compatible with restriction along continuous homomorphisms, since they arise from a morphism of orthogonal spaces. By the same argument as for orthogonal K-spectra in (3.3.14), transferring from a closed subgroup L to K annihilates the action of the Weyl group on $\pi_0^L(Y)$.

The next proposition is a global analog of Theorem 3.3.15 (i). We call an orthogonal spectrum *globally connective* if its G-equivariant homotopy groups vanish in negative dimensions, for all compact Lie groups G.

Proposition 4.1.11. *Let Y be an orthogonal space. Then the suspension spectrum $\Sigma^\infty_+ Y$ is globally connective. Moreover, for every compact Lie group K the equivariant homotopy group $\pi_0^K(\Sigma^\infty_+ Y)$ is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_L^K(\sigma^L(x)) \ ,$$

where L runs through all conjugacy classes of closed subgroups of K with finite

Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0^L(Y)$.

Proof We consider the functor on the product poset $s(\mathcal{U}_K)^2$ sending (U, V) to the set $[S^V, S^V \wedge Y(U)_+]^K$. The diagonal is cofinal in $s(\mathcal{U}_K)^2$, so the induced map

$$\begin{aligned}\pi_0^K(\Sigma_+^\infty Y) \ &= \ \operatorname{colim}_{V\in s(\mathcal{U}_G)}[S^V, S^V \wedge Y(V)_+]^K \\ &\longrightarrow \ \operatorname{colim}_{(V,U)\in s(\mathcal{U}_K)^2}[S^V, S^V \wedge Y(U)_+]^K\end{aligned}$$

is an isomorphism. The target can be calculated in two steps, hence the group we are after is isomorphic to

$$\operatorname{colim}_{U\in s(\mathcal{U}_K)}\left(\operatorname{colim}_{V\in s(\mathcal{U}_K)}[S^V, S^V \wedge Y(U)_+]^K\right) \ = \ \operatorname{colim}_{U\in s(\mathcal{U}_K)}\pi_0^K\left(\Sigma_+^\infty Y(U)\right) .$$

We may thus show that the latter group is free abelian with the specified basis.

Theorem 3.3.15 (i) identifies the equivariant homotopy group $\pi_0^K\left(\Sigma_+^\infty Y(U)\right)$ as the free abelian group with basis the classes $\operatorname{tr}_L^K(\sigma^L(x))$, where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0(Y(U)^L)$. Passage to the colimit over $U \in s(\mathcal{U}_K)$ yields the proposition. □

We let G be a compact Lie group and V a G-representation. We recall from (1.5.11) the tautological class

$$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V})$$

in the G-equivariant homotopy set of the semifree orthogonal space $\mathbf{L}_{G,V}$. The *stable tautological class* is

$$e_{G,V} \ = \ \sigma^G(u_{G,V}) \ \in \ \pi_0^G(\Sigma_+^\infty \mathbf{L}_{G,V}) . \tag{4.1.12}$$

Explicitly, $e_{G,V}$ is the homotopy class of the G-map

$$S^V \ \longrightarrow \ S^V \wedge (\mathbf{L}(V,V)/G)_+ \ = \ (\Sigma_+^\infty \mathbf{L}_{G,V})(V) , \quad v \ \longmapsto v \wedge (\mathrm{Id}_V \cdot G) .$$

In the following corollary we index certain homotopy classes by pairs (L, α) consisting of a closed subgroup L of a compact Lie group K and a continuous homomorphism $\alpha : L \longrightarrow G$. The *conjugate* of (L, α) by a pair $(k, g) \in K \times G$ is the pair $(L^k, c_g \circ \alpha \circ c_k^{-1})$ consisting of the conjugate subgroup L^k and the composite homomorphism

$$L^k \ \xrightarrow{c_k^{-1}} \ L \ \xrightarrow{\alpha} \ G \ \xrightarrow{c_g} \ G .$$

Corollary 4.1.13. *Let G and K be compact Lie groups and V a faithful G-representation. Then the homotopy group $\pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$ is a free abelian group*

with basis given by the classes

$$\mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

as (L, α) *runs over a set of representatives of all* $(K \times G)$*-conjugacy classes of pairs consisting of a closed subgroup* L *of* K *with finite Weyl group and a continuous homomorphism* $\alpha : L \longrightarrow G$.

Proof The map

$$\mathrm{Rep}(K, G) \longrightarrow \pi_0^K(\mathbf{L}_{G,V}), \quad [\alpha : K \longrightarrow G] \longmapsto \alpha^*(u_{G,V})$$

is bijective according to Proposition 1.5.12 (ii). Proposition 4.1.11 thus says that $\pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$ is a free abelian group with a basis given by the elements

$$\mathrm{tr}_L^K(\sigma^L(\alpha^*(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(\sigma^G(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and α runs through a set of representatives of the $W_K L$-orbits of the set $\mathrm{Rep}(L, G)$. The claim follows because $(K \times G)$-conjugacy classes of such pairs (L, α) biject with pairs consisting of a conjugacy class of subgroups (L) and a $W_K L$-equivalence class in $\mathrm{Rep}(L, G)$. □

Example 4.1.14. We discuss a specific example of Corollary 4.1.13, with $G = A_3$ the alternating group on three letters and $K = \Sigma_3$ the symmetric group on three letters. The group Σ_3 has four conjugacy classes of subgroups, with representatives $\Sigma_3, A_3, (12)$ and e. The groups $\Sigma_3, (12)$ and e admit only trivial homomorphisms to A_3, whereas the alternating group A_3 also has two automorphisms. None of the three endomorphisms of A_3 are conjugate, so the set $\mathrm{Rep}(A_3, A_3)$ has three elements. However, the Weyl group $W_{\Sigma_3} A_3$ has two elements, and its action realizes the two automorphisms of A_3. So while $\pi_0^{A_3}(B_{\mathrm{gl}} A_3) \cong \mathrm{Rep}(A_3, A_3)$ has three elements, it only contributes two generators to the stable group $\pi_0^{\Sigma_3}(\Sigma_+^\infty B_{\mathrm{gl}} A_3)$. A basis for the free abelian group $\pi_0^{\Sigma_3}(\Sigma_+^\infty B_{\mathrm{gl}} A_3)$ is thus given by the classes

$$p_{\Sigma_3}^*(1), \quad \mathrm{tr}_{A_3}^{\Sigma_3}(e_{A_3}), \quad \mathrm{tr}_{A_3}^{\Sigma_3}(p_{A_3}^*(1)), \quad \mathrm{tr}_{(12)}^{\Sigma_3}(p_{(12)}^*(1)) \quad \text{and} \quad \mathrm{tr}_e^{\Sigma_3}(p_e^*(1)).$$

Here the faithful A_3-representation V is unspecified and we write $e_{A_3} = e_{A_3,V} \in \pi_0^{A_3}(\Sigma_+^\infty B_{\mathrm{gl}} A_3)$ for the stable tautological class. Moreover $p_H : H \longrightarrow e$ denotes the unique homomorphism to the trivial group and $1 = \mathrm{res}_e^{A_3}(e_{A_3})$ is the restriction of the class e_{A_3} to the trivial group.

Now we discuss how multiplicative features related to the smash product and the homotopy group pairings work out for global homotopy types. In Construction 4.1.6 we associated an orthogonal space $\Omega^\bullet X$ to every orthogonal spectrum X. This functor is compatible with the smash product of orthogonal

spectra and the box product of orthogonal spaces, in the sense of a lax symmetric monoidal transformation

$$(\Omega^\bullet X) \boxtimes (\Omega^\bullet Y) \longrightarrow \Omega^\bullet(X \wedge Y) . \tag{4.1.15}$$

This morphism is associated with a bimorphism from $(\Omega^\bullet X, \Omega^\bullet Y)$ to $\Omega^\bullet(X \wedge Y)$ with (V, W)-component the composite

$$\begin{aligned} \mathrm{map}_*(S^V, X(V)) \times \mathrm{map}_*(S^W, Y(W)) &\xrightarrow{\wedge} \mathrm{map}_*(S^{V\oplus W}, X(V) \wedge Y(W)) \\ &\xrightarrow{\mathrm{map}_*(S^{V\oplus W}, i_{V,W})} \mathrm{map}_*(S^{V\oplus W}, (X \wedge Y)(V \oplus W)) . \end{aligned}$$

The morphism is unital, associative and symmetric. Finally, the homotopy group pairing (3.5.13) for $k = l = 0$ coincides with the composite

$$\begin{aligned} \pi_0^G(X) \times \pi_0^G(Y) \ = \ \pi_0^G(\Omega^\bullet X) \times \pi_0^G(\Omega^\bullet Y) &\xrightarrow{\times} \pi_0^G(\Omega^\bullet X \boxtimes \Omega^\bullet Y) \\ &\xrightarrow{(4.1.15)} \pi_0^G(\Omega^\bullet(X \wedge Y)) \ = \ \pi_0^G(X \wedge Y) . \end{aligned}$$

Example 4.1.16 (Orthogonal ring spectra and orthogonal monoid spaces)**.** The monoidal morphism (4.1.15) for the functor $\Omega^\bullet$ is unital, associative and symmetric. In particular the orthogonal space $\Omega^\bullet R$ associated with an orthogonal ring spectrum becomes an orthogonal monoid space via the composite

$$(\Omega^\bullet R) \boxtimes (\Omega^\bullet R) \longrightarrow \Omega^\bullet(R \wedge R) \xrightarrow{\Omega^\bullet \mu} \Omega^\bullet R ,$$

and this passage preserves commutativity of multiplications. The bimorphism corresponding to the induced product on $\Omega^\bullet R$ thus has as (V, W)-component the composite

$$\begin{aligned} \mathrm{map}_*(S^V, R(V)) \times \mathrm{map}_*(S^W, R(W)) &\xrightarrow{-\wedge-} \mathrm{map}_*(S^{V\oplus W}, R(V) \wedge R(W)) \\ &\xrightarrow{\mathrm{map}_*(S^{V\oplus W}, \mu_{V,W})} \mathrm{map}_*(S^{V\oplus W}, R(V \oplus W)) . \end{aligned}$$

The suspension spectrum functor (see Construction 4.1.7) takes the box product of orthogonal spaces to the smash product of orthogonal spectra. In more detail: for all inner product spaces V and W the maps

$$\begin{aligned} (\Sigma^\infty_+ X)(V) \wedge (\Sigma^\infty_+ Y)(W) \ &= (S^V \wedge X(V)_+) \wedge (S^W \wedge Y(W)_+) \\ &\cong S^{V\oplus W} \wedge (X(V) \times Y(W))_+ \xrightarrow{S^{V\oplus W} \wedge i_{V,W}} \\ &\quad S^{V\oplus W} \wedge (X \boxtimes Y)(V \oplus W)_+ \ = \ (\Sigma^\infty_+ (X \boxtimes Y))(V \oplus W) \end{aligned}$$

form a bimorphism, so they correspond to a morphism of orthogonal spectra

$$(\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y) \longrightarrow \Sigma^\infty_+ (X \boxtimes Y) . \tag{4.1.17}$$

Proposition 4.1.18. *For all orthogonal spaces X and Y the morphism* (4.1.17) *is an isomorphism. Together with the unique isomorphism* $\Sigma^\infty_+ * \cong \mathbb{S}$ *this makes* Σ^∞_+ *a strong symmetric monoidal functor from the category of orthogonal spaces to the category of orthogonal spectra.*

Proof We consider the composite

$$X \boxtimes Y \xrightarrow{\eta_X \boxtimes \eta_Y} \Omega^\bullet(\Sigma^\infty_+ X) \boxtimes \Omega^\bullet(\Sigma^\infty_+ Y) \xrightarrow{(4.1.15)} \Omega^\bullet((\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y)) ,$$

where $\eta : \mathrm{Id} \longrightarrow \Omega^\bullet \circ \Sigma^\infty_+$ is the adjunction unit. This composite is adjoint to a morphism of orthogonal spectra $\Sigma^\infty_+(X \boxtimes Y) \longrightarrow (\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y)$. We omit the verification that this morphism is indeed inverse to the morphism (4.1.17). □

Construction 4.1.19 (Orthogonal ring spectra from orthogonal monoid spaces)**.** The suspension spectrum $\Sigma^\infty_+ M$ of an orthogonal monoid space M becomes an orthogonal ring spectrum via the multiplication map

$$(\Sigma^\infty_+ M) \wedge (\Sigma^\infty_+ M) \xrightarrow[\cong]{(4.1.17)} \Sigma^\infty_+(M \boxtimes M) \xrightarrow{\Sigma^\infty_+ \mu_M} \Sigma^\infty_+ M .$$

If the multiplication of M is commutative, then so is the resulting multiplication on $\Sigma^\infty_+ M$. The functor pair $(\Sigma^\infty_+, \Omega^\bullet)$ is again an adjoint pair when viewed as functors between the categories of orthogonal monoid spaces and orthogonal ring spectra.

This construction includes spherical monoid ring spectra: if M is a topological monoid, then the constant orthogonal space with value M inherits an associative and unital product from M. The suspension spectrum of such a constant multiplicative functor is the monoid ring spectrum.

Construction 4.1.20. In Construction 3.5.12 we introduced pairings on the equivariant homotopy groups of orthogonal G-spectra. Now we consider two orthogonal spectra X and Y. In the global context, the equivariant homotopy groups also support inflation maps, which we can use to define another pairing

$$\boxtimes \ : \ \pi_k^G(X) \ \times \ \pi_l^K(Y) \ \longrightarrow \ \pi_{k+l}^{G\times K}(X \wedge Y) . \qquad (4.1.21)$$

Here G and K are compact Lie groups and $k, l \in \mathbb{Z}$. We define this pairing as the composite

$$\pi_k^G(X) \times \pi_l^K(Y) \xrightarrow{p_G^* \times p_K^*} \pi_k^{G\times K}(X) \times \pi_l^{G\times K}(Y) \xrightarrow{\times} \pi_{k+l}^{G\times K}(X \wedge Y) ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections to the two factors. Theorem 3.5.14 and the additivity of inflation maps imply that this pairing is biadditive, and that it satisfies certain associativity, commutativity and restriction properties. We do take the time to spell out the most important properties of these pairings in the next theorem.

As G and K vary through all compact Lie groups, the $\boxtimes$-pairings form a bimorphism of global functors in the sense of Construction 4.2.17 below. Moreover, the passage from the 'diagonal' pairings to the 'external' pairings (4.1.21) can be reversed by taking $K = G$ and restricting to the diagonal; suitably formalized, diagonal and external pairings contain the same amount of information. We refer to Remark 4.2.20 below for more details.

Theorem 4.1.22. *Let G, K and L be compact Lie groups and X, Y and Z orthogonal spectra.*

(i) (Biadditivity) *The product $\boxtimes : \pi_k^G(X) \times \pi_l^K(Y) \longrightarrow \pi_{k+l}^{G\times K}(X \wedge Y)$ is biadditive.*

(ii) (Unitality) *Let $1 \in \pi_0^e(\mathbb{S})$ denote the class represented by the identity of S^0. The product is unital in the sense that $1 \boxtimes x = x = x \boxtimes 1$ under the identifications $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$ and $e \times G \cong G \cong G \times e$.*

(iii) (Associativity) *For all $x \in \pi_k^G(X)$, $y \in \pi_l^K(Y)$ and $z \in \pi_m^L(Z)$ the relation*

$$x \boxtimes (y \boxtimes z) \;=\; (x \boxtimes y) \boxtimes z$$

holds in $\pi_{k+l+m}^{G\times K\times L}(X \wedge Y \wedge Z)$.

(iv) (Commutativity) *For all $x \in \pi_k^G(X)$ and $y \in \pi_l^K(Y)$ the relation*

$$\tau_*^{X,Y}(x \boxtimes y) \;=\; (-1)^{kl} \cdot \tau_{G,K}^*(y \boxtimes x)$$

holds in $\pi_{l+k}^{G\times K}(Y \wedge X)$, where $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetry isomorphism of the smash product and $\tau_{G,K} : G \times K \longrightarrow K \times G$ interchanges the factors.

(v) (Restriction) *For all $x \in \pi_k^G(X)$ and $y \in \pi_l^K(Y)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$\alpha^*(x) \boxtimes \beta^*(y) \;=\; (\alpha \boxtimes \beta)^*(x \boxtimes y)$$

holds in $\pi_{k+l}^{\bar{G}\times\bar{K}}(X \wedge Y)$.

(vi) (Transfer) *For all closed subgroups $H \leq G$ and $L \leq K$ the square*

$$\begin{tikzcd}
\pi_k^H(X) \times \pi_l^L(Y) \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{tr}_H^G \times \mathrm{tr}_L^K"'] & \pi_{k+l}^{H\times L}(X \wedge Y) \arrow[d, "\mathrm{tr}_{H\times L}^{G\times K}"] \\
\pi_k^G(X) \times \pi_l^K(Y) \arrow[r, "\boxtimes"'] & \pi_{k+l}^{G\times K}(X \wedge Y)
\end{tikzcd}$$

commutes.

Proof Parts (i) through (v) follow from the respective parts of Theorem 3.5.14

by naturality. For part (vi) we start with two special cases, namely $L = K$ and $H = G$. The two proofs are analogous, so we only treat the case $L = K$:

$$\begin{aligned} \mathrm{tr}_{H\times K}^{G\times K}(x \boxtimes y) &= \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x) \times p_K^*(y)) = \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x) \times \mathrm{res}_{H\times K}^{G\times K}(p_K^*(y))) \\ &= \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x)) \times p_K^*(y) = p_G^*(\mathrm{tr}_H^G(x)) \times p_K^*(y) = \mathrm{tr}_H^G(x) \boxtimes y \end{aligned}$$

We slightly abuse notation by writing p_K for both the projections of $H \times K$ and of $G \times K$ to K. The third equation is the reciprocity relation of Theorem 3.5.14 (vi). The fourth equation is the compatibility of transfers with inflations (Proposition 3.2.32 (ii)).

The general case is now obtained by combining the two special cases:

$$\begin{aligned} \mathrm{tr}_H^G(x) \boxtimes \mathrm{tr}_L^K(y) &= \mathrm{tr}_{H\times K}^{G\times K}(x \boxtimes \mathrm{tr}_L^K(y)) \\ &= \mathrm{tr}_{H\times K}^{G\times K}(\mathrm{tr}_{H\times L}^{H\times K}(x \boxtimes y)) = \mathrm{tr}_{H\times L}^{G\times K}(x \boxtimes y) \end{aligned} \qquad \square$$

For all orthogonal spectra (i.e., all global homotopy types), the collection of equivariant homotopy groups $\{\pi_0^G(X)\}_G$ come with restriction and transfer maps, and this data together forms a ‘global functor’, compare Definition 4.2.2 below. The geometric fixed-point homotopy groups have fewer natural operations, and they do *not* allow restriction to subgroups. However, geometric fixed-points still have inflation maps, i.e., restriction maps along epimorphisms. Indeed, in Construction 3.3.4 we defined inflation maps on geometric fixed-point homotopy groups, associated with a continuous epimorphism $\alpha : K \longrightarrow G$ between compact Lie groups. When X is an orthogonal spectrum, representing a global homotopy type, then $\alpha^*(X_G) = X_K$, and the inflation maps become homomorphisms

$$\alpha^* \ : \ \Phi_k^G(X) \ \longrightarrow \ \Phi_k^K(X) .$$

These inflation maps between the geometric fixed-point homotopy groups are clearly natural in the orthogonal spectrum.

Construction 4.1.23 (Semifree orthogonal spectra)**.** Given a compact Lie group G and a G-representation V, the functor

$$\mathrm{ev}_{G,V} \ : \ \mathcal{S}p \ \longrightarrow \ G\mathbf{T}_*$$

that sends an orthogonal spectrum X to the based G-space $X(V)$ has a left adjoint

$$F_{G,V} \ : \ G\mathbf{T}_* \ \longrightarrow \ \mathcal{S}p . \tag{4.1.24}$$

The semifree orthogonal spectrum generated by a based G-space A in level V is

$$F_{G,V}A \ = \ \mathbf{O}(V, -) \wedge_G A ;$$

the value at an inner product space W is thus given by

$$(F_{G,V}A)(W) \;=\; \mathbf{O}(V,W) \wedge_G A\ .$$

We note that $F_{G,V}A$ consists of a single point in all levels below the dimension of V. The 'freeness' property of $F_{G,V}A$ is a consequence of the enriched Yoneda lemma, see Remark C.2 or [90, Sec. 1.9]; it means that for every orthogonal spectrum X and every based G-map $f : A \longrightarrow X(V)$ there is a unique morphism $f^\flat : F_{G,V}A \longrightarrow X$ of orthogonal spectra such that the composite

$$A \xrightarrow{\mathrm{Id}\wedge -} \mathbf{O}(V,V) \wedge_G A = (F_{G,V}A)(V) \xrightarrow{f^\flat(V)} X(V)$$

is f. Indeed, the map $f^\flat(W)$ is the composite

$$\mathbf{O}(V,W) \wedge_G A \xrightarrow{\mathrm{Id}\wedge_G f} \mathbf{O}(V,W) \wedge_G X(V) \xrightarrow{\mathrm{act}} X(W)\ .$$

Remark 4.1.25 (Semifree spectra as global Thom spectra)**.** The underlying non-equivariant stable homotopy type of $F_{G,V}$ is the Thom spectrum of the negative of the bundle

$$\mathbf{L}(V,\mathbb{R}^\infty) \times_G V \;\longrightarrow\; \mathbf{L}(V,\mathbb{R}^\infty)/G \;=\; BG\ ,$$

the vector bundle over BG associated with the G-representation V. So one should think of the semifree orthogonal spectrum $F_{G,V} = F_{G,V}S^0$ as the 'global Thom spectrum' associated with a 'virtual global vector bundle', namely the negative of the vector bundle over $B_{\mathrm{gl}}G$ associated with the G-representation V.

The special case $G = O(m)$ of the orthogonal group with $V = \nu_m$, i.e., $\mathbb{R}^m$ with the tautological $O(m)$-action, will feature prominently in the rank filtration of the global Thom spectrum **mO** in Section 6.1. Non-equivariantly, $F_{O(m),\nu_m}$ is the Thom spectrum of the negative of the tautological m-plane bundle over the Grassmannian $Gr_m(\mathbb{R}^\infty)$; the traditional notation for this Thom spectrum is $MTO(m)$ or simply $MT(m)$. Indeed,

$$F_{O(m),\nu_m}(\mathbb{R}^{m+n}) \;=\; \mathbf{O}(\mathbb{R}^m,\mathbb{R}^{m+n})/O(m)$$

is the Thom space of the orthogonal complement of the tautological m-plane bundle over $\mathbf{L}(\mathbb{R}^m,\mathbb{R}^{m+n})/O(m) = Gr_m(\mathbb{R}^{m+n})$, and this is precisely the $(m+n)$th space of $MTO(m)$, see for example [59, Sec. 3.1]. Similarly, the non-equivariant homotopy type underlying $F_{SO(m),\nu_m}$ is an oriented version of $MTO(m)$, which is usually denoted $MTSO(m)$ or sometimes $MT(m)^+$.

Example 4.1.26 (Smash products of semifree orthogonal spectra)**.** The smash product of two semifree orthogonal spectra is again a semifree orthogonal spectrum. In more detail, we consider

- two compact Lie groups G and K,
- a G-representation V and a K-representation W, and
- a based G-space A and a based K-space B.

Then $V \oplus W$ is a $(G \times K)$-representation and $A \wedge B$ is a $(G \times K)$-space via

$$(g,k)\cdot(v,w) \;=\; (gv,kw) \quad \text{and} \quad (g,k)\cdot(a\wedge b) \;=\; ga\wedge kb\ .$$

We claim that the smash product $(F_{G,V}A) \wedge (F_{K,W}B)$ is canonically isomorphic to the semifree orthogonal spectrum generated by the $(G \times K)$-space $A \wedge B$ in level $V \oplus W$. Indeed, a morphism

$$(F_{G,V}A) \wedge (F_{K,W}B) \;\longrightarrow\; F_{G\times K,V\oplus W}(A\wedge B) \tag{4.1.27}$$

is obtained by the universal property (3.5.2) from the bimorphism with (U, U')-component

$$\begin{aligned}(F_{G,V}A)(U)\wedge(F_{K,W}B)(U') \;&=\; (\mathbf{O}(V,U)\wedge_G A)\wedge(\mathbf{O}(W,U')\wedge_K B)\\ &\xrightarrow{\oplus} \mathbf{O}(V\oplus W, U\oplus U')\wedge_{G\times K}(A\wedge B)\\ &= ((F_{G\times K,V\oplus W})(A\wedge B))(U\oplus U')\ .\end{aligned}$$

In the other direction, a morphism $F_{G\times K,V\oplus W}(A \wedge B) \longrightarrow F_{G,V}A \wedge F_{G,W}B$ is freely generated by the $(G \times K)$-map

$$A\wedge B \;\longrightarrow\; (F_{G,V}A)(V)\wedge(F_{K,W}B)(W) \;\xrightarrow{i_{V,W}}\; (F_{G,V}A\wedge F_{K,W}B)(V\oplus W)\ .$$

These two maps are inverse to each other.

In Proposition 1.1.26 (ii) we have seen that for every compact Lie group G, every G-representation V and every faithful G-representation W the restriction morphism of orthogonal spaces $\rho_{V,W}/G : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,W}$ is a global equivalence. One consequence is that the semifree orthogonal space $\mathbf{L}_{G,W}$ has a well-defined unstable global homotopy type, independent of which faithful G-representation is used. Another consequence is that the induced morphism

$$\Sigma^\infty_+\rho_{V,W}/G \;:\; \Sigma^\infty_+\mathbf{L}_{G,V\oplus W} \;\longrightarrow\; \Sigma^\infty_+\mathbf{L}_{G,W}$$

of suspension spectra is a global equivalence of orthogonal spectra, by Corollary 4.1.9. For an inner product space W, the untwisting homeomorphisms (3.1.2) descend to homeomorphisms on G-orbit spaces

$$\mathbf{O}(V,W)\wedge_G S^V \;\cong\; S^W\wedge\mathbf{L}(V,W)/G_+\ .$$

As W varies, these form the ‘untwisting isomorphism’ of orthogonal spectra

$$F_{G,V}S^V \;\cong\; \Sigma^\infty_+\mathbf{L}_{G,V}\ .$$

So suspension spectra of semifree orthogonal spaces are semifree orthogonal

spectra. We will now prove a generalization of the fact that $\Sigma^\infty_+ \rho_{V,W}/G$ is a global equivalence for these global Thom spectra. Given G-representations V and W, we define a restriction morphism of orthogonal spectra

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^V \longrightarrow F_{G,W} \tag{4.1.28}$$

as the adjoint of the based G-map

$$S^V \longrightarrow \mathbf{O}(W, V\oplus W)/G = F_{G,W}(V\oplus W) , \quad v \longmapsto ((v,0), i)\cdot G ,$$

where $i : W \longrightarrow V\oplus W$ is the embedding of the second summand. The value of $\lambda_{G,V,W}$ at an inner product space U is then

$$\begin{aligned} \lambda_{G,V,W}(U) \ : \ \mathbf{O}(V\oplus W, U)\wedge_G S^V &\longrightarrow \quad \mathbf{O}(W,U)/G \\ [(u,\varphi)\wedge v] \qquad &\longmapsto \ (u+\varphi(v), \varphi\circ i)\cdot G . \end{aligned}$$

Theorem 4.1.29. *Let G be a compact Lie group, V a G-representation and W a faithful G-representation. Then the morphism*

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^V \longrightarrow F_{G,W}$$

is a global equivalence of orthogonal spectra.

Proof To simplify the notation we abbreviate the restriction morphism of orthogonal spaces to

$$\rho = \rho_{V,W} \ : \ \mathbf{L}(V\oplus W, -) \longrightarrow \mathbf{L}(W,-) .$$

We let K be another compact Lie group and $U \in s(\mathcal{U}_K)$ a finite-dimensional K-subrepresentation of the complete K-universe $\mathcal{U}_K$. In a first step we produce a K-representation $U' \in s(\mathcal{U}_K)$ with $U \subseteq U'$ and a continuous $(K\times G)$-equivariant map

$$h \ : \ \mathbf{L}(W,U) \longrightarrow \mathbf{L}(V\oplus W, U')$$

such that the lower right triangle in the diagram

$$\begin{tikzcd} \mathbf{L}(V\oplus W, U) \arrow[r, "i_*"] \arrow[d, "\rho(U)"'] & \mathbf{L}(V\oplus W, U') \arrow[d, "\rho(U')"] \\ \mathbf{L}(W,U) \arrow[r, "i_*"'] \arrow[ur, dashed, "h"] & \mathbf{L}(W,U') \end{tikzcd} \tag{4.1.30}$$

commutes, and the upper left triangle commutes up to $(K\times G)$-equivariant fiberwise homotopy over $\mathbf{L}(W,U')$, where $i : U \longrightarrow U'$ is the inclusion.

Since G acts faithfully on W (and hence on $V\oplus W$), both $\mathbf{L}(W,\mathcal{U}_K)$ and $\mathbf{L}(V\oplus W, \mathcal{U}_K)$ are universal spaces for the same family of subgroups of $K\times G$, namely the family $\mathcal{F}(K;G)$ of graph subgroups, compare Proposition 1.1.26 (i). If Γ is

the graph of a continuous homomorphism $\alpha : L \longrightarrow G$ defined on some closed subgroup L of K, then the Γ-fixed-points of $\mathbf{L}(W, \mathcal{U}_K)$ are given by

$$\mathbf{L}(W, \mathcal{U}_K)^\Gamma \;=\; \mathbf{L}^L(\alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K)) ,$$

the space of L-equivariant linear isometric embeddings from $\alpha^* W$. The same is true for $V \oplus W$ instead of W, and so the Γ-fixed-point map $\rho(\mathcal{U}_K)^\Gamma : \mathbf{L}(V \oplus W, \mathcal{U}_K)^\Gamma \longrightarrow \mathbf{L}(W, \mathcal{U}_K)^\Gamma$ is the restriction map

$$\mathbf{L}^L(\alpha^* V \oplus \alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K)) \;\longrightarrow\; \mathbf{L}^L(\alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K))$$

to the summand $\alpha^* W$. This map is a locally trivial fiber bundle, hence a Serre fibration. We conclude that the restriction map $\rho(\mathcal{U}_K) : \mathbf{L}(V \oplus W, \mathcal{U}_K) \longrightarrow \mathbf{L}(W, \mathcal{U}_K)$ is both a $(K \times G)$-weak equivalence and a $(K \times G)$-fibration.

Since $\mathbf{L}(W, U)$ is cofibrant as a $(K \times G)$-space (by Proposition 1.1.19 (ii)), the $(K \times G)$-map $i_* : \mathbf{L}(W, U) \longrightarrow \mathbf{L}(W, \mathcal{U}_K)$ thus admits a $(K \times G)$-equivariant lift $h : \mathbf{L}(W, U) \longrightarrow \mathbf{L}(V \oplus W, \mathcal{U}_K)$ such that $\rho(\mathcal{U}_K) \circ h = i_*$. Since the space $\mathbf{L}(W, U)$ is compact and $\mathbf{L}(V \oplus W, \mathcal{U}_K)$ is the filtered union of the closed subspaces $\mathbf{L}(V \oplus W, U')$ for $U' \in s(\mathcal{U}_K)$, the lift h lands in the subspace $\mathbf{L}(V \oplus W, U')$ for suitably large $U' \in s(\mathcal{U}_K)$, and we may assume that $U \subseteq U'$.

The two maps

$$h \circ \rho(U) \,,\; i_* \;:\; \mathbf{L}(V \oplus W, U) \;\longrightarrow\; \mathbf{L}(V \oplus W, U')$$

become equal after applying $\rho(U') : \mathbf{L}(V \oplus W, U') \longrightarrow \mathbf{L}(V, U')$, hence the composites with $i_* : \mathbf{L}(V \oplus W, U') \longrightarrow \mathbf{L}(V \oplus W, \mathcal{U}_K)$ become equal after applying the $(K \times G)$-equivariant acyclic fibration $\rho(\mathcal{U}_K) : \mathbf{L}(V \oplus W, \mathcal{U}_K) \longrightarrow \mathbf{L}(W, \mathcal{U}_K)$. Since $\mathbf{L}(V \oplus W, U)$ is also $(K \times G)$-cofibrant, there is a fiberwise $(K \times G)$-equivariant homotopy between $h \circ \rho(U)$ and i_* in $\mathbf{L}(V \oplus W, \mathcal{U}_K)$. Again by compactness, the homotopy has image in $\mathbf{L}(V \oplus W, U'')$ for suitably large $U'' \in s(\mathcal{U}_K)$. So after increasing U', if necessary, we have proved the claim subsumed in the diagram (4.1.30).

Now we lift the data produced in the first step to the Thom spaces of the orthogonal complement bundles. The diagram (4.1.30) is covered by morphisms of $(K \times G)$-vector bundles:

$$\begin{tikzcd}
(U' - U) \times \xi(V \oplus W, U) \times V \arrow[r, "\bar{\imath}"] \arrow[d, "(U'-U) \times \bar{\rho}(U)"'] & \xi(V \oplus W, U') \times V \arrow[d, "\bar{\rho}(U')"] \\
(U' - U) \times \xi(W, U) \arrow[r, "\bar{\imath}"'] \arrow[ur, dashed, "\bar{h}"] & \xi(W, U')
\end{tikzcd} \tag{4.1.31}$$

The maps on the total spaces of the bundles are defined as follows: the right vertical map is

$$\bar{\rho}(U') \;:\; \xi(V \oplus W, U') \times V \;\longrightarrow\; \xi(W, U') \,, \quad ((u', \varphi),\, v) \;\longmapsto\; (u' + \varphi(v),\, \varphi|_W) \,.$$

The map $\bar{\rho}(U)$ is defined in the same way. The lower horizontal map is

$$\bar{\imath} \ : \ (U' - U) \times \xi(W, U) \ \longrightarrow \ \xi(W, U') \ , \quad (u', (u, \varphi)) \ \longmapsto \ (u' + u, \ \varphi) \ .$$

The upper horizontal map is defined in the same way, but with $V \oplus W$ instead of W and multiplied by the identity of V. These four outer morphisms in (4.1.31) are all fiberwise linear isomorphisms; so each of these four bundle maps expresses the source bundle as a pullback of the target bundle. In particular, the square

$$\begin{tikzcd} \xi(V \oplus W, U') \times V \arrow[r, "\bar{\rho}(U')"] \arrow[d] & \xi(W, U') \arrow[d] \\ \mathbf{L}(V \oplus W, U') \arrow[r, "\rho(U')"'] & \mathbf{L}(W, U') \end{tikzcd} \tag{4.1.32}$$

is a pullback; so the composite

$$(U' - U) \times \xi(W, U) \ \longrightarrow \ \mathbf{L}(W, U) \ \xrightarrow{h} \ \mathbf{L}(V \oplus W, U')$$

and the map of total spaces $\bar{\imath} : (U' - U) \times \xi(W, U) \longrightarrow \xi(W, U')$ assemble into a map

$$\bar{h} \ : \ (U' - U) \times \xi(W, U) \ \longrightarrow \ \xi(V \oplus W, U')$$

that covers h and is a fiberwise linear isomorphism.

In (4.1.31) (as in (4.1.30)) the outer square and the lower right triangle commute, but the upper left triangle does *not* commute. We will now show that the upper left triangle commutes up to homotopy of $(K \times G)$-equivariant bundle maps. For this purpose we let

$$H \ : \ \mathbf{L}(V \oplus W, U) \times [0, 1] \ \longrightarrow \ \mathbf{L}(V \oplus W, U')$$

be a $(K \times G)$-equivariant homotopy from the map i_* to $h \circ \rho(U)$, such that $\rho(U') \circ H : \mathbf{L}(V \oplus W, U) \times [0, 1] \longrightarrow \mathbf{L}(W, U')$ is the constant homotopy from $\rho(U') \circ i_* = i_* \circ \rho(U)$ to itself. Again because the square (4.1.32) is a pullback, the composite

$$\xi(V \oplus W, U) \times V \times (U' - U) \times [0, 1] \ \longrightarrow \ \mathbf{L}(V \oplus W, U) \times [0, 1] \ \xrightarrow{H} \ \mathbf{L}(V \oplus W, U')$$

and the map of total spaces

$$\begin{aligned} (U' - U) \times \xi(V \oplus W, U) \times V \times [0, 1] \ &\xrightarrow{\text{proj}} \ (U' - U) \times \xi(V \oplus W, U) \times V \\ &\xrightarrow{\bar{\rho}(U') \circ \bar{\imath} = \bar{\imath} \circ \bar{\rho}(U)} \ \xi(W, U') \end{aligned}$$

assemble into a map

$$\bar{H} \ : \ (U' - U) \times \xi(V \oplus W, U) \times V \times [0, 1] \ \longrightarrow \ \xi(V \oplus W, U') \times V$$

that covers the homotopy H. This lift $\bar{H}$ is a $(K \times G)$-equivariant homotopy of vector bundle morphisms, and for every $t \in [0, 1]$, the relation

$$\bar{\rho}(U') \circ \bar{H}(-, t) \ = \ \bar{\rho}(U') \circ \bar{\imath} \ = \ \bar{\rho}(U') \circ (\bar{h} \circ \bar{\rho}(U))$$

holds by definition of $\bar{H}$. For $t = 0$ this shows that $\bar{H}$ starts with $\bar{\imath} : (U' - U) \times \xi(V \oplus W, U) \times V \longrightarrow \xi(V \oplus W, U') \times V$; for $t = 1$ this shows that $\bar{H}$ ends with $\bar{h} \circ \bar{\rho}(U)$, one more time because (4.1.32) is a pullback. We conclude that $\bar{H}$ makes the upper left triangle in (4.1.31) commute up to equivariant homotopy of vector bundle maps.

Passing to Thom spaces in (4.1.31) gives a diagram of $(K \times G)$-equivariant based maps:

$$\begin{tikzcd}
S^{U'-U} \wedge \mathbf{O}(V \oplus W, U) \wedge S^V \arrow[r, "\sigma_{U,U'-U} \wedge S^V"] \arrow[d, "S^{U'-U} \wedge \lambda_{V,W}(U)"'] & \mathbf{O}(V \oplus W, U') \wedge S^V \arrow[d, "\lambda_{V,W}(U')"] \\
S^{U'-U} \wedge \mathbf{O}(W, U) \arrow[r, "\sigma_{U,U'-U}"'] \arrow[ur, dashed, "\bar{h}"] & \mathbf{O}(W, U')
\end{tikzcd}$$

Again, the lower right triangle commutes, and the upper left triangle commutes up to $(K \times G)$-equivariant based homotopy. We pass to G-orbit spaces and obtain a diagram of based K-spaces

$$\begin{tikzcd}
S^{U'-U} \wedge (F_{G,V \oplus W} S^V)(U) \arrow[r, "\sigma_{U,U'-U}"] \arrow[d, "S^{U'-U} \wedge \lambda_{G,V,W}(U)"'] & (F_{G,V \oplus W} S^V)(U') \arrow[d, "\lambda_{G,V,W}(U')"] \\
S^{U'-U} \wedge F_{G,W}(U) \arrow[r, "\sigma_{U,U'-U}"'] \arrow[ur, "\bar{h}/G"] & F_{G,W}(U')
\end{tikzcd}$$

whose lower right triangle commutes, and whose upper left triangle commutes up to K-equivariant based homotopy. Since we had started with an arbitrary K-subrepresentation $U \in s(\mathcal{U}_K)$, this implies that for every based K-space A and K-representation $\bar{U}$ the map on colimits

$$\begin{aligned}
&\operatorname{colim}_{U \in s(\mathcal{U}_K)} [S^U \wedge A, (F_{G,V \oplus W} S^V)(U \oplus \bar{U})]^K \\
&\qquad \longrightarrow \quad \operatorname{colim}_{U \in s(\mathcal{U}_K)} [S^U \wedge A, F_{G,W}(U \oplus \bar{U})]^K
\end{aligned}$$

induced by the morphism $\lambda_{G,V,W}$ is bijective. For $A = S^k$ and $\bar{U} = 0$ this shows that $\pi_k^K(\lambda_{G,V,W})$ is an isomorphism. For $A = S^0$ and $\bar{U} = \mathbb{R}^k$ this shows that $\pi_{-k}^K(\lambda_{G,V,W})$ is an isomorphism. So $\lambda_{G,V,W}$ is a global equivalence. □

4.2 Global functors

This section is devoted to the category of *global functors*, the natural home of the collection of equivariant homotopy groups of a global stable homotopy

type. The category of global functors is a symmetric monoidal abelian category with enough injectives and projectives that plays the same role for global homotopy theory that is played by the category of abelian groups in ordinary homotopy theory, or by the category of G-Mackey functors for G-equivariant homotopy theory.

We introduce global functors in Definition 4.2.2 as additive functors on the *global Burnside category*, the category of natural operations between equivariant stable homotopy groups. The abstract definition ensures that equivariant homotopy groups of orthogonal spectra are tautologically global functors. A key result is Theorem 4.2.6 that describes explicit bases of the morphism groups of the global Burnside category in terms of transfers and restriction operations. This calculation is the key to comparing our notion of global functors to other kinds of global Mackey functors, as well as for all concrete calculations with global functors. Example 4.2.8 lists interesting examples of global functors: the Burnside ring global functor, represented global functors, constant global functors, the representation ring global functor, and Borel type global functors. Many more examples of global functors are discussed in the remaining sections of this book.

An abstract way to motivate the appearance of global functors is as follows. One can consider the globally connective (or globally coconnective) orthogonal spectra, i.e., those where all equivariant homotopy groups vanish in negative dimensions (or in positive dimensions). Then the full subcategories of globally connective or globally coconnective spectra define a non-degenerate t-structure on the triangulated global stable homotopy category, and the heart of this t-structure is (equivalent to) the abelian category of global functors; we refer the reader to Theorem 4.4.9 below for details.

In the global context, the external pairing (4.1.21) of equivariant homotopy groups gives rise to a symmetric monoidal structure on the global Burnside category, see Theorem 4.2.15. Hence the abelian category of global functors can also be endowed with a Day type convolution product, the box product of global functors, see Construction 4.2.17.

Construction 4.2.1 (Burnside category)**.** We define the pre-additive *Burnside category* $\mathbf{A}$. The objects of $\mathbf{A}$ are all compact Lie groups; morphisms from a group G to K are defined as

$$\mathbf{A}(G, K) \ = \ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ ,$$

the set of natural transformations of functors, from orthogonal spectra to sets, between the equivariant homotopy group functors π_0^G and π_0^K. Composition in the category $\mathbf{A}$ is composition of natural transformations.

It is not a priori clear that the natural transformations from π_0^G to π_0^K form a set (as opposed to a proper class), but this follows from Proposition 4.2.5 below. The Burnside category $\mathbf{A}$ is skeletally small: isomorphic compact Lie groups are also isomorphic as objects in the category $\mathbf{A}$, and every compact Lie group is isomorphic to a closed subgroup of an orthogonal group $O(n)$. The functor π_0^K is abelian group-valued, so the set $\mathbf{A}(G, K)$ is an abelian group under objectwise addition of transformations. Proposition 2.2.12, applied to the category of orthogonal spectra, shows that set-valued natural transformations between the two reduced additive functors π_0^G and π_0^K are automatically additive. So composition in the Burnside category is additive in each variable, and $\mathbf{A}$ is indeed a pre-additive category.

Definition 4.2.2. A *global functor* is an additive functor from the Burnside category $\mathbf{A}$ to the category of abelian groups. A morphism of global functors is a natural transformation. We write $\mathcal{GF}$ for the category of global functors.

We discuss various explicit examples of interesting global functors in Example 4.2.8.

Example 4.2.3. The definition of the Burnside category $\mathbf{A}$ is made so that the collection of equivariant homotopy groups of an orthogonal spectrum is tautologically a global functor. Explicitly, the homotopy group global functor $\underline{\pi}_0(X)$ of an orthogonal spectrum X is defined on objects by

$$\underline{\pi}_0(X)(G) \ = \ \pi_0^G(X)$$

and on morphisms by evaluating natural transformations at X. It is less obvious that conversely every global functor is isomorphic to the global functor $\underline{\pi}_0(X)$ of some orthogonal spectrum X; we refer the reader to Remark 4.4.12 below for the construction of Eilenberg-Mac Lane spectra from global functors.

As a category of additive functors out of a skeletally small pre-additive category, the category $\mathcal{GF}$ of global functors has some immediate properties that we collect in the following proposition.

Proposition 4.2.4. *The category $\mathcal{GF}$ of global functors is a Grothendieck abelian category with enough injectives and projectives.*

Proof Any category of additive functors out of a skeletally small additive category is Grothendieck abelian with objectwise notion of exactness, see [162, IV Prop. 7.2] and [162, V Ex. 1.2]. A set of projective generators is given by the represented global functors $\mathbf{A}(G, -)$ where G runs through a set of representatives of the isomorphism classes of compact Lie groups, see [162, IV Cor. 7.5]. A set of injective cogenerators is given similarly by the global functors

$$\mathrm{Hom}(\mathbf{A}(-, K), \mathbb{Q}/\mathbb{Z}) \ : \ \mathbf{A} \longrightarrow \mathcal{A}b \ . \qquad \square$$

As we shall explain in Construction 4.2.17 below, the category $\mathcal{GF}$ has a closed symmetric monoidal product $\Box$ that arises as a convolution product for a certain symmetric monoidal structure on the Burnside category $\mathbf{A}$.

Our definition of the Burnside category is made so that every orthogonal spectrum X gives rise to a homotopy group global functor without further ado, but it is not clear from the definition how to describe the morphism groups of $\mathbf{A}$ explicitly. Our next aim is to show that each morphism group $\mathbf{A}(G, K)$ is a free abelian group with an explicit basis given by certain composites of a restriction and a transfer morphism. This calculation has two ingredients: We identify natural transformations from π_0^G to π_0^K with the group $\pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$, and then we exploit the explicit calculation of the latter group in Corollary 4.1.13.

Proposition 4.2.5. *Let G and K be compact Lie groups and V a faithful G-representation. Then evaluation at the stable tautological class* (4.1.12) *is a bijection*

$$\mathbf{A}(G, K) \;=\; \mathrm{Nat}^{Sp}(\pi_0^G, \pi_0^K) \;\longrightarrow\; \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})\,, \quad \tau \;\longmapsto\; \tau(e_{G,V})$$

to the 0th K-equivariant homotopy group of the orthogonal spectrum $\Sigma_+^\infty \mathbf{L}_{G,V}$. In other words, the morphism of global functors

$$\mathbf{A}(G, -) \;\longrightarrow\; \underline{\pi}_0(\Sigma_+^\infty \mathbf{L}_{G,V})$$

classified by the stable tautological class $e_{G,V}$ is an isomorphism.

Proof We apply the representability result of Proposition 1.5.13 to the category of orthogonal spectra and the adjoint functor pair $(\Sigma_+^\infty, \Omega^\bullet)$. If G is a compact Lie group, V a G-representation and W a faithful G-representation, then the restriction morphism $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,W}$ is a global equivalence of orthogonal spaces (Proposition 1.1.26). So the induced morphism of unreduced suspension spectra $\Sigma_+^\infty \rho_{G,V,W}$ is a global equivalence by Corollary 4.1.9; in particular, the morphism of Rep-functors $\underline{\pi}_0(\Sigma_+^\infty \rho_{G,V,W})$ is an isomorphism. So Proposition 1.5.13 applies and shows that evaluation at the tautological class is bijective. This proves the claim. $\Box$

Now we name a basis of the group $\mathbf{A}(G, K)$. For a pair (L, α) consisting of a closed subgroup L of K and a continuous group homomorphism $\alpha : L \longrightarrow G$ we define

$$[L, \alpha] \;=\; \mathrm{tr}_L^K \circ \alpha^* \;\in\; \mathbf{A}(G, K)\,,$$

the natural transformation whose value at X is the composite

$$\pi_0^G(X) \xrightarrow{\alpha^*} \pi_0^L(X) \xrightarrow{\mathrm{tr}_L^K} \pi_0^K(X)$$

of restriction along α with transfer from L to K.

If L has infinite index in its normalizer, then the transfer map tr_L^K, and hence also the element $[L,\alpha]$, is zero by Example 3.2.25. The *conjugate* of (L,α) by a pair $(k,g)\in K\times G$ of group elements is the pair $(L^k, c_g\circ\alpha\circ c_k^{-1})$ consisting of the conjugate subgroup L^k and the composite homomorphism

$$L^k \xrightarrow{c_k^{-1}} L \xrightarrow{\alpha} G \xrightarrow{c_g} G\ .$$

Since inner automorphisms induce the identity on equivariant homotopy groups (compare Proposition 3.1.16),

$$\mathrm{tr}_{L^k}^K \circ (c_g\circ\alpha\circ c_k^{-1})^* \ = \ \mathrm{tr}_{L^k}^K \circ k_\star^{-1}\circ\alpha^*\circ g_\star \ = \ k_\star^{-1}\circ \mathrm{tr}_L^K\circ\alpha^*\circ g_\star \ = \ \mathrm{tr}_L^K\circ\alpha^*\ .$$

So the transformation $[L,\alpha]$ only depends on the conjugacy class of (L,α), i.e.,

$$[L^k, c_g\circ\alpha\circ c_k^{-1}] \ = \ [L,\alpha] \quad \text{in } \mathbf{A}(G,K).$$

Theorem 4.2.6. *Let K and G be compact Lie groups. The morphism group $\mathbf{A}(G,K)$ in the Burnside category is a free abelian group with basis the transformations $[L,\alpha]$, where (L,α) runs over all $(K\times G)$-conjugacy classes of pairs consisting of*

- *a closed subgroup $L\leq K$ whose Weyl group $W_K L$ is finite, and*
- *a continuous group homomorphism $\alpha : L\longrightarrow G$.*

Proof We let V be any faithful G-representation. By Proposition 4.1.13 the composite

$$\mathbb{Z}\{[L,\alpha]\mid |W_K L|<\infty,\ \alpha: L\longrightarrow G\} \ \longrightarrow\ \mathrm{Nat}(\pi_0^G,\pi_0^K) \ \xrightarrow{\ \mathrm{ev}\ }\ \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$$

is an isomorphism, where the first map takes $[L,\alpha]$ to $\mathrm{tr}_L^K\circ\alpha^*$ and the second map is evaluation at the stable tautological class $e_{G,V}$. The evaluation map is an isomorphism by Proposition 4.2.5, so the first map is an isomorphism, as claimed. □

Theorem 4.2.6 amounts to a complete calculation of the Burnside category, because we know how to express the composite of two operations, each given in the basis of Theorem 4.2.6, as a sum of basis elements. Indeed, restrictions are contravariantly functorial and transfers are transitive, and we also know how to expand a transfer followed by a restriction: every group homomorphism is the composite of an epimorphism and a subgroup inclusion. Inflations commute with transfers according to Proposition 3.2.32, and the restriction of a transfer can be rewritten via the double coset formula as in Theorem 3.4.9.

Theorem 4.2.6 tell us what data is necessary to specify a global functor $M:\mathbf{A}\longrightarrow\mathcal{A}b$. For this, one needs to give the values $M(G)$ at all compact Lie groups G, restriction maps $\alpha^*: M(G)\longrightarrow M(L)$ for all continuous group homomorphisms $\alpha: L\longrightarrow G$ and transfer maps $\mathrm{tr}_L^K : M(L)\longrightarrow M(K)$ for all

closed subgroup inclusions $L \leq K$. This data has to satisfy the same kind of relations that the restriction and transfer maps for equivariant homotopy groups satisfy, namely:

- the restriction maps are contravariantly functorial;
- inner automorphisms induce the identity;
- transfers are transitive and tr_K^K is the identity;
- the transfer tr_L^K is zero if the Weyl group $W_K L$ is infinite;
- transfer along an inclusion $H \leq G$ interacts with inflation along an epimorphism $\alpha : K \longrightarrow G$ according to
$$\alpha^* \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_L^K \circ (\alpha|_L)^* \;:\; M(H) \;\longrightarrow\; M(K)\,,$$
where $L = \alpha^{-1}(H)$; and
- for all pairs of closed subgroups H and K of G, the double coset formula holds, see Theorem 3.4.9.

The hypothesis that inner automorphisms act as the identity implies that the restriction map α^* only depends on the homotopy class of α. Indeed, suppose that $\alpha, \alpha' : K \longrightarrow G$ are homotopic through continuous group homomorphisms. Then α and α' belong to the same path component of the space $\mathrm{hom}(K, G)$ of continuous homomorphisms, so they are conjugate by an element of G, compare Proposition A.25.

This explicit description allows us to relate our notion of global functor to other 'global' versions of Mackey functors. For example, our category of global functors is equivalent to the category of *functors with regular Mackey structure* in the sense of Symonds [170, §3, p. 177]. Our global functors are *not* equivalent to the global Mackey functors in the sense of tom Dieck [179, Ch. VI (8.14), Ex. 5]; indeed, in the indexing category Ω for tom Dieck's global Mackey functors the group $\mathrm{Hom}_\Omega(G, K)$ has a $\mathbb{Z}$-basis indexed by $(G \times K)$-conjugacy classes $[L, \alpha]$ where the Weyl group $W_K L$ is allowed to be infinite. As we shall explain in Remark 4.2.16 below, global functors defined on finite groups are equivalent to *inflation functors* in the sense of Webb [185] and to 'global $(\emptyset, \infty)$-Mackey functors' in the sense of Lewis [98].

Example 4.2.7 (Sphere spectrum)**.** The *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}(V) = S^V$, the one-point compactification of the inner product space V. The orthogonal group acts as the one-point compactification of its action on V. The structure map $\sigma_{V,W} : S^V \wedge S^W \longrightarrow S^{V \oplus W}$ is the canonical homeomorphism. The equivariant homotopy groups of the sphere spectrum are the equivariant stable stems. The sphere spectrum is the suspension spectrum of the constant one-point orthogonal space $\mathbf{L}_{e,0}$,
$$\mathbb{S} \;\cong\; \Sigma^\infty_+ \mathbf{L}_{e,0}\,.$$

The trivial representation is faithful as a representation of the trivial group, so $\mathbf{L}_{e,0} = B_{\mathrm{gl}}e$ is a global classifying space for the trivial group. The class $1 \in \pi_0(\mathbb{S})$ represented by the identity of S^0 is the stable tautological class $e_{e,0}$ (compare (4.1.12)). By Corollary 4.1.13 the group $\pi_0^K(\mathbb{S})$ is a free abelian group with basis the classes $\mathrm{tr}_L^K(p_L^*(1))$ where L runs over all conjugacy classes of closed subgroups of K with finite Weyl group, and where $p_L : L \longrightarrow e$ is the unique homomorphism. For finite groups, this is originally due to Segal [152], and for general compact Lie groups to tom Dieck, as a corollary to his splitting theorem (see Satz 2 and Satz 3 of [177]). By Proposition 4.2.5, the action on the unit $1 \in \pi_0(\mathbb{S})$ is an isomorphism of global functors

$$\mathbb{A} \ = \ \mathbf{A}(e, -) \ \longrightarrow \ \underline{\pi}_0(\mathbb{S})$$

from the Burnside ring global functor $\mathbb{A}$ to the 0th homotopy global functor of the sphere spectrum.

Example 4.2.8. (i) The *Burnside ring global functor* is the represented global functor $\mathbb{A} = \mathbf{A}(e, -)$ of morphisms out of the trivial group e. By Theorem 4.2.6, the value $\mathbb{A}(K) = \mathbf{A}(e, K)$ at a compact Lie group K is a free abelian group with basis the set of conjugacy classes of closed subgroups $L \leq K$ with finite Weyl group. When K is finite, then the Weyl group condition is vacuous and $\mathbb{A}(K)$ is canonically isomorphic to the Burnside ring of K, by sending the operation $[L, p_L] = \mathrm{tr}_L^K \circ p_L^* \in \mathbb{A}(K)$ to the class of the K-set K/L (where $p_L : L \longrightarrow e$ is the unique homomorphism). As we discussed in Example 4.2.7, the Burnside ring global functor $\mathbb{A}$ is realized by the sphere spectrum $\mathbb{S}$. More generally, the represented functors $\mathbf{A}(G, -)$ are other examples of global functors, and we have seen in Proposition 4.2.5 that the represented global functor $\mathbf{A}(G, -)$ is realized by the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$.

(ii) In [178, Sec. 5.5], tom Dieck gives a very different construction of the Burnside ring global functor. We let G be a compact Lie group. A *G-ENR* is a G-space equivariantly homeomorphic to a G-retract of a G-invariant open subset of a finite-dimensional G-representation. The acronym 'ENR' stands for *euclidean neighborhood retract*. Examples of G-ENRs are smooth compact G-manifolds and finite G-CW-complexes.

Tom Dieck calls two compact G-ENRs X and Y *equivalent* if for every closed subgroup H of G the Euler characteristics of the H-fixed-point spaces X^H and Y^H coincide; here Euler characteristics are taken with respect to compactly supported Alexander-Spanier cohomology, and there is some work involved in showing that a compact G-ENR has a well-defined Euler characteristic. Then $A(G)$ is defined as the set of equivalence classes of compact G-ENRs. The set $A(G)$ is naturally a commutative ring, with addition induced by disjoint union and multiplication induced by cartesian product of G-ENRs.

Tom Dieck shows in [178, Prop. 5.5.1] that $A(G)$ is a free abelian group with basis the classes of the homogeneous spaces G/H, where H runs over the conjugacy classes of closed subgroups with finite Weyl group. Moreover, the class of a general compact G-ENR X is expressed in terms of this basis by the formula

$$[X] \;=\; \sum\nolimits_{(H)} \chi^{\mathrm{AS}}(G\backslash X_{(H)}) \cdot [G/H]\;;$$

the sum is over conjugacy classes of closed subgroups, $X_{(H)}$ is the orbit type subspace, and χ^{AS} is the Euler characteristic based on Alexander-Spanier cohomology. A compact G-ENR has only finitely many orbit types, so the sum is in fact finite. Restriction of scalars along a continuous homomorphism $\alpha : K \longrightarrow G$ induces a ring homomorphism $\alpha^* : A(G) \longrightarrow A(K)$. If H is a closed subgroup of G, then extension of scalars – sending an H-ENR Y to $G \times_H Y$ – induces an additive transfer map $\mathrm{tr}_H^G : A(H) \longrightarrow A(G)$ that satisfies reciprocity with respect to restriction from G to H. By explicit comparison of bases, the homomorphisms

$$A(G) \;\longrightarrow\; \mathbb{A}(G)\,, \quad [G/H] \;\longmapsto\; \mathrm{tr}_H^G \circ p_H^*$$

define an isomorphism of global functors.

(iii) Given an abelian group M, the *constant global functor* $\underline{M}$ is given by $\underline{M}(G) = M$ and all restriction maps are identity maps. The transfer $\mathrm{tr}_H^G : \underline{M}(H) \longrightarrow \underline{M}(G)$ is multiplication by the Euler characteristic of the homogeneous space G/H. In particular, if H has finite index in G, then tr_H^G is multiplication by the index $[G : H]$. In this example, the double coset formula is a special case of the Euler characteristic formula (3.4.7), namely for the K-manifold $B = G/H$.

There is a well-known point-set level model of an Eilenberg-Mac Lane spectrum $\mathcal{H}M$ that we recall in Construction 5.3.8 below. However, contrary to what one may expect, the 0th homotopy group global functor $\underline{\pi}_0(\mathcal{H}M)$ is *not* isomorphic to the constant global functor $\underline{M}$. More precisely, the restriction map $p_G^* : \pi_0^e(\mathcal{H}M) \longrightarrow \pi_0^G(\mathcal{H}M)$ is an isomorphism for finite groups G, but not for general compact Lie groups, see Example 5.3.14.

(iv) The *unitary representation ring global functor* $\mathbf{RU}$ assigns to a compact Lie group G the unitary representation ring $\mathbf{RU}(G)$, i.e., the Grothendieck group of finite-dimensional complex G-representations, with product induced by tensor product of representations. The restriction maps $\alpha^* : \mathbf{RU}(G) \longrightarrow \mathbf{RU}(K)$ are induced by restriction of representations along a continuous homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [150, § 2]. If H is a subgroup of finite index of G, then this induction sends the

class of an H-representation to the induced G-representation $\mathrm{map}^H(G, V)$; in general, induction may send actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for **RU** was proved by Snaith [160, Thm. 2.4]. We look more closely at the representation ring global functor in Example 5.3.18, and we show in Theorem 6.4.24 that **RU** is realized by the global K-theory spectrum **KU** (see Construction 6.4.9).

(v) Given any generalized cohomology theory E (in the non-equivariant sense), we can define a global functor $\underline{E}$ by setting

$$\underline{E}(G) \;=\; E^0(BG)\ ,$$

the 0th E-cohomology of a classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer map for a subgroup inclusion $H \leq G$ comes from the stable transfer map (i.e., Becker-Gottlieb transfer)

$$\Sigma^\infty_+ BG \;\longrightarrow\; \Sigma^\infty_+ BH\ .$$

The double coset formula was proved in this context by Feshbach [53, Thm. II.11]. We will show in Proposition 4.5.22 that the global functor $\underline{E}$ is realized by an orthogonal spectrum bE, the 'global Borel theory' associated with E.

Remark 4.2.9. If we fix a finite group G and let H run through all subgroups of G, then the collection of H-equivariant homotopy groups $\pi_0^H(X)$ of an orthogonal spectrum X forms a *G-Mackey functor* (see Definition 3.4.15), with respect to restriction to subgroups, conjugation and transfer maps. As we already discussed in Remark 4.1.2, not all G-Mackey functors arise this way. To illustrate this we compare Mackey functors for the group $C_3 = \{1, \tau, \tau^2\}$ with three elements to additive functors on the full subcategory of **A** spanned by the group e and C_3. Generating operations can be displayed as follows:

$$\begin{tikzcd} F(C_3) \arrow[loop left, "\alpha^*"] \arrow[r, "\mathrm{res}"'] & F(e) \arrow[l, bend right, "\mathrm{tr}"'] \arrow[l, bend left, "p^*"] \end{tikzcd} \qquad\qquad \begin{tikzcd} F(C_3) \arrow[r, "\mathrm{res}"'] & F(e) \arrow[l, bend right, "\mathrm{tr}"'] \arrow[loop right, "\tau"] \end{tikzcd}$$

global functor on C_3 and e $\qquad\qquad$ C_3-Mackey functor

Here $\mathrm{res} = \mathrm{res}^{C_3}_e$ and $\mathrm{tr} = \mathrm{tr}^{C_3}_e$ are the restriction and transfer maps that are present in both cases. A global functor also comes with inflation maps along the epimorphism $p : C_3 \longrightarrow e$ and along the automorphism $\alpha : C_3 \longrightarrow C_3$ with $\alpha(\tau) = \tau^2$, and the relations are

$$\mathrm{res} \circ p^* \;=\; \mathrm{Id} \qquad \text{and} \qquad \mathrm{res} \circ \mathrm{tr} \;=\; 3 \cdot \mathrm{Id}$$

as well as $\alpha^* \circ \alpha^* = \mathrm{Id}$, $\alpha^* \circ p^* = p^*$, $\mathrm{res} \circ \alpha^* = \mathrm{res}$ and $\alpha^* \circ \mathrm{tr} = \mathrm{tr}$. In contrast,

C_3-Mackey functors have an additional action of C_3 (the Weyl group of e in C_3) on $F(e)$, and this action satisfies the relation

$$\mathrm{res} \circ \mathrm{tr} \ = \ \mathrm{Id} \ + \tau \ + \ \tau^2 \ .$$

We will now use the exterior homotopy group pairings (4.1.21)

$$\boxtimes \ : \ \pi_0^G(X) \ \times \ \pi_0^K(Y) \ \longrightarrow \ \pi_0^{G\times K}(X \wedge Y)$$

to define a biadditive functor

$$\times \ : \ \mathbf{A} \times \mathbf{A} \ \longrightarrow \ \mathbf{A}$$

that is given on objects by the product of Lie groups, and that extends to a symmetric monoidal structure on the Burnside category $\mathbf{A}$.

Proposition 4.2.10. *Let G, G', K and K' be compact Lie groups. Given operations $\tau \in \mathbf{A}(G, K)$ and $\psi \in \mathbf{A}(G', K')$, there is a unique operation*

$$\tau \times \psi \ \in \ \mathbf{A}(G \times G', \ K \times K')$$

with the following property: for all orthogonal spectra X and Y and all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^{G'}(Y)$ the relation

$$(\tau \times \psi)(x \boxtimes y) \ = \ \tau(x) \boxtimes \psi(y) \tag{4.2.11}$$

holds in $\pi_0^{K\times K'}(X \wedge Y)$.

Proof We choose a faithful G-representation V and a faithful G'-representation V', which have associated stable tautological classes (4.1.12)

$$e_{G,V} \ \in \ \pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V}) \qquad \text{and} \qquad e_{G',V'} \ \in \ \pi_0^{G'}(\Sigma^\infty_+ \mathbf{L}_{G',V'}) \ .$$

Combining the isomorphism (4.1.17) with the one from Example 1.3.3 shows that the orthogonal spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{G',V'}$ is isomorphic to $\Sigma^\infty_+ \mathbf{L}_{G\times G', V\oplus V'}$ in a way that matches the class $e_{G,V} \boxtimes e_{G',V'}$ with the class $e_{G\times G', V\oplus V'}$. Proposition 4.2.5 then shows that the pair $(\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{G',V'}, e_{G,V} \boxtimes e_{G',V'})$ represents the functor $\pi_0^{G\times G'}$. There is thus a unique operation $\tau \times \psi \in \mathbf{A}(G \times G', K \times K')$ that satisfies

$$(\tau \times \psi)(e_{G,V} \boxtimes e_{G',V'}) \ = \ \tau(e_{G,V}) \boxtimes \psi(e_{G',V'}) \tag{4.2.12}$$

in $\pi_0^{K\times K'}(\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{G',V'})$.

The relation (4.2.12) is a special case of (4.2.11), and it remains to show that the operation $\tau \times \psi$ satisfies the relation (4.2.11) in complete generality. As we already argued in the proof of Proposition 4.2.5, there is a G-representation W and a morphism of orthogonal spectra

$$f \ : \ \Sigma^\infty_+ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ X$$

that satisfies $f_*(e_{G,V\oplus W}) = x$. Similarly, there is a G'-representation W' and a morphism of orthogonal spectra

$$f' \ : \ \Sigma^\infty_+ \mathbf{L}_{G',V'\oplus W'} \longrightarrow Y$$

that satisfies $f'_*(e_{G',V'\oplus W'}) = y$. The morphism

$$\Sigma^\infty_+ \rho_{G,V,W} \ : \ \Sigma^\infty_+ \mathbf{L}_{G,V\oplus W} \longrightarrow \Sigma^\infty_+ \mathbf{L}_{G,V} \quad \text{satisfies} \quad (\Sigma^\infty_+ \rho_{G,V,W})(e_{G,V\oplus W}) = e_{G,V}\ ,$$

and similarly for the triple (G', V', W'). Naturality then yields

$$\begin{aligned}
&(\Sigma^\infty_+ \rho_{G,V,W} \wedge \Sigma^\infty_+ \rho_{G',V',W'})_*((\tau \times \psi)(e_{G,V\oplus W} \boxtimes e_{G',V'\oplus W'}))\\
&\qquad = (\tau \times \psi)((\Sigma^\infty_+ \rho_{G,V,W})_*(e_{G,V\oplus W}) \boxtimes (\Sigma^\infty_+ \rho_{G',V',W'})_*(e_{G',V'\oplus W'}))\\
&\qquad = (\tau \times \psi)(e_{G,V} \boxtimes e_{G',V'})\\
&\overset{(4.2.12)}{=} \tau(e_{G,V}) \boxtimes \psi(e_{G',V'})\\
&\qquad = \tau((\Sigma^\infty_+ \rho_{G,V,W})_*(e_{G,V\oplus W})) \boxtimes \psi((\Sigma^\infty_+ \rho_{G',V',W'})_*(e_{G',V'\oplus W'}))\\
&\qquad = (\Sigma^\infty_+ \rho_{G,V,W} \wedge \Sigma^\infty_+ \rho_{G',V',W'})_*(\tau(e_{G,V\oplus W}) \boxtimes \psi(e_{G',V'\oplus W'}))\ .
\end{aligned}$$

The morphism $\Sigma^\infty_+ \rho_{G,V,W} \wedge \Sigma^\infty_+ \rho_{G',V',W'}$ is isomorphic to $\Sigma^\infty_+ \rho_{G\times G',V\oplus V',W\oplus W'}$. The morphism $\rho_{G\times G',V\oplus V',W\oplus W'}$ is a global equivalence of orthogonal spaces (by Proposition 1.1.26 (ii)), and so the morphism $\Sigma^\infty_+ \rho_{G,V,W} \wedge \Sigma^\infty_+ \rho_{G',V',W'}$ is a global equivalence of orthogonal spectra (by Corollary 4.1.9). So in particular it induces an isomorphism on $\pi_0^{K\times K'}$, and we can conclude that

$$(\tau \times \psi)(e_{G,V\oplus W} \boxtimes e_{G',V'\oplus W'}) = \tau(e_{G,V\oplus W}) \boxtimes \psi(e_{G',V'\oplus W'})\ .$$

Now the relation (4.2.11) follows by simple naturality:

$$\begin{aligned}
(\tau \times \psi)(x \boxtimes y) &= (\tau \times \psi)(f_*(e_{G,V\oplus W}) \boxtimes f'_*(e_{G',V'\oplus W'}))\\
&= (\tau \times \psi)((f \wedge f')_*(e_{G,V\oplus W} \boxtimes e_{G',V'\oplus W'}))\\
&= (f \wedge f')_*((\tau \times \psi)(e_{G,V\oplus W} \boxtimes e_{G',V'\oplus W'}))\\
&= (f \wedge f')_*(\tau(e_{G,V\oplus W}) \boxtimes \psi(e_{G',V'\oplus W'}))\\
&= f_*(\tau(e_{G,V\oplus W})) \boxtimes f'_*(\psi(e_{G',V'\oplus W'}))\\
&= \tau(f_*(e_{G,V\oplus W})) \boxtimes \psi(f'_*(e_{G',V'\oplus W'})) = \tau(x) \boxtimes \psi(y) \qquad \square
\end{aligned}$$

Example 4.2.13. We calculate the product of two generating operations in the Burnside category. We recall that for a pair (L, α) consisting of a closed subgroup L of K and a continuous homomorphism $\alpha : L \longrightarrow G$ we defined

$$[L, \alpha] = \mathrm{tr}_L^K \circ \alpha^* \in \mathbf{A}(G, K)\ .$$

By Theorem 4.2.6, a certain subset of these operations forms a basis of the

abelian group $\mathbf{A}(G,K)$. Using parts (v) and (vi) of Theorem 4.1.22 we deduce

$$\begin{aligned}[L\times L',\alpha\times\alpha'](x\boxtimes y) &= \mathrm{tr}_{L\times L'}^{K\times K'}((\alpha\times\alpha')^*(x\boxtimes y))\\ &= \mathrm{tr}_{L\times L'}^{K\times K'}(\alpha^*(x)\boxtimes(\alpha')^*(y))\\ &= \mathrm{tr}_L^K(\alpha^*(x))\boxtimes\mathrm{tr}_{L'}^{K'}((\alpha')^*(y)) \;=\; [L,\alpha](x)\boxtimes[L',\alpha'](y)\,.\end{aligned}$$

So the operation $[L\times L',\alpha\times\alpha']$ has the property that characterizes the operation $[L,\alpha]\times[L',\alpha']$. Hence the monoidal product in $\mathbf{A}$ satisfies

$$[L,\alpha]\times[L'\alpha'] \;=\; [L\times L',\alpha\times\alpha']\,. \tag{4.2.14}$$

Now we are ready for the monoidal structure of the Burnside category.

Theorem 4.2.15. *Let G, G', K and K' be compact Lie groups.*

(i) *The map*

$$\times \;:\; \mathbf{A}(G,K)\times\mathbf{A}(G',K') \;\longrightarrow\; \mathbf{A}(G\times G',K\times K')$$

is biadditive.

(ii) *As the Lie groups vary, the maps of* (i) *form a functor* $\times:\mathbf{A}\times\mathbf{A}\longrightarrow\mathbf{A}$.

(iii) *The restriction operations along the group isomorphisms*

$$\begin{aligned}a_{G,G',G''} &\;:\; G\times(G'\times G'') \;\cong\; (G\times G')\times G''\,,\\ \tau_{G,G'} &\;:\; G\times G' \;\cong\; G'\times G \quad \textit{and} \quad G\times e \;\cong\; G \;\cong\; e\times G\end{aligned}$$

extend the functor $\times$ to a symmetric monoidal structure on the global Burnside category.

Proof (i) We show additivity in the first variable, the other case being analogous. The relation

$$\begin{aligned}((\tau\times\psi)+(\tau'\times\psi))(x\boxtimes y) &= (\tau\times\psi)(x\boxtimes y)+(\tau'\times\psi)(x\boxtimes y)\\ &= (\tau(x)\boxtimes\psi(y))+(\tau'(x)\boxtimes\psi(y))\\ &= (\tau(x)+\tau'(x))\boxtimes\psi(y) \;=\; (\tau+\tau')(x)\boxtimes\psi(y)\end{aligned}$$

shows that the operation $(\tau\times\psi')+(\tau'\times\psi)$ has the property that characterizes the operation $(\tau+\tau')\times\psi$. So

$$(\tau+\tau')\times\psi \;=\; (\tau\times\psi)+(\tau'\times\psi)\,.$$

(ii) The relation

$$\begin{aligned}((\tau'\times\psi')\circ(\tau\times\psi))(x\boxtimes y) &= (\tau'\times\psi')((\tau\times\psi)(x\boxtimes y))\\ &= (\tau'\times\psi')(\tau(x)\boxtimes\psi(y))\\ &= \tau'(\tau(x))\boxtimes\psi'(\psi(y)) \;=\; (\tau'\circ\tau)(x)\boxtimes(\psi'\circ\psi)(y)\end{aligned}$$

shows that the operation $(\tau' \times \psi') \circ (\tau \times \psi)$ has the property that characterizes the operation $(\tau' \circ \tau) \times (\psi' \circ \psi)$. So

$$(\tau' \circ \tau) \times (\psi' \circ \psi) \;=\; (\tau' \times \psi') \circ (\tau \times \psi)\;.$$

A similar (but even shorter) argument shows that $\mathrm{Id}_G \times \mathrm{Id}_{G'} = \mathrm{Id}_{G\times G'}$.

(iii) We start with naturality of the associativity isomorphism. We consider operations $\tau \in \mathbf{A}(G,K)$, $\psi \in \mathbf{A}(G',K')$ and $\kappa \in \mathbf{A}(G'',K'')$. The relation

$$\begin{aligned}(\tau \times (\psi \times \kappa))(a^*_{G,G',G''}((x \boxtimes y) \boxtimes z)) \;&=\; ((\tau \times (\psi \times \kappa))(x \boxtimes (y \boxtimes z)))\\ &=\; \tau(x) \boxtimes (\psi(y) \boxtimes \kappa(z))\\ &=\; a^*_{K,K',K''}((\tau(x) \boxtimes \psi(y)) \boxtimes \kappa(z))\\ &=\; a^*_{K,K',K''}(((\tau \times \psi) \times \kappa)((x \boxtimes y) \boxtimes z))\end{aligned}$$

shows that the operation $(a^*_{K,K',K''})^{-1} \circ (\tau \times (\psi \times \kappa)) \circ a^*_{G,G',G''}$ has the property that characterizes the operation $(\tau \times \psi) \times \kappa$. So

$$(\tau \times (\psi \times \kappa)) \circ a^*_{G,G',G''} \;=\; a^*_{K,K',K''} \circ ((\tau \times \psi) \times \kappa)\;.$$

The arguments for naturality of the unit and symmetry isomorphisms are similar, and we omit them.

The unit, associativity (pentagon) and symmetry (hexagon) coherence relations in $\mathbf{A}$ follow from the corresponding coherence relations for the product of groups, and the fact that passage from group homomorphisms to restriction operations is functorial. □

Remark 4.2.16. The full subcategory $\mathbf{A}_{\mathcal{F}in}$ of the Burnside category $\mathbf{A}$ spanned by *finite* groups has a different, more algebraic description, as we shall now recall. This alternative description is in terms of 'bisets', and is often taken as the definition in algebraic treatments of global functors (which are then sometimes called 'biset functors'). The category of 'global functors on finite groups', i.e., additive functors from $\mathbf{A}_{\mathcal{F}in}$ to abelian groups, is thus equivalent to the category of 'inflation functors' in the sense of Webb [185, p.271] and to the 'global $(\emptyset,\infty)$-Mackey functors' in the sense of Lewis [98].

We define the additive *biset category* $\mathbb{A}^{\mathrm{c}}$. The objects of $\mathbb{A}^{\mathrm{c}}$ are all finite groups. The abelian group $\mathbb{A}^{\mathrm{c}}(G,K)$ of morphisms from a group G to K is the Grothendieck group of finite K-G-bisets where the right G-action is free. In the special case $G = e$ of the trivial group as source we obtain $\mathbb{A}^{\mathrm{c}}(e,K)$, the Burnside ring of finite K-sets. Composition

$$\circ \;:\; \mathbb{A}^{\mathrm{c}}(K,L) \times \mathbb{A}^{\mathrm{c}}(G,K) \;\longrightarrow\; \mathbb{A}^{\mathrm{c}}(G,L)$$

is induced by the balanced product over K, i.e., it is the biadditive extension of

$$(S,T) \;\longmapsto\; S \times_K T\;.$$

Here S has a left L-action and a commuting free right K-action, whereas T has a left K-action and a commuting free right G-action. The balanced product $S \times_K T$ than inherits a left L-action from S and a free right G-action from T. Since the balanced product is associative up to isomorphism, this defines a pre-additive category. So the category $\mathbb{A}^c$ is the 'group completion' of the category $\mathbb{A}^+_{\mathcal{F}in}$, the restriction of the effective Burnside category $\mathbb{A}^+$ (compare Construction 2.2.26) to finite groups.

We define additive maps

$$\Psi_{G,K} \ : \ \mathbf{A}(G,K) \ \longrightarrow \ \mathbb{A}^c(G,K)$$

that form an additive equivalence of categories (restricted to finite groups). The map $\Psi_{G,K}$ sends a basis element $[L,\alpha]$ to the class of the K-G-biset

$$K \times_{(L,\alpha)} G \ = \ K \times G/(kl,g) \sim (k,\alpha(l)g)$$

whose right G-action is free. Every transitive G-free K-G-biset is isomorphic to one of this form, and $K \times_{(L,\alpha)} G$ is isomorphic, as a K-G-biset, to $K \times_{(L',\alpha')} G$ if and only if (L,α) is conjugate to (L',α'). So the map $\Psi_{G,K}$ sends the basis of $\mathbf{A}(G,K)$ of Theorem 4.2.6 to a basis of $\mathbb{A}^c(G,K)$, and it is thus an isomorphism.

We claim that the maps $\Psi_{G,K}$ form a functor as G and K vary through all finite groups; this then shows that Ψ is an additive equivalence of categories from the full subcategory of $\mathbf{A}_{\mathcal{F}in}$ to $\mathbb{A}^c$. The functoriality boils down to the fact that in both categories restriction, inflation and transfer interact with each other in exactly the same way. We omit the details.

The restriction of the monoidal structure on the Burnside category to finite groups has an interpretation in terms of the cartesian product of bisets: under the equivalence of categories $\Psi : \mathbf{A}_{\mathcal{F}in} \cong \mathbb{A}^c$, it corresponds to the monoidal structure

$$\mathbb{A}^c(G,K) \times \mathbb{A}^c(G',K') \ \longrightarrow \ \mathbb{A}^c(G \times G', K \times K') \ , \ ([S],[S']) \ \longmapsto [S \times S'] \ .$$

Here S is a right free K-G-biset and S' is a right free K'-G'-biset; the cartesian product $S \times S'$ is then a right free $(K \times K')$-$(G \times G')$-biset. Indeed, the equivalence $\Psi : \mathbf{A}_{\mathcal{F}in} \cong \mathbb{A}^c$ sends the basis element $[L,\alpha] \in \mathbf{A}(G,K)$ to the class of the biset $K \times_{(L,\alpha)} G$. So the equivalence is monoidal because

$$(K \times_{(L,\alpha)} G) \times (K' \times_{(L',\alpha')} G') \quad \text{and} \quad (K \times K') \times_{(L \times L', \alpha \times \alpha')} G \times G'$$

are isomorphic as $(K \times K')$-$(G \times G')$-bisets.

Construction 4.2.17 (Box product of global functors)**.** Since global functors are additive functors on the Burnside category $\mathbf{A}$, the symmetric monoidal product on $\mathbf{A}$ gives rise to a symmetric monoidal convolution product on the category of global functors. This is a special case of the general construction

of Day [42] that we review in Appendix C. We now make this convolution product more explicit. We denote by $\mathbf{A}\otimes\mathbf{A}$ the pre-additive category whose objects are pairs of compact Lie groups, and with morphism groups

$$(\mathbf{A}\otimes\mathbf{A})((G,G'),(K,K')) \;=\; \mathbf{A}(G,K)\otimes\mathbf{A}(G',K')\;.$$

We let F, F' and F'' be global functors. We denote by $F\otimes F' : \mathbf{A}\otimes\mathbf{A}\longrightarrow\mathcal{A}b$ the objectwise tensor product given on objects by

$$(F\otimes F')(G,G') \;=\; F(G)\otimes F'(G')\;.$$

A *bimorphism* is a natural transformation

$$F\otimes F' \;\longrightarrow\; F''\circ\times$$

of additive functors on the category $\mathbf{A}\otimes\mathbf{A}$. Since the morphism groups in the Burnside category are generated by transfers and restrictions, this means more explicitly that a bimorphism is a collection of group homomorphisms

$$b_{G,G'} \;:\; F(G)\otimes F'(G') \;\longrightarrow\; F''(G\times G')$$

for all compact Lie groups G and G', such that for all continuous homomorphisms $\alpha : K\longrightarrow G$ and $\alpha' : K'\longrightarrow G'$ and for all closed subgroups $H\leq G$ and $H'\leq G'$ the diagram

$$\begin{tikzcd}
F(H)\otimes F'(H') \arrow[r, "\mathrm{tr}_H^G\otimes\mathrm{tr}_{H'}^{G'}"] \arrow[d, "b_{H,H'}"'] & F(G)\otimes F'(G') \arrow[r, "\alpha^*\otimes(\alpha')^*"] \arrow[d, "b_{G,G'}"'] & F(K)\otimes F'(K') \arrow[d, "b_{K,K'}"] \\
F''(H\times H') \arrow[r, "\mathrm{tr}_{H\times H'}^{G\times G'}"'] & F''(G\times G') \arrow[r, "(\alpha\times\alpha')^*"'] & F''(K\times K')
\end{tikzcd}$$

commutes. Here we exploited that the generating operations multiply as described in (4.2.14). Equivalently: for every compact Lie group G the maps $\{b_{G,G'}\}_{G'}$ form a morphism of global functors $F(G)\otimes F'(-)\longrightarrow F''(G\times -)$ and for every compact Lie group G' the maps $\{b_{G,G'}\}_G$ form a morphism of global functors $F(-)\otimes F'(G')\longrightarrow F''(-\times G')$.

A *box product* of F and F' is a universal example of a global functor with a bimorphism from F and F'. More precisely, a box product is a pair $(F\Box F', i)$ consisting of a global functor $F\Box F'$ and a universal bimorphism i, i.e., such that for every global functor F'' the map

$$\mathcal{GF}(F\Box F', F'') \;\longrightarrow\; \mathrm{Bimor}((F,F'),F'')\;,\quad f\longmapsto fi$$

is bijective. Box products exist by the general theory (see Proposition C.5), and they are unique up to preferred isomorphism (see Remark C.7). The universal property guarantees that given any collection of choices of box product $F\Box F'$

for all pairs of global functors, $F \Box F'$ extends to an additive functor in both variables. Moreover, there are preferred associativity and commutativity isomorphisms that make the box product a symmetric monoidal structure on the category of global functors, with the Burnside ring global functor $\mathbb{A}$ as a strict unit object, compare Theorem C.10. The box product of representable global functors is again representable, by Remark C.11.

Remark 4.2.18. We claim that the box product is right exact in both variables. To see this we recall from Remark C.6 that the values of a Day type convolution product can be described as an enriched coend. In our present situation this says that $F \Box M$ is a cokernel of a certain homomorphism of global functors

$$d \ : \ \bigoplus_{H,H',G,G'} \mathbf{A}(H \times H', -) \otimes \mathbf{A}(G, H) \otimes \mathbf{A}(G', H') \otimes F(G) \otimes M(G') \\ \longrightarrow \bigoplus_{G,G'} \mathbf{A}(G \times G', -) \otimes F(G) \otimes M(G') \ .$$

The left sum is indexed over all quadruples (H, H', G, G') in a set of representatives of isomorphism classes of compact Lie groups, the right sum is indexed over pairs (G, G') of such groups. The map d is the difference of two homomorphisms; one of them sums the tensor products of

$$\mathbf{A}(H \times H', -) \otimes \mathbf{A}(G, H) \otimes \mathbf{A}(G', H') \ \longrightarrow \ \mathbf{A}(G \times G', -) \ , \ \varphi \otimes \tau \otimes \tau' \ \longmapsto \ \varphi \circ (\tau \times \tau')$$

and the identity on $F(G) \otimes M(G')$. The other map sums the tensor product of the identity of the global functor $\mathbf{A}(H \times H', -)$ and the action maps $\mathbf{A}(G, H) \otimes F(G) \longrightarrow F(H)$ and $\mathbf{A}(G', H') \otimes M(G') \longrightarrow M(H')$, respectively. Cokernels of global functors are calculated objectwise, so the value $(F \Box M)(K)$ is a cokernel of the morphism of abelian groups that we obtain by plugging K into the free variable above. Theorem 4.2.6 describes explicit free generators for the morphism groups in the Burnside category; using this, the value $(F \Box M)(K)$ can be expanded into a cokernel of a morphism between two huge sums of tensor products of values of F and M.

Now we consider a short exact sequence of global functors

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0 \ .$$

This gives rise to a commutative diagram of global functors

$$\begin{tikzcd}
\bigoplus \mathbf{A}(H \times H', -) \otimes \mathbf{A}(G, H) \otimes \mathbf{A}(G', H') \otimes F(G) \otimes M(G') \arrow[r] \arrow[d] & \bigoplus \mathbf{A}(G \times G', -) \otimes F(G) \otimes M(G') \arrow[d] \\
\bigoplus \mathbf{A}(H \times H', -) \otimes \mathbf{A}(G, H) \otimes \mathbf{A}(G', H') \otimes F(G) \otimes M'(G') \arrow[r] \arrow[d] & \bigoplus \mathbf{A}(G \times G', -) \otimes F(G) \otimes M'(G') \arrow[d] \\
\bigoplus \mathbf{A}(H \times H', -) \otimes \mathbf{A}(G, H) \otimes \mathbf{A}(G', H') \otimes F(G) \otimes M''(G') \arrow[r] \arrow[d] & \bigoplus \mathbf{A}(G \times G', -) \otimes F(G) \otimes M''(G') \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns. The induced sequence of horizontal cokernels

$$F \Box M \longrightarrow F \Box M' \longrightarrow F \Box M'' \longrightarrow 0$$

is thus also exact.

While the box product of global functors shares many properties with the tensor product of modules over a commutative ring, the constructions differ fundamentally in one aspect: *projectives are not generally flat* in the category of global functors. In other words, for most projective global functors P, the functor $-\Box P$ does *not* send monomorphisms to monomorphisms. This kind of phenomenon has been analyzed in great detail by Lewis in [98]; Lewis' notation for the category $\mathbb{A}^c$ is $\mathcal{B}_*(\emptyset, \infty)$, our $\mathcal{F}in$-global functors are his 'global $(\emptyset, \infty)$-Mackey functors' and the category of $\mathcal{F}in$-global functors is denoted $\mathfrak{M}_*(\emptyset, \infty)$. Theorem 6.10 of [98] shows that the representable functor $\mathbf{A}_{C_p}$ is not flat, where C_p is a cyclic group of prime order p.

We now remark that bimorphisms of global functors can be identified with another kind of structure that we call 'diagonal products'.

Definition 4.2.19. Let X, Y and Z be global functors. A *diagonal product* is a natural transformation $X \otimes Y \longrightarrow Z$ of Rep-functors to abelian groups that satisfies reciprocity, where $X \otimes Y$ is the objectwise tensor product.

More explicitly, a diagonal product consists of additive maps

$$\nu_G \ : \ X(G) \otimes Y(G) \ \longrightarrow \ Z(G)$$

for every compact Lie group G that are natural for restriction along continuous homomorphisms and satisfy the reciprocity relation

$$\mathrm{tr}_H^G(\nu_H(x \otimes \mathrm{res}_H^G(y))) \ = \ \nu_G(\mathrm{tr}_H^G(x) \otimes y)$$

for all closed subgroups H of G and all classes $x \in X(H)$ and $y \in Y(G)$, and similarly in the other variable.

Remark 4.2.20. Any bimorphism $\mu : (X, Y) \longrightarrow Z$ gives rise to a diagonal product as follows. For a group G we define ν_G as the composite

$$X(G) \otimes Y(G) \xrightarrow{\mu_{G,G}} Z(G \times G) \xrightarrow{\Delta_G^*} Z(G)$$

where $\Delta_G : G \longrightarrow G \times G$ is the diagonal. For a group homomorphism $\alpha : K \longrightarrow G$ we have $\Delta_G \circ \alpha = (\alpha \times \alpha) \circ \Delta_K$, so the following diagram commutes:

$$\begin{tikzcd}
X(G) \otimes Y(G) \arrow[r, "\mu_{G,G}"] \arrow[d, "\alpha^* \otimes \alpha^*"'] & Z(G \times G) \arrow[r, "\Delta_G^*"] \arrow[d, "(\alpha \times \alpha)^*"] & Z(G) \arrow[d, "\alpha^*"] \\
X(K) \otimes Y(K) \arrow[r, "\mu_{K,K}"'] & Z(K \times K) \arrow[r, "\Delta_K^*"'] & Z(K)
\end{tikzcd}$$

Since there is only one double coset for the left Δ_G-action and the right $(H \times G)$-action on $G \times G$, the double coset formula becomes

$$\Delta_G^* \circ \mathrm{tr}_{H \times G}^{G \times G} \ = \ \mathrm{tr}_H^G \circ \Delta_H^* \circ \mathrm{res}_{H \times H}^{H \times G} \ .$$

We conclude that

$$\begin{aligned}
\mathrm{tr}_H^G(\nu_H(x \otimes \mathrm{res}_H^G(y))) \ &= \ \mathrm{tr}_H^G(\Delta_H^*(\mathrm{res}_{H \times H}^{H \times G}(\mu_{H,G}(x \otimes y)))) \\
&= \ \Delta_G^*(\mathrm{tr}_{H \times G}^{G \times G}(\mu_{H,G}(x \otimes y))) \\
&= \ \Delta_G^*(\mu_{G,G}(\mathrm{tr}_H^G(x) \otimes y)) \ = \ \nu_G(\mathrm{tr}_H^G(x) \otimes y) \ ,
\end{aligned}$$

the reciprocity relation for the diagonal product ν. The reciprocity in the other variable is similar.

Conversely, given a diagonal product ν, we define a bimorphism with (G, K)-component as the composite

$$X(G) \otimes Y(K) \xrightarrow{p_G^* \otimes p_K^*} X(G \times K) \otimes Y(G \times K) \xrightarrow{\nu_{G \times K}} Z(G \times K) \ ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. If the diagonal product ν was defined from an external product μ as above, then

$$\begin{aligned}
\nu_{G \times K} \circ (p_G^* \otimes p_K^*) \ &= \ \Delta_{G \times K}^* \circ \mu_{G \times K, G \times K} \circ (p_G^* \otimes p_K^*) \\
&= \ \Delta_{G \times K}^* \circ (p_G \times p_K)^* \circ \mu_{G,K} \ = \ \mu_{G,K}
\end{aligned}$$

because the composite $(p_G \times p_K) \circ \Delta_{G \times K}$ is the identity. So the external product can be recovered from the diagonal product.

Given homomorphisms $\alpha : G \longrightarrow G'$ and $\beta : K \longrightarrow K'$, we have $p_{G'} \circ (\alpha \times$

$\beta) = \alpha \circ p_G$ and $p_{K'} \circ (\alpha \times \beta) = \beta \circ p_K$, so the left part of the diagram

$$\begin{tikzcd}
X(G) \otimes Y(K) \arrow[r, "p_G^* \otimes p_K^*"] \arrow[d, "\alpha^* \otimes \beta^*"'] & X(G \times K) \otimes Y(G \times K) \arrow[r, "\nu_{G \times K}"] \arrow[d, "(\alpha\times\beta)^* \otimes (\alpha\times\beta)^*"] & Z(G \times K) \arrow[d, "(\alpha\times\beta)^*"] \\
X(G') \otimes Y(K') \arrow[r, "p_{G'}^* \otimes p_{K'}^*"'] & X(G' \times K') \otimes Y(G' \times K') \arrow[r, "\nu_{G' \times K'}"'] & Z(G' \times K')
\end{tikzcd}$$

commutes. The right part commutes by naturality of the diagonal product ν.

For naturality with respect to transfers we let H be a closed subgroup of G, and we consider classes $x \in X(H)$ and $y \in Y(K)$. Then

$$\begin{aligned}
\mathrm{tr}_{H \times K}^{G \times K}(\mu_{H,K}(x \otimes y)) &= \mathrm{tr}_{H \times K}^{G \times K}(\nu_{H \times K}(p_H^*(x) \otimes \mathrm{res}_{H \times K}^{G \times K}(p_K^*(y)))) \\
&= \nu_{G \times K}(\mathrm{tr}_{H \times K}^{G \times K}(p_H^*(x)) \otimes p_K^*(y)) \\
&= \nu_{G \times K}(p_G^*(\mathrm{tr}_H^G(x)) \otimes p_K^*(y)) \ = \ \mu_{G,K}(\mathrm{tr}_H^G(x) \otimes y) \,.
\end{aligned}$$

The second equality is reciprocity, the third is compatibility of transfer and inflation. The argument for transfer naturality in the K-variable is similar.

4.3 Global model structures for orthogonal spectra

In this section we establish the strong level and global model structures on the category of orthogonal spectra. Many arguments are parallel to the unstable analogs in Section 1.2, so there is a certain amount of repetition. The main model structure of interest for us is the *global model structure*, see Theorem 4.3.18. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. More generally, we consider a global family $\mathcal{F}$ and define the *$\mathcal{F}$-global model structure*, see Theorem 4.3.17 below, with weak equivalences the $\mathcal{F}$-equivalences, i.e., those morphisms inducing isomorphisms of G-equivariant homotopy groups for all G in $\mathcal{F}$. Proposition 4.3.24 shows that the global model structure is monoidal with respect to the smash product of orthogonal spectra; more generally, the $\mathcal{F}$-global model structure is monoidal, provided that $\mathcal{F}$ is closed under products.

As we explained in more detail in Construction 3.5.6, there is a functorial way to write an orthogonal spectrum X as a sequential colimit of spectra which are made from the information below a fixed level, the *skeleta*

$$\mathrm{sk}^m X \ = \ l_m(X^{\le m}) \,,$$

the extension of the restriction of X to $\mathbf{O}_{\le m}$. The skeleton comes with a natural morphism $i_m : \mathrm{sk}^m X \longrightarrow X$, the counit of the adjunction $(l_m, (-)^{\le m})$. The value $i_m(V) : (\mathrm{sk}^m X)(V) \longrightarrow X(V)$ is an isomorphism for all inner product spaces

V of dimension at most m. The word 'filtration' should be used with caution because the morphism i_m need not be injective. The *mth latching space* of X is the based $O(m)$-space $L_m X = (\mathrm{sk}^{m-1} X)(\mathbb{R}^m)$; it comes with a natural based $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) \ : \ L_m X \longrightarrow X(\mathbb{R}^m) \ ,$$

the *mth latching map*.

Example 4.3.1. We let G be a compact Lie group and V a G-representation of dimension n. Then the semifree orthogonal spectrum (4.1.24) $F_{G,V}A$ generated by a based G-space A in level V is 'purely n-dimensional' in the following sense. The space $(F_{G,V}A)_m$ is trivial for $m < n$, and hence the latching space $L_m(F_{G,V}A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(F_{G,V}A) \longrightarrow (F_{G,V}A)(\mathbb{R}^m)$ is an isomorphism. So the skeleton $\mathrm{sk}^m(F_{G,V}A)$ is trivial for $m < n$ and $\mathrm{sk}^m(F_{G,V}A) = F_{G,V}A$ is the entire spectrum for $m \geq n$.

Proposition C.23 is a fairly general construction of level model structures. We specialize the general recipe to the category of orthogonal spectra. For a morphism $f : X \longrightarrow Y$ of orthogonal spectra and $m \geq 0$ we have a commutative square of $O(m)$-spaces:

$$\begin{tikzcd} L_m X \arrow[r, "L_m f"] \arrow[d, "\nu_m^X"'] & L_m Y \arrow[d, "\nu_m^Y"] \\ X(\mathbb{R}^m) \arrow[r, "f(\mathbb{R}^m)"'] & Y(\mathbb{R}^m) \end{tikzcd}$$

We thus get a natural morphism of based $O(m)$-spaces

$$\nu_m f \ = \ f(\mathbb{R}^m) \cup \nu_m^Y \ : \ X(\mathbb{R}^m) \cup_{L_m X} L_m Y \ \longrightarrow \ Y(\mathbb{R}^m) \ .$$

Definition 4.3.2. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *flat cofibration* if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration for all $m \geq 0$. An orthogonal spectrum Y is *flat* if the morphism from the trivial spectrum to it is a flat cofibration, i.e., for every $m \geq 0$ the latching map $\nu_m : L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration.

Flatness as just defined is thus a special case of 'G-flatness' in the sense of Definition 3.5.7, for G a trivial compact Lie group.

We let $\mathcal{F}$ be a global family in the sense of Definition 1.4.1, i.e., a non-empty class of compact Lie groups that is closed under isomorphisms, closed subgroups and quotient groups. As in the unstable situation in Section 1.4, we now develop the *$\mathcal{F}$-level model structure* on the category of orthogonal

spectra, in which the $\mathcal{F}$-level equivalences are the weak equivalences. This model structure has a 'global' (or 'stable') version, see Theorem 4.3.17 below.

We recall that $\mathcal{F} \cap G$ denotes the family of those closed subgroups of a compact Lie group G that belong to the global family $\mathcal{F}$. Moreover, $\mathcal{F}(m) = \mathcal{F} \cap O(m)$ is the family of those closed subgroups of the orthogonal group $O(m)$ that belong to the global family $\mathcal{F}$.

Definition 4.3.3. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is

- an *$\mathcal{F}$-level equivalence* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-equivalence for all $m \geq 0$;
- an *$\mathcal{F}$-level fibration* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-fibration for all $m \geq 0$; and
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-cofibration or all $m \geq 0$.

In other words, $f : X \longrightarrow Y$ is an $\mathcal{F}$-level equivalence (or $\mathcal{F}$-level fibration) if for every $m \geq 0$ and every subgroup H of $O(m)$ that belongs to the family $\mathcal{F}$ the map $f(\mathbb{R}^m)^H : X(\mathbb{R}^m)^H \longrightarrow Y(\mathbb{R}^m)^H$ is a weak equivalence (or Serre fibration).

We let G be any group from the family $\mathcal{F}$ and V a faithful G-representation of dimension m. We let $\alpha : \mathbb{R}^m \longrightarrow V$ be a linear isometry and define a homomorphism $c_\alpha : G \longrightarrow O(m)$ by 'conjugation by α', i.e., we set

$$(c_\alpha(g))(x) \;=\; \alpha^{-1}(g \cdot \alpha(x))$$

for $g \in G$ and $x \in \mathbb{R}^m$. We restrict the $O(m)$-action on $X(\mathbb{R}^m)$ to a G-action along the homomorphism c_α. Then the map

$$X(\alpha) \;:\; c_\alpha^*(X(\mathbb{R}^m)) \;\longrightarrow\; X(V)$$

is a G-equivariant homeomorphism, natural in X; it restricts to a natural homeomorphism from $X(\mathbb{R}^m)^{\bar{G}}$ to $X(V)^G$, where $\bar{G} = c_\alpha(G)$ is the image of c_α. This implies:

Proposition 4.3.4. *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ a morphism of orthogonal spectra.*

(i) *The morphism f is an $\mathcal{F}$-level equivalence if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F} \cap G)$-equivalence.*

(ii) *The morphism f is an $\mathcal{F}$-level fibration if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F} \cap G)$-fibration.*

Now we are ready to establish the $\mathcal{F}$-level model structure.

Proposition 4.3.5. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a model structure, the $\mathcal{F}$*-level model structure*, on the category of orthogonal spectra. The $\mathcal{F}$-level model structure is topological and cofibrantly generated.*

Proof The first part is a special case of Proposition C.23, in the following way. We let $C(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of based $O(m)$-spaces, i.e., the based version of the model structure of Proposition B.7. With respect to these choices of model structures $C(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition C.23 become the $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations.

The consistency condition (see Definition C.22) becomes the following condition: for all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the $\mathcal{F}(m)$-projective model structure on based $O(m)$-spaces, every cobase change, in the category of based $O(m+n)$-spaces, of the map

$$\mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} i \;:\; \mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} A \;\longrightarrow\; \mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} B$$

is an $\mathcal{F}(m+n)$-weak equivalence. We show a stronger statement, namely that the functor

$$\mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} - \;:\; O(m)\mathbf{T}_* \;\longrightarrow\; O(m+n)\mathbf{T}_*$$

takes acyclic cofibrations in the $\mathcal{F}(m)$-projective model structure to acyclic cofibrations in the projective model structure on the category of based $O(m+n)$-spaces (i.e., the $\mathcal{A}ll$-projective model structure, where $\mathcal{A}ll$ is the family of all closed subgroups of $O(m+n)$). Since the functor under consideration is a left adjoint, it suffices to prove the claim for the generating acyclic cofibrations, i.e., the maps

$$(O(m)/H \times j_k)_+$$

for all $H \in \mathcal{F}(m)$ and all $k \geq 0$, where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion. Up to isomorphism, the functor takes this map to

$$O(m+n) \ltimes_{H \times O(n)} S^n \wedge (j_k)_+$$

where H acts trivially on S^n. Since the projective model structure on the category of based $O(m+n)$-spaces is topological, it suffices to show that $O(m+n) \ltimes_{H \times O(n)} S^n$ is cofibrant in this model structure. Since S^n is $O(n)$-equivariantly homeomorphic to the reduced mapping cone of the map $O(n)/O(n-1)_+ \longrightarrow S^0$, it suffices to show that the two $O(m+n)$-spaces

$$O(m+n) \times_{H \times O(n)} (O(n)/O(n-1)) \qquad \text{and} \qquad O(m+n)/(H \times O(n))$$

are cofibrant in the unbased sense. Both of these $O(m+n)$-spaces are homogeneous spaces, hence $O(m+n)$-cofibrant.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-level model structure. As before we denote by

$$G_m = F_{O(m),\mathbb{R}^m} : O(m)\mathbf{T}_* \longrightarrow \mathcal{S}p$$

the left adjoint to the evaluation functor $X \mapsto X(\mathbb{R}^m)$, i.e., the semifree functor (4.1.24) indexed by the tautological $O(m)$-representation. We let $I_{\mathcal{F}}$ be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.8). Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure by Proposition C.23 (iii). Similarly, we let $J_{\mathcal{F}}$ be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.9). Again by Proposition C.23 (iii), $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure. The $\mathcal{F}$-level model structure is topological by Proposition B.5, where we take $\mathcal{G}$ as the set of semifree orthogonal spectra $F_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all $H \in \mathcal{F}(m)$, and $\mathcal{Z} = \emptyset$ as the empty set. □

When $\mathcal{F} = \langle e \rangle$ is the minimal global family consisting of all trivial groups, then the $\langle e \rangle$-level equivalences (or $\langle e \rangle$-level fibrations) are the level equivalences (or level fibrations) of orthogonal spectra in the sense of Definition 6.1 of [107]. Hence the $\langle e \rangle$-cofibrations are the 'q-cofibrations' in the sense of [107, Def. 6.1]. For the minimal global family, the $\langle e \rangle$-level model structure thus specializes to the level model structure of [107, Thm. 6.5].

For easier reference we spell out the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family of all compact Lie groups. In this case $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$, and the $\mathcal{A}ll$-cofibrations specialize to the flat cofibrations. We introduce special names for the $\mathcal{A}ll$-level equivalences and the $\mathcal{A}ll$-level fibrations, analogous to the unstable situation in Section 1.2.

Definition 4.3.6. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *strong level equivalence* (or *strong level fibration*) if for every $m \geq 0$ the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-weak equivalence (or $O(m)$-fibration).

For the global family $\mathcal{F} = \mathcal{A}ll$, Proposition 4.3.5 specializes to:

Proposition 4.3.7. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the* strong level model structure*, on the category of orthogonal spectra. The strong level model structure is topological and cofibrantly generated.*

Now we introduce and discuss an important class of orthogonal spectra.

Definition 4.3.8. An orthogonal spectrum X is a *global Ω-spectrum* if for every compact Lie group G, every faithful G-representation W and an arbitrary G-representation V the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}_*(S^V, X(V \oplus W))$$

is a G-weak equivalence.

The global Ω-spectra will turn out to be the fibrant objects in the global model structure on orthogonal spectra, see Theorem 4.3.18 below. This means that global Ω-spectra abound, because every orthogonal spectrum admits a global equivalence to a global Ω-spectrum.

Remark 4.3.9. Global Ω-spectra are a very rich kind of structure, because they encode compatible equivariant infinite loop spaces for all compact Lie groups at once. For a global Ω-spectrum X and a compact Lie group G the associated orthogonal G-spectrum X_G is 'eventually an Ω-G-spectrum' in the sense that the Ω-G-spectrum condition of [108, III Def. 3.1] holds for all 'sufficiently large' (i.e., faithful) G-representations. However, if G is a non-trivial group, then the associated orthogonal G-spectrum X_G is in general *not* an Ω-G-spectrum since there is no control over the G-homotopy type of the values at non-faithful representations.

For every compact Lie group G and every faithful G-representation W, the G-space

$$X[G] \ = \ \Omega^W X(W)$$

is a 'genuine' equivariant infinite loop space, i.e., deloopable in the direction of every representation. Indeed, for every G-representation V, the composite

$$X[G] \ = \ \Omega^W X(W) \xrightarrow{\Omega^W(\tilde{\sigma}_{V,W})} \Omega^W(\Omega^V X(V \oplus W)) \ \cong \ \Omega^V(\Omega^W X(V \oplus W))$$

is a G-weak equivalence, so the global Ω-spectrum X provides a V-deloop $\Omega^W X(V \oplus W)$ of $X[G]$. The G-space $X[G]$ is also independent, up to G-weak equivalence, of the choice of faithful G-representation. Indeed, if $\bar{W}$ is another faithful G-representation, then the G-maps

$$\Omega^W X(W) \xrightarrow{\Omega^W(\tilde{\sigma}_{W',W})} \Omega^W(\Omega^{W'} X(W' \oplus W))$$
$$\cong \ \Omega^{W'}(\Omega^W, X(W \oplus W')) \xleftarrow{\Omega^{W'}(\tilde{\sigma}_{W,W'})} \Omega^{W'} X(W')$$

are G-weak equivalences.

As G varies, the equivariant infinite loop spaces $X[G]$ are closely related to each other. For example, if H is a closed subgroup of G, then any faithful

G-representation is also faithful as an H-representation. So $X[H]$ is H-weakly equivalent to the restriction of the G-equivariant infinite loop space $X[G]$.

Remark 4.3.10. Let X be a global Ω-spectrum. Specialized to the trivial group, the condition in Definition 4.3.8 says that X is in particular a non-equivariant Ω-spectrum in the sense that the adjoint structure map $\tilde{\sigma}_{\mathbb{R},W} : X(W) \longrightarrow \Omega X(\mathbb{R} \oplus W)$ is a weak equivalence of (non-equivariant) spaces for every inner product space W.

If X is a global Ω-spectrum, then so is the shifted spectrum $\mathrm{sh}\, X$ and the function spectrum $\mathrm{map}_*(A, X)$ for every cofibrant based space A. Indeed, if W is a faithful G-representation, then $W \oplus \mathbb{R}$, the sum with a trivial 1-dimensional representation, is also faithful. So the adjoint structure map

$$\tilde{\sigma}^X_{V,W\oplus\mathbb{R}} \ : \ X(W \oplus \mathbb{R}) \ \longrightarrow \ \Omega^V(X(V \oplus W \oplus \mathbb{R}))$$

is a G-weak equivalence for every G-representation V. But this map is also the adjoint structure map

$$\tilde{\sigma}^{\mathrm{sh}\, X}_{V,W} \ : \ (\mathrm{sh}\, X)(W) \ \longrightarrow \ \Omega^V((\mathrm{sh}\, X)(V \oplus W))$$

of the shifted spectrum. The argument for mapping spectra is similar.

Definition 4.3.11. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every G-representation V and every faithful G-representation W the square

$$\begin{tikzcd}
X(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(W)^G"'] & \mathrm{map}^G_*(S^V, X(V \oplus W)) \arrow[d, "\mathrm{map}^G_*(S^V, f(V\oplus W))"] \\
Y(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] & \mathrm{map}^G_*(S^V, Y(V \oplus W))
\end{tikzcd} \qquad (4.3.12)$$

is homotopy cartesian.

We state a useful criterion for checking when a morphism is a global fibration. We recall from Construction 3.1.31 the homotopy fiber $F(f)$ of a morphism $f : X \longrightarrow Y$ of orthogonal spectra along with the natural map $i : \Omega Y \longrightarrow F(f)$. We apply this for the shifted morphism $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$ and pre-compose with the morphism $\tilde{\lambda}_Y$, the adjoint to $\lambda_Y : Y \wedge S^1 \longrightarrow \mathrm{sh}\, Y$ defined in (3.1.23). We denote by $\xi(f)$ the resulting composite

$$Y \ \xrightarrow{\tilde{\lambda}_Y} \ \Omega(\mathrm{sh}\, Y) \ \xrightarrow{i} \ F(\mathrm{sh}\, f) \, .$$

We note that the orthogonal spectrum $F(\mathrm{Id}_{\mathrm{sh}\, X})$ has a preferred contraction, so part (ii) of the next proposition is a way to make precise that the sequence

$$X \ \xrightarrow{f} \ Y \ \xrightarrow{\xi(f)} \ F(\mathrm{sh}\, f)$$

is a ‘strong level homotopy fiber sequence’.

Proposition 4.3.13. *Let $f : X \longrightarrow Y$ be a strong level fibration of orthogonal spectra. Then the following two conditions are equivalent.*

(i) *The morphism f is a global fibration.*
(ii) *The strict fiber of f is a global Ω-spectrum and the commutative square*

$$\begin{tikzcd} X \arrow[r, "\xi(\mathrm{Id}_X)"] \arrow[d, "f"'] & F(\mathrm{Id}_{\mathrm{sh}\,X}) \arrow[d] \\ Y \arrow[r, "\xi(f)"'] & F(\mathrm{sh}\,f) \end{tikzcd}$$

is homotopy cartesian in the strong level model structure.

Proof (i)$\Longrightarrow$(ii) Since the square (4.3.12) is homotopy cartesian and the vertical maps are Serre fibrations, the induced map on strict fibers $(f^{-1}(*)(W))^G \longrightarrow \mathrm{map}_*^G(S^V, f^{-1}(*)(V \oplus W))$ is a weak equivalence. So the strict fiber is a global Ω-spectrum.

The square in (ii) factors as the composite of two commutative squares:

$$\begin{tikzcd} X \arrow[r, "\tilde{\lambda}_X"] \arrow[d, "f"'] & \Omega\,\mathrm{sh}\,X \arrow[r, "i"] \arrow[d, "\Omega\,\mathrm{sh}\,f"] & F(\mathrm{Id}_{\mathrm{sh}\,X}) \arrow[d] \\ Y \arrow[r, "\tilde{\lambda}_Y"'] & \Omega\,\mathrm{sh}\,Y \arrow[r, "i"'] & F(\mathrm{sh}\,f) \end{tikzcd}$$

Specializing the global fibration property (4.3.12) for $W = \mathbb{R}$ with trivial G-action shows that the left square is homotopy cartesian in the strong level model structure. For every continuous based map $g : A \longrightarrow B$ the square

$$\begin{tikzcd} \Omega A \arrow[r, "i"] \arrow[d, "\Omega g"'] & F(\mathrm{Id}_A) \arrow[d] \\ \Omega B \arrow[r, "i"'] & F(g) \end{tikzcd}$$

is homotopy cartesian; applying this to $g = ((\mathrm{sh}\,f)(V))^G$ for all G-representations V shows that the right square is homotopy cartesian in the strong level model structure. So the composite square is homotopy cartesian in the strong level model structure.

(ii)$\Longrightarrow$(i) We let G be a compact Lie group, V a G-representation, and W a

faithful G-representation. We consider the commutative diagram

$$\begin{tikzcd}
X(W)^G \arrow[r, "(\tilde\sigma_{V,W})^G"] \arrow[d, "f(W)^G"'] & \mathrm{map}_*^G(S^V, X(V\oplus W)) \arrow[d, "\mathrm{map}_*^G(S^V, f(V\oplus W))"] \\
Y(W)^G \arrow[r, "(\tilde\sigma_{V,W})^G"'] \arrow[d, "(\xi(f)(W))^G"'] & \mathrm{map}_*^G(S^V, Y(V\oplus W)) \arrow[d, "\mathrm{map}_*^G(S^V, \xi(f)(V\oplus W))"] \\
F(f)(W\oplus\mathbb{R})^G \arrow[r, "\sim", "(\tilde\sigma_{V,W})^G"'] & \mathrm{map}_*^G(S^V, F(f)(V\oplus W\oplus\mathbb{R}))
\end{tikzcd}$$

Since f is a strong level fibration, the natural morphism from the strict fiber to the homotopy fiber is a strong level equivalence. So the lower horizontal map is a weak equivalence because the strict fiber, and hence the homotopy fiber, is a global Ω-spectrum. Moreover, the two vertical columns are homotopy fiber sequences by hypothesis, so the upper square is homotopy cartesian, both with respect to the strong level model structure. $\square$

Definition 4.3.14. Let $\mathcal{F}$ be a global family.

- A morphism $f : X \longrightarrow Y$ of orthogonal spectra is an $\mathcal{F}$*-equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all G in $\mathcal{F}$ and all integers k.
- A morphism $f : X \longrightarrow Y$ of orthogonal spectra is an $\mathcal{F}$-global fibration if it is an $\mathcal{F}$-level fibration and for every compact Lie group G in $\mathcal{F}$, every G-representation V and every faithful G-representation W the square (4.3.12) is homotopy cartesian.
- An orthogonal spectrum X is an $\mathcal{F}$-Ω*-spectrum* if for every compact Lie group G in $\mathcal{F}$, every G-representation V and every faithful G-representation W the adjoint structure map

$$\tilde\sigma_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}_*(S^V, X(V\oplus W))$$

is a G-weak equivalence.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then an $\mathcal{A}ll$-equivalence is just a global equivalence in the sense of Definition 4.1.3, and the $\mathcal{A}ll$-global fibrations are the global fibrations in the sense of Definition 4.3.11. Also, an $\mathcal{A}ll$-Ω-spectrum is the same as a global Ω-spectrum in the sense of Definition 4.3.8. When $\mathcal{F} = \langle e\rangle$ is the minimal global family of all trivial groups, then the $\langle e\rangle$-equivalences are just the traditional non-equivariant stable equivalences of orthogonal spectra, also known as 'π_*-isomorphisms'. The $\langle e\rangle$-Ω-spectra are just the traditional non-equivariant Ω-spectra.

The following proposition collects various useful relations between $\mathcal{F}$-equivalences, $\mathcal{F}$-level equivalences and $\mathcal{F}$-Ω-spectra.

Proposition 4.3.15. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-level equivalence of orthogonal spectra is an $\mathcal{F}$-equivalence.*
(ii) *Let X be an $\mathcal{F}$-Ω-spectrum. Then for every G in $\mathcal{F}$, every faithful G-representation V and every $k \geq 0$ the stabilization map*

$$[S^{V\oplus\mathbb{R}^k}, X(V)]^G \longrightarrow \pi_k^G(X), \quad [f] \longmapsto \langle f \rangle$$

is bijective.
(iii) *Let X be an $\mathcal{F}$-Ω-spectrum such that $\pi_k^G(X) = 0$ for every integer k and all G in $\mathcal{F}$. Then for every group G in $\mathcal{F}$ and every faithful G-representation V the space $X(V)$ is G-weakly contractible.*
(iv) *Every $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra is an $\mathcal{F}$-level equivalence.*
(v) *Every $\mathcal{F}$-equivalence that is also an $\mathcal{F}$-global fibration is an $\mathcal{F}$-level equivalence.*

Proof (i) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-level equivalence, and we need to show that the map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all G in $\mathcal{F}$. We start with the case $k = 0$. We let G be a group from $\mathcal{F}$ and V a finite-dimensional faithful G-subrepresentation of the complete G-universe $\mathcal{U}_G$. By Proposition 4.3.4 (i) the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence. Since the representation sphere S^V can be given a G-CW-structure, the induced map on G-homotopy classes

$$[S^V, f(V)]^G : [S^V, X(V)]^G \longrightarrow [S^V, Y(V)]^G$$

is bijective. The faithful G-representations are cofinal in the poset $s(\mathcal{U}_G)$, so taking the colimit over $V \in s(\mathcal{U}_G)$ shows that $\pi_0^G(f) : \pi_0^G(X) \longrightarrow \pi_0^G(Y)$ is an isomorphism. For $k > 0$ we exploit the fact that $\pi_k^G(X)$ is naturally isomorphic to $\pi_0^G(\Omega^k X)$ and the k-fold loop of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence. For $k < 0$ we exploit the fact that $\pi_k^G(X)$ is naturally isomorphic to $\pi_0^G(\mathrm{sh}^{-k} X)$ and that every shift of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence.

(ii) We may assume that V is a G-invariant subspace of the complete G-universe $\mathcal{U}_G$ used to define $\pi_k^G(X)$. Since V is faithful, so is every G-representation that contains V. So the directed system whose colimit is $\pi_k^G(X)$ consists of isomorphisms 'above V'. Hence the canonical map

$$[S^{V\oplus\mathbb{R}^k}, X(V)]^G \longrightarrow \pi_k^G(X)$$

is bijective.

(iii) We adapt an argument of Lewis-May-Steinberger [100, I 7.12] to our

context. Every $\mathcal{F}$-Ω-spectrum is in particular a non-equivariant Ω-spectrum; every non-equivariant Ω-spectrum with trivial homotopy groups is level-wise weakly contractible, so this takes care of trivial groups.

Now we let G be a non-trivial group in $\mathcal{F}$. We argue by a nested induction over the 'size' of G: we induct over the dimension of G and, for fixed dimension, over the cardinality of the finite set of path components of G. Every proper closed subgroup H of G either has strictly smaller dimension than G, or the same dimension but fewer path components. So we know by induction that the fixed-point space $X(V)^H$ is weakly contractible for every proper closed subgroup H of G. So it remains to analyze the G-fixed-points of $X(V)$.

We let $W = V - V^G$ be the orthogonal complement of the fixed subspace V^G of V. Then W is a faithful G-representation with trivial fixed-points. The cofiber sequence of G-CW-complexes

$$S(W)_+ \longrightarrow D(W)_+ \longrightarrow S^W$$

induces a fiber sequence of equivariant mapping spaces

$$\mathrm{map}^G_*(S^W, X(V)) \longrightarrow \mathrm{map}^G(D(W), X(V)) \longrightarrow \mathrm{map}^G(S(W), X(V)) .$$

Since $W^G = 0$, the G-fixed-points $S(W)^G$ are empty, so any G-CW-structure on $S(W)$ uses only equivariant cells of the form $G/H \times D^n$ for proper subgroups H of G. But for proper subgroups H, all the fixed-point spaces $X(V)^H$ are weakly contractible by induction. Hence the space $\mathrm{map}^G(S(W), X(V))$ is weakly contractible. Since the unit disc $D(W)$ is equivariantly contractible, the space $\mathrm{map}^G(D(W), X(V))$ is homotopy equivalent to $X(V)^G$, and we conclude that evaluation at the fixed-point $0 \in W$ is a weak equivalence

$$\mathrm{map}^G_*(S^W, X(V)) \simeq X(V)^G .$$

We have $X(V) = X(W \oplus V^G) = (\mathrm{sh}^{V^G} X)(W)$. The stabilization map

$$\begin{aligned} \pi_k\left(\mathrm{map}^G_*(S^W, X(V))\right) &= [S^{W\oplus\mathbb{R}^k}, (\mathrm{sh}^{V^G} X)(W)]^G \\ &\longrightarrow \pi^G_k(\mathrm{sh}^{V^G} X) \cong \pi^G_{k-\dim(V^G)}(X) \end{aligned}$$

is bijective for all $k \geq 0$, by part (ii). Since we assumed that the G-equivariant homotopy groups of X vanish, the space $\mathrm{map}^G_*(S^W, X(V))$ is path connected and has vanishing homotopy groups, i.e., it is weakly contractible. So $X(V)^G$ is weakly contractible, and this completes the proof of (iii).

(iv) Let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra. We let F denote the homotopy fiber of f, and we let G be a group from $\mathcal{F}$. For every G-representation V the G-space $F(V)$ is the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. So F is again an $\mathcal{F}$-Ω-spectrum. The long exact sequence of homotopy groups (see Proposition 3.1.36) implies that $\pi^H_*(F) = 0$ for all H in $\mathcal{F}$.

If G acts faithfully on V, then by the $\mathcal{F}$-Ω-spectrum property, the space $X(V)$ is G-weakly equivalent to $\Omega X(V \oplus \mathbb{R})$ and similarly for Y. So the map $f(V)$ is G-weakly equivalent to

$$\Omega f(V \oplus \mathbb{R}) \ : \ \Omega X(V \oplus \mathbb{R}) \ \longrightarrow \ \Omega X(V \oplus \mathbb{R}) \, .$$

Hence we have a homotopy fiber sequence of G-spaces

$$X(V) \ \xrightarrow{f(V)} \ Y(V) \ \longrightarrow \ F(V \oplus \mathbb{R}) \, .$$

Since F is an $\mathcal{F}$-Ω-spectrum with vanishing equivariant homotopy groups for groups in $\mathcal{F}$, the space $F(V \oplus \mathbb{R})$ is G-weakly contractible by part (iii). So $f(V)$ is a G-weak equivalence. The morphism f is then an $\mathcal{F}$-level equivalence by the criterion of Proposition 4.3.4 (i).

(v) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence and an $\mathcal{F}$-global fibration. Then the strict fiber $f^{-1}(*)$ of f is an $\mathcal{F}$-Ω-spectrum with trivial G-equivariant homotopy groups for all $G \in \mathcal{F}$. So $f^{-1}(*)$ is $\mathcal{F}$-level equivalent to the trivial spectrum by part (iii). Since f is an $\mathcal{F}$-level fibration, the embedding $f^{-1}(*) \longrightarrow F(f)$ of the strict fiber into the homotopy fiber is an $\mathcal{F}$-level equivalence, so the homotopy fiber $F(f)$ is $\mathcal{F}$-level equivalent to the trivial spectrum. The homotopy cartesian square of Proposition 4.3.13 (ii) then shows that f is an $\mathcal{F}$-level equivalence. □

Our next result, in fact the main result of this section, is the $\mathcal{F}$-global model structure. This model structure is cofibrantly generated, and we spell out an explicit set of generating cofibrations and generating acyclic cofibrations. The set $I_{\mathcal{F}}$ was defined in the proof of Proposition 4.3.5 as the set of morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.8). Similarly, the set $J_{\mathcal{F}}$ is the set of morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (B.9).

Given any compact Lie group G and G-representations V and W we recall from (4.1.28) the morphism

$$\lambda_{G,V,W} \ : \ F_{G,V \oplus W} S^V \ \longrightarrow \ F_{G,W} \, .$$

If the representation W is faithful, then this morphism is a global equivalence by Theorem 4.1.29. We set

$$K_{\mathcal{F}} \ = \ \bigcup_{G,V,W} \mathcal{Z}(\lambda_{G,V,W}) \, , \tag{4.3.16}$$

the set of all pushout products of sphere inclusions $i_m : \partial D^m \longrightarrow D^m$ with the mapping cylinder inclusions of the morphisms $\lambda_{G,V,W}$ (compare Construction 1.2.15); here the union is over a set of representatives of the isomorphism

classes of triples (G, V, W) consisting of a compact Lie group G in the family $\mathcal{F}$, a G-representation V and a faithful G-representation W.

Theorem 4.3.17 ($\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $\mathcal{F}$-cofibrations form a model structure on the category of orthogonal spectra, the $\mathcal{F}$*-global model structure.

(ii) *The fibrant objects in the $\mathcal{F}$-global model structure are the $\mathcal{F}$-Ω-spectra.*

(iii) *A morphism of orthogonal spectra is:*

- *an acyclic fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$;*
- *a fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

(iv) *The $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

(v) *The adjoint functor pair*

$$\Sigma^\infty_+ \,:\, spc \rightleftarrows \mathcal{S}p \,:\, \Omega^\bullet$$

is a Quillen pair for the two $\mathcal{F}$-global model structures on orthogonal spaces and orthogonal spectra.

Proof The category of orthogonal spectra is complete and cocomplete (MC1), the $\mathcal{F}$-equivalences satisfy the 2-out-of-3 property (MC2) and the classes of $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $\mathcal{F}$-cofibrations are closed under retracts (MC3). The $\mathcal{F}$-level model structure (Proposition 4.3.5) shows that every morphism of orthogonal spectra can be factored as $f \circ i$ for an $\mathcal{F}$-cofibration i followed by an $\mathcal{F}$-level equivalence f that is also an $\mathcal{F}$-level fibration. For every G in $\mathcal{F}$, every G-representation V and every faithful G-representation W, both vertical maps in the commutative square (4.3.12) are then weak equivalences, so the square is homotopy cartesian. The morphism f is thus an $\mathcal{F}$-global fibration and an $\mathcal{F}$-equivalence, so this provides one of the factorizations as required by MC5.

The morphism $\lambda_{G,V,W}$ represents the map

$$(\tilde{\sigma}_{V,W})^G \;:\; X(W)^G \;\longrightarrow\; \mathrm{map}^G_*(S^V, X(V \oplus W))^G \;;$$

by Proposition 1.2.16, the right lifting property with respect to the union $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ thus characterizes the $\mathcal{F}$-global fibrations. We apply the small object argument (see for example [48, 7.12] or [80, Thm. 2.1.14]) to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$. All morphisms in $J_{\mathcal{F}}$ are flat cofibrations and $\mathcal{F}$-level equivalences; $\mathcal{F}$-level equivalences are $\mathcal{F}$-equivalences by Proposition 4.3.15 (i). Since $F_{G,V\oplus W}S^V$ and $F_{G,W}$ are flat, the morphisms in $K_{\mathcal{F}}$ are also flat cofibrations, and they are

$\mathcal{F}$-equivalences because the morphisms $\lambda_{G,V,W}$ are. The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal spectra as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in $J_{\mathcal{F}} \cup K_{\mathcal{F}}$, and q has the right lifting property with respect to $J_{\mathcal{F}} \cup K_{\mathcal{F}}$; in particular, the morphism q is an $\mathcal{F}$-global fibration. All morphisms in $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ are $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations, hence also h-cofibrations (by Proposition A.30 applied to the strong level model structure). By Corollary 4.1.5 (or rather its modification for $\mathcal{F}$-equivalences), the class of h-cofibrations that are simultaneously $\mathcal{F}$-equivalences is closed under coproducts, cobase change and sequential composition. So the morphism i is an $\mathcal{F}$-cofibration and an $\mathcal{F}$-equivalence.

Now we show the lifting properties MC4. By Proposition 4.3.15 (v) a morphism that is both an $\mathcal{F}$-equivalence and an $\mathcal{F}$-global fibration is an $\mathcal{F}$-level equivalence, and hence an acyclic fibration in the $\mathcal{F}$-level model structure. So every morphism that is simultaneously an $\mathcal{F}$-equivalence and an $\mathcal{F}$-global fibration has the right lifting property with respect to $\mathcal{F}$-cofibrations. Now we let $j : A \longrightarrow B$ be an $\mathcal{F}$-cofibration that is also an $\mathcal{F}$-equivalence and we show that it has the left lifting property with respect to $\mathcal{F}$-global fibrations. We factor $j = q \circ i$, via the small object argument for $J_{\mathcal{F}} \cup K_{\mathcal{F}}$, where $i : A \longrightarrow W$ is a $(J_{\mathcal{F}} \cup K_{\mathcal{F}})$-cell complex and $q : W \longrightarrow B$ is an $\mathcal{F}$-global fibration. Then q is an $\mathcal{F}$-equivalence since j and i are, so q is an acyclic fibration in the $\mathcal{F}$-level model structure, by the above. Since j is an $\mathcal{F}$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to all $\mathcal{F}$-global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified sets of generating flat cofibrations $I_{\mathcal{F}}$ and generating acyclic cofibrations $J_{\mathcal{F}} \cup K_{\mathcal{F}}$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations. So the $\mathcal{F}$-model structure is cofibrantly generated.

Left properness of the $\mathcal{F}$-global model structure follows from Corollary 3.1.39 (i) and the fact that $\mathcal{F}$-cofibrations are h-cofibrations. Right properness follows from Corollary 3.1.39 (ii) and the fact that $\mathcal{F}$-global fibrations are in particular $\mathcal{F}$-level fibrations.

The $\mathcal{F}$-global model structure is topological by Proposition B.5; here we take $\mathcal{G}$ to be the set of semifree orthogonal spectra $G_m(O(m)/H)$ for all $m \geq 0$ and all $H \in \mathcal{F}(m)$, and we take $\mathcal{Z}$ as the set of mapping cylinder inclusions $c(\lambda_{G,V,W})$ of the morphisms $\lambda_{G,V,W}$ indexed as in the set $K_{\mathcal{F}}$.

Part (v) is straightforward from the definitions. □

In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-equivalences are the (non-equivariant) stable equivalences of orthogonal spectra, and the $\langle e \rangle$-global model structure coincides with the stable model structure established by Mandell, May, Shipley and the author in [107, Thm. 9.2]. For easier reference we spell out the special case $\mathcal{F} = \mathcal{A}ll$ for the maximal family of all compact Lie groups, resulting in the *global model structure* on the category of orthogonal spectra.

Theorem 4.3.18 (Global model structure)**.** *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spectra. The fibrant objects in the global model structure are the global Ω-spectra. The global model structure is proper, topological and cofibrantly generated.*

The same proof as in the unstable situation in Corollary 1.4.10 applies to prove the following characterization of $\mathcal{F}$-equivalences.

Corollary 4.3.19. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spectra and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *The morphism can be written as $f = w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*
(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the $\mathcal{F}$-level model structure and every $\mathcal{F}$-Ω-spectrum Y the induced map*

$$[f^c, Y] \; : \; [B^c, Y] \longrightarrow [A^c, Y]$$

on homotopy classes of morphisms is bijective.

Remark 4.3.20 (Mixed global model structures)**.** Cole's 'mixing theorem' for model structures [38] allows to construct many more $\mathcal{F}$-model structures on the category of orthogonal spectra. We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the $\mathcal{E}$-global model structure is a fibration in the $\mathcal{F}$-global model structure. By Cole's theorem [38, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spectra. By [38, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts

of all composites $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal spectrum is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal spectrum [38, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [38]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structures on the category of orthogonal spectra that are all Quillen equivalent. In the extreme case, the $\mathcal{A}ll$-mixed $\langle e \rangle$-model structure is the $\mathbb{S}$-model structure of Stolz [163, Prop. 1.3.10].

Remark 4.3.21 ($\mathcal{F}in$-global homotopy theory via symmetric spectra)**.** We denote by $\mathcal{F}in$ the global family of finite groups. The $\mathcal{F}in$-global stable homotopy theory has another very natural model, namely the category of symmetric spectra in the sense of Hovey, Shipley and Smith [81]. In [72, 73] Hausmann has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric spectra with the global model structure. Symmetric spectra cannot model global homotopy types for all compact Lie groups, basically because compact Lie groups of positive dimensions do not have any faithful permutation representations.

One of the points of model category structures is that they facilitate the analysis of and constructions in the homotopy category (which only depends on the class of weak equivalences). An example of this is the existence and constructions of products and coproducts in a homotopy category. We take the time to make this explicit for the $\mathcal{F}$-global stable homotopy category, for which we write $\mathcal{GH}_{\mathcal{F}}$.

Proposition 4.3.22. *Let $\mathcal{F}$ be a global family.*

(i) *The coproduct (i.e., wedge) of every family of orthogonal spectra is also a coproduct in the homotopy category $\mathcal{GH}_{\mathcal{F}}$.*
(ii) *Let $\{X_i\}_{i\in I}$ be a family of orthogonal spectra such that the canonical map*

$$\pi_k^G(\prod\nolimits_{i\in I} X_i) \longrightarrow \prod\nolimits_{i\in I} \pi_k^G(X_i)$$

is an isomorphism for every compact Lie group G in $\mathcal{F}$ and every integer k. Then the product $\prod_{i\in I} X_i$ of orthogonal spectra is also a product of $\{X_i\}_{i\in I}$ in the homotopy category $\mathcal{GH}_{\mathcal{F}}$.

In particular, the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$ has all set indexed coproducts and products.

Proof (i) Since the $\mathcal{F}$-equivalences are part of the $\mathcal{F}$-global model structure, general model category theory guarantees that coproducts in $\mathcal{GH}_{\mathcal{F}}$ can be constructed by taking the point-set level coproduct of $\mathcal{F}$-cofibrant approximations, see for example [80, Ex. 1.3.11]. Since equivariant homotopy groups take wedges of orthogonal spectra to direct sums (Corollary 3.1.37 (i)), any wedge of $\mathcal{F}$-equivalences is again an $\mathcal{F}$-equivalence. So the point-set level wedge maps by an $\mathcal{F}$-equivalence to the wedge of the $\mathcal{F}$-cofibrant approximations, and these two are isomorphic in $\mathcal{GH}_{\mathcal{F}}$.

(ii) This part is essentially dual to (i), with the caveat that infinite products of $\mathcal{F}$-equivalences need not be $\mathcal{F}$-equivalences in general. To construct a product in $\mathcal{GH}_{\mathcal{F}}$ of the given family, the abstract recipe is to choose $\mathcal{F}$-equivalences $f_i : X_i \longrightarrow X_i^{\mathcal{F}}$ to $\mathcal{F}$-Ω-spectra, and then form the point-set level product of the replacements. For all groups G in $\mathcal{F}$ we consider the commutative square:

$$\begin{tikzcd}
\pi_k^G\left(\prod_{i\in I} X_i\right) \arrow[r] \arrow[d, "(\prod f_i)_*"'] & \prod_{i\in I} \pi_k^G(X_i) \arrow[d, "\prod (f_i)_*"] \\
\pi_k^G\left(\prod_{i\in I} X_i^{\mathcal{F}}\right) \arrow[r] & \prod_{i\in I} \pi_k^G(X_i^{\mathcal{F}})
\end{tikzcd}$$

The upper map is an isomorphism by hypothesis, and the right map is an isomorphism since each f_i is an $\mathcal{F}$-equivalence. The lower map is also an isomorphism: since all $X_i^{\mathcal{F}}$ are $\mathcal{F}$-Ω-spectra, the colimit system

$$\begin{aligned}
\pi_k^G\left(\prod\nolimits_{i\in I} X_i^{\mathcal{F}}\right) &= \operatorname{colim}_{V\in s(\mathcal{U}_G)} \left[S^{V\oplus\mathbb{R}^k}, \prod\nolimits_{i\in I} X_i^{\mathcal{F}}(V)\right]^G \\
&= \operatorname{colim}_{V\in s(\mathcal{U}_G)} \left(\prod\nolimits_{i\in I} [S^{V\oplus\mathbb{R}^k}, X_i^{\mathcal{F}}(V)]^G\right)
\end{aligned}$$

consists of isomorphisms starting at any *faithful* G-representation V. Since faithful representations are cofinal in the poset $s(\mathcal{U}_G)$, in this particular situation the colimit commutes with the product. Altogether we can conclude that in our situation the morphism

$$\prod f_i \; : \; \prod\nolimits_{i\in I} X_i \longrightarrow \prod\nolimits_{i\in I} X_i^{\mathcal{F}}$$

is an $\mathcal{F}$-equivalence. Since the right-hand side is a product in $\mathcal{GH}_{\mathcal{F}}$ of the family $\{X_i\}_{i\in I}$, so is the left-hand side. □

We turn to the interaction of the smash product of orthogonal spectra with the level and global model structures. Given two morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ of orthogonal spectra, the *pushout product* morphism is defined as

$$f\Box g = (f\wedge Y)\cup(B\wedge g) \; : \; A\wedge Y\cup_{A\wedge X} B\wedge X \longrightarrow B\wedge Y .$$

We introduce another piece of notation that is convenient for the discussion of

the monoidal properties: when I and I' are two sets of morphisms of orthogonal spectra, then we define

$$I \sqsubset I'$$

to mean that every morphism in I is isomorphic to a morphism in I'. Moreover, we write $I \square I'$ for the set of pushout product morphisms $f \square g$ for all $f \in I$ and all $g \in I'$. If $\mathcal{E}$ and $\mathcal{F}$ are global families, then we denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$.

The sets $I_{\mathcal{F}}$ and $J_{\mathcal{F}}$ of generating cofibrations and of generating acyclic cofibrations for the $\mathcal{F}$-level model structure were defined in the proof of Proposition 4.3.5. They consist of the morphisms

$$F_{H,\mathbb{R}^m} \wedge (i_k)_+ \qquad \text{and} \qquad F_{H,\mathbb{R}^m} \wedge (j_k)_+$$

for all $k \geq 0$, all $m \geq 0$ and all compact Lie groups H in $\mathcal{F}(m)$. Here we used that the orthogonal spectrum $F_{G,V}(G/H_+)$ is isomorphic to $F_{H,V}$. The set $K_{\mathcal{F}}$ of acyclic cofibrations for the $\mathcal{F}$-global model structure was defined in (4.3.16).

Proposition 4.3.23. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families. Then the following relations hold for the sets of generating cofibrations and acyclic cofibrations:*

$$I_{\mathcal{E}} \square I_{\mathcal{F}} \sqsubset I_{\mathcal{E}\times\mathcal{F}}\ , \quad I_{\mathcal{E}} \square J_{\mathcal{F}} \sqsubset J_{\mathcal{E}\times\mathcal{F}} \quad \textit{and} \quad I_{\mathcal{E}} \square K_{\mathcal{F}} \sqsubset K_{\mathcal{E}\times\mathcal{F}}\ .$$

Proof We start with two key observations concerning the generating cofibrations $i_k : \partial D^k \longrightarrow D^k$ and the generating acyclic cofibrations $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ for the model structure of spaces: the pushout product $i_k \square i_m$ of two sphere inclusions is homeomorphic to the map i_{k+m}; similarly, the pushout product $i_k \square j_m$ is homeomorphic to the map j_{k+m}.

The first relation $I_{\mathcal{E}} \square I_{\mathcal{F}} \sqsubset I_{\mathcal{E}\times\mathcal{F}}$ is then a consequence of the compatibilities between smash products and the isomorphism (4.1.27) for the smash product of two semifree orthogonal spectra:

$$(F_{G,V} \wedge (i_k)_+) \square (F_{K,W} \wedge (i_m)_+) \cong (F_{G,V} \wedge F_{K,W}) \wedge (i_k \square i_m)_+ \cong F_{G\times K, V\oplus W} \wedge (i_{k+m})_+$$

The second relation $I_{\mathcal{E}} \square J_{\mathcal{F}} \sqsubset J_{\mathcal{E}\times\mathcal{F}}$ is proved in the same way:

$$(F_{G,V} \wedge (i_k)_+) \square (F_{K,W} \wedge (j_m)_+) \cong (F_{G,V} \wedge F_{K,W}) \wedge (i_k \square j_m)_+ \cong F_{G\times K, V\oplus W} \wedge (j_{k+m})_+$$

For the third relation we recall that $c_{K,U,W}$ is the mapping cylinder inclusion of the global equivalence

$$\lambda_{K,U,W} \ :\ F_{K,U\oplus W} S^U \longrightarrow F_{K,W}\ .$$

The claim $I_{\mathcal{E}} \square K_{\mathcal{F}} \sqsubset K_{\mathcal{E}\times\mathcal{F}}$ then follows from

$$\begin{aligned} (F_{G,V} \wedge (i_k)_+) \square (c_{K,U,W} \square (i_m)_+) &\cong (F_{G,V} \wedge c_{K,U,W}) \square (i_k \square i_m)_+ \\ &\cong c_{G\times K, V\oplus U, W} \wedge (i_{k+m})_+\ . \end{aligned}$$

□

Certain pushout product properties are now formal consequences. In the special case where $\mathcal{E} = \mathcal{F} = \langle e \rangle = \mathcal{E} \times \mathcal{F}$ are the trivial global families, part (iii) of the previous proposition specializes to Proposition 12.6 of [107].

Proposition 4.3.24. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of a flat cofibration with an $\mathcal{F}$-cofibration that is also an $\mathcal{F}$-equivalence is a flat cofibration and global equivalence.*
(iii) *Let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Then the $\mathcal{F}$-global model structure satisfies the pushout product property with respect to the smash product of orthogonal spectra.*

Proof (i) It suffices to show the claim for a set of generating cofibrations, where it follows from the relation $I_{\mathcal{E}} \Box I_{\mathcal{F}} \sqsubset I_{\mathcal{E} \times \mathcal{F}}$ established in Proposition 4.3.23.

(ii) Again it suffices to check the pushout product of any generating flat cofibration with a generating acyclic cofibration for the $\mathcal{F}$-global model structure. For generators, the claim follows from the relation

$$\begin{aligned} I_{\mathcal{A}ll} \Box (J_{\mathcal{F}} \cup K_{\mathcal{F}}) \;&=\; (I_{\mathcal{A}ll} \Box J_{\mathcal{F}}) \cup (I_{\mathcal{A}ll} \Box K_{\mathcal{F}}) \\ &\sqsubset\; J_{\mathcal{A}ll \times \mathcal{F}} \cup K_{\mathcal{A}ll \times \mathcal{F}} \;=\; J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll} \end{aligned}$$

established in Proposition 4.3.23.

(iii) The pushout product of two $\mathcal{F}$-cofibrations is an $\mathcal{F}$-cofibration by part (i) and the hypothesis that $\mathcal{F}$ is multiplicative. Since $\mathcal{F}$-cofibrations are in particular flat cofibrations, the pushout product of two $\mathcal{F}$-cofibrations one of which is also an $\mathcal{F}$-equivalence is another $\mathcal{F}$-equivalence by part (ii). □

The sphere spectrum $\mathbb{S}$ is the unit object for the smash product of orthogonal spectra, and it is 'free', i.e., $\langle e \rangle$-cofibrant. Thus $\mathbb{S}$ is cofibrant in the $\mathcal{F}$-global model structure for every global family $\mathcal{F}$. So if $\mathcal{F}$ is multiplicative, then with respect to the smash product, the $\mathcal{F}$-global model structure is a symmetric monoidal model category in the sense of [80, Def. 4.2.6]. A corollary is that the homotopy category $\mathcal{GH}_{\mathcal{F}}$, i.e., the localization of the category of orthogonal spectra at the class of $\mathcal{F}$-equivalences, inherits a closed symmetric monoidal structure, compare [80, Thm. 4.3.3]. The *derived smash product*

$$\wedge^{\mathbb{L}}_{\mathcal{F}} \;:\; \mathcal{GH}_{\mathcal{F}} \times \mathcal{GH}_{\mathcal{F}} \;\longrightarrow\; \mathcal{GH}_{\mathcal{F}} \,, \tag{4.3.25}$$

i.e., the induced symmetric monoidal product on $\mathcal{GH}_{\mathcal{F}}$, is any total left derived functor of the smash product.

Corollary 4.3.26. *For every multiplicative global family $\mathcal{F}$, the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$ is closed symmetric monoidal under the derived smash product* (4.3.25).

The value of the derived smash product at a pair (X, Y) of orthogonal spectra can be calculated as

$$X \wedge^{\mathbb{L}}_{\mathcal{F}} Y \;=\; X^c \wedge Y^c \ ,$$

where $X^c \longrightarrow X$ and $Y^c \longrightarrow Y$ are cofibrant replacements in the $\mathcal{F}$-global model structure, i.e., $\mathcal{F}$-equivalences with $\mathcal{F}$-cofibrant sources. By the following 'flatness theorem', it actually suffices to 'resolve' only one of the factors, and it is enough to require the source of the 'resolution' to be flat (as opposed to being $\mathcal{F}$-cofibrant).

Theorem 4.3.27. *Let $\mathcal{F}$ be a global family.*

(i) *Smashing with a flat orthogonal spectrum preserves $\mathcal{F}$-equivalences.*
(ii) *Smashing with any orthogonal spectrum preserves $\mathcal{F}$-equivalences between flat orthogonal spectra.*

Proof (i) We let G be a compact Lie group. Every cofibration of $O(m)$-spaces is also a cofibration of $(G \times O(m))$-spaces when we let G act trivially. So if we let G act trivially on a flat orthogonal spectrum X, then it is G-flat in the sense of Definition 3.5.7. Now we let $f : Y \longrightarrow Z$ be an $\mathcal{F}$-equivalence of orthogonal spectra. If G belongs to the global family $\mathcal{F}$, then f is in particular a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra with respect to the trivial G-actions. So $X \wedge f : X \wedge Y \longrightarrow X \wedge Z$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra by Theorem 3.5.10, because X is G-flat. This proves that $X \wedge f$ is again an $\mathcal{F}$-equivalence.

(ii) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between flat orthogonal spectra. We choose a global equivalence $\varphi : A^\flat \longrightarrow A$ with flat source and consider the commutative square:

$$\begin{tikzcd}
A^\flat \wedge X \arrow[r, "\varphi \wedge X"] \arrow[d, "A^\flat \wedge f"'] & A \wedge X \arrow[d, "A \wedge f"] \\
A^\flat \wedge Y \arrow[r, "\varphi \wedge Y"'] & A \wedge Y
\end{tikzcd}$$

The two horizontal morphisms are global equivalences by part (i), because X and Y are flat. The left vertical morphism is an $\mathcal{F}$-equivalence by part (i), because $A^\flat$ is flat. So $A \wedge f$ is also an $\mathcal{F}$-equivalence. □

Now we prove another important relationship between the global model structures and the smash product, the *monoid axiom* [146, Def. 3.3]. As in the

unstable situation in Proposition 1.4.13 we only discuss the weaker form of the monoid axiom with sequential (as opposed to more general transfinite) compositions.

Proposition 4.3.28 (Monoid axiom)**.** *We let $\mathcal{F}$ be a global family. For every flat cofibration $i : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal spectrum Y the morphism*

$$i \wedge Y \; : \; A \wedge Y \longrightarrow B \wedge Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions.

Proof Given Theorem 4.3.27, this is a standard argument, similar to the proofs of the monoid axiom in the non-equivariant or the G-equivariant context, compare [107, Prop. 12.5], [163, Prop. 1.3.10], [108, III Prop. 7.4] or [163, Prop. 2.3.27]. Every flat cofibration is an h-cofibration (Corollary A.30 (iii)), and h-cofibrations are closed under smashing with any orthogonal spectrum, so $i \wedge Y$ is an h-cofibration. Since i is a h-cofibration and $\mathcal{F}$-equivalence, its cokernel B/A is $\mathcal{F}$-stably contractible by the long exact homotopy group sequence (Corollary 3.1.38). But B/A is also flat as a cokernel of a flat cofibration, so the spectrum $(B/A) \wedge Y$ is $\mathcal{F}$-stably contractible by Theorem 4.3.27. Since $i \wedge Y$ is an h-cofibration with cokernel isomorphic to $(B/A) \wedge Y$, the long exact homotopy group sequence then shows that $i \wedge Y$ is an $\mathcal{F}$-equivalence.

The proof that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions is the same as for for global equivalences in Corollary 4.1.5. □

Every $\mathcal{F}$-cofibration is in particular a flat cofibration. So the monoid axiom implies the monoid axiom in the $\mathcal{F}$-global model structure. If the global family $\mathcal{F}$ is closed under products, Theorem [146, Thm. 4.1] applies to the $\mathcal{F}$-global model structure and shows the following lifting results. The additional claims in part (i) about the behavior of the forgetful functor on the cofibrations are proved as in the unstable analog in Corollary 1.4.15.

Corollary 4.3.29. *Let R be an orthogonal ring spectrum and $\mathcal{F}$ a multiplicative global family.*

(i) *The category of R-modules admits the $\mathcal{F}$-*global model structure *in which a morphism is an equivalence (or fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (or $\mathcal{F}$-global fibration). This model structure is cofibrantly generated. Every cofibration in this $\mathcal{F}$-*

global model structure is an h-cofibration of underlying orthogonal spectra. If the underlying orthogonal spectrum of R is $\mathcal{F}$-cofibrant, then every cofibration of R-modules is an $\mathcal{F}$-cofibration of underlying orthogonal spectra.

(ii) *If R is commutative, then with respect to $\wedge_R$ the $\mathcal{F}$-global model structure of R-modules is a monoidal model category that satisfies the monoid axiom.*

(iii) *If R is commutative, then the category of R-algebras admits the $\mathcal{F}$*-global model structure *in which a morphism is an equivalence (or fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (or $\mathcal{F}$-global fibration). Every cofibrant R-algebra is also cofibrant as an R-module.*

Strictly speaking, Theorem 4.1 of [146] does not apply verbatim to the $\mathcal{F}$-global model structure because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to sequential compositions of h-cofibrations, and this suffices to run the countable small object argument (compare also Remark 2.4 of [146]).

Proposition 4.3.30. *Let R be an orthogonal ring spectrum and N a right R-module that is cofibrant in the $\mathcal{A}ll$-global model structure of Corollary* 4.3.29 (i). *Then for every global family $\mathcal{F}$, the functor $N \wedge_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spectra.*

Proof The argument is completely parallel to the unstable precursor in Proposition 1.4.16. We call a right R-module N *homotopical* if the functor $N \wedge_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spectra. Since the $\mathcal{A}ll$-global model structure on the category of right R-modules is obtained by lifting the global model structure of orthogonal spectra along the free and forgetful adjoint functor pair, every cofibrant right R-module is a retract of an R-module that arises as the colimit of a sequence

$$* = M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_k \longrightarrow \dots \tag{4.3.31}$$

in which each M_k is obtained from M_{k-1} as a pushout

$$\begin{tikzcd} A_k \wedge R \arrow[r, "f_k \wedge R"] \arrow[d] & B_k \wedge R \arrow[d] \\ M_{k-1} \arrow[r] & M_k \end{tikzcd}$$

for some flat cofibration $f_k : A_k \longrightarrow B_k$ between flat orthogonal spectra. For

example, f_k can be chosen as a wedge of morphisms in the set $I_{\mathcal{A}ll}$ of generating flat cofibrations. We show by induction on k that each module M_k is homotopical. The induction starts with the trivial R-module, where there is nothing to show. Now we suppose that M_{k-1} is homotopical, and we claim that then M_k is homotopical as well. To see this we consider an $\mathcal{F}$-equivalence of left R-modules $\varphi : X \longrightarrow Y$. Then $M_k \wedge_R \varphi$ is obtained by passing to horizontal pushouts in the following commutative diagram of orthogonal spectra:

$$\begin{tikzcd}
M_{k-1} \wedge_R X \arrow[d, "M_{k-1}\wedge_R\varphi"'] & A_k \wedge X \arrow[l] \arrow[r, "f_k\wedge X"] \arrow[d, "A_k\wedge\varphi"] & B_k \wedge X \arrow[d, "B_k\wedge\varphi"] \\
M_{k-1} \wedge_R Y & A_k \wedge Y \arrow[l] \arrow[r, "f_k\wedge Y"'] & B_k \wedge Y
\end{tikzcd}$$

Here we have exploited that $(A_k \wedge R) \wedge_R X$ is naturally isomorphic to $A_k \wedge X$. In the diagram, the left vertical morphism is an $\mathcal{F}$-equivalence by hypothesis. The middle and right vertical morphisms are $\mathcal{F}$-equivalences because smash product with a flat orthogonal spectrum preserves $\mathcal{F}$-equivalences (Theorem 4.3.27 (i)). Moreover, since the morphism f_k is a flat cofibration, it is an h-cofibration (by Corollary A.30 (iii)), and so the morphisms $f_k \wedge X$ and $f_k \wedge Y$ are h-cofibrations. Corollary 3.1.39 (i) then implies that the induced morphism on horizontal pushouts $M_k \wedge_R \varphi$ is again an $\mathcal{F}$-equivalence.

Now we let M be a colimit of the sequence (4.3.31). Then $M \wedge_R X$ is a colimit of the sequence $M_k \wedge_R X$. Moreover, since $f_k : A_k \longrightarrow B_k$ is an h-cofibration, so is the morphism $f_k \wedge R$, and hence also its cobase change $M_{k-1} \longrightarrow M_k$. So the sequence whose colimit is $M \wedge_R X$ consists of h-cofibrations, which are in particular level-wise closed embeddings. The same is true for $M \wedge_R Y$. Since each M_k is homotopical and colimits of orthogonal spectra along closed embeddings are homotopical (see Proposition 3.1.41 (ii)), we conclude that the morphism $M \wedge_R \varphi : M \wedge_R X \longrightarrow M \wedge_R Y$ is an $\mathcal{F}$-equivalence, so that M is homotopical. Since $\mathcal{F}$-equivalences are closed under retracts, the class of homotopical R-modules is closed under retracts, and so every cofibrant right R-module is homotopical. □

For later use we record a positive version of the global model structure of Theorem 4.3.18. This positive variant is the one that can be lifted to the category of ultra-commutative ring spectra, see Theorem 5.4.3 below

Definition 4.3.32. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive cofibration* if it is a flat cofibration and the map $f(0) : X(0) \longrightarrow Y(0)$ is a homeomorphism. An orthogonal spectrum is a *positive global Ω-spectrum* if for every compact Lie group G, every G-representation V and every faithful

G-representation W with $W \neq 0$ the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(W) \longrightarrow \operatorname{map}_*(S^V, X(V \oplus W))$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positive global Ω-spectrum is a global Ω-spectrum (in the absolute sense) if in addition the adjoint structure map $\tilde{\sigma}_{0,\mathbb{R}}$: $X(0) \longrightarrow \Omega X(\mathbb{R})$ is a non-equivariant weak equivalence.

Proposition 4.3.33 (Positive global model structure)**.** *The global equivalences and positive cofibrations are part of a proper, cofibrantly generated, topological model structure, the* positive global model structure *on the category of orthogonal spectra. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a fibration in the positive global model structure if and only if for every compact Lie group G, every G-representation V and every faithful G-representation W with* $W \neq 0$ *the map $f(W)^G : X(W)^G \longrightarrow Y(W)^G$ is a Serre fibration and the square*

$$\begin{tikzcd} X(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(W)^G"'] & \operatorname{map}_*^G(S^V, X(V \oplus W)) \arrow[d, "\operatorname{map}_*^G(S^V, f(V\oplus W))"] \\ Y(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] & \operatorname{map}_*^G(S^V, Y(V \oplus W)) \end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positive global Ω-spectra. The model structure is monoidal with respect to the smash product of orthogonal spectra.

Proof We start by establishing a *positive strong level model structure*. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive strong level equivalence* (or *positive strong level fibration*) if for every inner product space V with $V \neq 0$ the map $f(V) : X(V) \longrightarrow Y(V)$ is an $O(V)$-weak equivalence (or $O(V)$-fibration). We claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a model structure on the category of orthogonal spectra.

The proof is another application of the general construction of level model structures in Proposition C.23. Indeed, we let $C(0)$ be the degenerate model structure on the category $\mathbf{T}_*$ of based spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m \geq 1$ we let $C(m)$ be the projective model structure (for the family of all closed subgroups) on the category of based $O(m)$-spaces. With respect to these choices of model structures $C(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition C.23 precisely become

the positive strong level equivalences, positive strong level fibrations and positive cofibrations. The consistency condition (Definition C.22) is now strictly weaker than for the strong level model structure, so it holds. The positive strong level model structure is topological by Proposition B.5, where we take $\mathcal{G}$ as the set of semifree orthogonal spectra $F_{H,\mathbb{R}^m}$ for all $m \geq 1$ and all closed subgroups H of $O(m)$.

We obtain the positive global model structure for orthogonal spectra by 'mixing' the positive strong level model structure with the global model structure of Theorem 4.3.18. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [38, Thm. 2.1], which is our first claim. By [38, Cor. 3.7] (or rather its dual formulation), an orthogonal spectrum is fibrant in the positive global model structure if it is equivalent in the positive strong level model structure to a global Ω-spectrum; this is equivalent to being a positive global Ω-spectrum. The proof that the positive global model structure is proper and topological is similar as for the global model structure. The proof of the pushout product property is as in the absolute global model structure (see Proposition 4.3.24); the only new ingredient is that the class of generators $F_{G,V}$ with $V \neq 0$ for the positive cofibrations is closed under the smash product of orthogonal spectra.

Finally, the positive global model structure is cofibrantly generated: we can simply take the same sets of generating cofibrations and generating acyclic cofibrations as for the global model structure, *except* that we omit all morphisms freely generated in level 0. □

4.4 Triangulated global stable homotopy categories

As the homotopy category of a stable model structure, the global stable homotopy category $\mathcal{GH}$ comes with the structure of a triangulated category. The shift functor is the suspension of orthogonal spectra, and the distinguished triangles arise from mapping cone sequences. In this section we collect the aspects of global stable homotopy theory that are best expressed in terms of the triangulated structure.

More generally, we work in the triangulated $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$, where $\mathcal{F}$ is any global family. Theorem 4.4.3 identifies a set of compact generators of $\mathcal{GH}_{\mathcal{F}}$, namely the suspension spectra of global classifying spaces of the groups in $\mathcal{F}$. A consequence is Brown representability for cohomological and homological functors out of $\mathcal{GH}_{\mathcal{F}}$, compare Corollary 4.4.5. Theorem 4.4.9 shows that the classes of $\mathcal{F}$-connective and $\mathcal{F}$-coconnective

spectra form a non-degenerate t-structure on $\mathcal{GH}_{\mathcal{F}}$; moreover, taking 0th equivariant homotopy groups is an equivalence from the heart of this t-structure to the category of $\mathcal{F}$-global functors. Immediate consequences are global Postnikov sections and the existence of Eilenberg-Mac Lane spectra that realize global functors. Proposition 4.4.15 establishes another connection between the smash product and the algebra of global functors: for globally connective orthogonal spectra the box product of global functors calculates the 0th homotopy groups of a derived smash product.

The last topic of this section are certain distinguished triangles in the global stable homotopy category that arise from special representations, namely when a compact Lie group G acts faithfully and transitively on the unit sphere. The stabilizer of a unit vector is then a closed subgroup H such that G/H is diffeomorphic to a sphere, and Theorem 4.4.21 exhibits a distinguished triangle that relates the global classifying space of G to certain semifree orthogonal spectra of G and H. The main examples of this situation are the tautological representations of the groups $O(m)$, $SO(m)$, $U(m)$, $SU(m)$ and $Sp(m)$; these will show up again in Section 6.1 in the rank filtrations of global Thom spectra.

The $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$ is the homotopy category of a stable model structure, so it is naturally a triangulated category, for example by [80, Sec. 7.1] or [143, Thm. A.12]. The shift functor is modeled by the point-set level suspension of orthogonal spectra. More precisely, the suspension functor $-\wedge S^1 : \mathcal{S}p \longrightarrow \mathcal{S}p$ of Construction 3.1.21 preserves $\mathcal{F}$-equivalences, so it descends to a functor on the $\mathcal{F}$-global stable homotopy category

$$-\wedge S^1 \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

for which we use the same name. The distinguished triangles are defined from mapping cone sequences, i.e., a triangle is distinguished if and only if it is isomorphic, in $\mathcal{GH}_{\mathcal{F}}$, to a sequence of the form

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ i\ } Cf \xrightarrow{\ p\ } X \wedge S^1$$

for some morphism of orthogonal spectra $f : X \longrightarrow Y$; here the morphisms i and p were defined in (3.1.33).

Example 4.4.1 (Shift preserves distinguished triangles)**.** The shift functor $\mathrm{sh} : \mathcal{S}p \longrightarrow \mathcal{S}p$ of Construction 3.1.21 preserves $\mathcal{F}$-equivalences, so it descends to a functor on the $\mathcal{F}$-global stable homotopy category

$$\mathrm{sh} \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

for which we use the same name. Moreover, shifting commutes with smashing with a based space *on the nose*, i.e., $(\mathrm{sh}\, X) \wedge A = \mathrm{sh}(X \wedge A)$; so we can (and will) leave out parentheses in such expressions. Since the suspension functor

on $\mathcal{GH}_{\mathcal{F}}$ is induced by smashing with S^1, the shift functor commutes with the suspension functor, again on the nose, both on the point-set level and also on the level of the $\mathcal{F}$-global stable homotopy category. We will now argue that shifting also preserves distinguished triangles on the nose; equivalently, the derived shift is an exact functor of triangulated categories if we equip it with the identity isomorphism $\mathrm{sh} \circ (- \wedge S^1) = (- \wedge S^1) \circ \mathrm{sh}$.

To prove our claim we consider a distinguished triangle in $\mathcal{GH}_{\mathcal{F}}$:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A \wedge S^1$$

The morphism $\lambda_A : A \wedge S^1 \longrightarrow \mathrm{sh}\, A$ is a natural global equivalence by Proposition 4.1.4 (i), so all vertical morphisms in the following diagram are isomorphisms in $\mathcal{GH}_{\mathcal{F}}$:

$$\begin{tikzcd}
A \wedge S^1 \arrow[r, "f \wedge S^1"] \arrow[d, "\lambda_A"'] & B \wedge S^1 \arrow[r, "g \wedge S^1"] \arrow[d, "\lambda_B"'] & C \wedge S^1 \arrow[r, "-h \wedge S^1"] \arrow[d, "\lambda_C"] & A \wedge S^1 \wedge S^1 \arrow[d, "\lambda_A \wedge S^1"] \\
\mathrm{sh}\, A \arrow[r, "\mathrm{sh}\, f"'] & \mathrm{sh}\, B \arrow[r, "\mathrm{sh}\, g"'] & \mathrm{sh}\, C \arrow[r, "\mathrm{sh}\, h"'] & \mathrm{sh}\, A \wedge S^1
\end{tikzcd}$$

The claim now follows from the following three observations:

- the suspension functor in any triangulated category preserves distinguished triangles *up to a sign*; so the upper triangle above is distinguished;
- the left and middle squares commute by naturality of the λ-morphisms; and
- the right square commutes because the two morphisms $\lambda_{A \wedge S^1}, \lambda_A \wedge S^1 : A \wedge S^1 \wedge S^1 \longrightarrow \mathrm{sh}\, A \wedge S^1$ differ by the twist involution of $S^1 \wedge S^1$; since this involution has degree -1, we obtain

$$(\mathrm{sh}\, h) \circ \lambda_C \;=\; \lambda_{A \wedge S^1} \circ (h \wedge S^1) \;=\; -(\lambda_A \wedge S^1) \circ (h \wedge S^1)\,.$$

We recall the notion of compactly generated triangulated categories. Compact generation has strong formal consequences, see Theorem 4.4.4 below. Theorem 4.4.3 shows that the $\mathcal{F}$-global stable homotopy category enjoys this special property.

Definition 4.4.2. Let $\mathcal{T}$ be a triangulated category with infinite sums. An object C of $\mathcal{T}$ is *compact* if for every family $\{X_i\}_{i \in I}$ of objects the canonical map

$$\bigoplus_{i \in I} \mathcal{T}(C, X_i) \longrightarrow \mathcal{T}(C, \oplus_{i \in I} X_i)$$

is an isomorphism. A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a set of *weak generators* if the following condition holds: if the groups $\mathcal{T}(G[k], X)$ are trivial for all $k \in \mathbb{Z}$ and all $G \in \mathcal{G}$, then X is a zero object. The triangulated category $\mathcal{T}$ is *compactly generated* if it has sums and a set of compact weak generators.

If G is from a global family $\mathcal{F}$, then the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ takes $\mathcal{F}$-equivalences to isomorphisms. So the universal property of a localization provides a unique factorization

$$\pi_0^G \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{A}b$$

through the $\mathcal{F}$-global stable homotopy category. We will abuse notation and use the same symbol for the equivariant homotopy group functor on the category of orthogonal spectra and for its 'derived' functor defined on $\mathcal{GH}_{\mathcal{F}}$. This abuse of notation is mostly harmless, but there is one point where it can create confusion, namely in the context of infinite products; we refer the reader to Remark 4.4.6 for this issue.

We recall from Definition 1.1.27 that the global classifying space $B_{\mathrm{gl}}G$ of a compact Lie group G is the semifree orthogonal space $\mathbf{L}_{G,V} = \mathbf{L}(V, -)/G$, for some faithful G-representation V. The choice of faithful representation is omitted from the notation because the global homotopy type of $B_{\mathrm{gl}}G$ does not depend on it. The suspension spectrum of $B_{\mathrm{gl}}G$ comes with a stable tautological class

$$e_G \ = \ e_{G,V} \in \ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}}G)$$

defined in (4.1.12).

In the proof of the next theorem we will start using the shorthand notation

$$[\![X, Y]\!]_{\mathcal{F}} \ = \ \mathcal{GH}_{\mathcal{F}}(X, Y)$$

for the abelian group of morphisms in the triangulated $\mathcal{F}$-global stable homotopy category.

Theorem 4.4.3. *Let $\mathcal{F}$ be a global family and G a compact Lie group in $\mathcal{F}$.*

(i) *The pair $(\Sigma^\infty_+ B_{\mathrm{gl}}G, e_G)$ represents the functor $\pi_0^G : \mathcal{GH}_{\mathcal{F}} \longrightarrow$* (sets).
(ii) *The orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is a compact object of the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$.*
(iii) *As G varies through a set of representatives of isomorphism classes of groups in $\mathcal{F}$, the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators for the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$.*

In particular, the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$ is compactly generated.

Proof (i) We need to show that for every orthogonal spectrum X the map

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \ \longrightarrow \ \pi_0^G(X) \ , \quad f \longmapsto f_*(e_G)$$

is bijective. Since both sides take $\mathcal{F}$-equivalences in X to bijections, we can assume that X is an $\mathcal{F}$-Ω-spectrum, and hence fibrant in the $\mathcal{F}$-global model

structure. For G in the family $\mathcal{F}$, the orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is $\mathcal{F}$-cofibrant. So the localization functor induces a bijection

$$\mathcal{S}p(\Sigma^\infty_+ B_{\mathrm{gl}}G, X)/\text{homotopy} \longrightarrow [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}}$$

from the set of homotopy classes of morphisms of orthogonal spectra to the set of morphisms in $\mathcal{GH}_{\mathcal{F}}$.

We let V be the faithful G-representation that is implicit in the definition of the global classifying space $B_{\mathrm{gl}}G$. By the freeness property of $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$, morphisms from $\Sigma^\infty_+ B_{\mathrm{gl}}G$ to X biject with based G-maps $S^V \longrightarrow X(V)$, and similarly for homotopies. The composite

$$[S^V, X(V)]^G \xrightarrow{\cong} [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \xrightarrow{f \mapsto f_*(e_G)} \pi_0^G(X)$$

is the stabilization map, and hence bijective by Proposition 4.3.15 (ii). Since the left map and the composite are bijective, so is the evaluation map at the stable tautological class.

(ii) By Proposition 4.3.22 the wedge of any family of orthogonal spectra is a coproduct in $\mathcal{GH}_{\mathcal{F}}$. The vertical maps in the commutative square

$$\begin{tikzcd}
\bigoplus_{i\in I}[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, X_i]\!]_{\mathcal{F}} \arrow[r] \arrow[d] & {[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, \bigoplus_{i\in I} X_i]\!]_{\mathcal{F}}} \arrow[d] \\
\bigoplus_{i\in I}\pi_0^G(X_i) \arrow[r] & \pi_0^G\left(\bigvee_{i\in I} X_i\right)
\end{tikzcd}$$

are evaluation at the stable tautological class, and hence isomorphisms by part (i). The lower horizontal map is an isomorphism by Corollary 3.1.37 (i), hence so is the upper horizontal map. This shows that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is compact as an object of the triangulated category $\mathcal{GH}_{\mathcal{F}}$.

(iii) If X is an orthogonal spectrum such that the group $[\![\Sigma^\infty_+ B_{\mathrm{gl}}G[k],\, X]\!]$ is trivial for every G in $\mathcal{F}$ and all integers k, then X is $\mathcal{F}$-equivalent to the trivial orthogonal spectrum by part (i); so X is a zero object in $\mathcal{GH}_{\mathcal{F}}$. This proves that the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators for $\mathcal{GH}_{\mathcal{F}}$ as G varies over $\mathcal{F}$. □

A covariant functor E from a triangulated category $\mathcal{T}$ to the category of abelian groups is called *homological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(A) \xrightarrow{E(f)} E(B) \xrightarrow{E(g)} E(C) \xrightarrow{E(h)} E(A[1])$$

is exact. A contravariant functor E from $\mathcal{T}$ to the category of abelian groups is called *cohomological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the

sequence of abelian groups

$$E(A[1]) \xrightarrow{E(h)} E(C) \xrightarrow{E(g)} E(B) \xrightarrow{E(f)} E(A)$$

is exact.

Theorem 4.4.4 (Brown representability)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category.*

(i) *Every cohomological functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ that takes sums in $\mathcal{T}$ to products of abelian groups is representable, i.e., isomorphic to the functor $\mathcal{T}(-, X)$ for some object X of $\mathcal{T}$.*
(ii) *Every homological functor $\mathcal{T} \longrightarrow \mathcal{A}b$ that takes products in $\mathcal{T}$ to products of abelian groups is representable, i.e., isomorphic to the functor $\mathcal{T}(Y, -)$ for some object Y of $\mathcal{T}$.*
(iii) *An exact functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ to another triangulated category has a right adjoint if and only if it takes sums in $\mathcal{T}$ to sums in $\mathcal{S}$.*
(iv) *An exact functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ to another triangulated category has a left adjoint if and only if it takes products in $\mathcal{T}$ to products in $\mathcal{S}$.*

A proof of part (i) of this form of Brown representability can be found in [127, Thm. 3.1] or [92, Thm. A]. A proof of part (ii) of this form of Brown representability can be found in [128, Thm. 8.6.1] or [92, Thm. B]. Part (iii) is a formal consequence of part (i): if F preserves sums, then for every object X of $\mathcal{S}$ the functor

$$\mathcal{S}(F(-), X) \ : \ \mathcal{T}^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

is cohomological and takes sums to products. Hence the functor is representable by an object RX in $\mathcal{T}$ and an isomorphism

$$\mathcal{T}(A, RX) \ \cong \ \mathcal{S}(FA, X) \ ,$$

natural in A. Once this representing data is chosen, the assignment $X \mapsto RX$ extends canonically to a functor $R : \mathcal{S} \longrightarrow \mathcal{T}$ that is right adjoint to F. In much the same way, part (iv) is a formal consequence of part (ii).

We let $\mathcal{T}$ be a triangulated category with sums. A *localizing subcategory* of $\mathcal{T}$ is a full subcategory $\mathcal{X}$ which is closed under sums and under extensions in the following sense: if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle in $\mathcal{T}$ such that two of the objects A, B or C belong to $\mathcal{X}$, then so does the third. A set of *compact* objects is a set of weak generators in the sense of Definition 4.4.2 if and only if the smallest localizing subcategory

containing the set is all of $\mathcal{T}$, see for example [148, Lemma 2.2.1]. Hence Theorem 4.4.3 entitles us to the following corollary.

Corollary 4.4.5. *Let $\mathcal{F}$ be a global family and $\mathcal{S}$ a triangulated category.*

(i) *Every localizing subcategory of the $\mathcal{F}$-global stable homotopy category that contains the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for every group G of $\mathcal{F}$ is all of $\mathcal{GH}_{\mathcal{F}}$.*
(ii) *Every cohomological functor on $\mathcal{GH}_{\mathcal{F}}$ that takes sums to products is representable.*
(iii) *Every homological functor on $\mathcal{GH}_{\mathcal{F}}$ that takes products to products is representable.*
(iv) *An exact functor $F : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{S}$ has a right adjoint if and only if it preserves sums.*
(v) *An exact functor $F : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{S}$ has a left adjoint if and only if it preserves products.*

Remark 4.4.6 (Equivariant homotopy groups of infinite products)**.** In Corollary 3.1.37 (ii) we showed that for every compact Lie group G the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ preserves *finite* products. However, it is *not* true that π_0^G, as a functor on the category of orthogonal spectra, preserves infinite products in general.

On the other hand, the 'derived' functor $\pi_0^G : \mathcal{GH} \longrightarrow \mathcal{A}b$ is representable by the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} G$, so it preserves infinite products. This is no contradiction because an infinite product of orthogonal spectra is *not* in general a product in the global homotopy category. To calculate a product in $\mathcal{GH}$ of a family $\{X_i\}_{i\in I}$ of orthogonal spectra, one has to choose stable equivalences $f_i : X_i \longrightarrow X_i^{\mathrm{f}}$ to global Ω-spectra. For an infinite indexing set, the morphism

$$\prod\nolimits_{i\in I} f_i \ : \ \prod\nolimits_{i\in I} X_i \ \longrightarrow \ \prod\nolimits_{i\in I} X_i^{\mathrm{f}}$$

may fail to be a global equivalence, and then the target, but not the source, of this map is a product in $\mathcal{GH}$ of the family $\{X_i\}_{i\in I}$. So when considering infinite products it is important to be aware of our abuse of notation and to remember that the symbol π_0^G has two different meanings.

The preferred set of compact generators $\{\Sigma^\infty_+ B_{\mathrm{gl}} G\}$ of the global stable homotopy category has another special property, it is 'positive' in the following sense: for all compact Lie groups G and K and all $n > 0$ the group

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}} G,\ \Sigma^\infty_+ B_{\mathrm{gl}} K[n]]\!] \ \cong\ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} K[n]) \ \cong\ \pi_{-n}^G(\Sigma^\infty_+ B_{\mathrm{gl}} K)$$

is trivial because every orthogonal suspension spectrum is globally connective (see Proposition 4.1.11). A set of positive compact generators in this sense has strong implications, as we shall now explain.

A *t-structure* as introduced by Beilinson, Bernstein and Deligne in [10,

Def. 1.3.1] axiomatizes the situation in the derived category of an abelian category given by cochain complexes whose cohomology vanishes in positive or negative dimensions, respectively. We are mainly interested in spectra, where a homological (as opposed to *co*homological) grading is more common. So we adapt the definition of a t-structure to homological notation.

Definition 4.4.7. A *t-structure* on a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ of full subcategories satisfying the following three conditions, where $\mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n]$ and $\mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n]$.

(a) For all $X \in \mathcal{T}_{\geq 0}$ and all $Y \in \mathcal{T}_{\leq -1}$ we have $\mathcal{T}(X, Y) = 0$.
(b) $\mathcal{T}_{\geq 0} \subset \mathcal{T}_{\geq -1}$ and $\mathcal{T}_{\leq 0} \supset \mathcal{T}_{\leq -1}$.
(c) For every object X of $\mathcal{T}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{T}_{\geq 0}$ and $B \in \mathcal{T}_{\leq -1}$.

The t-structure is *non-degenerate* if every object in the intersection $\bigcap_{n \in \mathbb{Z}} \mathcal{T}_{\geq n}$ is a zero object and every object in the intersection $\bigcap_{n \in \mathbb{Z}} \mathcal{T}_{\leq n}$ is a zero object.

Some of the basic results of Beilinson, Bernstein and Deligne about t-structures are (in our homological notation):

- the inclusion $\mathcal{T}_{\geq n} \longrightarrow \mathcal{T}$ has a right adjoint $\tau_{\geq n} : \mathcal{T} \longrightarrow \mathcal{T}_{\geq n}$, and the inclusion $\mathcal{T}_{\leq n} \longrightarrow \mathcal{T}$ has a left adjoint $\tau_{\leq n} : \mathcal{T} \longrightarrow \mathcal{T}_{\leq n}$ [10, Prop. 1.3.3];
- given choices of adjoints as above, then for all $m \leq n$ there is a preferred natural isomorphism of functors between $\tau_{\geq m} \circ \tau_{\leq n}$ and $\tau_{\leq n} \circ \tau_{\geq m}$ [10, Prop. 1.3.5];
- the *heart*

$$\mathcal{H} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0} ,$$

viewed as a full subcategory of $\mathcal{T}$, is an abelian category and $\tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{T} \longrightarrow \mathcal{H}$ is a homological functor [10, Thm. 1.3.6].

We will now show that the global stable homotopy category has a preferred t-structure whose heart is equivalent to the category of global functors.

Definition 4.4.8. Let $\mathcal{F}$ be a global family. An orthogonal spectrum X is $\mathcal{F}$-*connective* if the homotopy group $\pi_n^G(X)$ is trivial for every group G in $\mathcal{F}$ and every $n < 0$. An orthogonal spectrum X is $\mathcal{F}$-*coconnective* if the homotopy group $\pi_n^G(X)$ is trivial for every group G in $\mathcal{F}$ and every $n > 0$.

An $\mathcal{F}$-*global functor* is an additive functor to abelian groups from the full subcategory of the Burnside category with objects the global family $\mathcal{F}$. A morphism of $\mathcal{F}$-global functors is a natural transformation. We write $\mathcal{GF}_{\mathcal{F}}$ for the category of $\mathcal{F}$-global functors.

Theorem 4.4.9. *For every global family $\mathcal{F}$, the classes of $\mathcal{F}$-connective spectra and $\mathcal{F}$-coconnective spectra form a non-degenerate t-structure on $\mathcal{GH}_{\mathcal{F}}$. The heart of this t-structure consists of those orthogonal spectra X such that $\pi_n^G(X) = 0$ for all $G \in \mathcal{F}$ and all $n \neq 0$. The functor*

$$\underline{\pi}_0 \ : \ \mathscr{H} \ \longrightarrow \ \mathcal{GF}_{\mathcal{F}}$$

is an equivalence of categories from the heart of the t-structure to the category of $\mathcal{F}$-global functors.

Proof We deduce this from the more general arguments of Beligiannis and Reiten [11, Ch. III] who systematically investigate torsion pairs and t-structures in triangulated categories that are generated by compact objects. By Theorem 4.4.3 the set

$$\mathcal{P} \ = \ \{\Sigma_+^\infty B_{\mathrm{gl}} G\}_{[G]\in\mathcal{F}}$$

is a set of compact weak generators for the triangulated category $\mathcal{GH}_{\mathcal{F}}$, where $[G]$ indicates that we choose representatives from the isomorphism classes of compact Lie groups in $\mathcal{F}$.

We let $\mathcal{Y}$ be the class of orthogonal spectra Y such that

$$[\![P[n], Y]\!]_{\mathcal{F}} \ = \ 0$$

for all $P \in \mathcal{P}$ and all $n \geq 0$. The representability result of Theorem 4.4.3 (i) shows that this is precisely the class of those Y such that the group $\pi_n^G(Y)$ vanishes for all $G \in \mathcal{F}$ and all $n \geq 0$. So $\mathcal{Y}[1]$ is the class of $\mathcal{F}$-coconnective orthogonal spectra. We let $\mathcal{X}$ be the 'left orthogonal' to $\mathcal{Y}$, i.e., the class of orthogonal spectra X such that $[\![X, Y]\!]_{\mathcal{F}} = 0$ for all $Y \in \mathcal{Y}$. Since the objects of $\mathcal{P}$ are compact in $\mathcal{GH}_{\mathcal{F}}$ by Theorem 4.4.3 (ii), [11, Thm. III.2.3] shows that the pair $(\mathcal{X}, \mathcal{Y})$ is a 'torsion pair' in the sense of [11, Def. I.2.1]. This simply means that the pair $(\mathcal{X}, \mathcal{Y}[1])$ is a t-structure in the sense of Definition 4.4.7, see [11, Prop. I.2.13].

It remains to show that $\mathcal{X}$ coincides with the class of $\mathcal{F}$-connective orthogonal spectra. This needs the 'positivity property' of the set $\mathcal{P}$ of compact generators. Representability (Theorem 4.4.3 (i)) and the suspension isomorphism provide an isomorphism

$$[\![\Sigma_+^\infty B_{\mathrm{gl}} G, \Sigma_+^\infty B_{\mathrm{gl}} K[n]]\!]_{\mathcal{F}} \ \cong \ \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}} K[n]) \ \cong \ \pi_{-n}^G(\Sigma_+^\infty B_{\mathrm{gl}} K) \, .$$

The spectrum $\Sigma_+^\infty B_{\mathrm{gl}} K$ is globally connective by Proposition 4.1.11, so this latter group vanishes for all $n \geq 1$ and all $G, K \in \mathcal{F}$. So [11, Prop. III.2.8] shows that $\mathcal{X}$ coincides with the class of those orthogonal spectra X such that $[\![\Sigma_+^\infty B_{\mathrm{gl}} G, X[n]]\!]_{\mathcal{F}} = 0$ for all $G \in \mathcal{F}$ and $n \geq 1$. Since the group $[\![\Sigma_+^\infty B_{\mathrm{gl}} G, X[n]]\!]_{\mathcal{F}}$ is isomorphic to $\pi_{-n}^G(X)$, this shows that $\mathcal{X}$ is precisely the class of $\mathcal{F}$-connective

orthogonal spectra. The t-structure is non-degenerate because spectra with trivial $\mathcal{F}$-equivariant homotopy groups are zero objects in $\mathcal{GH}_{\mathcal{F}}$.

We denote by End($\mathcal{P}$) the 'endomorphism category' of the set $\mathcal{P}$, i.e., the full pre-additive subcategory of $\mathcal{GH}_{\mathcal{F}}$ with object set $\mathcal{P}$. By an End($\mathcal{P}$)-module we mean an additive functor

$$\mathrm{End}(\mathcal{P})^{\mathrm{op}} \longrightarrow \mathcal{A}b$$

from the opposite category of End($\mathcal{P}$). The tautological functor

$$\mathcal{GH}_{\mathcal{F}} \longrightarrow \text{mod-}\mathrm{End}(\mathcal{P}) \tag{4.4.10}$$

takes an object X to the restriction of the contravariant Hom-functor $[\![-, X]\!]_{\mathcal{F}}$ to the full subcategory End($\mathcal{P}$). Because $\mathcal{P}$ is a set of positive, compact generators for the triangulated category $\mathcal{GH}_{\mathcal{F}}$, [11, Thm. III.3.4] shows that the restriction of the tautological functor (4.4.10) to the heart is an equivalence of categories

$$\mathscr{H} \xrightarrow{\cong} \text{mod-}\mathrm{End}(\mathcal{P}) .$$

So to establish the last claim it suffices to show that End($\mathcal{P}$) is anti-equivalent to the full subcategory of the Burnside category $\mathbf{A}$ with object class $\mathcal{F}$, in such a way that the tautological functor corresponds to the functor $\underline{\pi}_0$. The equivalence $\mathrm{End}(\mathcal{P})^{\mathrm{op}} \longrightarrow \mathbf{A}_{\mathcal{F}}$ is given by the inclusion on objects, and on morphisms by the isomorphisms

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}} G, \Sigma^\infty_+ B_{\mathrm{gl}} K]\!]_{\mathcal{F}} \cong \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} K) \cong \mathbf{A}(K, G)$$

specified in Theorem 4.4.3 (i) and Proposition 4.2.5. □

Remark 4.4.11 (Postnikov sections)**.** For the standard t-structure on the global stable homotopy category (i.e., Theorem 4.4.9 for $\mathcal{F} = \mathcal{A}ll$) the truncation functor

$$\tau_{\leq n} \ : \ \mathcal{GH} \longrightarrow \mathcal{GH}_{\leq n} ,$$

left adjoint to the inclusion, provides 'global Postnikov sections': for every orthogonal spectrum X the spectrum $\tau_{\leq n} X$ satisfies $\underline{\pi}_k(\tau_{\leq n} X) = 0$ for $k > n$ and the adjunction unit $X \longrightarrow X_{\leq n}$ induces an isomorphism on the global functor $\underline{\pi}_k$ for every $k \leq n$.

Remark 4.4.12 (Eilenberg-Mac Lane spectra)**.** In the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family, Theorem 4.4.9 in particular provides an Eilenberg-Mac Lane spectrum for every global functor M, i.e., an orthogonal spectrum HM such that $\underline{\pi}_k(HM) = 0$ for all $k \neq 0$ and such that the global functor $\underline{\pi}_0(HM)$ is isomorphic to M; moreover, these properties characterize HM up

to preferred isomorphism in $\mathcal{GH}$. Indeed, a choice of inverse to the equivalence $\underline{\pi}_0$ of Theorem 4.4.9 , composed with the inclusion of the heart, provides an Eilenberg-Mac Lane functor

$$H : \mathcal{GF} \longrightarrow \mathcal{GH}$$

to the global stable homotopy category.

We let $\mathcal{T}$ be a triangulated category with infinite sums and we let C be a class of objects of $\mathcal{T}$. We denote by $\langle C\rangle_+$ the smallest class of objects of $\mathcal{T}$ that contains C, is closed under sums (possibly infinite) and is *closed under cones* in the following sense: if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle such that A and B belong to the class, then so does C. Any non-empty class of objects that is closed under cones contains all zero objects (because a zero object is a cone of any identity morphism) and is closed under suspension (because $A[1]$ is a cone of the morphism from A to a zero object).

Proposition 4.4.13. *The class $\langle \Sigma^\infty_+ B_{\mathrm{gl}} G\colon\ G$ compact Lie$\rangle_+$ in the triangulated category $\mathcal{GH}$ coincides with the class of globally connective spectra.*

Proof The spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ are globally connective by Proposition 4.1.11. The class of globally connective orthogonal spectra is closed under sums because equivariant homotopy groups commute with sums (Corollary 3.1.37 (i)). The class of globally connective orthogonal spectra is closed under cones by the long exact homotopy group sequence of Proposition 3.1.36. So the class $\langle \Sigma^\infty_+ B_{\mathrm{gl}} G\rangle_+$ is contained in the class of globally connective spectra.

For the converse we choose a set of representatives of the isomorphism classes of compact Lie groups and set

$$\mathcal{P} = \{(\Sigma^\infty_+ B_{\mathrm{gl}} G)[k]\}_{[G],k\geq 0} ,$$

indexed by the chosen representatives. We let X be any orthogonal spectrum. By induction on n we construct objects A_n in $\langle \mathcal{P}\rangle_+$ and morphisms $i_n : A_n \longrightarrow A_{n+1}$ and $u_n : A_n \longrightarrow X$ such that $u_{n+1} i_n = u_n$. We start with

$$A_0 = \bigoplus_{P\in\mathcal{P},x\in[\![P,X]\!]} P .$$

Then A_0 belongs to $\langle \mathcal{P}\rangle_+$ and the canonical morphism $u_0 : A_0 \longrightarrow X$ (i.e., the morphism x on the summand indexed by x) induces a surjection $[\![P,u_0]\!] : [\![P,A_0]\!] \longrightarrow [\![P,X]\!]$ for all $P \in \mathcal{P}$.

In the inductive step we suppose that A_n and $u_n : A_n \longrightarrow X$ have already been constructed. We define

$$D_n \;=\; \bigoplus_{P \in \mathcal{P}, x \in \ker[\![P, u_n]\!]} P\ ,$$

which comes with a tautological morphism $\tau : D_n \longrightarrow A_n$, again given by x on the summand indexed by x. We choose a distinguished triangle

$$D_n \xrightarrow{\tau} A_n \xrightarrow{i_n} A_{n+1} \longrightarrow D_n[1]\ .$$

Since D_n and A_n belong to the class $\langle \mathcal{P} \rangle_+$, we also have $A_{n+1} \in \langle \mathcal{P} \rangle_+$. Since $u_n \tau = 0$ (by definition), we can choose a morphism $u_{n+1} : A_{n+1} \longrightarrow X$ such that $u_{n+1} i_n = u_n$. This completes the inductive construction.

Now we choose a homotopy colimit $(A, \{\varphi_n : A_n \longrightarrow A\}_n)$ of the sequence of morphisms $i_n : A_n \longrightarrow A_{n+1}$, i.e., a distinguished triangle

$$\bigoplus\nolimits_{n \geq 0} A_n \xrightarrow{1-\mathrm{sh}} \bigoplus\nolimits_{n \geq 0} A_n \longrightarrow A \longrightarrow \bigoplus\nolimits_{n \geq 0} A_n[1]\ .$$

Since all the objects A_n are in $\langle \mathcal{P} \rangle_+$, so is A. Since a homotopy colimit in $\mathcal{GH}$ is a weak colimit, we can choose a morphism $u : A \longrightarrow X$ such that $u\varphi_n = u_n$ for all $n \geq 0$. The map

$$[\![P, u_0]\!] \;=\; [\![P, u]\!] \circ [\![P, \varphi_0]\!] \;:\; [\![P, A_0]\!] \;\longrightarrow\; [\![P, X]\!]$$

is surjective for $P \in \mathcal{P}$, hence so is $[\![P, u]\!] : [\![P, A]\!] \longrightarrow [\![P, X]\!]$.

We claim that $[\![P, u]\!]$ is also injective for all $P \in \mathcal{P}$. We let $\alpha : P \longrightarrow A$ be a morphism such that $u\alpha = 0$. Since P is compact, there is an $n \geq 0$ and a morphism $\alpha' : P \longrightarrow A_n$ such that $\alpha = \varphi_n \alpha'$. Then $u_n \alpha' = u\varphi_n \alpha' = u\alpha = 0$. So α' indexes one of the summands of D_n. Thus α' factors through the tautological morphism $\tau : D_n \longrightarrow A_n$ as $\alpha' = \tau\alpha''$, and hence

$$\alpha \;=\; \varphi_n \alpha' \;=\; \varphi_{n+1} i_n \tau \alpha'' \;=\; 0\ ,$$

since i_n and τ are consecutive morphisms in a distinguished triangle. So $[\![P, u]\!]$ is also injective, hence bijective. By the choice of the set $\mathcal{P}$ and the natural isomorphism

$$[\![(\Sigma_+^\infty B_{\mathrm{gl}} G)[k], A]\!] \;\cong\; \pi_k^G(A)\ ,$$

this shows that the morphism $u : A \longrightarrow X$ induces isomorphisms on all equivariant homotopy groups in non-negative dimensions.

So far the spectrum X was arbitrary. If we now assume that X is globally connective, then the morphism $u : A \longrightarrow X$ is a global equivalence (because A is globally connective by the previous paragraph). So X is isomorphic in $\mathcal{GH}$ to A, and hence $X \in \langle \mathcal{P} \rangle_+$. □

Given orthogonal spectra X and Y, the external product maps (4.1.21)

$$\boxtimes \;:\; \pi_0^G(X) \otimes \pi_0^K(Y) \;\longrightarrow\; \pi_0^{G\times K}(X \wedge Y)$$

form a bimorphism of global functors by Theorem 4.1.22. The box product of global functors was introduced in Construction 4.2.17. The universal property of the box product produces a morphism of global functors

$$\underline{\pi}_0(X) \,\Box\, \underline{\pi}_0(Y) \;\longrightarrow\; \underline{\pi}_0(X \wedge Y)\,. \tag{4.4.14}$$

We recall from (4.3.25) that the symmetric monoidal derived smash product $\wedge^{\mathbb{L}}$ on the global stable homotopy category is obtained as the total left derived functor of the smash product of orthogonal spectra. When applied to flat replacements of X and Y, the morphism (4.4.14) becomes the morphism of the following proposition.

Proposition 4.4.15. *For all globally connective spectra X and Y the orthogonal spectrum $X \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism*

$$\underline{\pi}_0(X) \,\Box\, \underline{\pi}_0(Y) \;\longrightarrow\; \underline{\pi}_0(X \wedge^{\mathbb{L}} Y)$$

is an isomorphism of global functors.

Proof We fix a compact Lie group K and let $\mathcal{X}$ be the class of globally connective orthogonal spectra X such that $X \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K$ is globally connective and the natural morphism

$$\underline{\pi}_0(X) \,\Box\, \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K) \;\longrightarrow\; \underline{\pi}_0(X \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K)$$

is an isomorphism of global functors. The class $\mathcal{X}$ is closed under sums and we claim that it is also closed under cones. We let

$$A \;\longrightarrow\; B \;\longrightarrow\; C \;\longrightarrow\; \Sigma A$$

be a distinguished triangle in $\mathcal{GH}$ such that A and B belong to $\mathcal{X}$. Since A is globally connective the global functor $\underline{\pi}_{-1}(A)$ vanishes; since A belongs to $\mathcal{X}$ the global functor $\underline{\pi}_{-1}(A \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K)$ vanishes. Since $-\Box\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K)$ is right exact (by Remark 4.2.18), the upper row in the commutative diagram

$$\begin{tikzcd}
\underline{\pi}_0(A)\Box\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] \arrow[d] & \underline{\pi}_0(B)\Box\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] \arrow[d] & \underline{\pi}_0(C)\Box\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] \arrow[d] & 0 \\
\underline{\pi}_0(A \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] & \underline{\pi}_0(B \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] & \underline{\pi}_0(C \wedge^{\mathbb{L}} \Sigma^\infty_+ B_{\mathrm{gl}} K) \arrow[r] & 0
\end{tikzcd}$$

is exact. The lower row is exact because smashing with $\Sigma^\infty_+ B_{\mathrm{gl}} K$ preserves distinguished triangles. The two left vertical maps are isomorphisms because

A and B belong to the class $\mathcal{X}$. So the right vertical map is an isomorphism and C belongs to $\mathcal{X}$ as well. Moreover, we have

$$\begin{aligned}\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} G) \Box \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K) \;&\cong\; \mathbf{A}(G,-) \Box \mathbf{A}(K,-) \\ &\cong\; \mathbf{A}(G \times K, -) \;\cong\; \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}(G \times K))\end{aligned}$$

by Proposition 4.2.5 and Remark C.11, and

$$(\Sigma^\infty_+ B_{\mathrm{gl}} G) \wedge (\Sigma^\infty_+ B_{\mathrm{gl}} K) \;\cong\; \Sigma^\infty_+ (B_{\mathrm{gl}} G \boxtimes B_{\mathrm{gl}} K) \;\cong\; \Sigma^\infty_+ B_{\mathrm{gl}}(G \times K)$$

by Proposition 4.1.18, and by (1.3.5). This shows that the class $\mathcal{X}$ is closed under sums and cones and contains the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for all compact Lie groups G. Proposition 4.4.13 then shows that $\mathcal{X}$ is the class of all globally connective orthogonal spectra. This proves the proposition in the special case $Y = \Sigma^\infty_+ B_{\mathrm{gl}} K$

Now we perform the same argument in the other variable. We fix a globally connective spectrum X and let $\mathcal{Y}$ denote the class of globally connective orthogonal spectra Y such that $X \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism of the proposition is an isomorphism of global functors. The class $\mathcal{Y}$ is again closed under sums and cones, by the same arguments as above. Moreover, for every compact Lie group K the suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} K$ belongs to the class $\mathcal{Y}$ by the previous paragraph. Again Proposition 4.4.13 shows that $\mathcal{Y}$ is the class of all globally connective orthogonal spectra. □

The semifree orthogonal spectrum $F_{G,V} = \mathbf{O}(V,-)/G$ generated by a G-representation V was defined in Construction 4.1.23. Now we identify the functor represented by $F_{G,V}$ in the global stable homotopy category. The element

$$(0, \mathrm{Id}_V) \cdot G \;\in\; \mathbf{O}(V,V)/G = \; F_{G,V}(V)$$

is a G-fixed-point, so the G-map

$$S^V \;\longrightarrow\; \mathbf{O}(V,V)/G \wedge S^V \;=\; (F_{G,V} \wedge S^V)(V)\,, \quad v \;\longmapsto\; (0, \mathrm{Id}_V) \cdot G \wedge (-v)$$

represents an equivariant homotopy class

$$a_{G,V} \;\in\; \pi_0^G(F_{G,V} \wedge S^V)\,. \tag{4.4.16}$$

The $\underline{\pi}_*$-isomorphism of orthogonal G-spectra $\lambda^V_E : E \wedge S^V \longrightarrow \mathrm{sh}^V E$ was defined in (3.1.23); its value at an inner product space U is the opposite structure map $\sigma^{\mathrm{op}}_{U,V} : E(U) \wedge S^V \longrightarrow E(U \oplus V) = (\mathrm{sh}^V E)(U)$. The following lemma is a direct consequence of Proposition 3.1.25 (i).

Lemma 4.4.17. *The morphism $\lambda^V_{F_{G,V}} : F_{G,V} \wedge S^V \longrightarrow \mathrm{sh}^V F_{G,V}$ takes the element $a_{G,V}$ to the class in $\pi_0^G(\mathrm{sh}^V F_{G,V})$ that is represented by the G-fixed point $(0, \mathrm{Id}_V) \cdot G$ of $(\mathrm{sh}^V F_{G,V})(0)$.*

We emphasize that the representative of $a_{G,V}$ involves the involution $S^{-\mathrm{Id}} : S^V \longrightarrow S^V$, which represents a certain unit in the ring $\pi_0^G(\mathbb{S})$ that squares to 1. Lemma 4.4.17 is our reason for choosing this particular normalization. We could remove the involution from the definition of $a_{G,V}$, thereby changing the class $a_{G,V}$ into $\varepsilon_V(a_{G,V})$, but then the unit would instead appear in other formulas later on.

Theorem 4.4.18. *Let G be a compact Lie group and V a faithful G-representation. Then the pair $(F_{G,V}, a_{G,V})$ represents the functor*

$$\mathcal{GH} \longrightarrow (\text{sets}) , \quad E \longmapsto \pi_0^G(E \wedge S^V) .$$

The semifree orthogonal spectrum $F_{G,V}$ is a compact object of the global stable homotopy category $\mathcal{GH}$.

Proof We define

$$b_{G,V} = (\lambda^V_{F_{G,V}})_*(a_{G,V}) \in \pi_0^G(\mathrm{sh}^V F_{G,V}) ;$$

Lemma 4.4.17 shows that the class $b_{G,V}$ is represented by the G-fixed-point

$$(0, \mathrm{Id}_V) \cdot G \in \mathbf{O}(V, V)/G = (\mathrm{sh}^V F_{G,V})(0) .$$

Since the morphism $\lambda^V_{F_{G,V}}$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, we may thus show that the pair $(F_{G,V}, b_{G,V})$ represents the functor

$$\mathcal{GH} \longrightarrow (\text{sets}) , \quad E \longmapsto \pi_0^G(\mathrm{sh}^V E) ,$$

i.e., for every orthogonal spectrum E the map

$$[\![F_{G,V}, E]\!] \longrightarrow \pi_0^G(\mathrm{sh}^V E) , \quad f \longmapsto (\mathrm{sh}^V f)_*(b_{G,V})$$

is bijective. Since both sides take global equivalences in E to bijections, we can assume that E is a global Ω-spectrum. The orthogonal spectrum $F_{G,V}$ is flat, and hence cofibrant in the global model structure. So the localization functor induces a bijection

$$\mathcal{S}p(F_{G,V}, E)/\text{homotopy} \longrightarrow [\![F_{G,V}, E]\!]$$

from the set of homotopy classes of morphisms of orthogonal spectra to the set of morphisms in $\mathcal{GH}$. By the freeness property, morphisms from $F_{G,V}$ to E biject with G-fixed-points of $E(V)$, and homotopies correspond to paths of G-fixed-points. The composite

$$\pi_0(E(V)^G) \xrightarrow{\cong} [\![F_{G,V}, E]\!] \xrightarrow{f \mapsto (\mathrm{sh}^V f)_*(b_{G,V})} \pi_0^G(\mathrm{sh}^V E)$$

is the stabilization map of the G-Ω-spectrum $\mathrm{sh}^V E$, and hence bijective. Since the left map and the composite are bijective, so is the evaluation map at the class $b_{G,V}$.

Now we prove that $F_{G,V}$ is a compact object in $\mathcal{GH}$. In the commutative square

$$\begin{tikzcd}
\bigoplus_{i\in I} [\![F_{G,V}, X_i]\!] \arrow[r] \arrow[d] & [\![F_{G,V}, \bigoplus_{i\in I} X_i]\!] \arrow[d] \\
\bigoplus_{i\in I} \pi_0^G(X_i \wedge S^V) \arrow[r] & \pi_0^G\left(\bigvee_{i\in I} X_i \wedge S^V\right)
\end{tikzcd}$$

both vertical maps are evaluation at the class $a_{G,V}$, which are isomorphisms by part (i). The lower horizontal map is an isomorphism by Corollary 3.1.37 (i), hence so is the upper horizontal map. This shows that $F_{G,V}$ is compact as an object of the triangulated category $\mathcal{GH}$. □

Now we construct certain distinguished triangles in the global stable homotopy category that arise from special representations of compact Lie groups G, namely when G acts faithfully and transitively on the unit sphere. Equivalently, we are looking for ways to present spheres as homogeneous spaces. The distinguished triangles relate two semifree orthogonal spectra for the compact Lie group G and a semifree orthogonal spectrum for a closed subgroup H that occurs as the stabilizer of a unit vector in the transitive faithful G-representation. The main case we care about is $G = O(m)$ acting tautologically on $\mathbb{R}^m$ with stabilizer group $H = O(m-1)$. This special case will become relevant in Section 6.1 when we analyze the rank filtration of the global Thom spectrum **mO**. Similarly, we can let $G = SO(m)$ act tautologically on $\mathbb{R}^m$ with stabilizer group $H = SO(m-1)$, and this shows up in the rank filtration of the global Thom spectrum **mSO**; here we need $m \geq 2$, since $SO(1)$ does not act transitively on $S(\mathbb{R})$. Other examples are $U(m)$ or $SU(m)$ (the latter for $m \geq 2$) acting on the underlying $\mathbb{R}$-vector space of $\mathbb{C}^m$, with stabilizer groups $U(m-1)$ and $SU(m-1)$, respectively. Similarly, we can consider the tautological representation of $Sp(m)$ on the underlying $\mathbb{R}$-vector space of $\mathbb{H}^m$ with stabilizer group $Sp(m-1)$. These examples show up in the rank filtrations of the global Thom spectra **mU**, **mSU** and **mSp**. There are also more exotic examples, such as the exceptional Lie group G_2, the group of $\mathbb{R}$-algebra automorphism of the octonions $\mathbb{O}$. Here we take V as the tautological 7-dimensional representation on the imaginary octonions, with stabilizer group isomorphic to $SU(3)$. For a complete list of examples we refer the reader to [13, Ch. 7.B, Ex. 7.13] and the references given therein.

Construction 4.4.19. We consider a representation V of a compact Lie group G such that G acts faithfully and transitively on the unit sphere $S(V)$. We choose a unit vector $v \in S(V)$ and let H be the stabilizer group of v. Then

the underlying H-representation of V decomposes as

$$V = L \oplus (\mathbb{R} \cdot v)$$

where L is the orthogonal complement of v. We use the letter L for this complement because it essentially 'is' the tangent representation $T_{eH}(G/H)$. More precisely, the differential at eH of the smooth G-equivariant embedding

$$G/H \longrightarrow V , \quad gH \longmapsto gv$$

is an H-equivariant linear isomorphism from $T_{eH}(G/H)$ onto L.

We define a based G-map

$$r : S^V \longrightarrow \mathbf{O}(L, V)/H = F_{H,L}(V) \tag{4.4.20}$$

$$\text{by} \qquad r(g \cdot t \cdot v) = g \cdot ((t^2 - 1)/t \cdot v, \mathrm{incl}) \cdot H ,$$

where $g \in G$ and $t \in [0, \infty]$. We let

$$T : \Sigma^\infty_+ B_{\mathrm{gl}} G = \Sigma^\infty_+ \mathbf{L}_{G,V} \longrightarrow F_{H,L}$$

denote the adjoint of r. We define a morphism of orthogonal spectra

$$i : F_{H,L} \longrightarrow \mathrm{sh}\, F_{G,V}$$

as the adjoint of the H-fixed-point

$$\psi^{-1} \cdot G \in \mathbf{O}(V, L \oplus \mathbb{R})/G = (\mathrm{sh}\, F_{G,V})(L) ,$$

where the H-equivariant isometry $\psi : L \oplus \mathbb{R} \cong V$ is defined as $\psi(x, t) = x + t \cdot v$. The value of the morphism i at an inner product space W is then the map

$$\begin{aligned} i(W) : F_{H,L}(W) = \mathbf{O}(L, W)/H &\longrightarrow \mathbf{O}(V, W \oplus \mathbb{R})/G = (\mathrm{sh}\, F_{G,V})(W) \\ (w, \varphi) \cdot H &\longmapsto ((w, 0), (\varphi \oplus \mathbb{R}) \circ \psi^{-1}) \cdot G . \end{aligned}$$

We define a morphism of orthogonal spectra $a : F_{G,V} \longrightarrow \Sigma^\infty_+ \mathbf{L}_{G,V} = \Sigma^\infty_+ B_{\mathrm{gl}} G$ as the adjoint of the G-fixed-point

$$0 \wedge \mathrm{Id} \cdot G \in S^V \wedge \mathbf{L}(V, V)/G_+ = (\Sigma^\infty_+ \mathbf{L}_{G,V})(V) .$$

The morphism $\lambda_X : X \wedge S^1 \longrightarrow \mathrm{sh}\, X$ was defined in (3.1.23) and is a global equivalence by Proposition 4.1.4 (i). The stable tautological class

$$e_G = e_{G,V} \in \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G)$$

was defined in (4.1.12). In the next theorem we abuse notation and use the same symbols for morphisms of orthogonal spectra and their images in the global stable homotopy category.

Theorem 4.4.21. *Let G be a compact Lie group and V a G-representation such that G acts faithfully and transitively on the unit sphere $S(V)$; let H be the stabilizer group of a unit vector of V.*

(i) *The sequence*

$$F_{G,V} \xrightarrow{a} \Sigma^\infty_+ B_{\mathrm{gl}}G \xrightarrow{T} F_{H,L} \xrightarrow{-\lambda^{-1}_{F_{G,V}}\circ i} F_{G,V} \wedge S^1$$

is a distinguished triangle in the global stable homotopy category.

(ii) *The morphism T satisfies the relation*

$$T_*(e_G) \ = \ \mathrm{Tr}_H^G(a_{H,L})$$

in the group $\pi_0^G(F_{H,L})$.

Proof We consider the based map $q : G/H_+ \longrightarrow S^0$ that sends G/H to the non-basepoint. We identify the mapping cone of q with the sphere S^V via the G-equivariant homeomorphism

$$h \ : \ Cq \ \cong \ S^V$$

that is induced by the map

$$G/H \times [0,1] \ \longrightarrow \ S^V\ , \quad (gH, x) \ \longmapsto \ g\cdot (1-x)/x \cdot v\ .$$

Under this identification the mapping cone inclusion $i : S^0 \longrightarrow Cq$ becomes the inclusion $\iota : \{0,\infty\} = S^0 \longrightarrow S^V$, and the projection $p : Cq \longrightarrow G/H_+ \wedge S^1$ becomes the map

$$p\circ h^{-1} \ : \ S^V \ \longrightarrow \ G/H_+ \wedge S^1\ , \quad g\cdot t\cdot v \ \longmapsto \ gH \wedge (1-t^2)/t\ ,$$

where $g \in G$ and $t \in [0,\infty]$. The semifree functor $F_{G,V}$ takes mapping cone sequences of based G-spaces to mapping cone sequences of orthogonal spectra, so the sequence

$$F_{G,V}(G/H_+) \xrightarrow{F_{G,V}q} F_{G,V} \xrightarrow{F_{G,V}\iota} F_{G,V}S^V \xrightarrow{F_{G,V}(p\circ h^{-1})} F_{G,V}(G/H_+ \wedge S^1)$$

is a distinguished triangle in the global stable homotopy category.

We define a morphism $\Lambda : F_{G,V}(G/H_+ \wedge S^1) \longrightarrow F_{H,L}$ as the adjoint of the G-map

$$\begin{aligned} G/H_+ \wedge S^1 \ &\longrightarrow \ \mathbf{O}(L,V)/H \ = \ F_{H,L}(V) \\ gH \ \wedge t \ &\longmapsto \ g\cdot(-t\cdot v, \mathrm{incl})\cdot H\ . \end{aligned}$$

The morphism Λ is the composite of an isomorphism $F_{G,V}(G/H_+ \wedge S^1) \cong F_{H,L\oplus\mathbb{R}}S^1$ and the morphism $\lambda_{H,L,\mathbb{R}} : F_{H,L\oplus\mathbb{R}}S^1 \longrightarrow F_{H,L}$ defined in (4.1.28).

The morphism $\lambda_{H,L,\mathbb{R}}$ is a global equivalence by Theorem 4.1.29, hence Λ is a global equivalence as well. The composite

$$\Sigma^\infty_+ B_{\mathrm{gl}} G \xrightarrow{\mathrm{untwist}^{-1}} F_{G,V} S^V \xrightarrow{F_{H,L}(p\circ h^{-1})} F_{G,V}(G/H_+ \wedge S^1) \xrightarrow{\Lambda} F_{H,L}$$

coincides with the morphism T, by direct inspection of the effects at the inner product space V.

Our next claim is that the following diagram of orthogonal spectra commutes up to homotopy:

$$\begin{tikzcd} F_{G,V}(G/H_+) \wedge S^1 \arrow[r, "(F_{G,V}q)\wedge S^1"] \arrow[d, "\Lambda"'] & F_{G,V} \wedge S^1 \arrow[d, "\lambda_{F_{G,V}}"] \\ F_{H,L} \arrow[r, "i"'] & \mathrm{sh}\, F_{G,V} \end{tikzcd} \tag{4.4.22}$$

The representing property of $F_{G,V}$ reduces this to evaluating the square at V and showing that the two resulting G-maps from $G/H_+ \wedge S^1$ to $(\mathrm{sh}\, F_{G,V})(V)$ are equivariantly homotopic. Since G-maps out of G/H correspond to H-fixed-points, this in turn reduces to the claim that the two maps

$$S^1 \longrightarrow (\mathbf{O}(V, V \oplus \mathbb{R})/G)^H = ((\mathrm{sh}\, F_{G,V})(V))^H$$

that send $t \in S^1$ to

$$((-t \cdot v, 0), i_0) \cdot G \quad \text{and} \quad ((0, t), i_1) \cdot G$$

are based homotopic, where $i_0, i_1 : V \longrightarrow V \oplus \mathbb{R}$ are given by

$$i_0(x) = (x - \langle x, v\rangle \cdot v,\ \langle x, v\rangle) \quad \text{and} \quad i_1(x) = (x, 0)\ .$$

A homotopy that witnesses this is

$$\begin{aligned} S^1 \times [0,1] &\longrightarrow (\mathbf{O}(V, V \oplus \mathbb{R})/G)^H \\ (t, s) &\longmapsto ((-t \cdot \cos(\pi s/2) \cdot v, t \cdot \sin(\pi s/2)),\ i_s) \cdot G\ , \end{aligned}$$

with

$$i_s : V \longrightarrow V \oplus \mathbb{R}\ ,\ i_s(x) = (x + (\sin(\pi s/2) - 1) \cdot \langle x, v\rangle \cdot v, \cos(\pi s/2) \cdot \langle x, v\rangle)\ .$$

This shows the claim that the square (4.4.22) is homotopy commutative.

So altogether we have produced a commutative diagram in the global stable homotopy category:

$$\begin{tikzcd} F_{G,V} \arrow[r, "F_{G,V}\iota"] \arrow[d, equal] & F_{G,V} S^V \arrow[r, "F_{G,V}(p\circ h^{-1})"] \arrow[d, "\mathrm{untwist}"', "\cong"] & F_{G,V}(G/H_+ \wedge S^1) \arrow[r, "-(F_{G,V}q\wedge S^1)"] \arrow[d, "\sim"', "\Lambda"] & F_{G,V} \wedge S^1 \arrow[d, "\sim"', "\lambda_{F_{G,V}}"] \\ F_{G,V} \arrow[r, "a"'] & \Sigma^\infty_+ B_{\mathrm{gl}} G \arrow[r, "T"'] & F_{H,L} \arrow[r, "-i"'] & \mathrm{sh}\, F_{G,V} \end{tikzcd}$$

The upper sequence is obtained by rotating a distinguished triangle, so it is distinguished. Since all vertical morphisms are isomorphisms in $\mathcal{GH}$, we conclude that the sequence $(a, T, -\lambda^{-1}_{F_{G,V}} \circ i)$ is a distinguished triangle in $\mathcal{GH}$.

It remains to analyze the effect of the morphism T on the stable tautological class e_G. We consider the wide G-equivariant embedding

$$G/H \longrightarrow V , \quad gH \longmapsto gv .$$

This embedding was already used to identify the tangent space $T_{eH}(G/H)$ at the preferred coset with the subspace L inside V; the inclusion $L \longrightarrow V$ corresponds to the differential at eH. The orthogonal complement of L is precisely the line spanned by the vector v, and the subgroup H acts trivially on this complement.

As described more generally in Construction 3.2.7, the wide embedding gives rise to a G-equivariant Thom-Pontryagin collapse map (3.2.10)

$$c \,:\, S^V \longrightarrow G/H_+ \wedge S^1 .$$

In the more general context of an arbitrary closed subgroup, the target of the collapse map is $G \ltimes_H S^{V-L}$; but in our situation H acts trivially on $\mathbb{R}\cdot v = V - L$, so we can (and will) identify $V - L$ with $\mathbb{R}$ via $x \mapsto \langle x, v\rangle$. The collapse map c sends the origin and all vectors of length at least 2 to the basepoint; if $x \in V$ satisfies $0 < |x| < 2$, then $x = g \cdot t \cdot v$ for some $g \in G$ and $t \in (0, 2)$. The collapse map c sends such a vector x to

$$gH \wedge \frac{t-1}{1-|t-1|} \,\in\, G/H_+ \wedge S^1 .$$

Now we calculate the class $\mathrm{Tr}^G_H(a_{H,L})$. We factor the external transfer $G \ltimes_H - : \pi^G_0(E \wedge S^L) \longrightarrow \pi^G_0(G \ltimes_H E)$ as the composite

$$\pi^H_0(E \wedge S^L) \xrightarrow{(\lambda^L_E)_*} \pi^H_0(\mathrm{sh}^L E) \xrightarrow{G \boxtimes_H -} \pi^G_0(G \ltimes_H E) .$$

Here $\lambda^L_E : E \wedge S^L \longrightarrow \mathrm{sh}^L E$ is the morphism of orthogonal H-spectra defined in (3.1.23); the second homomorphism is a variation of the external transfer, and is defined as follows. We let U be an H-representation and $f : S^U \longrightarrow E(U \oplus L) = (\mathrm{sh}^L E)(U)$ an H-equivariant based map that represents a class in $\pi^H_0(\mathrm{sh}^L E)$. By enlarging U, if necessary, we can assume that it is the underlying H-representation of a G-representation. We denote by $f \diamond \mathbb{R}$ the composite H-map

$$S^U \wedge S^1 \xrightarrow{f \wedge S^1} E(U \oplus L) \wedge S^1 \xrightarrow{\sigma^{\mathrm{op}}_{U \oplus L, \mathbb{R}}} E(U \oplus L \oplus \mathbb{R}) \xrightarrow[\cong]{E(U \oplus \psi)} E(U \oplus V) .$$

The class $G \boxtimes_H \langle f \rangle$ in $\pi_0^G(G \ltimes_H E)$ is then represented by the composite G-map

$$S^{U\oplus V} \xrightarrow{S^U \wedge c} S^U \wedge (G/H_+ \wedge S^1) \xrightarrow[\cong]{\text{shear}} G \ltimes_H (S^U \wedge S^1)$$
$$\xrightarrow{G \ltimes_H (f \diamond \mathbb{R})} G \ltimes_H E(U \oplus V) \cong (G \ltimes_H E)(U \oplus V) .$$

The relation $G \boxtimes_H ((\lambda^L_E)_* \langle f \rangle) = G \ltimes_H \langle f \rangle$ is straightforward from the definitions.

By Lemma 4.4.17, the class $(\lambda^L_{F_{H,L}})_*(a_{H,L})$ is represented by the H-fixed-point

$$(0, \mathrm{Id}_L) \cdot H \in \mathbf{O}(L, L)/H = (\mathrm{sh}^L F_{H,L})(0) .$$

So the class $\mathrm{Tr}^G_H(a_{H,L}) = \mathrm{act}_*(G \boxtimes_H ((\lambda^L_{F_{H,L}})_*(a_{H,L})))$ is represented by the composite G-map

$$S^V \xrightarrow{c} G/H_+ \wedge S^1 \xrightarrow{G \ltimes_H ((0,\mathrm{Id}_L) H \diamond \mathbb{R})} G \ltimes_H (\mathbf{O}(L, V)/H) \xrightarrow{\text{act}} \mathbf{O}(L, V)/H .$$

Expanding all the definitions identifies this composite as the map

$$S^V \longrightarrow \mathbf{O}(L, V)/H , \quad g \cdot t \cdot v \longmapsto g \cdot (\xi(t) \cdot v, \mathrm{incl}) \cdot H ,$$

for $g \in G$ and $t \in [0, \infty]$, with

$$\xi : [0, \infty] \longrightarrow S^1 , \quad \xi(t) = \begin{cases} \frac{t-1}{1-|t-1|} & \text{if } t \in (0, 2), \text{ and} \\ * & \text{else.} \end{cases}$$

The function ξ is homotopic, relative to $\{0, \infty\}$, to the function sending t to $(t^2 - 1)/t$. Any relative homotopy between these two functions induces a based G-equivariant homotopy between the representative of the class $\mathrm{Tr}^G_H(a_{H,L})$ and the map r defined in (4.4.20). So r itself is a representative of the class $\mathrm{Tr}^G_H(a_{H,L})$. Since $T : \Sigma^\infty_+ B_{\mathrm{gl}} G \longrightarrow F_{H,L}$ is adjoint to the G-map r, we deduce the desired relation $\mathrm{Tr}^G_H(a_{H,L}) = [r] = T_*(e_G)$. □

Part (ii) of Theorem 4.4.21 can be interpreted as saying that for global stable homotopy types, the morphism of orthogonal spectra $T : \Sigma^\infty_+ B_{\mathrm{gl}} G \longrightarrow F_{H,L}$ represents the dimension shifting transfer from H to G. Indeed, the pair $(\Sigma^\infty_+ B_{\mathrm{gl}} G, e_G)$ represents the functor π^G_0 (by Theorem 4.4.3 (i)), and the pair $(F_{H,L}, a_{H,L})$ represents the functor $\pi^H_0(- \wedge S^L)$ (by Theorem 4.4.18). Naturality of transfers then promotes the relation $T_*(e_G) = \mathrm{Tr}^G_H(a_{H,L})$ to a commutative diagram:

$$\begin{tikzcd}
\llbracket F_{H,L}, E \rrbracket \arrow[r, "\llbracket T,E \rrbracket"] \arrow[d, "f \mapsto f_*(a_{H,L})"', "\cong"] & \llbracket \Sigma^\infty_+ B_{\mathrm{gl}} G, E \rrbracket \arrow[d, "\cong"', "f \mapsto f_*(e_G)"] \\
\pi^H_0(E \wedge S^L) \arrow[r, "\mathrm{Tr}^G_H"'] & \pi^G_0(E)
\end{tikzcd}$$

4.5 Change of families

In this section we compare the global stable homotopy categories for two different global families $\mathcal{F}$ and $\mathcal{E}$, where we suppose that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so we get a 'forgetful' functor on the homotopy categories

$$U = U^{\mathcal{E}}_{\mathcal{F}} \ : \ \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$$

from the universal property of localizations. The global model structures are stable, so the two global homotopy categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have preferred triangulated structures, and the forgetful functor is canonically an exact functor of triangulated categories. We show in Theorem 4.5.1 that this forgetful functor has a left and a right adjoint, both fully faithful, and that this data is part of a recollement of triangulated categories. If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the smash product of orthogonal spectra can be derived to symmetric monoidal products on $\mathcal{GH}_{\mathcal{E}}$ and on $\mathcal{GH}_{\mathcal{F}}$ (see Corollary 4.3.26). The forgetful functor is strongly monoidal with respect to these derived smash products. Indeed, the derived smash product in $\mathcal{GH}_{\mathcal{E}}$ can be calculated by flat approximation up to $\mathcal{E}$-equivalence; every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so these flat approximations can also be used to calculate the derived smash product in $\mathcal{GH}_{\mathcal{F}}$. Theorem 4.5.1 below also exhibits symmetric monoidal structures on the two adjoints of the forgetful functor.

The special case with $\mathcal{F}$ the trivial global family allows a quick calculation of the Picard group of $\mathcal{GH}_{\mathcal{E}}$, with the possibly disappointing answer that it is free abelian of rank 1, generated by the suspension of the global sphere spectrum (see Theorem 4.5.5). Propositions 4.5.8 and 4.5.16 provide characterizations of the global homotopy types in the image of the left and right adjoints. As we explain in Example 4.5.19, the 'absolute' right adjoint from the non-equivariant to the full global stable homotopy category models Borel cohomology theories. Construction 4.5.21 exhibits a particularly nice lift of the global Borel functor to the point-set level, i.e., a lax symmetric monoidal endofunctor of the category of orthogonal spectra.

In Theorem 4.5.24 we relate the global stable homotopy category to the G-equivariant stable homotopy category G-$\mathcal{SH}$ for a fixed compact Lie group G. There is another forgetful functor $\mathcal{GH} \longrightarrow G$-$\mathcal{SH}$ which is an exact functor of triangulated categories and has both a left adjoint and a right adjoint. Here, however, the adjoints are *not* fully faithful as soon as the group G is non-trivial.

The global family $\mathcal{F}in$ of finite groups is an important example to which the discussion of this section applies. We will show that rationally, the associated $\mathcal{F}in$-global stable homotopy category admits an algebraic model: Theorem 4.5.29 provides a chain of Quillen equivalences between the category of

orthogonal spectra with the rational $\mathcal{F}in$-global model structure and the category of chain complexes of rational global functors on finite groups. Under this equivalence, the homotopy group global functor for spectra corresponds to the homology group global functor for complexes.

The algebraic model can be simplified further. Rational G-Mackey functors for a fixed finite group naturally split into contributions indexed by conjugacy classes of subgroups, see Proposition 3.4.18. While the analogous global context is *not* semisimple, the category of rational $\mathcal{F}in$-global functors is equivalent to a simpler one, namely contravariant functors from the category Out of finite groups and conjugacy classes of epimorphisms to $\mathbb{Q}$-vector spaces, by Theorem 4.5.35 below. So rational $\mathcal{F}in$-global stable homotopy theory is also modeled by the complexes of such functors. Under the composite equivalence, the geometric fixed-point homotopy group functors for spectra corresponds to the homology group Out-functors for complexes, see Corollary 4.5.37.

Theorem 4.5.1. *Let $\mathcal{F}$ and $\mathcal{E}$ be two global families such that $\mathcal{F} \subseteq \mathcal{E}$.*

(i) *The forgetful functor*

$$U \, : \, \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$$

has a left adjoint L and a right adjoint R, and both adjoints are fully faithful.

(ii) *If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the right adjoint has a preferred lax symmetric monoidal structure $(RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB) \longrightarrow R(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$.*

(iii) *If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the left adjoint has a preferred strong symmetric monoidal structure, i.e., a natural isomorphism between $L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$ and $(LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$.*

Proof By Proposition 4.3.22 the categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have infinite sums and infinite products, and the forgetful functor preserves both.

(i) As spelled out in Corollary 4.4.5 (v), the existence of the left adjoint is a formal consequence of the fact that $\mathcal{GH}_{\mathcal{E}}$ is compactly generated and that the functor U preserves products. But instead of arguing by hand that U preserves products, we give an alternative construction of the left adjoint by model category theory. Indeed, it is immediate from the definitions of $\mathcal{F}$-equivalences and $\mathcal{F}$-global fibrations that the identity functor is a right Quillen functor from the $\mathcal{E}$-global to the $\mathcal{F}$-global model structure. So its derived functor has a (derived) left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$, see for example [134, I.4, Thm. 3] or [80, Lemma 1.3.10]. Since the right Quillen functor (i.e., the identity) preserves all weak equivalences, the adjunction unit $A \longrightarrow U(LA)$ is an isomorphism in the $\mathcal{F}$-global homotopy category. So the left adjoint is fully faithful.

By Proposition 4.3.22 sums in both $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ are represented by the wedge of orthogonal spectra. So the forgetful functor U preserves sums. As spelled out in Corollary 4.4.5 (iv), the existence of the right adjoint is then a formal consequence of the fact that $\mathcal{GH}_{\mathcal{E}}$ is compactly generated.

The fact that R is fully faithful is also a formal consequence of properties of the adjoint pair (L, U). As we saw above, the adjunction unit $\eta_A : A \longrightarrow U(LA)$ is an isomorphism for every object A of $\mathcal{GH}_{\mathcal{F}}$. So the map

$$(\eta_A)^* \ : \ [\![U(LA), X]\!]_{\mathcal{F}} \ \longrightarrow \ [\![A, X]\!]_{\mathcal{F}}$$

is bijective for every object X of $\mathcal{GH}_{\mathcal{F}}$. The adjunction (U, R) lets us rewrite the left-hand side as $[\![LA, RX]\!]_{\mathcal{E}}$, and the adjunction (L, U) lets us rewrite this further to $[\![A, U(RX)]\!]_{\mathcal{F}}$. Under these substitutions, the map $(\eta_A)^*$ becomes the map induced by the adjunction counit $\epsilon_X \ : \ U(RX) \ \longrightarrow \ X$. This adjunction counit is thus a natural isomorphism, and so R is also fully faithful.

(ii) The lax monoidal structure of the right adjoint R is a formal consequence of the strong monoidal structure of the forgetful functor U. Indeed, for every pair of orthogonal spectra A and B this strong monoidal structure and the adjunction counits provide a morphism

$$U\Big((RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB)\Big) \ \cong \ U(RA) \wedge^{\mathbb{L}}_{\mathcal{F}} U(RB) \ \xrightarrow{\epsilon_A \wedge^{\mathbb{L}} \epsilon_B} \ A \wedge^{\mathbb{L}}_{\mathcal{F}} B$$

whose adjoint $(RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB) \ \longrightarrow \ R(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$ is associative, commutative and unital.

(iii) The strong symmetric monoidal structure on U and the adjunction units provide a morphism

$$A \wedge^{\mathbb{L}}_{\mathcal{F}} B \ \xrightarrow{\eta_A \wedge^{\mathbb{L}} \eta_B} \ U(LA) \wedge^{\mathbb{L}}_{\mathcal{F}} U(LB) \ \cong \ U\Big((LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)\Big)$$

whose adjoint $\lambda_{A,B} \ : \ L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B) \ \longrightarrow \ (LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$ is associative, commutative and unital. We claim that the morphism $\lambda_{A,B}$ is in fact an isomorphism in $\mathcal{GH}_{\mathcal{E}}$ (and hence it can be turned around). We can assume that A and B are $\mathcal{F}$-cofibrant so that $LA = A$ and $LB = B$. Since $\mathcal{F}$ is multiplicative, the point-set level smash product $A \wedge B$ is again $\mathcal{F}$-cofibrant by Proposition 4.3.24 (i). So the value of the left adjoint L on $A \wedge B$ is also given by $A \wedge B$. □

Remark 4.5.2 (Recollements)**.** Theorem 4.5.1 implies that for all pairs of nested global families $\mathcal{F} \subseteq \mathcal{E}$ the diagram of triangulated categories and exact functors

$$\begin{tikzcd} \mathcal{GH}(\mathcal{E};\mathcal{F}) \arrow[r, "i_*"] & \mathcal{GH}_{\mathcal{E}} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{GH}_{\mathcal{F}} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

is a *recollement* in the sense of [10, Sec. 1.4]. Here $\mathcal{GH}(\mathcal{E};\mathcal{F})$ denotes the '$\mathcal{E}$-global homotopy category with support outside $\mathcal{F}$', i.e., the full subcategory of $\mathcal{GH}_{\mathcal{E}}$ of spectra all of whose $\mathcal{F}$-equivariant homotopy groups vanish. The functor $i_* : \mathcal{GH}(\mathcal{E};\mathcal{F}) \longrightarrow \mathcal{GH}_{\mathcal{E}}$ is the inclusion, and i^* (or $i^!$) is a left adjoint (or right adjoint) of i_*.

Remark 4.5.3. In Theorem 4.5.1 the left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ of the forgetful functor $U : \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$ is obtained as the total left derived functor of the identity functor on orthogonal spectra with respect to change of model structure from the $\mathcal{F}$-global to the $\mathcal{E}$-global model structure. So one can calculate the value of the left adjoint L on an orthogonal spectrum X by choosing any cofibrant replacement in the $\mathcal{F}$-global model structure, i.e., an $\mathcal{F}$-equivalence $X_{\mathcal{F}} \longrightarrow X$ with $\mathcal{F}$-cofibrant source. The global homotopy type (so in particular the $\mathcal{E}$-global homotopy type) of $X_{\mathcal{F}}$ is then well-defined. Indeed, since the identity is a left Quillen functor from the $\mathcal{F}$-global to the global model structure (the special case $\mathcal{E} = \mathcal{A}ll$), every acyclic cofibration in the $\mathcal{F}$-global model structure is a global equivalence. Ken Brown's lemma (see the proof of [29, I.4 Lemma 1], [80, Lemma 1.1.12] or [78, Cor. 7.7.2]) then implies that every $\mathcal{F}$-equivalence between $\mathcal{F}$-cofibrant orthogonal spectra is a global equivalence.

It seems worth spelling out the extreme case when $\mathcal{F} = \langle e \rangle$ is the minimal global family of trivial groups and when $\mathcal{E} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups: An orthogonal spectrum X is $\langle e \rangle$-cofibrant if for every $m \geq 0$ the latching morphism $L_m X \longrightarrow X(\mathbb{R}^m)$ is an $O(m)$-cofibration and $O(m)$ acts freely on the complement of the image. These are precisely the orthogonal spectra called 'q-cofibrant' in [107]. So every non-equivariant stable equivalence between q-cofibrant orthogonal spectra is a global equivalence.

The formal properties of the change-of-family functors established in Theorem 4.5.1 facilitate an easy and rather formal argument to identify the Picard group of the global stable homotopy category. The result is that there are no 'exotic' invertible objects, i.e., the only smash invertible objects of $\mathcal{GH}$ are the suspensions and desuspensions of the global sphere spectrum. The same is true more generally for the $\mathcal{F}$-global stable homotopy category relative to any multiplicative global family $\mathcal{F}$, see Theorem 4.5.5 below.

We recall that an object X of a monoidal category $(\mathcal{C}, \Box, I)$ is *invertible* if there is another object Y such that both $X \Box Y$ and $Y \Box X$ are isomorphic to the unit object I. If the isomorphism classes of invertible objects form a set, then the *Picard group* $\mathrm{Pic}(\mathcal{C})$ is this set, with the group structure induced by the monoidal product. Every strong monoidal functor between monoidal categories takes invertible objects to invertible objects, and thus induces a group homomorphism of Picard groups.

Part of the calculation of $\mathrm{Pic}(\mathcal{GH})$ involves a very general argument that we spell out explicitly.

Proposition 4.5.4. *Let $(C, \Box, I)$ be a monoidal category, $P : C \longrightarrow C$ a strong monoidal functor and $\epsilon : P \longrightarrow \mathrm{Id}_C$ a monoidal transformation such that $\epsilon_I : PI \longrightarrow I$ is an isomorphism. Then the induced endomorphism*

$$\mathrm{Pic}(P) \ : \ \mathrm{Pic}(C) \ \longrightarrow \ \mathrm{Pic}(C)$$

is the identity.

Proof We let X be any invertible object, and Y an inverse of X. Since ϵ is a monoidal transformation, the composite

$$(PX)\Box(PY) \ \xrightarrow{\cong} \ P(X\Box Y) \ \xrightarrow{\epsilon_{X\Box Y}} \ X\Box Y$$

agrees with the morphism $\epsilon_X\Box\epsilon_Y$, where the first isomorphism is the strong monoidal structure on P. Since $X\Box Y$ is isomorphic to the unit object and ϵ_I is an isomorphism, the morphism $\epsilon_{X\Box Y}$ is also an isomorphism. So the composite $\epsilon_X\Box\epsilon_Y$ is an isomorphism. Since

$$\epsilon_X\Box\epsilon_Y \ = \ (X\Box\epsilon_Y)\circ(\epsilon_X\Box PY) \ = \ (\epsilon_X\Box Y)\circ(PX\Box\epsilon_Y)$$

we conclude that $\epsilon_X\Box PY$ has a left inverse and $\epsilon_X\Box Y$ has a right inverse. Since Y and PY are invertible objects, the functors $-\Box Y$ and $-\Box PY$ are equivalences of categories, so already $\epsilon_X : PX \longrightarrow X$ has both a left and a right inverse. So ϵ_X is an isomorphism, and hence $\mathrm{Pic}(P)[X] = [PX] = [X]$. □

Now we have all necessary ingredients to determine the Picard group of the $\mathcal{F}$-global stable homotopy category.

Theorem 4.5.5. *For every multiplicative global family $\mathcal{F}$, the Picard group of the $\mathcal{F}$-global stable homotopy category is free abelian of rank 1, generated by the suspension of the global sphere spectrum.*

Proof The forgetful functor

$$U \ = \ U^{\mathcal{F}}_{\langle e\rangle} \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\langle e\rangle} \ = \ \mathcal{SH}$$

to the non-equivariant stable homotopy category and its left adjoint

$$L \ : \ \mathcal{SH} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

both have strong monoidal structures, the latter by Theorem 4.5.1 (iii). Moreover, the adjunction counit $\epsilon : LU \longrightarrow \mathrm{Id}$ is a monoidal transformation, and the morphism $\epsilon_{\mathbb{S}} : L(U\mathbb{S}) \longrightarrow \mathbb{S}$ is an isomorphism. We apply Proposition 4.5.4

to the composite endofunctor LU of $\mathcal{GH}_{\mathcal{F}}$ and conclude that the composite homomorphism

$$\mathrm{Pic}(\mathcal{GH}_{\mathcal{F}}) \xrightarrow{\mathrm{Pic}(U)} \mathrm{Pic}(\mathcal{SH}) \xrightarrow{\mathrm{Pic}(L)} \mathrm{Pic}(\mathcal{GH}_{\mathcal{F}})$$

is the identity. In particular, the homomorphism $\mathrm{Pic}(U)$ induced by the forgetful functor is injective. The Picard group of the non-equivariant stable homotopy category is free abelian of rank 1, generated by the suspension of the non-equivariant sphere spectrum. This generator is the image of the suspension of the global sphere spectrum. So the homomorphism $\mathrm{Pic}(U)$ is surjective, hence an isomorphism. So $\mathrm{Pic}(\mathcal{GH}_{\mathcal{F}})$ is also free abelian of rank 1, generated by the suspension of the global sphere spectrum. □

Now we develop criteria that characterize global homotopy types in the essential image of one of the adjoints to a forgetful change-of-family functor. The following terminology is convenient here.

Definition 4.5.6. Let $\mathcal{F}$ be a global family. An orthogonal spectrum is *left induced* from $\mathcal{F}$ if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$. Similarly, an orthogonal spectrum is *right induced* from $\mathcal{F}$ if it is in the essential image of the right adjoint $R_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$.

We start with a criterion, for certain 'reflexive' global families, that characterizes the left induced homotopy types in terms of geometric fixed-points.

Definition 4.5.7. A global family $\mathcal{F}$ is *reflexive* if for every compact Lie group K there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among continuous homomorphisms from K to groups in $\mathcal{F}$.

In other words, $\mathcal{F}$ is reflexive if and only if the inclusion into the category of all compact Lie groups has a left adjoint. As always with adjoints, the universal pair (uK, p) is then unique up to unique isomorphism under K. Moreover, the universal homomorphism $p : K \longrightarrow uK$ is necessarily surjective. Indeed, the image of p is a closed subgroup of uK, hence also in the global family $\mathcal{F}$. So if the image of p were strictly smaller than K, then p would not be initial among morphisms into groups from $\mathcal{F}$. Some examples of reflexive global families are the minimal global family $\langle e \rangle$ of trivial groups, the global family $\mathcal{F}in$ of finite groups and the global family of abelian compact Lie groups. The maximal family of all compact Lie groups is also reflexive, but in this case the following proposition has no content.

A reflexive global family $\mathcal{F}$ is in particular multiplicative. Indeed, for $G, K \in \mathcal{F}$ the projections $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ factor through

continuous homomorphisms $q_G : u(G \times K) \longrightarrow G$ and $q_K : u(G \times K) \longrightarrow K$, respectively. The composite

$$G \times K \xrightarrow{p} u(G \times K) \xrightarrow{(q_G, q_K)} G \times K$$

is then the identity, so the universal homomorphism $p : G \times K \longrightarrow u(G \times K)$ is injective. Since $u(G \times K)$ belongs to $\mathcal{F}$, so does $G \times K$.

Proposition 4.5.8. *Let $\mathcal{F}$ be a reflexive global family. Then an orthogonal spectrum X is left induced from $\mathcal{F}$ if and only if for every compact Lie group K the inflation map*

$$p^* \ : \ \Phi_*^{uK}(X) \ \longrightarrow \ \Phi_*^K(X)$$

associated with the universal morphism $p : K \longrightarrow uK$ is an isomorphism between the geometric fixed-point homotopy groups for uK and K.

Proof We let $\mathcal{X}$ be the full subcategory of $\mathcal{GH}$ consisting of the orthogonal spectra X such that for every compact Lie group K the inflation map $p^* : \Phi_*^{uK}(X) \longrightarrow \Phi_*^K(X)$ is an isomorphism. We need to show that $\mathcal{X}$ coincides with the class of spectra left induced from $\mathcal{F}$.

Geometric fixed-point homotopy groups commute with sums and take exact triangles to long exact sequences. So $\mathcal{X}$ is closed under sums and triangles, i.e., it is a localizing subcategory of the global homotopy category. Now we claim that for every group G in $\mathcal{F}$ the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ belongs to $\mathcal{X}$. Since $p : K \longrightarrow uK$ is initial among morphisms into groups from $\mathcal{F}$, pre-composition with p is a bijection between the sets of conjugacy classes of homomorphisms into G; moreover, the image of a homomorphism $\alpha : uK \longrightarrow G$ agrees with the image of $\alpha \circ p : K \longrightarrow G$, because p is surjective. Proposition 1.5.12 (i) identifies the fixed-points of the orthogonal space $B_{\mathrm{gl}}G$ as a disjoint union, over conjugacy classes of homomorphisms, of centralizers of images. So the restriction map along p is a weak equivalence of fixed-points spaces

$$p^* \ : \ \left((B_{\mathrm{gl}}G)(\mathcal{U}_{uK})\right)^{uK} \ = \ \left((B_{\mathrm{gl}}G)(\mathcal{U}_{uK})\right)^{K} \ \simeq \ \left((B_{\mathrm{gl}}G)(\mathcal{U}_{K})\right)^{K} \ .$$

Geometric fixed-points commute with suspension spectra (see Example 3.3.3), in the sense of an isomorphism

$$\Phi_*^K(\Sigma_+^\infty B_{\mathrm{gl}}G) \ \cong \ \pi_*^e\left(\Sigma_+^\infty \left((B_{\mathrm{gl}}G)(\mathcal{U}_K)\right)^K\right) \ ,$$

natural for inflation maps. So together this implies the claim for the suspension spectrum of $B_{\mathrm{gl}}G$.

Now we have shown that $\mathcal{X}$ is a localizing subcategory of the global stable homotopy category that contains the suspension spectra of global classifying

spaces of all groups in $\mathcal{F}$. The left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is fully faithful and $\mathcal{GH}_{\mathcal{F}}$ is generated by the suspension spectra of the global classifying spaces in $\mathcal{F}$ (by Theorem 4.4.3). So $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is an equivalence onto the full triangulated subcategory generated by the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for all $G \in \mathcal{F}$. So the image of L is contained in $\mathcal{X}$.

Now suppose that conversely X is an orthogonal spectrum in $\mathcal{X}$. The adjunction counit $\epsilon_X : L(UX) \longrightarrow X$ is an $\mathcal{F}$-equivalence, so it induces isomorphisms of geometric fixed-point groups for all groups in $\mathcal{F}$. By the hypothesis on X and naturality of the inflation maps p^*, the morphism ϵ_X induces isomorphisms of geometric fixed-point homotopy groups for all compact Lie groups. So ϵ_X is a global equivalence, and in particular X is left induced from $\mathcal{F}$. □

Remark 4.5.9. The same proof as in Proposition 4.5.8 yields the following relative version of the proposition. We let $\mathcal{F} \subset \mathcal{E}$ be global families and assume that $\mathcal{F}$ is *reflexive relative to* $\mathcal{E}$, i.e., for every compact Lie group K from the family $\mathcal{E}$ there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among homomorphisms to groups in $\mathcal{F}$. Then an orthogonal spectrum X is in the essential image of the relative left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every compact Lie group K in $\mathcal{E}$ the universal morphism $p : K \longrightarrow uK$ induces isomorphisms

$$p^* \ : \ \Phi^{uK}_*(X) \ \longrightarrow \ \Phi^K_*(X)$$

between the geometric fixed-point homotopy groups of uK and K.

Example 4.5.10. The minimal global family $\mathcal{F} = \langle e \rangle$ of trivial groups is reflexive, and the unique morphism $K \longrightarrow e$ to any trivial group is universal. So Proposition 4.5.8 characterizes the global homotopy types in the essential image of the left adjoint $L : \mathcal{SH} = \mathcal{GH}_{\langle e \rangle} \longrightarrow \mathcal{GH}$ from the non-equivariant stable homotopy category to the global stable homotopy category: an orthogonal spectrum X is left induced from the trivial family if and only if for every compact Lie group K the unique homomorphism $p_K : K \longrightarrow e$ induces an isomorphism

$$p_K^* \ : \ \Phi^e_*(X) \ \longrightarrow \ \Phi^K_*(X) \, .$$

The geometric fixed-point homotopy groups $\Phi^e_*(X)$ with respect to the trivial group are isomorphic to $\pi^e_*(X)$, the stable homotopy groups of the underlying non-equivariant spectrum. So the global homotopy types in the essential image of the left adjoint $L : \mathcal{SH} \longrightarrow \mathcal{GH}$ are precisely the orthogonal spectra with ‘constant geometric fixed-points’.

Here are some specific examples of left induced global homotopy types.

Example 4.5.11 (Suspension spectra)**.** The global sphere spectrum $\mathbb{S}$ and the

suspension spectrum of every based space are left induced from the trivial global family $\langle e \rangle$. Indeed, geometric fixed-points commute with suspension spectra in the following sense: if A has trivial G-action, then

$$\pi_*(\Sigma^\infty A) \;\cong\; \Phi^G_*(\Sigma^\infty A)\,,$$

compare Example 3.3.3. So the suspension spectrum $\Sigma^\infty A$ has 'constant geometric fixed-points', and it is left induced from the trivial family by the criterion of Example 4.5.10.

Example 4.5.12 (Global classifying spaces and semifree orthogonal spectra)**.** If G is a compact Lie group from a global family $\mathcal{F}$, then the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ is left induced from $\mathcal{F}$. To see this, we can refer to the proof of Proposition 4.5.8; alternatively, we may show that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is $\mathcal{F}$-cofibrant, i.e., has the left lifting property with respect to morphisms that are both $\mathcal{F}$-level equivalences and $\mathcal{F}$-level fibrations. We recall that $B_{\mathrm{gl}}G = \mathbf{L}_{G,V} \;=\; \mathbf{L}(V,-)/G$ is a semifree orthogonal space, where V is any faithful G-representation. So morphisms $\Sigma^\infty_+ B_{\mathrm{gl}}G \longrightarrow X$ of orthogonal spectra biject with continuous based G-maps $S^V \longrightarrow X(V)$; since S^V can be given the structure of a based G-CW-complex, it has the left lifting property with respect to G-weak equivalences that are also G-fibrations, and the claim follows by adjointness.

The same kind of reasoning shows that the semifree orthogonal spectra $F_{G,V}$ introduced in Construction 4.1.23 are left induced from $\mathcal{F}$ whenever G belongs to $\mathcal{F}$ and V is a faithful G-representation.

Example 4.5.13 ($\mathbf{\Gamma}$-spaces)**.** We let $\mathbf{\Gamma}$ denote the category whose objects are the based sets $n_+ \;=\; \{0,1,\dots,n\}$, with basepoint 0, and with morphisms all based maps. A $\mathbf{\Gamma}$-*space* is a functor from $\mathbf{\Gamma}$ to the category of based spaces which is reduced (i.e., the value at 0_+ is a one-point space).

A $\mathbf{\Gamma}$-space $F\;:\;\mathbf{\Gamma} \longrightarrow \mathbf{T}_*$ can be extended to a continuous functor on the category of based spaces by a coend construction

$$F(K) \;=\; \int^{n_+\in\mathbf{\Gamma}} F(n_+)\times K^n\;; \tag{4.5.14}$$

here K is a based space and we use the fact that $K^n = \mathrm{map}_*(n_+, K)$ is contravariantly functorial in n_+. We refer to the extended functor as the *prolongation* of F and denote it by the same letter. This abuse of notation is justified by the fact that the value of the prolongation at n_+ is canonically homeomorphic to the original value, see Remark B.23. The coend can be calculated by a familiar quotient space construction in the ambient category of all topological spaces, compare Proposition B.26: $F(K)$ can be obtained from the disjoint union of the spaces $F(n_+)\times K^n$, for $n \geq 0$, by dividing out the equivalence relation generated

by

$$(F(\alpha)(x);\, k_1, \dots, k_n) \;\sim\; (x;\, k_{\alpha(1)}, \dots, k_{\alpha(m)})$$

for all $x \in F(m_+)$, all $(k_1, \dots, k_n)$ in K^n, and all morphisms $\alpha : m_+ \longrightarrow n_+$ in $\mathbf{\Gamma}$. Here $k_{\alpha(i)}$ is to be interpreted as the basepoint of K whenever $\alpha(i) = 0$. The non-obvious fact, proved in Proposition B.26 (ii), is that this quotient space is automatically compactly generated, and hence a coend in the category $\mathbf{T}_*$.

We write $[x;\, k_1, \dots, k_n]$ for the equivalence class of a tuple $(x;\, k_1, \dots, k_n) \in F(n_+) \times K^n$ in $F(K)$. The prolongation is continuous and comes with a continuous, based *assembly map*

$$\alpha \;:\; K \wedge F(L) \;\longrightarrow\; F(K \wedge L)\,,\; \alpha(k \wedge [x;\, l_1, \dots, l_n]) \;=\; [x;\, k \wedge l_1, \dots, k \wedge l_n]\,.$$

The assembly map is natural in all three variables and associative and unital.

We can now define an orthogonal spectrum $F(\mathbb{S})$ by evaluating the $\mathbf{\Gamma}$-space F on spheres. In other words, the value of $F(\mathbb{S})$ at an inner product space V is

$$F(\mathbb{S})(V) \;=\; F(S^V)$$

and the structure map $\sigma_{V,W} : S^V \wedge F(\mathbb{S})(W) \longrightarrow F(\mathbb{S})(V \oplus W)$ is the assembly map for $K = S^V$ and $L = S^W$, followed by the effect of F on the canonical homeomorphism $S^V \wedge S^W \cong S^{V \oplus W}$. The $O(V)$-action on $F(\mathbb{S})(V)$ is via the action on S^V and the continuous functoriality of F.

Proposition 4.5.15. *Let F be a $\mathbf{\Gamma}$-space and G a compact Lie group.*

(i) *The projection $p : G \longrightarrow \pi_0 G = \bar{G}$ to the group of path components induces an isomorphism of geometric fixed-point homotopy groups*

$$p^* \;:\; \Phi_*^{\bar{G}}(F(\mathbb{S})) \;\longrightarrow\; \Phi_*^{G}(F(\mathbb{S}))$$

of the orthogonal spectrum $F(\mathbb{S})$.

(ii) *The orthogonal spectrum $F(\mathbb{S})$ obtained by evaluation of F on spheres is left induced from the global family $\mathcal{F}in$ of finite groups.*

Proof (i) If G is connected, then for every based G-space K the map $F(K^G) \longrightarrow (F(K))^G$ induced by the fixed-point inclusion $K^G \longrightarrow K$ is a homeomorphism by Proposition B.42, where G acts trivially on the $\mathbf{\Gamma}$-space F. We can calculate G-fixed-points by first taking fixed-points with respect to the normal subgroup G° (the path component of the identity) and then fixed-points with respect to the quotient $\bar{G} = G/G^\circ = \pi_0 G$. So for $k \geq 0$, the G-geometric fixed-points of

the orthogonal spectrum $F(\mathbb{S})$ can be rewritten as

$$\begin{aligned}\Phi^G_k(F(\mathbb{S})) &= \operatorname{colim}_{V\in s(\mathcal{U}_G)} [S^{V^G\oplus\mathbb{R}^k}, F(S^V)^G] \\ &\cong \operatorname{colim}_{V\in s(\mathcal{U}_G)} [(S^{V^{G^\circ}\oplus\mathbb{R}^k})^{\bar G}, F(S^{V^{G^\circ}})^{\bar G}] \\ &\cong \operatorname{colim}_{W\in s(\mathcal{U}_{\bar G})} [S^{W^{\bar G}\oplus\mathbb{R}^k}, F(S^W)^{\bar G}] = \Phi^{\bar G}_k(F(\mathbb{S}))\ .\end{aligned}$$

The third step uses that $(\mathcal{U}_G)^{G^\circ}$ is a complete universe for the finite group $\bar G$ and as V runs through $s(\mathcal{U}_G)$, the G°-fixed-points V^{G° exhaust $(\mathcal{U}_G)^{G^\circ}$. The composite isomorphism is inverse to the inflation map p^*. The argument for $k < 0$ is similar.

(ii) The global family $\mathcal{F}in$ of finite groups is reflexive, and for every compact Lie group K the projection $K \longrightarrow \pi_0 K$ to the finite group of path components is universal with respect to $\mathcal{F}in$. Part (i) verifies the geometric fixed-point criterion, so by Proposition 4.5.8 the orthogonal spectrum $F(\mathbb{S})$ is left induced from the global family of finite groups. □

Now we look more closely at right induced global homotopy types. For a global family $\mathcal{F}$ and a compact Lie group G we denote by $\mathcal{F}\cap G$ the family of those closed subgroups of G that belong to $\mathcal{F}$, and $E(\mathcal{F}\cap G)$ is a universal G-space for the family $\mathcal{F}\cap G$. We also need the equivariant cohomology theory represented by an orthogonal spectrum X. If A is a G-space, we define the G-cohomology $X^k_G(A)$ as

$$X^k_G(A) = [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, X[k]]\!]\ ,$$

the group of degree k maps in $\mathcal{GH}$ from the suspension spectrum of the semifree orthogonal space $\mathbf{L}_{G,V}A$ to X. Here V is an implicitly chosen faithful G-representation. By the adjunction between the global stable homotopy category and the G-equivariant stable homotopy category that we discuss in Theorem 4.5.24 below, the group $X^k_G(A)$ is isomorphic to the value at A of the G-cohomology theory represented by the underlying G-spectrum X_G.

Proposition 4.5.16. *An orthogonal spectrum X is right induced from a global family $\mathcal{F}$ if and only if for every compact Lie group G and every cofibrant G-space A the map*

$$X^*_G(A) \longrightarrow X^*_G(A\times E(\mathcal{F}\cap G))$$

induced by the projection $A\times E(\mathcal{F}\cap G)\longrightarrow A$ is an isomorphism.

Proof For every G-space A the projection from $A\times E(\mathcal{F}\cap G)$ to A is an $(\mathcal{F}\cap G)$-equivalence; moreover, if A is cofibrant, then the source is $(\mathcal{F}\cap G)$-projective, so the suspension spectrum of the semifree orthogonal space

$$\Sigma^\infty_+ \mathbf{L}_{G,V}(A\times E(\mathcal{F}\cap G))$$

is left induced from the global family $\mathcal{F}$. This implies that

$$L_{\mathcal{F}}(U_{\mathcal{F}}(\Sigma^\infty_+ \mathbf{L}_{G,V} A)) \cong \Sigma^\infty_+ \mathbf{L}_{G,V}(A \times E(\mathcal{F} \cap G))$$

in the global stable homotopy category. Hence

$$\begin{aligned} X^k_G(A \times E(\mathcal{F} \cap G)) &= [\![\Sigma^\infty_+ \mathbf{L}_{G,V}(A \times E(\mathcal{F} \cap G)), X[k]]\!] \\ &\cong [\![L_{\mathcal{F}}(U_{\mathcal{F}}(\Sigma^\infty_+ \mathbf{L}_{G,V} A)), X[k]]\!] \cong [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, R_{\mathcal{F}}(U_{\mathcal{F}}(X))[k]]\!] \,. \end{aligned}$$

for every orthogonal spectrum X. Under this composite isomorphism, the map of the proposition becomes the map

$$X^k_G(A) = [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, X[k]]\!] \longrightarrow [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, R_{\mathcal{F}}(U_{\mathcal{F}}(X))[k]]\!]$$

induced by the adjunction unit $X \longrightarrow R_{\mathcal{F}}(U_{\mathcal{F}}(X))$.

If X is right induced from $\mathcal{F}$, then this adjunction unit is an isomorphism, hence so is the map $X^*_G(A) \longrightarrow X^*_G(A \times E(\mathcal{F} \cap G))$. If, on the other hand, this map is an isomorphism for all G-spaces A, then for $A = *$ we deduce that the map

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}} G, X[k]]\!] \longrightarrow [\![\Sigma^\infty_+ B_{\mathrm{gl}} G, R_{\mathcal{F}}(U_{\mathcal{F}}(X))[k]]\!]$$

is an isomorphism. Since the suspension spectrum of $B_{\mathrm{gl}}G$ represents π_0^G (by Theorem 4.4.3 (i)), this shows that the adjunction unit $X \longrightarrow R_{\mathcal{F}}(U_{\mathcal{F}}(X))$ is a global equivalence. So X is right induced from $\mathcal{F}$. □

Remark 4.5.17. Essentially the same proof also shows the following relative version of Proposition 4.5.16. We let $\mathcal{F} \subseteq \mathcal{E}$ be two nested global families. Then an orthogonal spectrum X is in the essential image of the relative right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every group G in $\mathcal{E}$ and every cofibrant G-space A the map

$$X^*_G(A) \longrightarrow X^*_G(A \times E(\mathcal{F} \cap G))$$

induced by the projection $A \times E(\mathcal{F} \cap G) \longrightarrow A$ is an isomorphism.

Example 4.5.18. We let X be a global Ω-spectrum with the property that for every inner product space V, the $O(V)$-space $X(V)$ is cofree, i.e., for some (hence any) universal free $O(V)$-space E the map

$$\mathrm{const} : X(V) \longrightarrow \mathrm{map}(E, X(V))$$

that sends a point to the corresponding constant map is an $O(V)$-weak equivalence. We claim that then the orthogonal spectrum X is right induced from the trivial global family $\langle e \rangle$. We use the criterion of Proposition 4.5.16 and show that for every compact Lie group G, every cofibrant G-space A and every integer k the map $X^k_G(\Pi)$ induced by the projection $\Pi : A \times EG \longrightarrow A$ is an isomorphism.

We start with the case $k = 0$. We let V be any faithful G-representation. The projection Π is a weak equivalence of underlying non-equivariant spaces, and source and target are cofibrant as G-spaces. So the G-map

$$S^V \wedge \Pi_+ \ : \ S^V \wedge (A \times EG)_+ \longrightarrow S^V \wedge A_+$$

is also a weak equivalence of underlying non-equivariant spaces, and source and target are G-cofibrant in the based sense. Since $X(V)$ is cofree as an $O(V)$-space, it is also cofree as a G-space, where G acts via the representation homomorphism $G \longrightarrow O(V)$. So the induced map

$$[S^V \wedge \Pi_+, X(V)]^G \ : \ [S^V \wedge A_+, X(V)]^G \longrightarrow [S^V \wedge (A \times EG)_+, X(V)]^G$$

is bijective. Since X is a global Ω-spectrum and the orthogonal suspension spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} A$ is flat, the localization map

$$\mathcal{S}p(\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ X)/\text{homotopy} \longrightarrow [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ X]\!] \ = \ X^0_G(A)$$

is bijective. By the freeness property, the left-hand side bijects with the set $[S^V \wedge A_+, X(V)]^G$. So by the previous paragraph the map $X^0_G(\Pi) : X^0_G(A) \longrightarrow X^0_G(A \times EG)$ is bijective.

For $k > 0$ we apply the same argument to the global Ω-spectrum $\mathrm{sh}^k X$ (which also has cofree levels) and exploit the natural isomorphism

$$(\mathrm{sh}^k X)^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ \mathrm{sh}^k X]\!] \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ X[k]]\!] \ = \ X^k_G(A) \ .$$

For $k < 0$ we apply the same argument to the global Ω-spectrum $\Omega^{-k} X$ (which also has cofree levels) and exploit the natural isomorphism

$$(\Omega^{-k} X)^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ \Omega^{-k} X]\!] \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\ X[k]]\!] \ = \ X^k_G(A) \ .$$

As we explain in more detail in Example 6.4.27 below, the global K-theory spectrum **KU** is right induced from the global family of finite cyclic groups. This fact is a reformulation of the generalization, due to Adams, Haeberly, Jackowski and May [1], of the Atiyah-Segal completion theorem.

Example 4.5.19 (Global Borel theories)**.** We let E be a non-equivariant generalized cohomology theory. Then we obtain a global functor $\underline{E}$ by setting

$$\underline{E}(G) \ = \ E^0(BG) \ ,$$

the 0th E-cohomology of the classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer map for a subgroup inclusion $H \subset G$ is the Becker-Gottlieb transfer [6]

$$\mathrm{tr} \ : \ \Sigma^\infty_+ BG \longrightarrow \Sigma^\infty_+ BH$$

associated with the fiber bundle $EG/H \longrightarrow EG/G = BG$ with fiber G/H, using that EG/H is a classifying space for H. Strictly speaking, in [6] Becker and Gottlieb only define a stable transfer map for locally trivial fiber bundles with smooth compact manifold fiber whenever the base is a finite CW-complex. To get the transfer above one approximates EG (and hence BG) by its finite-dimensional skeleta. The verification of the double coset formula for this global functor is due to Feshbach [53, Thm. II.11].

More generally, we can consider the Borel equivariant cohomology theory represented by E. For a compact Lie group G and a cofibrant G-space A, its value is

$$E^*(EG \times_G A) ,$$

the E-cohomology of the Borel construction (also known as homotopy orbit construction). Here EG is a universal free G-space, which is unique up to equivariant homotopy equivalence. We claim that these Borel cohomology theories associated with E are represented by a specific global homotopy type, namely the result of applying the right adjoint

$$R \ : \ \mathcal{SH} \ \longrightarrow \ \mathcal{GH}$$

to the forget functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ to any non-equivariant spectrum that represents E. To verify this claim we choose a faithful G-representation V and recall from Proposition 1.1.26 (i) that the G-space $\mathbf{L}(V, \mathbb{R}^\infty)$ is a universal free G-space. So for every G-space A the underlying non-equivariant homotopy type of the closed orthogonal space $\mathbf{L}_{G,V}A$ is

$$(\mathbf{L}_{G,V}A)(\mathbb{R}^\infty) \ = \ \mathbf{L}(V, \mathbb{R}^\infty) \times_G A \ = \ EG \times_G A \ .$$

The adjunction between U and R thus provides an isomorphism

$$\begin{aligned}(RE)^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V}A, RE]\!] \ &\cong \ \mathcal{SH}(U(\Sigma^\infty_+ \mathbf{L}_{G,V}A), E) \\ &= \ \mathcal{SH}(\Sigma^\infty_+(EG \times_G A), E) \ = \ E^0(EG \times_G A) \ .\end{aligned}$$

When A is a one-point G-space, then $EG \times_G * = BG$, and this bijection gives rise to a composite isomorphism

$$\pi_0^G(RE) \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V}, RE]\!] \ \cong \ E^0(BG) \ , \tag{4.5.20}$$

where the first one is inverse to evaluation at the stable tautological class $e_G = e_{G,V} \in \pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V})$. We claim that the isomorphisms (4.5.20) are compatible with restriction maps arising from continuous group homomorphisms $\alpha : K \longrightarrow G$. For this purpose we also choose a faithful K-representation W. This data gives rise to a composite morphism of global classifying spaces

$$B_{\mathrm{gl}}\alpha \ : \ B_{\mathrm{gl}}K \ = \ \mathbf{L}_{K,\alpha^*(V)\oplus W} \ \xrightarrow{\rho_{\alpha^*(V),W}/K} \ \mathbf{L}_{K,\alpha^*(V)} \ \xrightarrow{\text{proj}} \ \mathbf{L}_{G,V} \ = \ B_{\mathrm{gl}}G \ .$$

The first morphism restricts a linear isometric embedding from $\alpha^*(V) \oplus W$ to $\alpha^*(V)$, and the second morphism is the quotient map from K-orbits to G-orbits. On the underlying non-equivariant homotopy types (i.e., after evaluating at $\mathbb{R}^\infty$), the morphism $B_{\mathrm{gl}}\alpha$ classifies the homomorphism α. Moreover, the morphism has the 'correct' behavior on the unstable tautological classes. i.e.,

$$(B_{\mathrm{gl}}\alpha)_*(u_K) \;=\; \alpha^*(u_G) \quad \text{in } \pi_0^K(B_{\mathrm{gl}}G)\ ,$$

by direct verification from the definitions. The analogous relation for the stable tautological classes

$$\begin{aligned}(\Sigma^\infty_+ B_{\mathrm{gl}}\alpha)_*(e_K) \;&=\; (\Sigma^\infty_+ B_{\mathrm{gl}}\alpha)_*(\sigma^K(u_K)) \;=\; \sigma^K((B_{\mathrm{gl}}\alpha)_*(u_K)) \\ &=\; \sigma^K(\alpha^*(u_G)) \;=\; \alpha^*(\sigma^G(u_G)) \;=\; \alpha^*(e_G)\end{aligned}$$

in the stable group $\pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ then follows by naturality of the suspension maps $\sigma^K : \pi_0^K(Y) \longrightarrow \pi_0^K(\Sigma^\infty_+ Y)$ and their compatibility with restriction. This proves the compatibility of the isomorphisms (4.5.20) with restriction maps. Compatibility with transfers is essentially built in, as both the transfer in Construction 3.2.7 and the Becker-Gottlieb transfer in [6] are defined as the Thom-Pontryagin construction based on smooth equivariant embedding of G/H into a G-representation; we omit the formal proof. In any case, the group isomorphisms (4.5.20) together form an isomorphism of global functors between $\underline{\pi}_0(RE)$ and $\underline{E}$. This proves the claim that the 'global Borel theories' are precisely the ones right induced from non-equivariant stable homotopy theory.

Construction 4.5.21. We introduce a specific point-set level lift

$$b \;:\; \mathcal{S}p \;\longrightarrow\; \mathcal{S}p$$

of the right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ to the category of orthogonal spectra. Given an orthogonal spectrum E we define a new orthogonal spectrum bE as follows. For an inner product space V we set

$$(bE)(V) \;=\; \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V))\ ,$$

the space of all continuous maps from $\mathbf{L}(V, \mathbb{R}^\infty)$ to $E(V)$. The orthogonal group $O(V)$ acts on this mapping space by conjugation, through its actions on $\mathbf{L}(V, \mathbb{R}^\infty)$ and on $E(V)$. We define structure maps $\sigma_{V,W} : S^V \wedge (bE)(W) \longrightarrow (bE)(V \oplus W)$ as the composite

$$\begin{aligned} S^V \wedge \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), E(W)) \quad &\xrightarrow{\text{assembly}} \quad \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), S^V \wedge E(W)) \\ &\xrightarrow{\mathrm{map}(\mathrm{res}_W, \sigma^E_{V,W})} \mathrm{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), E(V \oplus W)) \end{aligned}$$

where $\mathrm{res}_W : \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(W, \mathbb{R}^\infty)$ restricts an isometric embedding

from $V \oplus W$ to W. In the functorial description of orthogonal spectra, the structure maps are given by

$$\begin{aligned} \mathbf{O}(V, W) \wedge \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V)) &\longrightarrow \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), E(W)) \\ (w, \varphi) \wedge \quad f \quad &\longmapsto \{ \psi \mapsto X(w, \varphi)(f(\psi \circ \varphi)) \} . \end{aligned}$$

The endofunctor b on the category of orthogonal spectra comes with a natural transformation

$$i_E \ : \ E \longrightarrow bE$$

whose value at an inner product space V sends a point $x \in E(V)$ to the constant map $\mathbf{L}(V, \mathbb{R}^\infty) \longrightarrow E(V)$ with value x. Since $\mathbf{L}(V, \mathbb{R}^\infty)$ is contractible, the morphism $i_E : E \longrightarrow bE$ is a non-equivariant level equivalence, hence a non-equivariant stable equivalence.

The next result shows that the global Borel construction b takes Ω-spectra to global Ω-spectra, and that the functor b realizes, in a certain precise way, the right adjoint to the forgetful functor from the global to the non-equivariant stable homotopy category. Since the morphism $i_E : E \longrightarrow bE$ is a non-equivariant stable equivalence, it becomes invertible in the non-equivariant stable homotopy category $\mathcal{SH}$. Part (ii) of the following proposition shows that the morphism $i_E^{-1} : bE \longrightarrow E$ is the counit of the adjunction (U, R).

Proposition 4.5.22. *Let E be an orthogonal Ω-spectrum.*

(i) *The orthogonal spectrum bE is a global Ω-spectrum and right induced from the trivial global family.*
(ii) *For every orthogonal spectrum A both of the two group homomorphisms*

$$[\![A, bE]\!] \xrightarrow{U} \mathcal{SH}(A, bE) \xrightarrow{(i_E)_*^{-1}} \mathcal{SH}(A, E) .$$

are bijective.

Proof (i) We let G be a compact Lie group and V and W two G-representations such that W is faithful. Since E is an Ω-spectrum, the adjoint structure map

$$\tilde{\sigma}^E_{V,W} \ : \ E(W) \longrightarrow \Omega^V E(V \oplus W)$$

is a non-equivariant weak equivalence. The G-space $\mathbf{L}(W, \mathbb{R}^\infty)$ is cofibrant by Proposition 1.1.19 (ii). Because W is a faithful G-representation, the induced G-action on $\mathbf{L}(W, \mathbb{R}^\infty)$ is free. So the induced map

$$\begin{aligned} \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), \tilde{\sigma}^E_{V,W}) \ : \ (bE)(W) \ &= \ \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), E(W)) \\ &\longrightarrow \ \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), \Omega^V E(V \oplus W)) \end{aligned}$$

is a G-weak equivalence. Moreover, the restriction map $\mathrm{res}_W : \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \longrightarrow$

$\mathbf{L}(W,\mathbb{R}^\infty)$ is a G-homotopy equivalence (by Proposition 1.1.26 (ii)), hence it induces another G-homotopy equivalence

$$\begin{aligned}\mathrm{map}(\mathrm{res}_W, \Omega^V E(V\oplus W)) \ : \ &\mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), \Omega^V E(V\oplus W)) \\ &\longrightarrow \ \mathrm{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty), \Omega^V E(V\oplus W))\end{aligned}$$

on mapping spaces. The target of this last map is G-homeomorphic to

$$\mathrm{map}_*(S^V, \mathrm{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty), E(V\oplus W))) \ = \ \Omega^V((bE)(V\oplus W)) \ ;$$

under this homeomorphism, the composite of the two G-weak equivalences becomes the adjoint structure map

$$\tilde{\sigma}^{bE}_{V,W} \ : \ (bE)(W) \ \longrightarrow \ \Omega^V((bE)(V\oplus W)) \ .$$

So we have shown that $\tilde{\sigma}^{bE}_{V,W}$ is a G-weak equivalence, and that means that bE is a global Ω-spectrum.

The $O(V)$-space $\mathbf{L}(V,\mathbb{R}^\infty)$ is a universal free $O(V)$-space by Proposition 1.1.26 (i). So the $O(V)$-space $(bE)(V) = \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty), E(V))$ is cofree. Since bE is also a global Ω-spectrum, the criterion of Example 4.5.18 shows that it is right induced from the trivial global family.

Part (ii) is a formal consequence of (i): since bE is right induced from the trivial global family, the forgetful functor induces a bijection $U : [\![A, bE]\!] \cong \mathcal{SH}(A, bE)$. Since the morphism $i_E : E \longrightarrow bE$ becomes an isomorphism in $\mathcal{SH}$, it induces another bijection on $\mathcal{SH}(A,-)$. □

We endow the functor b with a lax symmetric monoidal transformation

$$\mu_{E,F} \ : \ bE \wedge bF \ \longrightarrow \ b(E\wedge F) \ .$$

To construct $\mu_{E,F}$ we start from the $(O(V)\times O(W))$-equivariant maps

$$\begin{aligned}&\mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty), E(V))\wedge \mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), F(W)) \\ &\quad\xrightarrow{\ \wedge\ } \ \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty)\times \mathbf{L}(W,\mathbb{R}^\infty), E(V)\wedge F(W)) \\ &\quad\xrightarrow{\mathrm{map}(\mathrm{res}_{V,W}, i_{V,W})} \ \mathrm{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty), (E\wedge F)(V\oplus W))\end{aligned}$$

that constitute a bimorphism from (bE, bF) to $b(E\wedge F)$. Here

$$\mathrm{res}_{V,W} \ : \ \mathbf{L}(V\oplus W,\mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V,\mathbb{R}^\infty)\times \mathbf{L}(W,\mathbb{R}^\infty)$$

maps an embedding of $V\oplus W$ to its restrictions to V and W. The morphism $\mu_{E,F}$ is associated with this bimorphism via the universal property of the smash product (3.5.2).

Remark 4.5.23. Various ‘completion’ maps (also called ‘bundling maps’) fit in here as follows. For this we suppose that E is a commutative orthogonal ring

spectrum and a positive Ω-spectrum (in the non-equivariant sense). Then the morphism $i_E : E \longrightarrow bE$ is a kind of 'global completion map'. For every compact Lie group G it induces a ring homomorphism of G-equivariant homotopy groups

$$\pi_0^G(E) \longrightarrow \pi_0^G(bE) \cong E^0(BG) .$$

When $E = \mathbb{S}$ is the sphere spectrum and G is finite, Carlsson's theorem [35] (proving the Segal conjecture) shows that the map

$$\mathbb{A}(G) \cong \pi_0^G(\mathbb{S}) \longrightarrow \pi^0(BG)$$

is completion of the Burnside ring at the augmentation ideal. The sphere spectrum is the suspension spectrum of a global classifying space of the trivial group; more generally, for the global classifying space $B_{\mathrm{gl}}K$ of a finite group K the 'forgetful' map

$$\mathbf{A}(K, G) \cong \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}K) \longrightarrow \mathcal{SH}(\Sigma_+^\infty BG, \Sigma_+^\infty BK)$$

is again completion at the augmentation ideal of the Burnside ring $A(G)$, see [99, Thm. A].

Since the global Borel theory functor $b : \mathcal{S}p \longrightarrow \mathcal{S}p$ is lax symmetric monoidal, it takes orthogonal ring spectra to orthogonal ring spectra, in a way preserving commutativity and module structures. Since the transformation i_E is monoidal, it becomes a homomorphism of orthogonal ring spectra when E is an orthogonal ring spectrum. We let $\mathbb{S} \longrightarrow \mathbb{S}^{\mathrm{f}}$ be a 'positively fibrant replacement', i.e., a morphism of commutative orthogonal ring spectra that is a non-equivariant stable equivalence and whose target is a positive Ω-spectrum; such a replacement exists and is homotopically unique by the positive model structure for commutative orthogonal ring spectra of [107, Thm. 15.1]. The spectrum $\hat{\mathbb{S}} = b(\mathbb{S}^{\mathrm{f}})$ thus comes with a commutative ring spectrum structure, and we call it the *completed sphere spectrum*. Moreover, for every $\mathbb{S}^{\mathrm{f}}$-module spectrum E the map

$$\hat{\mathbb{S}} \wedge bE = b(\mathbb{S}^{\mathrm{f}}) \wedge bE \xrightarrow{\mu_{\mathbb{S}^{\mathrm{f}},E}} b(\mathbb{S}^{\mathrm{f}} \wedge E) \xrightarrow{b(\mathrm{act})} bE$$

makes the orthogonal spectrum bE a module spectrum over the completed sphere spectrum. Since $\hat{\mathbb{S}}$ is non-equivariantly stably equivalent to $\mathbb{S}$, this shows that for every group G the equivariant homotopy group

$$\pi_k^G(bE) \cong E^{-k}(BG)$$

is naturally a module over the commutative ring $\pi_0^G(\hat{\mathbb{S}})$.

For the global K-theory spectrum (compare Construction 6.4.9 below) and any compact Lie group G, the bundling map

$$\mathbf{RU}(G) \cong \pi_0^G(\mathbf{KU}) \longrightarrow KU^0(BG)$$

takes a virtual G-representation to the class of the associated virtual vector bundle over BG. The Atiyah-Segal completion theorem [5] shows that this map is completion at the augmentation ideal of the representation ring. For the Eilenberg-Mac Lane spectrum $\mathcal{H}\mathbb{Z}$ (see Construction 5.3.8 below), the global functor $G \mapsto H^0(BG;\mathbb{Z})$ is constant with value $\mathbb{Z}$; the map

$$\pi_0^G(\mathcal{H}\mathbb{Z}) \longrightarrow H^0(BG;\mathbb{Z})$$

is surjective and an isomorphism modulo torsion for all compact Lie groups whose identity path component is commutative (compare Example 5.3.14).

Now we fix a compact Lie group G and relate the global stable homotopy category to the G-equivariant stable homotopy category (based on a complete universe). We denote by G-$\mathcal{SH}$ the *G-equivariant stable homotopy category*, i.e., any localization of the category $G\mathcal{S}p$ of orthogonal G-spectra at the class of $\underline{\pi}_*$-isomorphisms. Various stable model structures have been established with $\underline{\pi}_*$-isomorphisms as weak equivalences, for example by Mandell-May [108, III Thm. 4.2], Stolz [163, Thm. 2.3.27] and Hill-Hopkins-Ravenel [77, Prop. B.63]. In particular, as for every stable model category, the homotopy category G-$\mathcal{SH}$ comes with a preferred structure of a triangulated category.

A functor

$$(-)_G \,:\, \mathcal{S}p \longrightarrow G\mathcal{S}p\,, \quad X \longmapsto X_G$$

from orthogonal spectra to orthogonal G-spectra is given by endowing an orthogonal spectrum with the trivial G-action. Since the trivial action functor takes global equivalences to $\underline{\pi}_*$-isomorphisms, the universal property of localizations provides a 'forgetful' functor on the homotopy categories

$$U = U_G \,:\, \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}\,.$$

Moreover, U is canonically an exact functor of triangulated categories. We will show now that the forgetful functor has both a left and a right adjoint.

The 'equivariant' smash product of orthogonal G-spectra is simply the smash product of the underlying non-equivariant orthogonal spectra with diagonal G-action. So the trivial action functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ is strong symmetric monoidal. The smash product of orthogonal spectra and of orthogonal G-spectra can be derived to symmetric monoidal products on $\mathcal{GH}$ and on G-$\mathcal{SH}$ (see Corollary 4.3.26). The forgetful functor is strongly monoidal with respect to these derived smash products. Indeed, the derived smash product in $\mathcal{GH}$ can be calculated by flat approximation up to global equivalence; a flat orthogonal spectrum, endowed with trivial G-action, is G-flat, and hence can be used to calculate the derived smash product G-$\mathcal{SH}$, by the flatness theorem (Theorem 3.5.10).

When $G = e$ is a trivial group, the next theorem reduces to the change of family functor of Theorem 4.5.1, with $\mathcal{E} = \mathcal{A}ll$ and $\mathcal{F} = \langle e \rangle$.

Theorem 4.5.24. *For every compact Lie group G the forgetful functor*

$$U \ : \ \mathcal{GH} \ \longrightarrow \ G\text{-}\mathcal{SH}$$

preserves sums and products, and it has a left adjoint and a right adjoint. The left adjoint has a preferred lax symmetric comonoidal structure. The right adjoint has a preferred lax symmetric monoidal structure.

Proof Sums in $\mathcal{GH}$ and G-$\mathcal{SH}$ are represented in both cases by the point-set level wedge. For $\mathcal{GH}$ we state this explicitly in Proposition 4.3.22 (i); for G-$\mathcal{SH}$ we can run the argument based on the stable model structure for orthogonal G-spectra established in [108, III Thm 4.2]. On the point-set level, the forgetful functor preserves wedges, so the derived forgetful functor preserves sums. As spelled out in Corollary 4.4.5 (iv), the existence of the right adjoint is a formal consequence of the fact that $\mathcal{GH}$ is compactly generated and that the functor U preserves sums.

The forgetful functor also preserves infinite products, but the argument here is slightly more subtle because products in $\mathcal{GH}$ are not generally represented by the point-set level product, and because equivariant homotopy groups do not in general commute with infinite point-set level products, compare Remark 4.4.6. We let $\{X_i\}_{i \in I}$ be a family of orthogonal spectra. By replacing each factor by a globally equivalent spectrum, if necessary, we can assume without loss of generality that each X_i is a global Ω-spectrum. Since global Ω-spectra are the fibrant objects in a model structure underlying $\mathcal{GH}$, the point-set level product $\prod_{i \in I} X_i$ then represents the product in $\mathcal{GH}$.

Even though X_i is a global Ω-spectrum, the underlying orthogonal G-spectrum $(X_i)_G$ need *not* be a G-Ω-spectrum. However, as we spell out in the proof of Proposition 4.3.22 (ii), the natural map

$$\pi_k^G\left(\prod\nolimits_{i \in I} X_i\right) \ \longrightarrow \ \prod\nolimits_{i \in I} \pi_k^G(X_i)$$

is an isomorphism for all integers k. Again we can run the argument of Proposition 4.3.22 (ii) in the stable model structure for orthogonal G-spectra [108, III Thm 4.2], and conclude that in this situation, the point-set level product is also a product in G-$\mathcal{SH}$. So the derived forgetful functor preserves products. The existence of the left adjoint is then again a formal consequence of the fact that $\mathcal{GH}$ is compactly generated, compare Corollary 4.4.5 (v).

The same formal argument as in part (iii) of Theorem 4.5.1 shows how to turn the strong monoidal structure of the forgetful functor U into a lax comonoidal structure $L(A \wedge^{\mathbb{L}} B) \longrightarrow (LA) \wedge^{\mathbb{L}} (LB)$ of the left adjoint. In contrast to Theorem 4.5.1 (iii), however, this morphism is usually *not* an isomorphism,

so we cannot turn it around into a monoidal structure on L. The same formal argument as in Theorem 4.5.1 (ii) constructs the lax monoidal structure on R from the strong monoidal structure of the forget functor U. □

Theorem 4.5.24 looks similar to the change-of-family Theorem 4.5.1, but there is one important difference: if the group G is non-trivial, then neither of the two adjoints to the forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$ is fully faithful.

Remark 4.5.25. We mention an alternative way to construct the two adjoints to the forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$, by exhibiting the point-set forgetful functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ as a left or right Quillen functor for suitable model structures. We sketch this for the left adjoint, where we can use the stable model structure of orthogonal G-spectra established by Mandell and May in [108, III Thm 4.2]. However, we cannot argue directly with the functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$, since it is *not* a right Quillen functor. Indeed, if it were a right Quillen functor, then it would preserve fibrant objects. However, a global Ω-spectrum is typically *not* a G-Ω-spectrum when given the trivial G-action.

What saves us is that a global Ω-spectrum is 'eventually' a G-Ω-spectrum, i.e., starting at faithful representations. This lets us modify $(-)_G$ into a right Quillen functor as follows. We choose a faithful G-representation V and let

$$\Omega^V \operatorname{sh}^V \ : \ \mathcal{S}p \ \longrightarrow \ G\mathcal{S}p$$

denote the functor that takes an orthogonal spectrum X to the orthogonal G-spectrum with Uth level

$$(\Omega^V \operatorname{sh}^V X)(U) \ = \ \operatorname{map}_*(S^V, X(U \oplus V)) \ .$$

We emphasize that the G-action on $\Omega^V \operatorname{sh}^V X$ is non-trivial, despite the fact that we started with an orthogonal spectrum without a G-action. A natural morphism of orthogonal G-spectra $\tilde{\lambda}^V_X : X \longrightarrow \Omega^V \operatorname{sh}^V X$ is given by the adjoint of the morphism $\lambda^V_X : X \wedge S^V \longrightarrow \operatorname{sh}^V X$ defined in (3.1.23); the morphism $\tilde{\lambda}^V_X$ is a $\underline{\pi}_*$-isomorphism by Proposition 3.1.25 (ii). In particular, the functor $\Omega^V \operatorname{sh}^V$ also takes global equivalences of orthogonal spectra to $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, and the derived functor of $\Omega^V \operatorname{sh}^V$ is naturally isomorphic to the forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$. The argument can then be completed by showing that the functor $\Omega^V \operatorname{sh}^V$ is a right Quillen functor from the global model structure on orthogonal spectra to the stable model structure on orthogonal G-spectra from [108, III Thm 4.2]. Then by general model category theory, the derived functor of $\Omega^V \operatorname{sh}^V$, and hence also the forgetful functor U, has a left adjoint.

The existence of the right adjoint to U can also be established by model category reasoning. For this one can use the $\mathbb{S}$-model structure on orthogonal

G-spectra constructed by Stolz [163, Thm. 2.3.27]. We leave it to the interested reader to verify that the cofibrant objects in the $\mathbb{S}$-model structure are precisely the G-flat orthogonal G-spectra in the sense of Definition 3.5.7. This shows that the forgetful functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ is a left Quillen functor from the global model structure on orthogonal spectra to the stable $\mathbb{S}$-model structure on orthogonal G-spectra.

The left adjoint $L : G\text{-}\mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor is an exact functor of triangulated categories that preserves infinite sums. The G-equivariant stable homotopy category is compactly generated by the unreduced suspension spectra of all the coset spaces G/H, for all closed subgroups H of G. So L is essentially determined by its values on these generators. The sequence of natural bijections

$$\mathcal{GH}(L(\Sigma^\infty_+ G/H), X) \cong G\text{-}\mathcal{SH}(\Sigma^\infty_+ G/H,\, UX) \cong \pi_0^H(X) \cong \mathcal{GH}(\Sigma^\infty_+ B_{\mathrm{gl}} H, X)$$

shows that the left adjoint L takes the unreduced suspension spectrum of the coset space G/H to the suspension spectrum of the global classifying space of H. In the special case $H = G$ the spectrum $\Sigma^\infty_+ G/H$ is the equivariant sphere spectrum $\mathbb{S}_G$, and we obtain that

$$L(\mathbb{S}_G) \cong \Sigma^\infty_+ B_{\mathrm{gl}} G\,.$$

Now

$$G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \cong \pi_0^G(\mathbb{S}) \cong \mathbb{A}(G) = \mathbf{A}(e, G)$$

is the Burnside ring, whereas

$$\mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G)) \cong \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G) \cong \mathbf{A}(G, G)$$

is the double Burnside ring. The map $L : G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \longrightarrow \mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G))$ corresponds to the ring homomorphism

$$\mathbb{A}(G) = \mathbf{A}(e, G) \longrightarrow \mathbf{A}(G, G)\,, \quad \mathrm{tr}_H^G \circ p_H^* \longmapsto \mathrm{tr}_H^G \circ \mathrm{res}_H^G$$

from the Burnside ring to the double Burnside ring; this homomorphism is never surjective unless G is trivial, so the left adjoint is not full.

Remark 4.5.26. The discussion in this section could be done relative to a global family $\mathcal{F}$, as long as $\mathcal{F}$ contains the compact Lie group G under consideration (and hence also all its subgroups). Indeed, if $\mathcal{F}$ contains G, then every $\mathcal{F}$-equivalence of orthogonal spectra is a $\underline{\pi}_*$-isomorphism of underlying orthogonal G-spectra. Hence the trivial action functor descends to a 'forgetful' functor on the homotopy categories

$$U_G^{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow G\text{-}\mathcal{SH}$$

by the universal property of localizations. The same arguments as in Theorem 4.5.24 show the existence of both adjoints to this forgetful functor, with the same kind of monoidal properties.

Theorem 4.5.24 discusses the maximal case of the global family $\mathcal{F} = \mathcal{A}ll$ of all compact Lie groups. The minimal case is the global family $\langle G\rangle$ generated by G, i.e., the class of compact Lie groups that are isomorphic to a quotient of a closed subgroup of G. All the forgetful functors $U_G^{\mathcal{F}}$ then factor as composites

$$\mathcal{GH}_{\mathcal{F}} \xrightarrow{U^{\mathcal{F}}_{\langle G\rangle}} \mathcal{GH}_{\langle G\rangle} \xrightarrow{U^{\langle G\rangle}_G} G\text{-}\mathcal{SH}$$

of a change-of-family functor and a family-to-group functor. The various adjoints then compose accordingly.

Whenever G is non-trivial, then the global homotopy category $\mathcal{GH}_{\langle G\rangle}$ associated with the global family generated by G is different from the G-equivariant stable homotopy category G-$\mathcal{SH}$. In other words, if G is non-trivial, then the forgetful family-to-group functor $U_G^{\langle G\rangle} : \mathcal{GH}_{\langle G\rangle} \longrightarrow G\text{-}\mathcal{SH}$ is *not* an equivalence, and neither of its adjoints is fully faithful.

In the rest of this section we turn to the global family $\mathcal{F}in$ of finite groups and describe the associated global stable homotopy category $\mathcal{GH}_{\mathcal{F}in}$ rationally. By our previous results, $\mathcal{GH}_{\mathcal{F}in}$ is a compactly generated triangulated category with a symmetric monoidal derived smash product. We call an object X of the category $\mathcal{GH}_{\mathcal{F}in}$ *rational* if the equivariant homotopy groups $\pi_k^G(X)$ are uniquely divisible (i.e., $\mathbb{Q}$-vector spaces) for all finite groups G. In this section we will give an algebraic model of the *rational $\mathcal{F}in$-global stable homotopy category*, i.e., the full subcategory $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ of rational spectra in $\mathcal{GH}_{\mathcal{F}in}$. Theorem 4.5.29 below shows that the homotopy types in $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ are determined by a chain complex of global functors, up to quasi-isomorphism. More precisely, we construct an equivalence of triangulated categories from $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ to the unbounded derived category of rational global functors on finite groups.

We let G and K be compact Lie groups. We recall from Proposition 4.2.5 that the evaluation map

$$\mathbf{A}(G, K) \longrightarrow \pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)\,, \quad \tau \longmapsto \tau(e_G)$$

is an isomorphism, where $e_G \in \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ is the stable tautological class. More precisely, the definition of the global classifying space $B_{\mathrm{gl}}G$ involves an implicit choice of faithful G-representation V that is omitted from the notation, and e_G is the class denote $e_{G,V}$ in (4.1.12). The unreduced suspension spectrum of every orthogonal space is globally connective (see Proposition 4.1.11), so the group $\pi_k^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ is trivial for $k < 0$.

Proposition 4.5.27. *Let G and K be finite groups. Then for every $k > 0$, the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion.*

Proof We show first that the geometric fixed-point group $\Phi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion for $k > 0$. This part of the argument needs G to be finite, but K could be any compact Lie group. Geometric fixed-points commute with suspension spectra, i.e., the groups $\Phi_*^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ are isomorphic to the non-equivariant stable homotopy groups of the fixed-point space $((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K$. Proposition 1.5.12 (i) identifies these fixed-points as

$$((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K \simeq \coprod_{[\alpha]\in\mathrm{Rep}(K,G)} BC(\alpha) ,$$

where the disjoint union is indexed by conjugacy classes of homomorphisms from K to G, and $C(\alpha)$ is the centralizer of the image of $\alpha : K \longrightarrow G$. Since G is finite, so are all the centralizers $C(\alpha)$, hence the classifying space $BC(\alpha)$ has no rational homology, hence no rational stable homotopy, in positive dimensions. So we conclude that the rationalized stable homotopy groups of the space $((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K$ vanish in positive dimensions.

If K is also finite, then the kth rationalized equivariant stable homotopy group of any orthogonal K-spectrum can be recovered from the kth rationalized geometric fixed-point homotopy groups for all subgroups L of K, as described in Corollary 3.4.28. So when both G and K are finite, then also the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion for all $k > 0$. □

The conclusion of Proposition 4.5.27 is no longer true if we drop the finiteness hypothesis on one of the two groups G or K. For example, for $G = e$ we have $\Sigma_+^\infty B_{\mathrm{gl}}G = \mathbb{S}$, and the dimension shifting transfer $\mathrm{Tr}_e^{U(1)}(1)$ is an element of infinite order in the group $\pi_1^{U(1)}(\mathbb{S})$. On the other hand, for $K = e$ the group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is the non-equivariant stable homotopy group of the ordinary classifying space BG. For $G = U(1)$ the group $\pi_k^e(\Sigma_+^\infty BU(1))$ contains a free summand of rank 1 whenever $k \geq 0$ is even.

Now we can establish the algebraic model for the rational $\mathcal{F}in$-global homotopy category. We let $\mathcal{A}$ be a pre-additive category, such as the $\mathcal{F}in$-Burnside category $\mathbf{A}_{\mathcal{F}in}$. We denote by $\mathcal{A}$-mod the category of additive functors from $\mathcal{A}$ to the category of $\mathbb{Q}$-vector spaces. This is an abelian category, and the represented functors $\mathcal{A}(a, -)$, for all objects a of $\mathcal{A}$, form a set of finitely presented projective generators of $\mathcal{A}$-mod. The category of $\mathbb{Z}$-graded chain complexes in the abelian category $\mathcal{A}$-mod then admits the *projective model structure* with the quasi-isomorphisms as weak equivalences. The fibrations in the projective model structure are those chain morphisms that are surjective in every chain complex degree and at every object of $\mathcal{A}$. This projective model structure for

complexes of $\mathcal{A}$-modules is a special case of [36, Thm. 5.1]. Indeed, the projective (in the usual sense) $\mathcal{A}$-modules together with the epimorphisms form a projective class (in the sense of [36, Def. 1.1]), and this class is determined (in the sense of [36, Sec. 5.2]) by the set of represented functors.

We also need the rational version of the $\mathcal{F}$-global model structure, for a global family $\mathcal{F}$. We call a morphism $f : X \longrightarrow Y$ of orthogonal spectra a *rational $\mathcal{F}$-equivalence* if the map

$$\mathbb{Q} \otimes \pi_k^G(f) \ : \ \mathbb{Q} \otimes \pi_k^G(X) \ \longrightarrow \ \mathbb{Q} \otimes \pi_k^G(Y)$$

is an isomorphism for all integers k and all groups G in the family $\mathcal{F}$.

Theorem 4.5.28 (Rational $\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family.*

(i) *The rational $\mathcal{F}$-equivalences and the $\mathcal{F}$-cofibrations are part of a model structure on the category of orthogonal spectra, the* rational $\mathcal{F}$-global model structure.
(ii) *The fibrant objects in the rational $\mathcal{F}$-global model structure are the $\mathcal{F}$-Ω-spectra X such that for all $G \in \mathcal{F}$ the equivariant homotopy groups $\pi_*^G(X)$ are uniquely divisible.*
(iii) *The rational $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Theorem 4.5.28 is obtained by Bousfield localization of the $\mathcal{F}$-global model structure on orthogonal spectra, and one can use a similar proof as for the rational stable model structure on sequential spectra in [147, Lemma 4.1]. We omit the details.

Theorem 4.5.29. *There is a chain of Quillen equivalences between the category of orthogonal spectra with the rational $\mathcal{F}in$-global model structure and the category of chain complexes of rational global functors on finite groups. In particular, this induces an equivalence of triangulated categories*

$$\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in} \xrightarrow{\cong} \mathcal{D}\big(\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}\big) \ .$$

The equivalence can be chosen so that the homotopy group global functor on the left-hand side corresponds to the homology global functor on the right-hand side.

Proof We prove this as a special case of the 'generalized tilting theorem' of Brooke Shipley and the author. Indeed, by Theorem 4.4.3 the suspension spectra of the global classifying spaces $B_{\mathrm{gl}}G$ are compact generators of the global homotopy category $\mathcal{GH}_{\mathcal{F}in}$ as G varies through all finite groups. So the rationalizations $(\Sigma_+^\infty B_{\mathrm{gl}}G)_{\mathbb{Q}}$ are compact generators of the rational global

homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$. If k is any integer, then the morphism vector spaces between two such objects are given by

$$\begin{aligned}[\![(\Sigma^\infty_+ B_{\mathrm{gl}}K)_{\mathbb{Q}}[k],\ (\Sigma^\infty_+ B_{\mathrm{gl}}G)_{\mathbb{Q}}]\!] \ &\cong\ \pi_k^K((\Sigma^\infty_+ B_{\mathrm{gl}}G)_{\mathbb{Q}})\ \cong\ \mathbb{Q}\otimes\pi_k^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)\\ &\cong\ \begin{cases}\mathbb{Q}\otimes\mathbf{A}(G,K) & \text{for } k=0, \text{ and}\\ 0 & \text{for } k\neq 0.\end{cases}\end{aligned}$$

The vanishing for $k > 0$ is Proposition 4.5.27; the vanishing for $k < 0$ is Proposition 4.1.11.

The rational $\mathcal{F}in$-global model structure on orthogonal spectra is topological (hence simplicial), cofibrantly generated, proper and stable; so we can apply the Tilting Theorem [148, Thm. 5.1.1]. This theorem yields a chain of Quillen equivalences between orthogonal spectra in the rational $\mathcal{F}in$-global model structure and the category of chain complexes of $\mathbb{Q}\otimes\mathbf{A}_{\mathcal{F}in}$-modules, i.e., additive functors from the rationalized Burnside category $\mathbb{Q}\otimes\mathbf{A}_{\mathcal{F}in}$ to abelian groups. This functor category is equivalent to the category of additive functors from $\mathbf{A}_{\mathcal{F}in}$ to $\mathbb{Q}$-vector spaces, and this proves the theorem. □

Remark 4.5.30. There is an important homological difference between global functors on finite groups and Mackey functors for one fixed finite group. Indeed, for a finite group G, the category of rational G-Mackey functors is equivalent to a product, indexed over conjugacy classes (H) of subgroups of G, of the module categories over the rational group rings $\mathbb{Q}[W_G H]$ of the Weyl groups, see Theorem 3.4.22 (ii). In particular, the abelian category of rational G-Mackey functors is semisimple, every object is projective and injective and the derived category is equivalent, by taking homology, to the category of graded rational G-Mackey functors.

There is no analog of this for rational $\mathcal{F}in$-global functors. For example, the rationalized augmentation

$$\mathbb{Q}\otimes\mathbb{A} = \mathbb{Q}\otimes\mathbf{A}(e,-)\ \longrightarrow\ \underline{\mathbb{Q}}$$

from the rationalized Burnside ring global functor to the constant global functor for the group $\mathbb{Q}$ does *not* split on finite groups. The new phenomenon is that any splitting would have to be natural for inflation maps. Let us be even more specific. In the constant global functor $\underline{\mathbb{Q}}$ we have

$$\mathrm{tr}_e^{C_2}(1)\ =\ 2\cdot p^*(1)\qquad\text{in}\qquad \underline{\mathbb{Q}}(C_2)=\mathbb{Q}\,,$$

where $p : C_2 \longrightarrow e$ is the unique group homomorphism. So for every morphism of global functors $\varphi : \underline{\mathbb{Q}} \longrightarrow N$ the image $\varphi(e)(1)$ of the unit element under the map $\varphi(e) : \mathbb{Q} \longrightarrow N(e)$ must satisfy

$$\mathrm{tr}_e^{C_2}(\varphi(e)(1))\ =\ 2\cdot p^*(\varphi(e)(1))\,.$$

But in the Burnside ring $\mathbb{A}(e)$, and also in its rationalization, 0 is the only element in the kernel of $\operatorname{tr}_e^{C_2} - 2 \cdot p^*$; so every morphism of global functors from $\underline{\mathbb{Q}}$ to $\mathbb{Q} \otimes \mathbb{A}$ is zero.

While the abelian category $\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}$ is not semisimple, we can still 'divide out transfers' and thereby replace $\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}$ by an equivalent, but simpler category. We let Out denote the category of finite groups and conjugacy classes of surjective group homomorphisms. We write

$$\text{mod-Out} \;=\; \mathcal{F}(\text{Out}^{\text{op}}, \mathcal{Ab})$$

for the category of 'right Out-modules', i.e., contravariant functors from Out to the category of abelian groups. To a $\mathcal{F}in$-global functor $M : \mathbf{A}_{\mathcal{F}in} \longrightarrow \mathcal{Ab}$ we can associate a right Out-module $\tau M : \text{Out}^{\text{op}} \longrightarrow \mathcal{Ab}$, the *reduced functor* as follows. On objects we set

$$(\tau M)(G) \;=\; \tau_G M \;=\; M(G)/t_G M \;,$$

the quotient of the group $M(G)$ by the subgroup $t_G M$ generated by the images of all transfer maps $\operatorname{tr}_H^G : M(H) \longrightarrow M(G)$ for all proper subgroups H of G. If $\alpha : K \longrightarrow G$ is a surjective group homomorphism and $H \leq G$ a proper subgroup, then $L = \alpha^{-1}(H)$ is a proper subgroup of K and the relation

$$\alpha^* \circ \operatorname{tr}_H^G \;=\; \operatorname{tr}_L^K \circ (\alpha|_L)^* \;:\; M(H) \longrightarrow M(K)$$

shows that the inflation map $\alpha^* : M(G) \longrightarrow M(K)$ passes to a homomorphism $\alpha^* : (\tau M)(G) \longrightarrow (\tau M)(K)$ of quotient groups. We will now argue that the reduction functor

$$\tau \;:\; \mathcal{GF}_{\mathcal{F}in} \;\longrightarrow\; \text{mod-Out}$$

is rationally an equivalence of categories, compare Theorem 4.5.35. By construction, the projection maps $M(G) \longrightarrow \tau M(G)$ form a natural transformation from the restriction of the global functor M to the category Out^{op} to τM.

Example 4.5.31. We let A be an abelian group and $\underline{A}$ the constant global functor with value A, compare Example 4.2.8 (iii). Then $(\tau\underline{A})(G) = A/cA$, where c is the greatest common divisor of the indices of all proper subgroups of G. If G is not a p-group for any prime p, then this greatest common divisor is 1. If G is a non-trivial p-group, then G has a proper subgroup of index p. So we have

$$(\tau\underline{A})(G) \;=\; \begin{cases} A & \text{if } G = e, \\ A/pA & \text{if } G \text{ is a non-trivial } p\text{-group, and} \\ 0 & \text{else.} \end{cases}$$

The inflation maps in $\tau\underline{A}$ are quotient maps.

Example 4.5.32. The Burnside ring $\mathbb{A}(K) = \mathbf{A}(e, K)$ of a finite group K is freely generated, as an abelian group, by the transfers $\mathrm{tr}_L^K(1)$ where L runs through representatives of the conjugacy classes of subgroups of K. So $(\tau\mathbb{A})(K)$ is free abelian of rank 1, generated by the class of the multiplicative unit 1. All restriction maps preserve the unit, so the reduced functor $\tau\mathbb{A}$ of the global Burnside ring functor is isomorphic to the constant functor $\mathrm{Out}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ with value $\mathbb{Z}$.

We denote by $\mathbf{A}_G = \mathbf{A}(G, -)$ the global functor represented by a finite group G. We will now present $\tau(\mathbf{A}_G)$ as an explicit quotient of a sum of representable Out-modules. For every closed subgroup H of G the restriction map res^G_H is a morphism in $\mathbf{A}(G, H)$. If H is finite, the Yoneda lemma provides a unique morphism

$$\mathrm{Out}_H \longrightarrow \tau(\mathbf{A}_G)$$

of Out-functors from the representable functor $\mathrm{Out}_H = \mathbb{Z}[\mathrm{Out}(-, H)]$ to $\tau(\mathbf{A}_G)$ that sends the identity of H to the class of res^G_H in $\tau(\mathbf{A}_G)(H)$. For every element $g \in G$ the conjugation isomorphism $c_g : H \longrightarrow H^g$ given by $c_g(\gamma) = g^{-1}\gamma g$ induces an isomorphism

$$g_\star \circ - \ : \ \mathbf{A}_G(H^g) \ \longrightarrow \ \mathbf{A}_G(H)$$

by post-composition. We have

$$g_\star \circ [\mathrm{res}^G_{H^g}] \ = \ [g_\star \circ \mathrm{res}^G_{H^g}] \ = \ [\mathrm{res}^G_H \circ g_\star] \ = \ [\mathrm{res}^G_H]$$

in $(\tau\mathbf{A}_G)(H)$; so the Yoneda lemma shows that the triangle of Out-functors

$$\begin{tikzcd}
\mathrm{Out}_H \arrow[rr, "c_g \circ -"] \arrow[dr] & & \mathrm{Out}_{H^g} \arrow[dl] \\
& \tau(\mathbf{A}_G) &
\end{tikzcd}$$

commutes. The direct sum of the transformations $\mathrm{Out}_H \longrightarrow \tau(\mathbf{A}_G)$ thus factors over a natural transformation

$$\Big(\bigoplus\nolimits_{H \leq G} \mathrm{Out}_H\Big)/G \ \longrightarrow \ \tau(\mathbf{A}_G)\,. \tag{4.5.33}$$

The source of this morphism can be rewritten if we choose representatives of the conjugacy classes of subgroups in G:

$$\bigoplus_{(H)} (\mathrm{Out}_H / W_G H) \ \longrightarrow \ \tau(\mathbf{A}_G)\,.$$

Now the sum is indexed by conjugacy classes.

Proposition 4.5.34. *For every finite group G, the morphism* (4.5.33) *is an isomorphism of* Out*-modules.*

Proof By Theorem 4.2.6 the abelian group $\mathbf{A}_G(K) = \mathbf{A}(G,K)$ is freely generated by the elements $\mathrm{tr}_L^K \circ \alpha^*$ where (L,α) runs through representatives of the $(K \times G)$-conjugacy classes of pairs consisting of a subgroup L of K and a homomorphism $\alpha : L \longrightarrow G$. So $(\tau \mathbf{A}_G)(K)$ is a free abelian group with basis the classes of α^* for all conjugacy classes of homomorphisms $\alpha : K \longrightarrow G$.

On the other hand, the group $(\mathrm{Out}_H / W_G H)(K)$ is free abelian with basis given by $W_G H$-orbits of conjugacy classes of epimorphisms $\alpha : K \longrightarrow H$. The map (4.5.33) sends the basis element represented by α to the basis element represented by the composite of α with the inclusion $H \longrightarrow G$. So the homomorphism (4.5.33) takes a basis of the source to a basis of the target, and is thus an isomorphism. □

We recalled in Proposition 3.4.18 above how the value of a G-Mackey functor M, for a finite group G, can rationally be recovered from the groups $(\tau M)(H)$ for all subgroups H of G: the map

$$\psi_G^M \ : \ M(G) \ \longrightarrow \ \Big(\prod\nolimits_{H \leq G} (\tau M)(H)\Big)^G$$

whose H-component is the composite

$$M(G) \ \xrightarrow{\mathrm{res}_H^G} \ M(H) \ \xrightarrow{\mathrm{proj}} (\tau M)(H)$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. When applied to a $\mathcal{F}in$-global functor M, we see that M can rationally be recovered from the Out-module τM. This is the key input to the following equivalence of categories. I suspect that the following result is well known, but I have been unable to find an explicit reference.

Theorem 4.5.35. *The restriction of the functor τ to the full subcategory of rational $\mathcal{F}in$-global functors is an equivalence of categories*

$$\tau \ : \ \mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in} \ \longrightarrow \ \text{mod-}\,\mathrm{Out}_{\mathbb{Q}} \ = \ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$$

onto the category of rational Out-*modules.*

Proof Since global functors are an enriched functor category and the functor

$$\tau \ : \ \mathcal{GF}_{\mathcal{F}in} \ \longrightarrow \ \text{mod-}\,\mathrm{Out}$$

commutes with colimits, τ has a right adjoint

$$\rho \ : \ \text{mod-}\,\mathrm{Out} \ \longrightarrow \ \mathcal{GF}_{\mathcal{F}in} \ .$$

If ρX is an Out-module, the value of the $\mathcal{F}in$-global functor ρX at a finite group G is necessarily given by

$$(\rho X)(G) \ = \ \text{mod-}\,\mathrm{Out}(\tau(\mathbf{A}_G), X) \ ,$$

the group of Out-module homomorphisms from $\tau(\mathbf{A}_G)$ to X. The global functoriality in G is via τ, i.e., as the composite

$$\begin{aligned}\mathbf{A}(G,K)\otimes(\rho X)(G) \ &\longrightarrow\ \mathcal{GF}(\mathbf{A}_K,\mathbf{A}_G)\otimes(\rho X)(G)\\ &\xrightarrow{\tau\otimes\mathrm{Id}}\ \text{mod-}\operatorname{Out}(\tau(\mathbf{A}_K),\tau(\mathbf{A}_G))\otimes(\rho X)(G)\ \xrightarrow{\circ}\ (\rho X)(K)\ .\end{aligned}$$

We rewrite the definition of $(\rho X)(G)$ using the description of the Out-module $\tau(\mathbf{A}_G)$ given in (4.5.33). Indeed, by Proposition 4.5.34, pre-composition with (4.5.33) induces an isomorphism

$$(\rho X)(G)\ \cong\ \text{mod-}\operatorname{Out}\Big(\Big(\bigoplus\nolimits_{H\le G}\ \operatorname{Out}_H\Big)/G, X\Big)\ \cong\ \Big(\prod\nolimits_{H\le G}X(H)\Big)^G\ .$$

So for every global functor M, this description shows that $\rho(\tau M)(G)$ is isomorphic to the target of the morphism ψ^M_G. A closer analysis reveals that for $X=\tau M$, the above isomorphism identifies ψ^M_G with the value of the adjunction unit $\eta : M\longrightarrow\rho(\tau M)$ at G. So Proposition 3.4.18 shows that for every rational $\mathcal{F}in$-global functor M the adjunction unit $\eta_M : M\longrightarrow\rho(\tau M)$ is an isomorphism. This implies that the restriction of the left adjoint τ to the category of rational $\mathcal{F}in$-global functors is fully faithful.

Now we consider a rational Out-module X. Then ρX is a rational $\mathcal{F}in$-global functor, so $\eta_{\rho X} : \rho X\longrightarrow\rho(\tau(\rho X))$ is an isomorphism by the previous paragraph. Since $\eta_{\rho X}$ is right inverse to $\rho(\epsilon_X)$, the morphism $\rho(\epsilon_X) : \rho(\tau(\rho X))\longrightarrow\rho X$ is an isomorphism of $\mathcal{F}in$-global functors. By Proposition 4.5.34 the represented Out-module Out_G is a direct summand of $\tau(\mathbf{A}_G)$. So the group

$$\text{mod-}\operatorname{Out}(\operatorname{Out}_G, X)\ \cong\ X(G)$$

is a direct summand of the group

$$\text{mod-}\operatorname{Out}(\tau(\mathbf{A}_G), X)\ \cong\ \mathcal{GF}(\mathbf{A}_G,\rho X)\ \cong\ (\rho X)(G)\ ,$$

and this splitting is natural for morphisms of Out-modules in X. In particular, the morphism $\epsilon_X(G) : (\tau(\rho X))(G)\longrightarrow X(G)$ is a direct summand of the morphism

$$\rho(\epsilon_X)(G)\ :\ (\rho(\tau(\rho X)))(G)\ \longrightarrow\ (\rho X)(G)\ .$$

The latter is an isomorphism by the previous paragraph, so the morphism $\epsilon_X(G)$ is also an isomorphism. This shows that for every rational Out-module X the adjunction counit $\epsilon_X : \tau(\rho X)\longrightarrow X$ is an isomorphism.

Altogether we have now seen that when restricted to rational objects on both sides, the unit and counit of the adjunction (τ,ρ) are isomorphisms. This proves the theorem. □

The rational equivalence τ of abelian categories prolongs to an equivalence

of derived categories by applying τ dimensionwise to chain complexes. The combination with the equivalence of triangulated categories of Theorem 4.5.29 is then a chain of two exact equivalences of triangulated categories

$$\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in} \underset{\cong}{\longrightarrow} \mathcal{D}\left(\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}\right) \underset{\cong}{\xrightarrow{\mathcal{D}(\tau)}} \mathcal{D}(\text{mod-}\,\mathrm{Out}_{\mathbb{Q}})\ . \tag{4.5.36}$$

The next proposition shows that this composite equivalence is an algebraic model for the geometric fixed-point homotopy groups.

For every orthogonal spectrum X and every compact Lie group G, the geometric fixed-point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ annihilates all transfers from proper subgroups by Proposition 3.3.11. So the geometric fixed-point map factors over a homomorphism

$$\bar{\Phi} \ :\ \tau(\underline{\pi}_k(X))(G) \ \longrightarrow\ \Phi_k^G(X)$$

that we called the reduced geometric fixed-point map above. The geometric fixed-point maps are compatible with inflations (Proposition 3.3.5 (iii)), so as G varies among finite groups, the reduced geometric fixed-point maps form a morphism of Out-modules. When we apply Proposition 3.4.26 to the underlying orthogonal G-spectrum of an orthogonal spectrum, it specializes to the following:

Corollary 4.5.37. *For every orthogonal spectrum X, every finite group G and every integer k the map*

$$\bar{\Phi} \ :\ \tau(\underline{\pi}_k(X))(G) \ \longrightarrow\ \Phi_k^G(X)$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. So for varying finite groups G, these maps form a rational isomorphism of Out*-functors $\tau(\underline{\pi}_k(X)) \cong \underline{\Phi}_k(X)$.*

As a corollary we obtain that the combined equivalence κ of triangulated categories (4.5.36) from the rational finite global homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ to the derived category of the abelian category mod- $\mathrm{Out}_{\mathbb{Q}}$ comes with a natural isomorphism

$$\Phi_*^G(X) \ \cong\ H_*(\kappa(X))\ ,$$

for every object X of $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$, between the geometric fixed-point homotopy groups and the homology Out-modules of $\kappa(X)$.

5
Ultra-commutative ring spectra

This chapter is devoted to ultra-commutative ring spectra, our model for extremely highly structured, multiplicative global stable homotopy types. On the point-set level, these objects are simply commutative orthogonal ring spectra; we use the term *'ultra-commutative'* to emphasize that we care about their homotopy theory with respect to multiplicative morphisms that are global equivalences. We refer to the introduction of Chapter 2 for further justification of the adjective 'ultra-commutative'. In short, the slogan "E_∞=commutative" is not true globally, and a strictly commutative multiplication encodes a large amount of additional structure that deserves a special name.

Section 5.1 introduces the formal setup for power operations on ultra-commutative ring spectra. We define global power functors as global Green functors equipped with additional power operations, satisfying a list of axioms reminiscent of the properties of the power maps $x \mapsto x^m$ in a commutative ring. We show in Theorem 5.1.11 that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R supports such power operations, and is an example of a global power functor.

Section 5.2 is primarily of algebraic nature, and gives both a monadic and a comonadic description of the category of global power functors. We introduce the comonad of 'exponential sequences' on the category of global Green functors, and show that its coalgebras are equivalent to global power functors. A formal consequence is that the category of global power functors has all limits and colimits, and that they are created in the category of global Green functors. We discuss localization of global Green functors and global power functors at a multiplicative subset of the underlying ring, including rationalization of global power functors.

In Section 5.3 we discuss various examples of global power functors, such as the Burnside ring global power functor, the global functor represented by an abelian compact Lie group, free global power functors, constant global power functors, and the complex representation ring global functor. In Section 5.4 we

set up the global model structure on the category of ultra-commutative ring spectra, compare Theorem 5.4.3. In Theorem 5.4.4 we calculate the algebra of natural operations on the 0th homotopy groups of ultra-commutative ring spectra: we show that these operations are freely generated by restrictions, transfers and power operations. Theorem 5.4.14 shows that every global power functor can be realized by an ultra-commutative ring spectrum.

5.1 Power operations

In this section we introduce the formal setup for encoding the power operations on ultra-commutative ring spectra. In Definition 5.1.6 we define global power functors, which are global Green functors equipped with additional power operations, satisfying a list of axioms reminiscent of the properties of the power maps $x \mapsto x^m$ in a commutative ring. Theorem 5.1.11 shows that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R supports power operations, and is an example of a global power functor. As we shall see in Theorem 5.4.4 below, all the natural operations on $\underline{\pi}_0(R)$ are generated by restrictions, transfers and power operations, so we are not missing any additional structure. Moreover, every global power functor is realized by an ultra-commutative Eilenberg-Mac Lane ring spectrum, see Theorem 5.4.14 below. For a different perspective of global power functors (restricted to finite groups), including the relationship to the concepts of λ-rings, τ-rings and β-rings, we refer the reader to Ganter's paper [60].

Definition 5.1.1. An *ultra-commutative ring spectrum* is a commutative orthogonal ring spectrum. We write *ucom* for the category of ultra-commutative ring spectra.

In Section 2.2 we introduced power operations and transfers on the equivariant homotopy sets of ultra-commutative monoids. For every ultra-commutative ring spectrum R, the orthogonal space $\Omega^\bullet R$ inherits a commutative multiplication, making it an ultra-commutative monoid (compare Example 4.1.16). Moreover, $\pi_0^G(\Omega^\bullet R) = \pi_0^G(R)$, so this endows the 0th stable equivariant homotopy groups $\pi_0^G(R)$ with multiplicative power operations and transfers, natural for homomorphisms of ultra-commutative ring spectra. Since these operations come from the multiplicative (as opposed to the 'additive' structure) of the ring spectrum, we now switch to a multiplicative notation and write

$$P^m \ : \ \pi_0^G(R) \longrightarrow \pi_0^{\Sigma_m \wr G}(R) \tag{5.1.2}$$

(instead of $[m]$) for the multiplicative power operations, and we write N_H^G (instead of tr_H^G) for multiplicative transfers. Multiplicative transfers are often re-

ferred to as *norm maps*, and that is also the terminology we will use. Since we will work with power operations a lot, we take the time to expand the definition: the operation P^m takes the class represented by a based G-map $f : S^V \longrightarrow R(V)$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} = (S^V)^{\wedge m} \xrightarrow{f^{\wedge m}} R(V)^{\wedge m} \xrightarrow{\mu_{V,\dots,V}} R(V^m)\ ,$$

where $\mu_{V,\dots,V}$ is the iterated, $(\Sigma_m \wr G)$-equivariant multiplication map of R.

Definition 5.1.3. A *global Green functor* is a commutative monoid in the category of global functors under the monoidal structure given by the box product of Construction 4.2.17. We write $\mathcal{G}l\mathcal{G}re$ for the category of global Green functors.

As we explain after Definition 4.2.19, the commutative multiplication on a global Green functor R can be made more explicit in two equivalent ways:

- as a commutative ring structure on the group $R(G)$ for every compact Lie group, subject to the requirement that all restrictions maps are ring homomorphisms and the transfer maps satisfy reciprocity;
- as a unit element $1 \in R(e)$ and biadditive, commutative, associative and unital external pairings $\times : R(G) \times R(K) \longrightarrow R(G \times K)$ that are morphisms of global functors in each variable separately.

We clarify next how the power operations of ultra-commutative ring spectra interact with the other structure on equivariant stable homotopy groups, such as the addition, restriction and transfer maps. For $m \geq 2$ the power operation P^m is *not* additive, but it satisfies various properties reminiscent of the map $x \mapsto x^m$ in a commutative ring. We formalize these properties into the concept of a *global power functor*. Conditions (i) through (vi) in the following definition express the fact that a global power functor has an underlying 'multiplicative' global power monoid, in the sense of Definition 2.2.8, if we forget the additive structure. The definition makes use of certain embeddings between products and wreath products:

$$\begin{array}{rcl} \Phi_{i,j}\ :\ (\Sigma_i \wr G) \times (\Sigma_j \wr G) & \longrightarrow & \Sigma_{i+j} \wr G \\ ((\sigma;\, g_1, \dots, g_i),\, (\sigma';\, g_{i+1}, \dots, g_{i+j})) & \longmapsto & (\sigma + \sigma';\, g_1, \dots, g_{i+j}) \end{array} \tag{5.1.4}$$

and

$$\begin{array}{rcl} \Psi_{k,m}\ :\ \Sigma_k \wr (\Sigma_m \wr G) & \longrightarrow & \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, h^1), \dots, (\tau_k;\, h^k)) & \longmapsto & (\sigma \natural (\tau_1, \dots, \tau_k);\, h^1 + \dots + h^k)\ . \end{array} \tag{5.1.5}$$

These monomorphisms were defined in Construction 2.2.3.

Definition 5.1.6. A *global power functor* is a global Green functor R equipped with maps

$$P^m \; : \; R(G) \longrightarrow R(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Unit) $P^m(1) = 1$ for the unit $1 \in R(e)$.

(ii) (Identity) $P^1 = \mathrm{Id}$ under the identification $G \cong \Sigma_1 \wr G$ sending g to $(1; g)$.

(iii) (Naturality) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and all $m \geq 1$ the relation

$$P^m \circ \alpha^* \; = \; (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps $R(G) \longrightarrow R(\Sigma_m \wr K)$.

(iv) (Multiplicativity) For all compact Lie groups G, all $m \geq 1$ and all classes $x, y \in R(G)$ the relation

$$P^m(x \cdot y) \; = \; P^m(x) \cdot P^m(y)$$

holds in the group $R(\Sigma_m \wr G)$.

(v) (Restriction) For all compact Lie groups G, all $m > k > 0$ and all $x \in R(G)$ the relation

$$\Phi^*_{k,m-k}(P^m(x)) \; = \; P^k(x) \times P^{m-k}(x)$$

holds in $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$.

(vi) (Transitivity) For all compact Lie groups G, all $k, m \geq 1$ and all $x \in R(G)$ the relation

$$\Psi^*_{k,m}(P^{km}(x)) \; = \; P^k(P^m(x))$$

holds in $R(\Sigma_k \wr (\Sigma_m \wr G))$.

(vii) (Additivity) For all compact Lie groups G, all $m \geq 1$, and all $x, y \in R(G)$ the relation

$$P^m(x + y) \; = \; \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}(P^k(x) \times P^{m-k}(y))$$

holds in $R(\Sigma_m \wr G)$, where $\mathrm{tr}_{k,m-k}$ is the transfer associated with the embedding $\Phi_{k,m-k} : (\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G) \longrightarrow \Sigma_m \wr G$ defined in (5.1.4). Here $P^0(x)$ is the multiplicative unit 1.

(viii) (Transfer) For every closed subgroup H of a compact Lie group G and every $m \geq 1$ the relation

$$P^m \circ \mathrm{tr}_H^G \; = \; \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \circ P^m$$

holds as maps $R(H) \longrightarrow R(\Sigma_m \wr G)$.

A *morphism* of global power functors is a morphism of global Green functors that also commutes with the power operations. We write $\mathcal{G}l\mathcal{P}ow$ for the category of global power functors.

In a global power functor the relation $P^m(0) = 0$ also holds for every $m \geq 1$ and all G. Indeed, the additivity and unit relations give

$$\sum_{k=1}^{m-1} \mathrm{tr}_{k,m-k}(P^k(0) \times 1) \ = \ P^m(0+1) - P^m(0) - P^m(1) \ = \ -P^m(0) \ .$$

Starting from $P^1(0) = 0$ this shows inductively that $P^m(0) = 0$.

Remark 5.1.7 (Global power functors versus global Tambara functors)**.** A global power functor gives rise to two underlying global power monoids, the *additive* and the *multiplicative* one. As we explained in Construction 2.2.29, applied to the multiplicative global power monoid, the power operations P^m lead to 'multiplicative transfers', $N_H^G : R(H) \longrightarrow R(G)$ that are called *norm maps*, for every subgroup H of finite index in G. For the convenience of the reader, we recall the construction of the norm maps. We suppose that $[G : H] = m$, and we choose an H-basis of G, i.e., an ordered m-tuple $\bar{g} = (g_1, \ldots, g_m)$ of elements in disjoint H-orbits such that

$$G \ = \ \bigcup\nolimits_{i=1}^m g_i H \ .$$

The wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of G, by the formula

$$(g_1, \ldots, g_m) \cdot (\sigma; h_1, \ldots, h_m) \ = \ (g_{\sigma(1)} h_1, \ldots, g_{\sigma(m)} h_m) \ .$$

We obtain a continuous homomorphism $\Psi_{\bar{g}} : G \longrightarrow \Sigma_m \wr H$ by requiring that

$$\gamma \cdot \bar{g} \ = \ \bar{g} \cdot \Psi_{\bar{g}}(\gamma) \ .$$

The norm $N_H^G : R(H) \longrightarrow R(G)$ is then the composite

$$R(H) \xrightarrow{P^m} R(\Sigma_m \wr H) \xrightarrow{\Psi_{\bar{g}}^*} R(G) \ .$$

Any other H-basis is of the form $\bar{g}\omega$ for a unique $\omega \in \Sigma_m \wr H$. We have $\Psi_{\bar{g}\omega} = c_\omega \circ \Psi_{\bar{g}}$ as maps $G \longrightarrow \Sigma_m \wr H$, where $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. Since inner automorphisms induce the identity in any Rep-functor, we have

$$\Psi_{\bar{g}}^* = \Psi_{\bar{g}\omega}^* \ : \ R(\Sigma_m \wr H) \ \longrightarrow \ R(G) \ .$$

So the norm N_H^G does not depend on the choice of basis $\bar{g}$.

The norms maps satisfy a number of important relations, by Proposition 2.2.30 applied to the multiplicative monoid of the global power functor R. There relations – turned into multiplicative notation – are as follows.

(i) (Transitivity) We have $N^G_G = \mathrm{Id}_{R(G)}$ and for nested subgroups $H \subseteq G \subseteq F$ of finite index the relation

$$N^F_G \circ N^G_H \;=\; N^F_H$$

holds as maps $R(H) \longrightarrow R(F)$.

(ii) (Multiplicative double coset formula) For every subgroup K of G (not necessarily of finite index) the relation

$$\mathrm{res}^G_K \circ N^G_H \;=\; \prod_{[g]\in K\backslash G/H} N^K_{K\cap {}^gH} \circ g_\star \circ \mathrm{res}^H_{K^g\cap H}$$

holds as maps $R(H) \longrightarrow R(K)$. Here $[g]$ runs over a set of representatives of the finite set of K-H-double cosets.

(iii) (Inflation) For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation

$$\alpha^* \circ N^G_H \;=\; N^K_L \circ (\alpha|_L)^*$$

holds as maps from $R(H) \longrightarrow R(K)$, where $L = \alpha^{-1}(H)$.

(iv) For every $m \geq 1$ the mth power operation can be recovered as

$$P^m \;=\; N^{\Sigma_m \wr G}_K \circ q^* \,, \tag{5.1.8}$$

where K is the subgroup of $\Sigma_m \wr G$ consisting of all $(\sigma;\, g_1, \dots, g_m)$ such that $\sigma(m) = m$ and $q : K \longrightarrow G$ is defined by $q(\sigma;\, g_1, \dots, g_m) = g_m$.

In particular, the power operations define the norm maps, but they can also be reconstructed from the norm maps. So the information in a global power functor could be packaged in an equivalent but different way using norm maps instead of power operations. The algebraic structure that arises then is the global analog of a *TNR-functor* in the sense of Tambara [171], nowadays also called a *Tambara functor*; here the acronym stands form 'Transfer, Norm and Restriction'. This observation can be stated as an equivalence of categories between global power functors and a certain category of 'global TNR functors'; we shall not pursue this further. Our reason for favoring power operations over norm maps is that they satisfy explicit and intuitive formulas with respect to the rest of the structure (restriction, transfer, sum, product,...). The norm maps also satisfy universal formulas when applied to sums and transfers, but I find them harder to describe and to remember.

For a fixed finite group G, Brun [31, Sec. 7.2] has constructed norm maps on the 0th equivariant homotopy group Mackey functor of every commutative orthogonal G-ring spectrum, and he showed that this structure is a TNR-functor. So when restricted to finite groups, the global power functor structure on $\underline{\pi}_0(R)$ for an ultra-commutative ring spectrum, obtained in the following Theorem

5.1.11, could also be deduced by using Brun's TNR-structure for the underlying orthogonal G-ring spectrum R_G, and then turning the norm maps into power operations as in (5.1.8). However, Brun's construction is rather indirect and this would hide the simple and explicit nature of the power operations.

In order to show that the 0th equivariant homotopy group functor of an ultra-commutative ring spectrum satisfies the transfer axiom (viii) of a global power functor, we study the interplay between power operations, the Wirthmüller isomorphism and the degree shifting transfer. To state the results we first have to generalize power operations from equivariant homotopy groups to equivariant homology theories.

We let R be an orthogonal spectrum, G a compact Lie group and A a based G-space. We define the G-equivariant R-homology group of A as the group

$$R_0^G(A) \;=\; \pi_0^G(R \wedge A) \;=\; \operatorname{colim}_{V \in s(\mathcal{U}_G)} \, [S^V, R(V) \wedge A]^G \; .$$

Every continuous group homomorphism $\alpha : K \longrightarrow G$ induces a restriction homomorphism

$$\alpha^* \; : \; R_0^G(A) \;\longrightarrow\; R_0^K(\alpha^*(A))$$

that generalizes the restriction homomorphism $\alpha^* : \pi_0^G(R) \longrightarrow \pi_0^K(R)$. Again α^* is defined by applying restriction of scalars to any representative of a given equivariant homology class.

Now we let R be an orthogonal ring spectrum (not necessarily commutative). Then the equivariant homology theories represented by R inherit multiplications in the form of bilinear maps

$$\times \; : \; R_0^G(A) \times R_0^G(B) \;\longrightarrow\; R_0^G(A \wedge B) \; .$$

We define this pairing simply as the composite

$$\begin{aligned} R_0^G(A) \times R_0^G(B) \;=\; \pi_0^G(R \wedge A) \times \pi_0^G(R \wedge B) \;&\xrightarrow{\;\times\;}\; \pi_0^G((R \wedge A) \wedge (R \wedge B)) \\ &\xrightarrow{\;\mu_*\;}\; \pi_0^G(R \wedge A \wedge B) \;=\; R_0^G(A \wedge B) \, , \end{aligned}$$

where the first map is the homotopy group pairing of Construction 3.5.12 and $\mu : (R \wedge A) \wedge (R \wedge B) \longrightarrow R \wedge A \wedge B$ stems from the multiplication of R. For $A = B = S^0$ this construction reduces to the pairings of equivariant homotopy groups (3.5.16).

Now we suppose that the ring spectrum R is ultra-commutative. Given a based G-space A, we write $A^{(m)} = A^{\wedge m}$ for its m-fold smash power, which is naturally a based $(\Sigma_m \wr G)$-space. Then we define the mth power operation

$$P^m \; : \; R_0^G(A) \;\longrightarrow\; R_0^{\Sigma_m \wr G}(A^{(m)})$$

by the obvious generalization of (5.1.2): the operation P^m takes the class represented by a based G-map $f : S^V \longrightarrow R(V) \wedge A$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} = (S^V)^{(m)} \xrightarrow{f^{(m)}} (R(V) \wedge A)^{(m)}$$
$$\xrightarrow{\text{shuffle}} R(V)^{(m)} \wedge A^{(m)} \xrightarrow{\mu_{V,\dots,V} \wedge A^{(m)}} R(V^m) \wedge A^{(m)} ,$$

where $\mu_{V,\dots,V}$ is the iterated multiplication map of R. We omit the straightforward verification that the power operations in equivariant R-homology are compatible with restriction maps: for every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and every based G-space A, the relation

$$P^m \circ \alpha^* \;=\; (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps from $R_0^G(A)$ to $R_0^{\Sigma_m \wr K}(\alpha^*(A)^{(m)})$, exploiting that $(\Sigma_m \wr \alpha)^*(A^{(m)}) = \alpha^*(A)^{(m)}$ as $(\Sigma_m \wr K)$-spaces.

The following proposition makes precise how the power operations in equivariant R-homology interact with the Wirthmüller isomorphism of Theorem 3.2.15. To give the precise statement we have to introduce additional notation. We let H be a closed subgroup of a compact Lie group G. As before we let $L = T_{eH}(G/H)$ denote the tangent H-representation, the tangent space of G/H at the distinguished coset eH. We write

$$\gamma \;:\; (G/H)^m \;\cong\; (\Sigma_m \wr G)/(\Sigma_m \wr H)\,,\; (g_1H,\dots,g_mH) \;\longmapsto\; (1;\, g_1,\dots,g_m)\cdot(\Sigma_m \wr H)$$

for the distinguished $(\Sigma_m \wr G)$-equivariant diffeomorphism. The differential of γ at $(eH,\dots,eH)$ is a $(\Sigma_m \wr H)$-equivariant linear isometry

$$(d\gamma)_{(eH,\dots,eH)} \;:\; L^m \;\cong\; T_{e(\Sigma_m \wr H)}\big((\Sigma_m \wr G)/(\Sigma_m \wr H)\big)\,.$$

In the next proposition and its corollaries, we will use this equivariant isometry to identify L^m with the tangent representation of $\Sigma_m \wr H$ inside $\Sigma_m \wr G$.

Proposition 5.1.9. *Let R be an ultra-commutative ring spectrum and H a closed subgroup of a compact Lie group G. Then the following diagram commutes*

$$\begin{tikzcd}
R_0^G(G/H_+) \arrow[r, "\mathrm{Wirth}_H^G", "\cong"'] \arrow[d, "P^m"'] & R_0^H(S^L) \arrow[d, "P^m"] \\
R_0^{\Sigma_m \wr G}((G/H)^m_+) \arrow[d, "\gamma_*"', "\cong"] & R_0^{\Sigma_m \wr H}((S^L)^{(m)}) \arrow[d, "\cong"] \\
R_0^{\Sigma_m \wr G}((\Sigma_m \wr G)/(\Sigma_m \wr H)_+) \arrow[r, "\cong", "\mathrm{Wirth}_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & R_0^{\Sigma_m \wr H}(S^{L^m})
\end{tikzcd}$$

where the horizontal maps are the respective Wirthmüller isomorphisms.

Proof We choose a slice as in the definition of the Wirthmüller map in Construction 3.2.1, i.e., a smooth embedding $s : D(L) \longrightarrow G$ that satisfies

$$s(0) = 1 , \quad s(h \cdot l) = h \cdot s(l) \cdot h^{-1} \qquad \text{and} \qquad s(-l) = s(l)^{-1}$$

for all $(l, h) \in D(L) \times H$, and such that the differential at 0 of the composite

$$D(L) \xrightarrow{s} G \xrightarrow{\text{proj}} G/H$$

is the identity. The collapse map

$$\lambda_H^G = l_H^G/H : G/H_+ \longrightarrow S^L$$

is then given by the formula

$$\lambda_H^G(gH) = \begin{cases} l/(1-|l|) & \text{if } g = s(l) \cdot h \text{ with } (l, h) \in D(L) \times H, \text{ and} \\ * & \text{if } g \text{ is not of this form.} \end{cases}$$

We define a slice for the pair $(\Sigma_m \wr G, \Sigma_m \wr H)$ from the slice s for the pair (G, H), namely as the smooth embedding

$$\bar{s} : D(L^m) \longrightarrow \Sigma_m \wr G , \quad \bar{s}(l_1, \dots, l_m) = (1; s(l_1), \dots, s(l_m)) .$$

Clearly, $\bar{s}(0, \dots, 0)$ is the multiplicative unit,

$$\begin{aligned} \bar{s}(-l_1, \dots, -l_m) &= (1; s(-l_1), \dots, s(-l_m)) \\ &= (1; s(l_1)^{-1}, \dots, s(l_m)^{-1}) = \bar{s}(-l_1, \dots, -l_m)^{-1} , \end{aligned}$$

and

$$\begin{aligned} \bar{s}((\sigma^{-1}; h_1, \dots, h_m) \cdot (l_1, \dots, l_m)) &= \bar{s}(h_{\sigma(1)} l_{\sigma(1)}, \dots, h_{\sigma(m)} l_{\sigma(m)}) \\ &= (1; s(h_{\sigma(1)} l_{\sigma(1)}), \dots, s(h_{\sigma(m)} l_{\sigma(m)})) \\ &= (1; h_{\sigma(1)} s(l_{\sigma(1)}) h_{\sigma(1)}^{-1}, \dots, h_{\sigma(m)} s(l_{\sigma(m)}) h_{\sigma(m)}^{-1}) \\ &= (\sigma^{-1}; h_1, \dots, h_m) \cdot (1; s(l_1), \dots, s(l_m)) \cdot (\sigma; h_{\sigma(1)}^{-1}, \dots, h_{\sigma(m)}^{-1}) \\ &= (\sigma^{-1}; h_1, \dots, h_m) \cdot \bar{s}(l_1, \dots, l_m) \cdot (\sigma^{-1}; h_1, \dots, h_m)^{-1} , \end{aligned}$$

for all $(l_1, \dots, l_m) \in D(L^m)$ and all $(\sigma^{-1}; h_1, \dots, h_m) \in \Sigma_m \wr H$. Finally, the differential of the composite

$$D(L^m) \xrightarrow{\bar{s}} \Sigma_m \wr G \xrightarrow{\text{proj}} (\Sigma_m \wr G)/(\Sigma_m \wr H) \xrightarrow{\gamma^{-1}} (G/H)^m$$

is the identity, so we have indeed defined a slice. We let

$$\lambda_{\Sigma_m \wr H}^{\Sigma_m \wr G} : (\Sigma_m \wr G)/(\Sigma_m \wr H)_+ \longrightarrow S^{L^m}$$

denote the collapse map based on the slice $\bar{s}$. The composite

$$(G/H)^m_+ \xrightarrow{\gamma_+} (\Sigma_m \wr G)/(\Sigma_m \wr H)_+ \xrightarrow{\lambda^{\Sigma_m \wr G}_{\Sigma_m \wr H}} S^{L^m}$$

sends a point $(s(l_1)H, \ldots, s(l_m)H)$ with $(l_1, \ldots, l_m) \in D(L^m)$ to the point

$$\left(1 - \sqrt{|l_1|^2 + \cdots + |l_m|^2}\right)^{-1} \cdot (l_1, \ldots, l_m) ;$$

all other points of $(G/H)^m$ are sent to the basepoint at infinity. A scaling homotopy thus witnesses that the following diagram commutes up to $(\Sigma_m \wr H)$-equivariant based homotopy:

$$\begin{tikzcd}
(G/H)^m_+ \arrow[r, "(\lambda^G_H)^{(m)}"] \arrow[d, "\gamma_+"', "\cong"] & (S^L)^{(m)} \arrow[d, "\cong"] \\
(\Sigma_m \wr G)/(\Sigma_m \wr H)_+ \arrow[r, "\lambda^{\Sigma_m \wr G}_{\Sigma_m \wr H}"'] & S^{L^m}
\end{tikzcd}$$

Now we contemplate the diagram:

$$\begin{tikzcd}
R^G_0(G/H_+) \arrow[r, "\mathrm{res}^G_H"] \arrow[d, "P^m"'] & R^H_0(G/H_+) \arrow[r, "(\lambda^G_H)_*"] \arrow[d, "P^m"'] & R^H_0(S^L) \arrow[d, "P^m"] \\
R^{\Sigma_m \wr G}_0((G/H)^m_+) \arrow[r, "\mathrm{res}^{\Sigma_m \wr G}_{\Sigma_m \wr H}"] \arrow[d, "(\gamma_+)_*"', "\cong"] & R^{\Sigma_m \wr H}_0((G/H)^m_+) \arrow[r, "((\lambda^G_H)^m)_*"] \arrow[d, "(\gamma_+)_*"', "\cong"] & R^{\Sigma_m \wr H}_0((S^L)^{(m)}) \arrow[d, "\cong"] \\
R^{\Sigma_m \wr G}_0((\Sigma_m \wr G)/(\Sigma_m \wr H)_+) \arrow[r, "\mathrm{res}^{\Sigma_m \wr G}_{\Sigma_m \wr H}"'] & R^{\Sigma_m \wr H}_0((\Sigma_m \wr G)/(\Sigma_m \wr H)_+) \arrow[r, "(\lambda^{\Sigma_m \wr G}_{\Sigma_m \wr H})_*"'] & R^{\Sigma_m \wr H}_0(S^{L^m})
\end{tikzcd}$$

The two squares on the left commute by compatibility of power operations with restriction and by naturality of restriction. The upper right square commutes by naturality of power operations. The lower right square commutes by the previous paragraph. Since the upper and lower horizontal composites are the respective Wirthmüller maps, this proves the proposition. □

A direct consequence of the previous proposition is that power operations are compatible with dimension shifting and degree zero transfer maps.

Corollary 5.1.10. *Let R be an ultra-commutative ring spectrum and H a closed subgroup of a compact Lie group G. Then the following two diagrams*

commute:

$$\begin{tikzcd}
R_0^H(S^L) \arrow[rr, "\mathrm{Tr}_H^G"] \arrow[d, "P^m"'] & & \pi_0^G(R) \arrow[dd, "P^m"] \\
R_0^{\Sigma_m \wr H}((S^L)^{(m)}) \arrow[d, "\cong"'] & & \\
R_0^{\Sigma_m \wr H}(S^{L^m}) \arrow[rr, "\mathrm{Tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & & \pi_0^{\Sigma_m \wr G}(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_0^H(R) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "P^m"'] & \pi_0^G(R) \arrow[d, "P^m"] \\
\pi_0^{\Sigma_m \wr H}(R) \arrow[r, "\mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & \pi_0^{\Sigma_m \wr G}(R)
\end{tikzcd}$$

Proof By Theorem 3.2.15, the external transfer

$$G \ltimes_H - \;:\; R_0^H(S^L) \;=\; \pi_0^H(R \wedge S^L) \;\longrightarrow\; \pi_0^G(R \wedge G/H_+) \;=\; R_0^G(G/H_+)$$

is inverse to the Wirthmüller isomorphism, up to the effect of the 'negative' map of S^L. The identification $(S^L)^{(m)} \cong S^{L^m}$ takes $(S^{-\mathrm{Id}_L})^{(m)}$ to $S^{-\mathrm{Id}_{L^m}}$, so the following diagram commutes by naturality of power operations:

$$\begin{tikzcd}
R_0^H(S^L) \arrow[r, "\varepsilon_L", "\cong"'] \arrow[d, "P^m"'] & R_0^H(S^L) \arrow[d, "P^m"] \\
R_0^{\Sigma_m \wr H}((S^L)^{(m)}) \arrow[d, "\cong"'] & R_0^{\Sigma_m \wr H}((S^L)^{(m)}) \arrow[d, "\cong"] \\
R_0^{\Sigma_m \wr H}(S^{L^m}) \arrow[r, "\cong", "\varepsilon_{L^m}"'] & R_0^{\Sigma_m \wr H}(S^{L^m})
\end{tikzcd}$$

Stacking this diagram next to the commutative diagram of Wirthmüller isomorphisms given in Proposition 5.1.9, and reading the composite diagram backwards yields the commutativity of the left part of the following diagram:

$$\begin{tikzcd}
R_0^H(S^L) \arrow[r, "G \ltimes_H -"'] \arrow[rr, bend left, "\mathrm{Tr}_H^G"] \arrow[d, "P^m"'] & R_0^G((G/H)_+) \arrow[r, "(p_+)_*"'] \arrow[d, "P^m"] & \pi_0^G(R) \arrow[d, "P^m"] \\
R_0^{\Sigma_m \wr H}((S^L)^{(m)}) \arrow[d, "\cong"'] & R_0^{\Sigma_m \wr G}((G/H)_+^m) \arrow[r, "(p_+^m)_*"'] \arrow[d, "\cong"', "\gamma_*"] & \pi_0^{\Sigma_m \wr G}(R) \arrow[d, equal] \\
R_0^{\Sigma_m \wr H}(S^{L^m}) \arrow[r, "\Sigma_m \wr G \ltimes_{\Sigma_m \wr H} -"] \arrow[rr, bend right, "\mathrm{Tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & R_0^{\Sigma_m \wr G}((\Sigma_m \wr G)/(\Sigma_m \wr H)_+) \arrow[r, "(p_+)_*"] & \pi_0^{\Sigma_m \wr G}(R)
\end{tikzcd}$$

The dimension shifting transfer $\mathrm{Tr}_H^G : R_0^H(S^L) \longrightarrow \pi_0^G(R)$ is the composite of the external transfer and the effect of the unique G-map $p : G/H \longrightarrow *$. So the

first claim follows from the commutativity of the right part of the above diagram. The degree zero transfer is obtained from the dimension shifting transfer by pre-composing with the effect of the map $S^0 \longrightarrow S^L$, the inclusion of the origin into the tangent representation. If we raise the inclusion of the origin of S^L to the mth power, the canonical homeomorphism $(S^L)^{(m)} \longrightarrow S^{L^m}$ identifies it with the inclusion of the origin of S^{L^m}. So the power operations are also compatible with degree zero transfers. □

Much of the next result is contained, at least implicitly, in Greenlees' and May's construction of norm maps [68, Sec. 7-9], simply because an ultra-commutative ring spectrum is an example of a '$\mathcal{GI}_*$-FSP' in the sense of [68, Def. 5.5].

Theorem 5.1.11. *Let R be an ultra-commutative ring spectrum. The power operations* (5.1.2) *make the global functor $\underline{\pi}_0(R)$ a global power functor.*

Proof The properties (i) through (vi) only involve the multiplication, power operations and restriction maps, so they are special cases of Proposition 2.2.14 for the multiplicative ultra-commutative monoid $\Omega^\bullet R$. The transfer relation (viii) is proved in Corollary 5.1.10. The most involved argument remaining is required for the additivity formula (vii), identifying the behavior of power operations on sums.

We first show an external version of the additivity relation. We consider two equivariant homology classes $x, y \in \pi_0^G(R)$. Since equivariant homotopy groups take wedges to direct sums there is a unique class

$$x \oplus y \ \in \ \pi_0^G(R \wedge \{1,2\}_+)$$

such that

$$p^1_*(x \oplus y) \ = \ x \qquad \text{and} \qquad p^2_*(x \oplus y) \ = \ y\ ,$$

where $p^1, p^2 : \{1,2\}_+ \longrightarrow S^0$ are the projections determined by $(p^i)^{-1}(0) = \{i\}$. We write $\Sigma_{k,m-k} \wr G$ for the image of the homomorphism $\Phi_{k,m-k} : (\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G) \longrightarrow \Sigma_m \wr G$ and let

$$\psi^k \ : \ (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G) \ \longrightarrow \ \{1,2\}^m$$

be the embedding of $(\Sigma_m \wr G)$-sets that sends the preferred coset $e(\Sigma_{k,m-k} \wr G)$ to the point

$$(\underbrace{1,\dots,1}_{k},\underbrace{2,\dots,2}_{m-k}) \ \in \ \{1,2\}^m\ .$$

Here $\Sigma_m \wr G$ acts on $\{1,2\}^m$ through the projection to Σ_m, by permutation of

coordinates. The orbits of the $(\Sigma_m \wr G)$-set $\{1,2\}^m$ are precisely the images of the maps $\psi^0, \ldots, \psi^m$. We will show the relation

$$P^m(x \oplus y) \;=\; \sum_{k=0}^{m} (\psi^k_+)_*((\Sigma_m \wr G) \ltimes_{\Sigma_{k,m-k}\wr G} (P^k(x) \times P^{m-k}(y))) \qquad (5.1.12)$$

in the group $\pi_0^{\Sigma_m\wr G}(R \wedge \{1,2\}^m_+)$. Since $\pi_0^{\Sigma_m\wr G}(R \wedge -)$ is additive on wedges, it suffices to show the relation after projection to each $(\Sigma_m \wr G)$-orbit of $\{1,2\}^m$. So we let

$$\bar{\psi}^k \;:\; \{1,2\}^m_+ \;\longrightarrow\; (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+$$

be the right inverse to ψ^k_+ that sends the other orbits to the basepoint, i.e., $\bar{\psi}^k \circ \psi^j_+$ is constant for $j \neq k$. The relation (5.1.12) thus follows if we can show

$$\bar{\psi}^k_*(P^m(x \oplus y)) \;=\; (\Sigma_m \wr G) \ltimes_{\Sigma_{k,m-k}\wr G} (P^k(x) \times P^{m-k}(y)) \qquad (5.1.13)$$

in the group

$$\pi_0^{\Sigma_m\wr G}(R \wedge (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+)$$

for all $0 \leq k \leq m$. We apply the Wirthmüller isomorphism, i.e., the composite

$$\pi_0^{\Sigma_m\wr G}(R \wedge (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+) \xrightarrow{\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_{k,m-k}\wr G}}$$
$$\pi_0^{\Sigma_{k,m-k}\wr G}(R \wedge (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+) \xrightarrow{(l_k)_*} \pi_0^{\Sigma_{k,m-k}\wr G}(R)\,.$$

Here $l_k : (\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+ \longrightarrow S^0$ is the projection to the preferred coset. We obtain

$$\begin{aligned}
(l_k)_*(\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_{k,m-k}\wr G}(\bar{\psi}^k_*(P^m(x \oplus y)))) \;&=\; (l_k \circ \bar{\psi}^k)_*(\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_{k,m-k}\wr G}(P^m(x \oplus y))) \\
&=\; ((p^1)^{(k)} \wedge (p^2)^{(m-k)})_*(P^k(x \oplus y) \times P^{m-k}(x \oplus y)) \\
&=\; P^k(p^1_*(x \oplus y)) \times P^{m-k}(p^2_*(x \oplus y)) \;=\; P^k(x) \times P^{m-k}(y)\,.
\end{aligned}$$

Since the Wirthmüller isomorphism is inverse to the external transfer (compare Theorem 3.2.15), this proves (5.1.13), and hence (5.1.12).

Now we obtain the additivity relation by naturality for the fold map $\nabla : \{1,2\}_+ \longrightarrow S^0$ with $\nabla(1) = \nabla(2) = 0$. Then

$$\begin{aligned}
P^m(x + y) \;&=\; P^m(\nabla_*(x \oplus y)) \;=\; (\nabla^{(m)})_*(P^m(x \oplus y)) \\
\overset{(5.1.12)}&{=}\; \sum_{k=0}^{m} (\nabla^{(m)} \circ \psi^k_+)_*((\Sigma_m \wr G) \ltimes_{\Sigma_{k,m-k}\wr G} (P^k(x) \times P^{m-k}(y))) \\
&=\; \sum_{k=0}^{m} \mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_{k,m-k}\wr G}(P^k(x) \times P^{m-k}(y))\,.
\end{aligned}$$

The third equation uses that the composite

$$(\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)_+ \xrightarrow{\psi^k_+} \{1,2\}^m_+ \xrightarrow{\nabla^{(m)}} (S^0)^{(m)} = S^0$$

sends all of $(\Sigma_m \wr G)/(\Sigma_{k,m-k} \wr G)$ to the non-basepoint. □

Remark 5.1.14 (Relation to classical power operations)**.** We owe an explanation how power operations for ultra-commutative ring spectra refine the classical power operations defined in the (non-equivariant) cohomology theory represented by an H_∞-ring spectrum. We recall from [34, I.§4] that an H_∞-structure is an algebra structure over the monad

$$L\mathbb{P} \ : \ \mathcal{SH} \longrightarrow \mathcal{SH}$$

on the stable homotopy category that can be obtained by deriving the 'symmetric algebra' monad

$$\mathbb{P} \ : \ \mathcal{S}p \longrightarrow \mathcal{S}p$$

on the category of orthogonal spectra (whose algebras are commutative orthogonal ring spectra). This is not the full truth, because Bruner, May, McClure and Steinberger use a different model for the stable homotopy category, so strictly speaking one would have to translate the relevant parts of [34] to the context of orthogonal spectra. If we did that, the derived functor of the m-symmetric power of orthogonal spectra would be modeled by the mth *extended power*

$$D_m X \ = \ (E\Sigma_m)_+ \wedge_{\Sigma_m} X^{\wedge m} \ .$$

Specifying an H_∞-structure on an orthogonal spectrum E thus amounts to specifying morphisms, in the non-equivariant stable homotopy category,

$$\mu_m \ : \ D_m E \longrightarrow E$$

from the mth extended power to E; the algebra structure over the monad $L\mathbb{P}$ then translates into a specific collection of relations among the morphisms μ_m that are spelled out in [34, Ch. I Def. 3.1].

For every space A, the H_∞-structure gives rise to power operations

$$\mathcal{P}_m \ : \ E^0(A) \longrightarrow E^0(B\Sigma_m \times A) \tag{5.1.15}$$

in E-cohomology defined in [34, Ch. I Def. 4.1] as the following composite:

$$\begin{aligned} E^0(A) \ = \ [\Sigma^\infty_+ A, E] \ &\xrightarrow{D_m} \ [D_m(\Sigma^\infty_+ A), D_m E] \ \xrightarrow{[D_m(\Sigma^\infty_+ A), \mu_n]} \ [D_m(\Sigma^\infty_+ A), E] \\ &\cong \ [\Sigma^\infty_+(D_m A), E] \ = \ E^0(D_m A) \ \xrightarrow{E^0(E\Sigma_m \times_{\Sigma_m} \Delta)} \ E^0(B\Sigma_m \times A) \ . \end{aligned}$$

Here $[-,-]$ denotes the morphism group in the stable homotopy category $\mathcal{SH}$, $D_m A = E\Sigma_m \times_{\Sigma_m} A^m$ is the space level extended power, and $\Delta : A \longrightarrow A^m$ is the

diagonal. Depending on the context, the power operations (5.1.15) are often processed further; in favorable cases, $E^0(B\Sigma_m \times A)$ can be explicitly described as a functor of $E^0(A)$, and the power operations can be translated into a specific kind of algebraic structure.

Now we let R be an ultra-commutative ring spectrum. Then the underlying H_∞-structure is given by the composite morphism

$$D_m R \;=\; (E\Sigma_m)_+ \wedge_{\Sigma_m} R^{\wedge m} \;\longrightarrow\; \Sigma_m \backslash R^{\wedge m} \;\xrightarrow{\text{mult}}\; R$$

where the first morphism collapses $E\Sigma_m$ to a point and the second map is induced by the iterated multiplication $R^{\wedge m} \longrightarrow R$. The definition (5.1.2) of the power operations on equivariant homotopy groups directly extends to power operations

$$P^m \;:\; R^0_G(A) \;\longrightarrow\; R^0_{\Sigma_m \wr G}(A^m)$$

in the equivariant cohomology of a G-space A. The operation P^m takes the class represented by a based G-map $f : S^V \wedge A_+ \longrightarrow R(V)$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} \wedge A^m_+ \;\cong\; (S^V \wedge A_+)^{\wedge m} \;\xrightarrow{f^{\wedge m}}\; R(V)^{\wedge m} \;\xrightarrow{\mu_{V,\dots,V}}\; R(V^m)\,,$$

where $\mu_{V,\dots,V}$ is the iterated, $(\Sigma_m \wr G)$-equivariant multiplication map of R. A forgetful homomorphism

$$R^0_G(A) \;\longrightarrow\; R^0(EG \times_G A)$$

is defined as the composite

$$\begin{aligned} R^0_G(A) \;=\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, R]\!] \;&\xrightarrow{U}\; [U(\Sigma^\infty_+ \mathbf{L}_{G,V} A), R] \\ &=\; [\Sigma^\infty_+(EG \times_G A), R] \;=\; R^0(EG \times_G A)\,, \end{aligned}$$

where $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ is the forgetful functor from the global stable homotopy category to the non-equivariant stable homotopy category. Then the following diagram commutes:

$$\begin{tikzcd}
R^0(A) \arrow[r, "D_m"] \arrow[d, "P^m"'] & {[D_m(\Sigma^\infty_+ A), D_m R]} \arrow[d, "(\mu_m)_*"] \\
R^0_{\Sigma_m}(A^m) \arrow[r, "U"] \arrow[d, "R^0_{\Sigma_m}(\Delta)"'] & R^0(E\Sigma_m \times_{\Sigma_m} A) \arrow[d, "R^0(E\Sigma_m \times_{\Sigma_m} \Delta)"] \\
R^0_{\Sigma_m}(A) \arrow[r, "U"'] & R^0(B\Sigma_m \times A)
\end{tikzcd}$$

The composite through the upper right corner is the H_∞-power operation $\mathcal{P}_m$;

so this diagram makes precise in which way power operations for ultra-commutative ring spectra refine power operations for H_∞-ring spectra.

Remark 5.1.16 (G_∞-ring spectra)**.** As we recalled in the previous remark, non-equivariant power operations on a ring valued cohomology theory already arise from a weaker structure than a commutative multiplication (or equivalently E_∞-multiplication): all that is needed is an H_∞-structure, compare [34, I.§4]. This suggests a global analog of an H_∞-structure that, for lack of better name, we call a *G_∞-structure*. For the formal definition we exploit the global model structure for ultra-commutative ring spectra, to be established in Theorem 5.4.3 below. This model structure is obtained by lifting the positive global model structure on the category of orthogonal spectra (see Proposition 4.3.33) along the forgetful functor. In particular, the free and forgetful functor form a Quillen adjoint functor pair:

$$\mathbb{P} \,:\, \mathcal{S}p \rightleftarrows ucom \,:\, U$$

Every Quillen adjoint pair between model categories derives to an adjoint functor pair between the homotopy categories, see [134, I.4, Thm. 3] or [80, Lemma 1.3.10]. In our situation this provides the derived adjunction:

$$L_{\mathrm{gl}}\mathbb{P} \,:\, \mathcal{GH} \rightleftarrows \mathrm{Ho}(ucom) \,:\, \mathrm{Ho}(U)$$

Since the forgetful functor is fully homotopical, it does not even have to be derived. The composite

$$\mathbb{G} \;=\; \mathrm{Ho}(U)\circ L_{\mathrm{gl}}\mathbb{P} \,:\, \mathcal{GH} \longrightarrow \mathcal{GH}$$

is then canonically a monad on the global stable homotopy category, whose algebras we call G_∞-ring spectra.

The underlying non-equivariant homotopy type of a G_∞-ring spectrum comes with an H_∞-structure, so we arrive at a square of forgetful functors between categories of structured ring spectra with different degrees of commutativity:

$$\begin{tikzcd}
\mathrm{Ho}(ucom) \arrow[r] \arrow[d] & (G_\infty\text{-ring spectra}) \arrow[d] \\
\mathrm{Ho}^{\pi_*\text{-iso}}(\text{comm. ring spectra}) \arrow[r] & (H_\infty\text{-ring spectra})
\end{tikzcd}$$

We emphasize that, like H_∞-ring spectra, the category of G_∞-ring spectra is not the homotopy category of any natural model category.

Example 5.1.17 (Units of an ultra-commutative ring spectrum)**.** In Example 2.2.16 we defined the naive units of an orthogonal monoid space. When R is an

orthogonal ring spectrum, then the naive units of the multiplicative orthogonal monoid space $\Omega^\bullet R$ satisfy

$$\pi_0^G((\Omega^\bullet R)^{n\times}) \;=\; \{x \in \pi_0^G(R) \mid \mathrm{res}^G_e(x) \text{ is a unit in } \pi_0^e(R)\} ,$$

the multiplicative submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. One should beware that these naive units may contain non-invertible elements, i.e., the orthogonal monoid space $(\Omega^\bullet R)^{n\times}$ need not be group-like.

When the ring spectrum R is ultra-commutative, then there is a more refined construction

$$GL_1(R) \;=\; (\Omega^\bullet R)^\times ,$$

the *global units* of R. Indeed, if R is ultra-commutative, then $\Omega^\bullet R$ is an ultra-commutative monoid, so we can form the 'true' global units, the homotopy fiber of the multiplication morphism, see Construction 2.5.18. Then $GL_1(R)$ is a group-like ultra-commutative monoid and for every compact Lie group G,

$$\pi_0^G(GL_1(R)) \;=\; (\pi_0^G(R))^\times ,$$

the multiplicative submonoid of units of the commutative ring $\pi_0^G(R)$. Moreover, the power operations in $\underline{\pi}_0(R)$ correspond to the power operations in $\underline{\pi}_0(GL_1(R))$.

In the non-equivariant context, $GL_1(R)$ is an infinite loop space, i.e., weakly equivalent to the 0th space of an Ω-spectrum of units. This fact has a global generalization as follows. As we hope to explain elsewhere, every ultra-commutative monoid M has a *global delooping* $\mathbf{B}M$, an orthogonal spectrum that is a $\mathcal{F}in$-global Ω-spectrum. It also comes with a natural morphism of orthogonal spaces $M \longrightarrow \Omega^\bullet(\mathbf{B}M)$ that is a $\mathcal{F}in$-global equivalence whenever M is group-like. Since $GL_1(R)$ is an ultra-commutative monoid, it has a global delooping

$$gl_1(R) \;=\; \mathbf{B}(GL_1(R)) .$$

Since the ultra-commutative monoid $GL_1(R)$ is group-like, the morphism

$$\xi \;:\; GL_1(R) \;\longrightarrow\; \Omega^\bullet(\mathbf{B}(GL_1(R))) \;=\; \Omega^\bullet(gl_1(R))$$

is a $\mathcal{F}in$-global equivalence of orthogonal spaces. For every compact Lie group G, this induces a map

$$\begin{aligned} (\pi_0^G(R))^\times \;&\cong\; \pi_0^G(GL_1(R)) \\ &\cong\; \pi_0^G((GL_1(R))^c) \xrightarrow{\underline{\pi}_0(\xi)} \pi_0^G(\Omega^\bullet(gl_1(R))) \;=\; \pi_0^G(gl_1(R)) . \end{aligned}$$

These maps are compatible with restriction along continuous homomorphisms

and they are bijective whenever G is finite. Moreover, the maps take the multiplication in $(\underline{\pi}_0(R))^\times$ to the addition in $\underline{\pi}_0(gl_1(R))$, and they match norm operations with finite index transfers

Remark 5.1.18 (Picard groups)**.** The global units $GL_1(R)$ of an ultra-commutative ring spectrum R ought to have an interesting delooping $\mathrm{pic}(R)$ that records the information about invertible modules over the equivariant ring spectra underlying R. At present I have no construction of this delooping as an ultra-commutative monoid, but I describe the evidence for expecting its existence.

For every compact Lie group G the underlying orthogonal G-ring spectrum R_G of R has a symmetric monoidal model category of modules, i.e., orthogonal G-spectra with an action by R (where G acts trivially on R). The equivalences we consider here are R-linear morphisms that are $\underline{\pi}_*$-isomorphisms of underlying orthogonal G-spectra; the construction of such a symmetric monoidal model category can be found in [108, III Thm. 7.6]. We let

$$\mathrm{Pic}(R)(G) \ = \ \mathrm{Pic}(\mathrm{Ho}(R_G\text{-mod}))$$

be the resulting Picard group, i.e., the set of isomorphism classes, in the homotopy category of R_G-modules, of objects that are invertible under the derived smash product. For a continuous group homomorphism $\alpha : K \longrightarrow G$ the restriction functor $\alpha^* : R_G\text{-mod} \longrightarrow R_K\text{-mod}$ derives to a strong symmetric monoidal functor

$$R\alpha^* \ : \ \mathrm{Ho}(R_G\text{-mod}) \ \longrightarrow \ \mathrm{Ho}(R_K\text{-mod}) \ .$$

So $R\alpha^*$ preserves invertibility and induces a group homomorphism

$$\alpha^* \ : \ \mathrm{Pic}(R)(G) \longrightarrow \ \mathrm{Pic}(R)(K) \ .$$

For a second homomorphism $\beta : L \longrightarrow K$ the functors $(R\beta^*) \circ (R\alpha^*)$ and $R(\alpha \circ \beta)^*$ are naturally isomorphic. Moreover, for every element $g \in G$ the restriction functor $R(c_g)^*$ is naturally isomorphic to the identity functor of $\mathrm{Ho}(R_G\text{-mod})$, via left multiplication by g. So the assignment $G \mapsto \mathrm{Pic}(R)(G)$ becomes a functor

$$\mathrm{Pic}(R) \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b \ .$$

But the ultra-commutativity gives more. For every finite index subgroup $H \leq G$, the norm construction of Hill, Hopkins and Ravenel derives to a strong symmetric monoidal functor

$$N_H^G \ : \ \mathrm{Ho}(R_H\text{-mod}) \ \longrightarrow \ \mathrm{Ho}(R_G\text{-mod}) \ ,$$

compare [77, Prop. B.105]. So also N_H^G preserves invertibility and induces a group homomorphism

$$N_H^G \ : \ \mathrm{Pic}(R)(H) \longrightarrow \ \mathrm{Pic}(R)(G) \ .$$

These norm maps are transitive and they extend the abelian Rep-monoid to a global power monoid Pic(R).

We expect that there is a 'natural' ultra-commutative monoid pic(R) such that $\underline{\pi}_0(\mathrm{pic}(R)) \cong \mathrm{Pic}(R)$ as global power monoids and such that $\Omega(\mathrm{pic}(R))$ is globally equivalent, as an ultra-commutative monoid, to $GL_1(R)$. The G-fixed-points $(\mathrm{pic}(R)(\mathcal{U}_G))^G$ ought to have the homotopy type, as an E_∞-space, of the nerve of the category of invertible R_G-modules and $\underline{\pi}_*$-isomorphisms. Despite the strong evidence for its existence, I cannot presently construct pic(R) as an ultra-commutative monoid in our formalism.

Example 5.1.19 (Free global power functors)**.** For a compact Lie group K we construct a *free global power functor* C_K generated by K. The underlying global functor is

$$C_K \;=\; \bigoplus\nolimits_{m\geq 0} \mathbf{A}(\Sigma_m \wr K, -)\ ,$$

the direct sum of the global functors represented by the wreath products $\Sigma_m \wr K$, including the trivial group $\Sigma_0 \wr K = e$. The multiplication $\mu : C_K \Box C_K \longrightarrow C_K$ that makes this a global Green functor restricted to the (m, n)-summand is the morphism

$$\mathbf{A}(\Sigma_m \wr K, -)\Box\mathbf{A}(\Sigma_n \wr K, -) \;\longrightarrow\; C_K$$

that corresponds, via the universal property of the box product, to the bimorphism with (G, G')-component

$$\begin{aligned}\mathbf{A}(\Sigma_m \wr K, G)\Box\mathbf{A}(\Sigma_n \wr K, G') &\xrightarrow{\ \times\ } \mathbf{A}((\Sigma_m \wr K) \times (\Sigma_n \wr K), G \times G')\\ &\xrightarrow{\mathbf{A}(\Phi^*_{m,n}, G\times G')} \mathbf{A}(\Sigma_{m+n} \wr K, G \times G') \xrightarrow{\ \mathrm{incl}\ } C_K(G \times G')\ ;\end{aligned}$$

here $\Phi^*_{m,n}$ is the restriction map associated with the embedding (5.1.4)

$$\Phi_{m,n} \ :\ (\Sigma_m \wr K) \times (\Sigma_n \wr K) \;\longrightarrow\; \Sigma_{m+n} \wr K\ .$$

The multiplication is associative because

$$\begin{aligned}\Phi_{k+m,n} \circ (\Phi_{k,m} \times (\Sigma_n \wr K)) \;=\; \Phi_{k,m+n} \circ ((\Sigma_k \wr K) \times \Phi_{m,n})\ :&\\ (\Sigma_k \wr K) \times (\Sigma_m \wr K) \times (\Sigma_n \wr K) \;\longrightarrow\; \Sigma_{k+m+n} \wr K\ .&\end{aligned}$$

The multiplication is commutative because the group homomorphisms

$$\Phi_{m,n}\ ,\ \Phi_{n,m} \circ \tau_{\Sigma_m\wr K, \Sigma_n\wr K} \ :\ (\Sigma_m \wr K) \times (\Sigma_n \wr K) \;\longrightarrow\; \Sigma_{m+n} \wr K$$

are conjugate, so they represent the same morphism in $\mathbf{A}((\Sigma_m \wr K) \times (\Sigma_n \wr K), \Sigma_{m+n} \wr K)$. The unit is the inclusion $\mathbf{A}(e, -) \longrightarrow C_K$ of the summand indexed by $m = 0$.

The global Green functor C_K can be made a global power functor in a unique way such that the relation

$$P^m(1_K) \;=\; 1_{\Sigma_m \wr K}$$

holds in the mth summand of $C_K(\Sigma_m \wr K)$, where $1_K \in \mathbf{A}(K,K)$ and $1_{\Sigma_m \wr K} \in \mathbf{A}(\Sigma_m \wr K, \Sigma_m \wr K)$ are the identity operations. Indeed, C_K is generated as a global functor by the classes $1_{\Sigma_m \wr K}$ for all $k \geq 0$, so there is at most one such global power structure, and every morphism of global power functors out of C_K is determined by its value on the class 1_K. The existence of a global power structure on C_K with this property could be justified purely algebraically, but we show it by realizing C_K by an ultra-commutative ring spectrum. As we shall make precise in Proposition 5.2.6 (ii) below, C_K is indeed *freely* generated, as a global power functor, by the class 1_K.

The unreduced suspension spectrum

$$\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}} K) \;\cong\; \bigvee\nolimits_{m \geq 0} \Sigma^\infty_+ B_{\mathrm{gl}}(\Sigma_m \wr K)$$

of the free ultra-commutative monoid (compare Example 2.1.5) generated by a global classifying space of K is an ultra-commutative ring spectrum. According to Proposition 4.2.5, its 0th homotopy group global functor is given additively by

$$\underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}} K)) \;\cong\; \bigoplus\nolimits_{m \geq 0} \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}(\Sigma_m \wr K)) \;\cong\; \bigoplus\nolimits_{m \geq 0} \mathbf{A}(\Sigma_m \wr K, -)\,.$$

Under this isomorphism, the stable tautological class $e_K \in \pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}} K)$ maps to the generator $1_K \in \mathbf{A}(K,K)$. The class e_K is the stabilization of the unstable tautological class $u_K \in \pi_0^K(B_{\mathrm{gl}} K)$, whose mth power is $[m](u_K) = u_{\Sigma_m \wr K}$ in $\pi_0^{\Sigma_m \wr K}(\mathbb{P}(B_{\mathrm{gl}} K))$, see (2.2.21). The stabilization map $\sigma : \underline{\pi}_0(B_{\mathrm{gl}} K) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} K)$ commutes with power operations, so this shows that

$$P^m(e_K) \;=\; P^m(\sigma^K(u_K)) \;=\; \sigma^{\Sigma_m \wr K}([m](u_K)) \;=\; \sigma^{\Sigma_m \wr K}(u_{\Sigma_m \wr K}) \;=\; e_{\Sigma_m \wr K}$$

in the group $\pi_0^{\Sigma_m \wr K}(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}} K))$.

We warn the reader that the previous example of the free global power functor C_K is *not* the symmetric algebra, with respect to the box product of global functors, of the represented global functor $\mathbf{A}(K,-)$. The issue is that the global functors

$$\mathbf{A}(K,-)^{\Box m}/\Sigma_m \;\cong\; \mathbf{A}(K^m,-)/\Sigma_m \qquad \text{and} \qquad \mathbf{A}(\Sigma_m \wr K, -)$$

are typically not isomorphic. The restriction map $\mathrm{res}^{\Sigma_m \wr K}_{K^m} \in \mathbf{A}(\Sigma_m \wr K, K^m)$ induces a morphism of represented global functors

$$- \circ \mathrm{res}^{\Sigma_m \wr K}_{K^m} \;:\; \mathbf{A}(K^m,-) \;\longrightarrow\; \mathbf{A}(\Sigma_m \wr K, -)$$

that equalizes the Σ_m-action on the source because every permutation of the factors of K^m becomes an inner automorphism in $\Sigma_m \wr K$. So the morphism factors over a morphism of global functors

$$\mathbf{A}(K^m, -)/\Sigma_m \longrightarrow \mathbf{A}(\Sigma_m \wr K, -)$$

which, however, is generally *not* an isomorphism (already for $K = e$ and $m = 2$). The box product symmetric algebra

$$\bigoplus_{m\geq 0} \mathbf{A}(K, -)^{\Box m}/\Sigma_m \cong \bigoplus_{m\geq 0} \mathbf{A}(K^m, -)/\Sigma_m$$

also has a universal property: it is the free global Green functor generated by K. However, this box product symmetric algebra does not seem to have natural power operations.

5.2 Comonadic description of global power functors

In this section we show that the category of global power functors is both monadic and comonadic over the category of global Green functors. We introduce the functor of exponential sequences and make it a comonad on the category of global Green functors. For a global Green functor R and a compact Lie group G, Construction 5.2.1 introduces the commutative ring $\exp(R; G)$ of exponential sequences. Construction 5.2.7 connects these rings for varying Lie groups G, making the entire data a new global Green functor $\exp(R)$, compare Proposition 5.2.8. Theorem 5.2.10 extends the functor of exponential sequences to a comonad on the category of global Green functors, and Theorem 5.2.13 shows that the category of its coalgebras is isomorphic to the category of global power functors. Proposition 5.2.21 shows that the category of global power functors is also monadic over the category of global Green functors. A formal consequence is that the category of global power functors has all limits and colimits, and that they are created in the category of global Green functors. So the relationship of global power functors to global Green functors is formally similar to the situation for λ-rings, which are both monadic and comonadic over the category of commutative rings. When restricted to finite groups, most of the results about the comonad of exponential sequences are contained in the PhD thesis of J. Singer [159], a former student of the author. Also for finite groups (as opposed to compact Lie groups), this comonadic description has independently been obtained by Ganter [60].

Proposition 5.2.17 discusses localization of global Green functors at a multiplicative subset of the underlying ring; while the structure of global Green functor always 'survives localization', this is not generally true for power operations. Theorem 5.2.18 exhibits a necessary and sufficient condition so that

a localization of a global power functor inherits power operations. For localization at a set of integer primes this condition is always satisfied (Example 5.2.19), so global power functors can in particular be rationalized.

Construction 5.2.1. We let R be a global Green functor and G a compact Lie group. We let

$$\exp(R;G) \ \subset \ \prod\nolimits_{m\geq 0} R(\Sigma_m \wr G)$$

be the set of *exponential sequences*, i.e., of those families $(x_m)_m$ that satisfy $x_0 = 1$ in $R(\Sigma_0 \wr G) = R(e)$ and

$$\Phi^*_{k,m-k}(x_m) \ = \ x_k \times x_{m-k}$$

in $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$ for all $0 < k < m$, where

$$\Phi_{k,m-k} \ : \ (\Sigma_k \wr G) \ \times \ (\Sigma_{m-k} \wr G) \ \longrightarrow \ \Sigma_m \wr G$$

is the monomorphism (5.1.4). We define a multiplication on the set $\exp(R;G)$ by coordinatewise multiplication in the rings $R(\Sigma_m \wr G)$, i.e.,

$$(x \cdot y)_m \ = \ x_m \cdot y_m \ .$$

We introduce another binary operation $\oplus$ on $\exp(R;G)$ by

$$(x \oplus y)_m \ = \ \sum_{k=0}^{m} \operatorname{tr}_{k,m-k}(x_k \times y_{m-k}) \ ,$$

where $x = (x_m)$, $y = (y_m)$ and $\operatorname{tr}_{k,m-k} : R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)) \longrightarrow R(\Sigma_m \wr G)$ is the transfer associated with the monomorphism $\Phi_{k,m-k}$.

Proposition 5.2.2. *Let R be a global Green functor and G a compact Lie group. Then the addition $\oplus$ and the componentwise multiplication make the set* $\exp(R;G)$ *of exponential sequences a commutative ring.*

Proof If x and y are exponential sequences, then the relation

$$\begin{aligned}\Phi^*_{k,m-k}(x_m \cdot y_m) \ &= \ \Phi^*_{k,m-k}(x_m) \cdot \Phi^*_{k,m-k}(y_m) \\ &= \ (x_k \times x_{m-k}) \cdot (y_k \times y_{m-k}) \ = \ (x_k \cdot y_k) \times (x_{m-k} \cdot y_{m-k})\end{aligned}$$

holds in $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$; so the product $x \cdot y$ is again exponential. The product is associative and commutative since all the multiplications in the rings $R(\Sigma_m \wr G)$ have this property. The exponential sequence $(1)_{m\geq 0}$ is a multiplicative unit.

Now we show that the sum, with respect to $\oplus$, of two exponential sequences is again exponential. The key step in this verification is an application of the double coset formula, for which we need to understand the $(\Sigma_i \times \Sigma_{m-i})$-$(\Sigma_k \times$

Σ_{m-k})-double cosets inside Σ_m. We parametrize these double cosets by pairs (a,b) of natural numbers satisfying

$$0 \le a \le i\ , \quad 0 \le b \le m-i \qquad \text{and} \qquad a+b=k\ . \tag{5.2.3}$$

For each such pair we define a permutation $\chi(a,b) \in \Sigma_m$ by

$$\chi(a,b)(j) \;=\; \begin{cases} j & \text{for } 1 \le j \le a, \\ j-a+i & \text{for } a+1 \le j \le a+b, \\ j-b & \text{for } a+b+1 \le j \le i+b, \\ j & \text{for } i+b+1 \le j \le m. \end{cases}$$

In other words, $\chi(a,b)$ is the unique $(k,m-k)$-shuffle such that

$$\chi(a,b)(\{1,\dots,k\}) \;=\; \{1,\dots,a\} \,\cup\, \{i+1,\dots,i+b\}\ .$$

The permutations $\chi(a,b)$ form a set of double coset representatives for the subgroups $\Sigma_i \times \Sigma_{m-i}$ and $\Sigma_k \times \Sigma_{m-k}$ inside Σ_m, for all pairs (a,b) subject to (5.2.3).

When applying the double coset formula we will need the relations

$$(\Sigma_i \times \Sigma_{m-i})^{\chi(a,b)} \,\cap\, (\Sigma_k \times \Sigma_{m-k}) \;=\; \Sigma_a \times \Sigma_b \times \Sigma_{i-a} \times \Sigma_{m-i-b}$$

and

$$(\Sigma_i \times \Sigma_{m-i}) \,\cap\, {}^{\chi(a,b)}(\Sigma_k \times \Sigma_{m-k}) \;=\; \Sigma_a \times \Sigma_{i-a} \times \Sigma_b \times \Sigma_{m-i-b}\ .$$

Thus the double coset formula becomes

$$\begin{aligned} &\Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x \times y)) \qquad\qquad (5.2.4)\\ &\quad= \sum_{a,b} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{(\Sigma_i\times\Sigma_{m-i})\cap{}^{\chi(a,b)}(\Sigma_k\times\Sigma_{m-k})} \left(\chi(a,b)_\star \left(\mathrm{res}^{\Sigma_k\times\Sigma_{m-k}}_{(\Sigma_i\times\Sigma_{m-i})^{\chi(a,b)}\cap(\Sigma_k\times\Sigma_{m-k})}(x\times y)\right)\right)\\ &\quad= \sum_{a,b} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_a\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}} \left(\chi(a,b)_\star \left(\mathrm{res}^{\Sigma_k}_{\Sigma_a\times\Sigma_b}(x)\times \mathrm{res}^{\Sigma_{m-k}}_{\Sigma_{i-a,m-i-b}}(y)\right)\right) \end{aligned}$$

The two sums run over all pairs (a,b) of natural numbers satisfying (5.2.3). Now we consider exponential sequences $x, y \in \exp(R;G)$ and calculate

$$\begin{aligned} \Phi^*_{i,m-i}((x\oplus y)_m) \;&=\; \sum_{k=0}^{m} \Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x_k \times y_{m-k}))\\ &\overset{(5.2.4)}{=}\; \sum_{a,b} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_i\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}} (\chi(a,b)_\star(x_a \times x_b \times y_{i-a} \times y_{m-i-b}))\\ &=\; \sum_{a,b} \mathrm{tr}_{a,i-a}(x_a \times y_{i-a}) \times \mathrm{tr}_{b,m-i-b}(x_b \times y_{m-i-b})\\ &=\; (x\oplus y)_i \times (x\oplus y)_{m-i}\ . \end{aligned}$$

Here the last two sums run over all pairs (a, b) of natural numbers satisfying $0 \leq a \leq i$ and $0 \leq b \leq m - i$. This shows that the sequence $x \oplus y$ is again exponential.

The following square of group monomorphisms commutes:

$$\begin{tikzcd}
(\Sigma_j \wr G) \times (\Sigma_k \wr G) \times (\Sigma_l \wr G) \arrow[r, "(\Sigma_j \wr G)\times\Phi_{k,l}"] \arrow[d, "\Phi_{j,k}\times(\Sigma_l \wr G)"'] & (\Sigma_j \wr G) \times (\Sigma_{k+l} \wr G) \arrow[d, "\Phi_{j,k+l}"] \\
(\Sigma_{j+k} \wr G) \times (\Sigma_l \wr G) \arrow[r, "\Phi_{j+k,l}"'] & \Sigma_{j+k+l} \wr G
\end{tikzcd}$$

So for all $x \in R(\Sigma_j \wr G)$, $y \in R(\Sigma_k \wr G)$, and $z \in R(\Sigma_l \wr G)$, the relation

$$\mathrm{tr}_{j,k+l}(x \times \mathrm{tr}_{k,l}(y \times z)) \;=\; \mathrm{tr}_{j+k,l}(\mathrm{tr}_{j,k}(x \times y) \times z)$$

holds in the group $R(\Sigma_{j+k+l} \wr G)$. By unraveling the definitions, this becomes the associativity of the operation $\oplus$.

Also, the following square of group monomorphisms commutes:

$$\begin{tikzcd}
(\Sigma_k \wr G) \times (\Sigma_l \wr G) \arrow[r, "\Phi_{k,l}"] \arrow[d, "\mathrm{twist}"'] & \Sigma_{k+l} \wr G \arrow[d, "c_\chi"] \\
(\Sigma_l \wr G) \times (\Sigma_k \wr G) \arrow[r, "\Phi_{l,k}"'] & \Sigma_{l+k} \wr G
\end{tikzcd}$$

Here $\chi = (\chi_{k,l}; 1, \ldots, 1)$, for the shuffle permutation $\chi_{k,l} \in \Sigma_{k+l}$. So for all $x \in R(\Sigma_k \wr G)$ and $y \in R(\Sigma_l \wr G)$, the relation

$$\mathrm{tr}_{l,k}(y \times x) \;=\; \chi_\star(\mathrm{tr}_{k,l}(x \times y)) \;=\; \mathrm{tr}_{k,l}(x \times y)$$

holds in the group $R(\Sigma_{k+l} \wr G)$. By unraveling the definitions, this implies the commutativity of the operation $\oplus$.

The sequence $\underline{0}$ with $\underline{0}_0 = 1$ and $\underline{0}_m = 0$ for $m \geq 1$ is a neutral element for $\oplus$. Given an exponential sequence x we define a sequence y inductively by $y_0 = 1$ and by

$$y_m \;=\; -\sum\nolimits_{k=1}^{m} \mathrm{tr}_{k,m-k}(x_k \times y_{m-k})$$

for $m \geq 1$. To see that the sequence y is again exponential, we show the relation

$$\Phi^*_{i,m-i}(y_m) \;=\; y_i \times y_{m-i}$$

by induction on m. The induction starts with $m = 1$, where there is nothing to

show. Now we assume that $m \geq 2$; then

$$\begin{aligned}
-\Phi^*_{i,m-i}(y_m) &= \sum\nolimits_{k=1}^m \Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x_k \times y_{m-k})) \\
\overset{(5.2.4)}{=} &\sum_{a+b\geq 1} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_a\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}} (\chi(a,b)_\star(x_a \times x_b \times y_{i-a} \times y_{m-i-b})) \\
&= \sum_{a,b\geq 1} \mathrm{tr}_{a,i-a}(x_a \times y_{i-a}) \times \mathrm{tr}_{b,m-i-b}(x_b \times y_{m-i-b}) \\
&+ \sum_{a=1}^{i} \mathrm{tr}_{a,i-a}(x_a \times y_{i-a}) \times y_{m-i} + \sum_{b=1}^{m-i} y_i \times \mathrm{tr}_{b,m-i-b}(x_b \times y_{m-i-b}) \\
&= y_i \times y_{m-i} - y_i \times y_{m-i} - y_i \times y_{m-i} = -y_i \times y_{m-i} \,.
\end{aligned}$$

In the sums, (a, b) runs over all pairs of natural numbers satisfying $a \leq i$ and $b \leq m - i$ plus the conditions attached to the summation symbols. The second equation uses the inductive hypothesis. This shows that the sequence y is again exponential. The relation $x \oplus y = \underline{0}$ holds by construction, so y is an inverse of x with respect to $\oplus$. Altogether this shows that $\exp(R; G)$ is an abelian group under $\oplus$.

It remains to show distributivity of multiplication over $\oplus$. We let $x, y, z \in \exp(R; G)$ be exponential sequences. Then

$$\begin{aligned}
((x \oplus y) \cdot z)_m &= \left(\sum\nolimits_{k=0}^m \mathrm{tr}_{k,m-k}(x_k \times y_{m-k})\right) \cdot z_m \\
&= \sum\nolimits_{k=0}^m \mathrm{tr}_{k,m-k}((x_k \times y_{m-k}) \cdot \Phi^*_{k,m-k}(z_m)) \\
&= \sum\nolimits_{k=0}^m \mathrm{tr}_{k,m-k}((x_k \times y_{m-k}) \cdot (z_k \times z_{m-k})) \\
&= \sum\nolimits_{k=0}^m \mathrm{tr}_{k,m-k}((x_k \cdot z_k) \times (y_{m-k} \cdot z_{m-k})) = ((x \cdot z) \oplus (y \cdot z))_m \,.
\end{aligned}$$

The second equation is reciprocity in the global Green functor R. □

The ring $\exp(R; G)$ is covariantly functorial in R: given a morphism of global Green functors $\varphi : R \longrightarrow S$, we define a map

$$\exp(\varphi; G) \;:\; \exp(R; G) \longrightarrow \exp(S; G) \quad \text{by} \quad (\exp(\varphi; G)(x))_m = \varphi(x_m) \,.$$

We omit the straightforward verifications that for $x \in \exp(R; G)$ the sequence $\exp(\varphi; G)(x)$ is again exponential, that $\exp(\varphi; G)$ is a ring homomorphism, and that $\exp(-; G)$ is compatible with identity and composition.

Now we turn to the functoriality of $\exp(R; G)$ in the Lie group variable, which is slightly more subtle. The ultimate aim is to make $\exp(R; -)$ into another global Green functor. Proposition 5.2.2 already provides the ring structures on the values, so the missing data is to give $\exp(R; -)$ the structure of a global functor. Our approach is indirect and not based on explicit formulas; rather we exploit the fact that the functor $R \mapsto \exp(R; G)$ is representable by

the underlying global Green functor of the free global power functor C_G discussed in Example 5.1.19. The next proposition will be used for establishing this representability property, see Proposition 5.2.6 below.

Proposition 5.2.5. *Let R and S be global power functors and $f : R \longrightarrow S$ a morphism of global Green functors. Then the collection of elements $x \in R(G)$, for varying compact Lie groups G, that satisfy*

$$f(P^m(x)) = P^m(f(x))$$

in $S(\Sigma_m \wr G)$ for all $m \geq 0$ form a global power subfunctor of R.

Proof This is a straightforward consequence of the various relations enjoyed by power operations. For the course of this proof we call a pair (G, x) consisting of a compact Lie group G and an element $x \in R(G)$ *good* if the relation $f(P^m(x)) = P^m(f(x))$ holds for all $m \geq 0$. Then the naturality relation $P^m \circ \alpha^* = (\Sigma_m \wr \alpha)^* \circ P^m$ for a continuous homomorphism $\alpha : K \longrightarrow G$ implies

$$\begin{aligned} f(P^m(\alpha^*(x))) &= f((\Sigma_m \wr \alpha)^*(P^m(x))) = (\Sigma_m \wr \alpha)^*(f^*(P^m(x))) \\ &= (\Sigma_m \wr \alpha)^*(P^m(f^*(x))) = P^m(\alpha^*(f^*(x))) = P^m(f^*(\alpha^*(x))) . \end{aligned}$$

So if (G, x) is good, then so is $(K, \alpha^*(x))$. By the analogous calculation, the transfer relation $P^m \circ \mathrm{tr}_H^G = \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \circ P^m$ for a closed subgroup H of G implies that if (H, x) is good, then so is $(G, \mathrm{tr}_H^G(x))$.

The relation $P^m(1) = 1$ and the fact $f(1) = 1$ show that $(e, 1)$ is good. Since good pairs are closed under restriction, the pair $(G, 1)$ is good for every compact Lie group G. The multiplicativity $P^m(x \cdot y) = P^m(x) \cdot P^m(y)$ and the hypothesis that f is multiplicative show that for all G, the good elements are closed under multiplication in the ring $R(G)$. Closure under restriction then implies that good elements are also closed under the external pairing

$$\times : R(G) \times R(K) \longrightarrow R(G \times K) .$$

The additivity relation

$$P^m(x + y) = \sum\nolimits_{k=0}^{m} \mathrm{tr}_{k,m-k}(P^k(x) \times P^{m-k}(y)) ,$$

the hypothesis that f is additive and the already established closure under transfers and external multiplication then shows that for all G, the good elements are closed under addition in the ring $R(G)$. The additivity relation and the fact that $P^m(1 + (-1)) = P^m(0) = 0$ imply that

$$P^m(-1) = -\sum_{k=1}^{m} \mathrm{tr}_{k,m-k}(P^k(1) \times P^{m-k}(-1)) = -\sum_{k=1}^{m} \mathrm{tr}_{k,m-k}(1 \times P^{m-k}(-1)) .$$

So induction on m and the previously established closure properties show that

$f(P^m(-1)) = P^m(f(-1))$, so $(e,-1)$ is good. Closure under restriction implies that also $(G,-1)$ is good for every compact Lie group G. Since the good elements are closed under multiplication and contain -1, they are closed under additive inverses. Since $(G,1)$ is good, so is $(G,1-1)=(G,0)$. This completes the proof that the good elements form a global Green subfunctor of R.

The closure property under power operations is then a consequence of the transitivity relation and closure under restriction: if (G,x) is good, then

$$\begin{aligned} f(P^k(P^m(x))) &= f(\Psi^*_{k,m}(P^{km}(x))) = \Psi^*_{k,m}(f(P^{km}(x))) \\ &= \Psi^*_{k,m}(P^{km}(f(x))) = P^k(P^m(f(x))) = P^k(f(P^m(x))) \end{aligned}$$

holds in $S(\Sigma_k \wr (\Sigma_m \wr G))$, for all $k \geq 0$. This proves that the pair $(\Sigma_m \wr G, P^m(x))$ is also good. Altogether this shows that the good elements form a global power subfunctor of R. □

The free global power functor C_K associated with a compact Lie group K was introduced in Example 5.1.19. The next proposition justifies the adjective 'free', and also shows that the underlying global Green functor of C_K represents the functor $\exp(-;K)$ of exponential sequences. We recall that

$$1_{\Sigma_m \wr K} \in \mathbf{A}(\Sigma_m \wr K, \Sigma_m \wr K) \subset C_K(\Sigma_m \wr K)$$

denotes the identity operation of the functor $\pi_0^{\Sigma_m \wr K}$. Two key relations among these elements in the global power functor C_K are

$$P^m(1_K) = 1_{\Sigma_m \wr K} \qquad \text{and} \qquad \Phi^*_{k,m-k}(1_{\Sigma_m \wr K}) = 1_{\Sigma_k \wr K} \times 1_{\Sigma_{m-k} \wr K} .$$

The second set of relations shows that the tuple $(1_{\Sigma_m \wr K})_{m \geq 0}$ is exponential, i.e., an element of $\exp(C_K;K)$. In fact, part (i) of the following proposition shows that it is a universal exponential element.

Proposition 5.2.6. *Let K be a compact Lie group.*

(i) *For every global Green functor R the map*

$$\begin{aligned} \epsilon_K \ : \ \mathcal{G}l\mathcal{G}re(C_K, R) &\longrightarrow \exp(R;K) \\ f &\longmapsto \exp(f;(1_{\Sigma_m \wr K})_{m\geq 0}) = (f(1_{\Sigma_m \wr K}))_{m \geq 0} \end{aligned}$$

is bijective.

(ii) *For every global power functor R the map*

$$\mathcal{G}l\mathcal{P}ow(C_K, R) \longrightarrow R(K)\ , \quad f \longmapsto f(1_K)$$

is bijective.

Proof (i) The underlying global functor of C_K is the direct sum of the represented global functors $\mathbf{A}(\Sigma_m \wr K, -)$, for $m \geq 0$. So the enriched Yoneda lemma (see Remark C.2) shows that evaluation at the universal elements is a bijection

$$\mathcal{GF}(C_K, R) \longrightarrow \prod_{m\geq 0} R(\Sigma_m \wr K) , \quad f \longmapsto (f(1_{\Sigma_m \wr K}))_{m\geq 0} .$$

A morphism of global functors $f : C_K \longrightarrow R$ is a morphism of global Green functors if and only if it is also multiplicative and unital. Unitality corresponds to the condition $f(1_{\Sigma_0 \wr K}) = f(\mathrm{Id}_e) = 1$ in $R(e)$. Multiplicativity means that the following square of global functors commutes:

$$\begin{tikzcd} C_K \Box C_K \arrow[r, "f\Box f"] \arrow[d, "\mathrm{mult}"'] & R \Box R \arrow[d, "\mathrm{mult}"] \\ C_K \arrow[r, "f"'] & R \end{tikzcd}$$

Since the box product of global functors is biadditive and the box product of two represented functors is represented (see Remark C.11), the global functor $C_K \Box C_K$ is isomorphic to

$$\bigoplus_{m,n\geq 0} \mathbf{A}(\Sigma_m \wr K, -)\Box\mathbf{A}(\Sigma_n \wr K, -) \;\cong\; \bigoplus_{m,n\geq 0} \mathbf{A}((\Sigma_m \wr K) \times (\Sigma_n \wr K), -) .$$

So commutativity of the above square can be tested by evaluation at the universal classes, by another application of the enriched Yoneda lemma. The multiplication of C_K takes the exterior product of the classes $1_{\Sigma_m \wr K}$ and $1_{\Sigma_n \wr K}$ to the class $\Phi^*_{m,n}(1_{\Sigma_{m+n} \wr K})$, so the multiplicativity condition becomes the exponential condition

$$\Phi^*_{m,n}(f(1_{\Sigma_{m+n} \wr K})) \;=\; f(1_{\Sigma_m \wr K}) \times f(1_{\Sigma_n \wr K}) .$$

(ii) Every morphism $f : C_K \longrightarrow R$ of global power functors is in particular a morphism of global Green functors. So by part (i), f is determined by the exponential sequence $(f(1_{\Sigma_m \wr K}))_{m\geq 0}$. The relation $P^m(1_K) = 1_{\Sigma_m \wr K}$ holds in the global power functor C_K; so since f also commutes with power operations, it is already determined by the element $f(1_K)$. This shows that the map in question is injective.

Now we show that evaluation at 1_K is also surjective. We let $y \in R(K)$ be any element. Then the sequence $(P^m(y))_{m\geq 0}$ is exponential, so part (i) provides a unique morphism of global Green functors $f : C_K \longrightarrow R$ satisfying

$$f(1_{\Sigma_m \wr K}) \;=\; P^m(y) \quad \text{in } R(\Sigma_m \wr K)$$

for all $m \geq 0$. We must show that f is also compatible with power operations. By Theorem 4.2.6 the abelian group

$$C_K(G) \;=\; \bigoplus_{m\geq 0} \mathbf{A}(\Sigma_m \wr K, G)$$

is generated by the classes

$$\mathrm{tr}_H^G(\alpha^*(P^m(1_K)))$$

for $m \geq 0$, and where (H, α) runs over pairs consisting of a closed subgroup H of G and a continuous homomorphism $\alpha : H \longrightarrow \Sigma_m \wr K$. In particular, C_K is generated, as a global power functor, by the single element 1_K in $C_K(K)$.

Proposition 5.2.5 shows that the collection of elements x of C_K that satisfy $f(P^m(x)) = P^m(f(x))$ for all $m \geq 0$ form a global power subfunctor of C_K. The element 1_K is among these because $f(P^m(1_K)) = f(1_{\Sigma_m \wr K}) = P^m(y) = P^m(f(1_K))$. Since 1_K generates C_K as a global power functor, all elements of C_K have this property. So f is a morphism of global power functors. □

Now we can make the rings $\exp(R; G)$ of exponential sequences into a global functor for varying G. As an auxiliary tool we introduce a functor

$$\Gamma \ : \ \mathbf{A}^{\mathrm{op}} \ \longrightarrow \ \mathcal{G}l\mathcal{P}ow \ .$$

The functor is given by $\Gamma(K) = C_K$ on objects; on morphisms, the freeness property of Proposition 5.2.6 (ii) allows us to define

$$\Gamma \ : \ \mathbf{A}(K, G) \ \longrightarrow \ \mathcal{G}l\mathcal{P}ow(C_G, C_K)$$

by the requirement

$$\Gamma(\tau)(1_G) \ = \ \tau \ \in \ \mathbf{A}(K, G) \ \subset \ C_K(G) \ .$$

The contravariant functoriality is rather formal: for $\psi \in \mathbf{A}(L, K)$ we have

$$\Gamma(\tau \circ \psi)(1_G) \ = \ \tau \circ \psi \ = \ \Gamma(\psi)(\tau) \ = \ (\Gamma(\psi) \circ \Gamma(\tau))(1_G) \ .$$

So $\Gamma(\tau \circ \psi) = \Gamma(\psi) \circ \Gamma(\tau)$ by freeness, i.e., Proposition 5.2.6 (ii). We emphasize that while $\mathbf{A}$ is a preadditive category, Γ is just a plain functor, i.e., not additive in any sense.

Construction 5.2.7. As before we consider a global Green functor R. For compact Lie groups G and K we define a map

$$\exp(R; -) \ : \ \mathbf{A}(G, K) \times \exp(R; G) \ \longrightarrow \ \exp(R; K) \ , \ (\tau, x) \ \longmapsto \ \exp(R; \tau)(x) \ .$$

The definition exploits the representability of $\exp(R; G)$, i.e., that the map

$$\epsilon_G \ : \ \mathcal{G}l\mathcal{G}re(C_G, R) \ \longrightarrow \ \exp(R; G)$$

sending f to the exponential sequence $(f(1_{\Sigma_m \wr G}))_{m \geq 0}$ is bijective, by Proposition 5.2.6 (i). We define $\exp(R; \tau)$ by requiring that for every morphism of global Green functors $f : C_G \longrightarrow R$ the following relation holds:

$$\exp(R; \tau)(\epsilon_G(f)) \ = \ \epsilon_K(f \circ \Gamma(\tau)) \ .$$

Since ϵ_G is bijective, this is a legitimate definition. It is rather straightforward to see that the assignments

$$G \longmapsto \exp(R;G) \qquad \text{and} \qquad \tau \longmapsto \exp(R;\tau)$$

define a functor from the global Burnside category to the category of sets. Indeed, for $\tau \in \mathbf{A}(K,G)$, $\psi \in \mathbf{A}(L,K)$ and a morphism of global Green functors $f : C_G \longrightarrow R$, we have

$$\begin{aligned} \exp(R;\tau\circ\psi)(\epsilon_G(f)) \ &= \ \epsilon_L(f\circ\Gamma(\tau\circ\psi)) \ = \ \epsilon_L(f\circ\Gamma(\psi)\circ\Gamma(\tau)) \\ &= \ \exp(R;\tau)(\epsilon_K(f\circ\Gamma(\psi))) \ = \ \exp(R;\tau)(\exp(R;\psi)(\epsilon_G(f))) \ . \end{aligned}$$

Since every exponential sequence in $\exp(R;G)$ is of the form $\epsilon_G(f)$, this proves that $\exp(R;\tau\circ\psi) = \exp(R;\tau)\circ\exp(R;\psi)$. It is not completely obvious, though, that this construction is additive in both variables, but we will show that in the next proposition.

The construction of exponential sequences is in fact a functor in two variables: for every operation $\tau \in \mathbf{A}(G,K)$ and all morphisms of global Green functors $f : C_G \longrightarrow R$ we have

$$\begin{aligned} ((\exp(\varphi;K)\circ\exp(R;\tau))(\epsilon_G(f)))_m \ &= \ ((\exp(\varphi;K)(\epsilon_K(f\circ\Gamma(\tau)))))_m \\ &= \ \varphi\left((\epsilon_K(f\circ\Gamma(\tau)))_m\right) \\ &= \ (\varphi\circ f\circ\Gamma(\tau))(1_{\Sigma_m\wr K}) \\ &= \ (\exp(S;\tau)(\epsilon_G(\varphi\circ f)))_m \\ &= \ ((\exp(S;\tau)\circ\exp(\varphi;G))(\epsilon_G(f)))_m \ . \end{aligned}$$

Every exponential sequence in $\exp(R;G)$ is of the form $\epsilon_G(f)$, so

$$\exp(\varphi;K)\circ\exp(R;\tau) \ = \ \exp(S;\tau)\circ\exp(\varphi;G) \ : \ \exp(R;G) \ \longrightarrow \ \exp(S;K) \ .$$

The next proposition makes the abstract definition of the map $\exp(R;\tau)$ more explicit by giving a concrete formula when τ is a restriction or a transfer.

Proposition 5.2.8. *Let R be a global Green functor.*

(i) *For every continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups the map*

$$\exp(R;\alpha^*) \ : \ \exp(R;G) \ \longrightarrow \ \exp(R;K)$$

is a ring homomorphism and satisfies

$$(\exp(R;\alpha^*)(x))_m \ = \ (\Sigma_m\wr\alpha)^*(x_m)$$

in $R(\Sigma_m\wr K)$ for all $x \in \exp(R;G)$.

(ii) *For every closed subgroup H of a compact Lie group G the map*

$$\exp(R;\mathrm{tr}_H^G) \ : \ \exp(R;H) \longrightarrow \exp(R;G)$$

is additive, satisfies reciprocity with respect to restriction from G to H and is given by

$$\Big(\exp(R;\mathrm{tr}_H^G)(x)\Big)_m \ = \ \mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(x_m)$$

for all $x\in\exp(R;H)$.

(iii) *For all compact Lie groups G and K, the map*

$$\exp(R;-) \ : \ \mathbf{A}(G,K)\times\exp(R;G) \longrightarrow \exp(R;K)$$

is biadditive, i.e., $\exp(R;-)$ *becomes a global functor in the Lie group variable.*

(iv) *As the Lie group G varies, the ring structures and the functoriality in the global Burnside category make* $\exp(R)$ *a global Green functor.*

Proof (i) We let $f : C_G \longrightarrow R$ be any morphism of global Green functors. Then

$$\begin{aligned}
\big(\exp(R;\alpha^*)(\epsilon_G(f))\big)_m \ &= \ (\epsilon_K(f\circ\Gamma(\alpha^*)))_m \ = \ (f\circ\Gamma(\alpha^*))(1_{\Sigma_m\wr K})\\
&= \ f(\Gamma(\alpha^*)(P^m(1_K))) \ = \ f(P^m(\Gamma(\alpha^*)(1_K)))\\
&= \ f(P^m(\alpha^*(1_G))) \ = \ f((\Sigma_m\wr\alpha)^*(P^m(1_G)))\\
&= \ f((\Sigma_m\wr\alpha)^*(1_{\Sigma_m\wr G})) \ = \ (\Sigma_m\wr\alpha)^*(f(1_{\Sigma_m\wr G}))\\
&= \ (\Sigma_m\wr\alpha)^*(\epsilon_G(f)_m)\ .
\end{aligned}$$

Since every exponential sequence in $\exp(R;G)$ is of the form $\epsilon_G(f)$, this proves the formula for $\exp(R;\alpha^*)$. Since the multiplication in $\exp(R;G)$ is coordinate-wise and the original restriction maps for R are ring homomorphisms, the formula for $\exp(R;\alpha^*)$ shows that $\exp(R;\alpha^*)$ is multiplicative and preserves the multiplicative unit. The additivity of $\exp(R;\alpha^*)$ uses the relation

$$(\Sigma_m\wr\alpha)^*\circ\mathrm{tr}_{k,m-k} \ = \ \mathrm{tr}_{k,m-k}\circ((\Sigma_k\wr\alpha)\times(\Sigma_{m-k}\wr\alpha))^* \qquad (5.2.9)$$

as maps from $R((\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G))$ to $R(\Sigma_m\wr K)$. To prove (5.2.9) we distinguish two cases. If α is surjective, then so is $\Sigma_m\wr\alpha$, and

$$(\Sigma_m\wr\alpha)^{-1}((\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G)) \ = \ (\Sigma_k\wr K)\times(\Sigma_{m-k}\wr K)\ .$$

So for epimorphisms, the relation (5.2.9) is a special case of compatibility of transfer with inflation. If H is a closed subgroup of G, then $\Sigma_m\wr G$ consists of a single double coset for the left $(\Sigma_m\wr H)$-action and right $((\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G))$-action, and

$$(\Sigma_m\wr H) \ \cap \ ((\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G)) \ = \ (\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)\ .$$

So the double coset formula specializes to

$$\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_m\wr H}\circ \mathrm{tr}_{k,m-k} \;=\; \mathrm{tr}_{k,m-k}\circ \mathrm{res}^{(\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}\ ,$$

which is precisely the relation (5.2.9) for the inclusion $H \longrightarrow G$. Every homomorphism factors as an epimorphism followed by a subgroup inclusion, so relation (5.2.9) follows in general. We can then conclude that $\exp(R;\alpha^*)$ is additive:

$$\begin{aligned}
(\exp(R;\alpha^*)(x\oplus y))_m \;&=\; (\Sigma_m\wr\alpha)^*((x\oplus y)_m) \;=\; \sum_{k=0}^{m}(\Sigma_m\wr\alpha)^*(\mathrm{tr}_{k,m-k}(x_k\times y_{m-k}))\\
&\overset{(5.2.9)}{=}\; \sum_{k=0}^{m}\mathrm{tr}_{k,m-k}\left(((\Sigma_k\wr\alpha)\times(\Sigma_{m-k}\wr\alpha))^*(x_k\times y_{m-k})\right)\\
&=\; \sum_{k=0}^{m}\mathrm{tr}_{k,m-k}\left((\Sigma_k\wr\alpha)^*(x_k)\times(\Sigma_{m-k}\wr\alpha)^*(y_{m-k})\right)\\
&=\; \sum_{k=0}^{m}\mathrm{tr}_{k,m-k}\left((\exp(R;\alpha^*)(x))_k\times(\exp(R;\alpha^*)(y))_{m-k}\right)\\
&=\; \big(\exp(R;\alpha^*)(x)\oplus\exp(R;\alpha^*)(y)\big)_m\ .
\end{aligned}$$

Part (ii) is similar to part (i). We let $f : C_H \longrightarrow R$ be any morphism of global Green functors. Then

$$\begin{aligned}
\Big(\exp(R;\mathrm{tr}_H^G)(\epsilon_H(f))\Big)_m \;&=\; \Big(\epsilon_G(f\circ\Gamma(\mathrm{tr}_H^G))\Big)_m \;=\; (f\circ\Gamma(\mathrm{tr}_H^G))(1_{\Sigma_m\wr G})\\
&=\; f(\Gamma(\mathrm{tr}_H^G)(P^m(1_G))) \;=\; f(P^m(\Gamma(\mathrm{tr}_H^G)(1_G)))\\
&=\; f(P^m(\mathrm{tr}_H^G(1_H))) \;=\; f(\mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}(P^m(1_H)))\\
&=\; f(\mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}(1_{\Sigma_m\wr H})) \;=\; \mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}(f(1_{\Sigma_m\wr H})) \;=\; \mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}(\epsilon_H(f)_m)\ .
\end{aligned}$$

Since every exponential sequence is of the form $\epsilon_H(f)$, this proves the formula for $\exp(R;\mathrm{tr}_H^G)$. To see that the map $\exp(R;\mathrm{tr}_H^G)$ is additive we observe that

$$\mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}\circ \mathrm{tr}_{k,m-k} \;=\; \mathrm{tr}_{k,m-k}\circ \mathrm{tr}^{(\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}\ ,$$

by transitivity of transfers. Thus

$$\begin{aligned}
(\exp(R;\mathrm{tr}_H^G)(x\oplus y))_m \;&=\; \sum_{k=0}^{m}\mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}(\mathrm{tr}_{k,m-k}(x_k\times y_{m-k}))\\
&=\; \sum_{k=0}^{m}\mathrm{tr}_{k,m-k}\left(\mathrm{tr}^{(\Sigma_k\wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k\wr H)\times(\Sigma_{m-k}\wr H)}(x_k\times y_{m-k})\right)\\
&=\; \sum_{k=0}^{m}\mathrm{tr}_{k,m-k}\left(\mathrm{tr}^{\Sigma_k\wr G}_{\Sigma_k\wr H}(x_k)\times \mathrm{tr}^{\Sigma_{m-k}\wr G}_{\Sigma_{m-k}\wr H}(y_{m-k})\right)\\
&=\; (\exp(R;\mathrm{tr}_H^G)(x)\oplus\exp(R;\mathrm{tr}_H^G)(y))_m\ .
\end{aligned}$$

The reciprocity for restriction and transfer is now a direct consequence of the reciprocity for the global Green functor R:

$$\begin{aligned}(\exp(R;\mathrm{tr}_H^G)(x)\cdot y)_m \ &= \ \mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(x_m)\cdot y_m\\ &= \ \mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(x_m\cdot \mathrm{res}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(y_m)) \ = \ (\exp(R;\mathrm{tr}_H^G)(x\cdot \mathrm{res}_H^G(y)))_m \ .\end{aligned}$$

(iii) We start with additivity of the map $\exp(R;-)$ in the variable τ. We let $\tau'\in\mathbf{A}(G,K)$ be another natural transformation. Then the morphism of global power functors $\Gamma(\tau+\tau') : C_K \longrightarrow C_G$ satisfies

$$\begin{aligned}\Gamma(\tau+\tau')(1_{\Sigma_m\wr K}) \ &= \ \Gamma(\tau+\tau')(P^m(1_K)) \ = \ P^m(\Gamma(\tau+\tau')(1_K))\\ &= \ P^m(\tau+\tau') \ = \ \sum_{k=0}^m \mathrm{tr}_{k,m-k}(P^k(\tau)\times P^{m-k}(\tau'))\\ &= \ \sum_{k=0}^m \mathrm{tr}_{k,m-k}(P^k(\Gamma(\tau)(1_K))\times P^{m-k}(\Gamma(\tau')(1_K)))\\ &= \ \sum_{k=0}^m \mathrm{tr}_{k,m-k}(\Gamma(\tau)(1_{\Sigma_k\wr K})\times \Gamma(\tau')(1_{\Sigma_{m-k}\wr K})) \ .\end{aligned}$$

For a morphism of global Green functors $f : C_G \longrightarrow R$ this implies

$$(f\circ\Gamma(\tau+\tau'))(1_{\Sigma_m\wr K}) \ = \ \sum_{k=0}^m \mathrm{tr}_{k,m-k}(f(\Gamma(\tau)(1_{\Sigma_k\wr K}))\times f(\Gamma(\tau')(1_{\Sigma_{m-k}\wr K}))) \ .$$

This establishes the mth component of the relation

$$\epsilon_K(f\circ\Gamma(\tau+\tau')) \ = \ \epsilon_K(f\circ\Gamma(\tau))\oplus\epsilon_K(f\circ\Gamma(\tau')) \ ,$$

and this in turn proves

$$\exp(R;\tau+\tau')(\epsilon_G(f)) \ = \ \exp(R;\tau)(\epsilon_G(f)) \ \oplus \ \exp(R;\tau')(\epsilon_G(f)) \ .$$

Now we come to additivity of the map in the exponential sequence, i.e., that for fixed $\tau\in\mathbf{A}(G,K)$ the map

$$\exp(R;\tau) \ : \ \exp(R;G) \ \longrightarrow \exp(R;K)$$

is additive with respect to $\oplus$. By Theorem 4.2.6 the abelian group $\mathbf{A}(G,K)$ is generated by operations of the form $\mathrm{tr}_L^K\circ\alpha^*$ where L is a closed subgroup of K and $\alpha : L\longrightarrow G$ a continuous homomorphism. Since we have already established additivity in τ, we can assume that $\tau = \mathrm{tr}_L^K\circ\alpha^*$ is one of the generating operations. Since

$$\exp(R;\mathrm{tr}_L^K\circ\alpha^*) \ = \ \exp(R;\mathrm{tr}_L^K)\circ\exp(R;\alpha^*) \ ,$$

this in turn follows from additivity for restrictions and transfers, which we showed in part (i) and (ii), respectively.

Part (iv) is just the summary of the properties in parts (i), (ii) and (iii). □

The previous proposition establishes the functor exp of exponential sequences as an endofunctor of the category of global Green functors. When iterating the construction, we encounter iterated wreath products, and we will save a substantial number of parentheses by using the short hand notation

$$\Sigma_k \wr \Sigma_m \wr G \ = \ \Sigma_k \wr (\Sigma_m \wr G) \ .$$

Now we make this endofunctor into a comonad. A natural transformation of global Green functors

$$\eta_R \ : \ \exp(R) \ \longrightarrow \ R$$

is given by $\eta(x) = x_1$, using the identification $G \cong \Sigma_1 \wr G$ via $g \mapsto (1; g)$. A natural transformation

$$\kappa_R \ : \ \exp(R) \ \longrightarrow \ \exp(\exp(R))$$

is given at a compact Lie group G by

$$(\kappa(x)_m)_k \ = \ \Psi^*_{k,m}(x_{km}) \ \in \ R(\Sigma_k \wr \Sigma_m \wr G) \ ;$$

here the restriction is along the monomorphism (5.1.5)

$$\begin{aligned} \Psi_{k,m} \ : \ \Sigma_k \wr \Sigma_m \wr G \ & \longrightarrow \ \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, h^1), \dots, (\tau_k;\, h^k)) \ & \longmapsto \ (\sigma \natural (\tau_1, \dots, \tau_k);\, h^1 + \dots + h^k) \ . \end{aligned}$$

The analog of the following for finite groups is Satz 2.17 in [159].

Theorem 5.2.10. *Let R be a global Green functor.*

(i) *For every compact Lie group G and for every exponential sequence $x \in \exp(R; G)$, the sequence $\kappa(x)$ is an element of* $\exp(\exp(R); G)$.

(ii) *As the group varies, the maps κ form a morphism of global Green functors* $\kappa_R : \exp(R) \longrightarrow \exp(\exp(R))$, *natural in R.*

(iii) *The natural transformations*

$$\eta \ : \ \exp \ \longrightarrow \ \mathrm{Id} \qquad \textit{and} \qquad \kappa \ : \ \exp \ \longrightarrow \ \exp \circ \exp$$

make the functor exp *a comonad on the category of global Green functors.*

Proof (i) Because the square of group homomorphisms

$$\begin{tikzcd} (\Sigma_j \wr \Sigma_m \wr G) \times (\Sigma_{k-j} \wr \Sigma_m \wr G) \arrow[r, "\Phi_{j,k-j}"] \arrow[d, "\Psi_{j,m} \times \Psi_{k-j,m}"'] & \Sigma_k \wr \Sigma_m \wr G \arrow[d, "\Psi_{k,m}"] \\ (\Sigma_{jm} \wr G) \times (\Sigma_{(k-j)m} \wr G) \arrow[r, "\Phi_{jm,(k-j)m}"'] & \Sigma_{km} \wr G \end{tikzcd}$$

commutes, the elements $\Psi^*_{k,m}(x_{km})$ satisfy

$$\begin{aligned}\Phi^*_{j,k-j}(\Psi^*_{k,m}(x_{km})) &= (\Psi_{j,m} \times \Psi_{k-j,m})^*(\Phi^*_{jm,(k-j)m}(x_{km}))\\ &= (\Psi_{j,m} \times \Psi_{k-j,m})^*(x_{jm} \times x_{(k-j)m})\\ &= \Psi^*_{j,m}(x_{jm}) \times \Psi^*_{k-j,m}(x_{(k-j)m}) \ .\end{aligned}$$

This shows that for fixed $m \geq 0$, the sequence $\kappa(x)_m = (\Psi^*_{k,m}(x_{km}))_{k\geq 0}$ is exponential, i.e., an element of the ring $\exp(R; \Sigma_m \wr G)$.

The square of group homomorphisms

$$\begin{tikzcd} \Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G)) \arrow[r, "\Sigma_k \wr \Phi_{i,m-i}"] \arrow[d, "\Delta"'] & \Sigma_k \wr \Sigma_m \wr G \arrow[dd, "\Psi_{k,m}"] \\ (\Sigma_k \wr \Sigma_i \wr G) \times (\Sigma_k \wr \Sigma_{m-i} \wr G) \arrow[d, "\Psi_{k,i} \times \Psi_{k,m-i}"'] & \\ (\Sigma_{ki} \wr G) \times (\Sigma_{k(m-i)} \wr G) \arrow[r, "\Phi_{ki,k(m-i)}"'] & \Sigma_{km} \wr G \end{tikzcd} \tag{5.2.11}$$

does *not* commute; we invite the reader to check the case $k = m = 2$, $i = 1$ and $G = e$, where the phenomenon is already visible in the fact that the square

$$\begin{tikzcd} \Sigma_2 \arrow[r, "\Sigma_2 \wr \Phi_{1,1}"] \arrow[d, "\Delta"'] & \Sigma_2 \wr \Sigma_2 \arrow[d, "\natural"] \\ \Sigma_2 \times \Sigma_2 \arrow[r, "+"'] & \Sigma_4 \end{tikzcd}$$

does *not* commute. However, the square (5.2.11) does commute up to conjugation by an element of $\Sigma_{km} \wr G$. Since inner automorphisms are invisible through the eyes of a Rep-functor, we conclude that the relation

$$\begin{aligned}\left(\Phi^*_{i,m-i}(\kappa(x)_m)\right)_k &= (\Sigma_k \wr \Phi_{i,m-i})^*(\Psi^*_{k,m}(x_{km}))\\ &= \Delta^*((\Psi_{k,i} \times \Psi_{k,m-i})^*(\Phi^*_{ki,k(m-i)}(x_{km})))\\ &= \Delta^*(\Psi^*_{k,i}(x_{ki}) \times \Psi^*_{k,m-i}(x_{k(m-i)}))\\ &= \Delta^*((\kappa(x)_i)_k \times (\kappa(x)_{m-i})_k) \ = \ (\kappa(x)_i \times \kappa(x)_{m-i})_k\end{aligned}$$

holds in $R(\Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G)))$. For varying $k \geq 0$, this shows that

$$\Phi^*_{i,m-i}(\kappa(x)_m) \ = \ \kappa(x)_i \times \kappa(x)_{m-i} \qquad \text{in} \quad \exp(R; (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G)).$$

In other words, the sequence $\kappa(x) = (\kappa(x)_m)_{m\geq 0}$ is itself exponential.

(ii) The relations $\kappa(1) = 1$ and $\kappa(x \cdot y) = \kappa(x) \cdot \kappa(y)$ are straightforward from the definitions, using that multiplication is defined coordinatewise and that the restriction map $\Psi^*_{k,m}$ is multiplicative and unital. The verification of

the additivity of $\kappa_R(G) : \exp(R; G) \longrightarrow \exp(\exp(R); G)$ is more involved, and forces us to confront the double coset formula for the subgroups

$$\Sigma_k \wr' \Sigma_m \ = \ \Psi_{k,m}(\Sigma_k \wr \Sigma_m \wr G) \quad \text{and} \quad \Sigma_i \times' \Sigma_{km-i} \ = \ \Phi_{i,km-i}((\Sigma_i \wr G) \times (\Sigma_{km-i} \wr G))$$

of the wreath product $\Sigma_{km} \wr G$. To a large extent, the group G acts like a dummy, which is why we omit it from the notation for $\Sigma_k \wr' \Sigma_m$ and $\Sigma_i \times' \Sigma_{km-i}$. We specify a bijection between the set of double cosets

$$\Sigma_k \wr' \Sigma_m \backslash \Sigma_{km} \wr G / \Sigma_i \times' \Sigma_{km-i}$$

and the set of tuples $(a_0, \dots, a_m)$ of natural numbers that satisfy

$$a_0 + \dots + a_m \ = \ k \qquad \text{and} \qquad \sum\nolimits_{j=0}^m j \cdot a_j \ = \ i \ . \tag{5.2.12}$$

For $1 \leq b \leq k$ we let $j_b \in \{1, \dots, m\}$ be the unique number such that

$$a_0 + \dots + a_{j_b - 1} \ < \ b \ \leq \ a_0 + \dots + a_{j_b - 1} + a_{j_b} \ .$$

We define

$$A(a_0, \dots, a_m) \ = \ \bigcup\nolimits_{b=1}^k \{(b-1)m + 1, \dots, (b-1)m + j_b\} \ \subseteq \ \{1, \dots, km\} \ ;$$

because

$$\sum\nolimits_{b=1}^k j_b \ = \ \sum\nolimits_{j=1}^m a_j \cdot j \ = \ i \ ,$$

the set $A(a_0, \dots, a_m)$ has exactly i elements. We let $\bar{\sigma} = \bar{\sigma}(a_0, \dots, a_m) \in \Sigma_{km}$ be any permutation such that

$$\{\bar{\sigma}(1), \dots, \bar{\sigma}(i)\} \ = \ A(a_0, \dots, a_m) \ .$$

Then the elements

$$\sigma(a_0, \dots, a_m) \ = \ (\bar{\sigma}(a_0, \dots, a_m);\ 1, \dots, 1) \ \in \ \Sigma_{km} \wr G$$

are a complete set of double coset representatives as $(a_0, \dots, a_m)$ ranges over those tuples that satisfy (5.2.12). Moreover,

$$(\Sigma_k \wr' \Sigma_m)^{\sigma(a_0, \dots, a_m)} \cap (\Sigma_i \times' \Sigma_{km-i}) \ = \ \left({\prod}'^{m}_{j=0} \Sigma_{a_j} \wr' \Sigma_j \right) \times' \left({\prod}'^{m}_{j=0} \Sigma_{a_j} \wr' \Sigma_{m-j} \right)$$

and

$$(\Sigma_k \wr' \Sigma_m) \cap {}^{\sigma(a_0, \dots, a_m)}(\Sigma_i \times' \Sigma_{km-i}) \ = \ {\prod}'^{m}_{j=0} (\Sigma_{a_j} \wr' \Sigma_j) \times' (\Sigma_{a_j} \wr' \Sigma_{m-j}) \ .$$

For $x_i \in R(\Sigma_i \wr G)$ and $y_{km-i} \in R(\Sigma_{km-i} \wr G)$, the double coset formula thus becomes the relation

$$\begin{aligned} &\Psi^*_{k,m}(\mathrm{tr}_{i,km-i}(x_i \times y_{km-i})) \\ &\quad = \sum \mathrm{tr}^{\Sigma_k \wr \Sigma_m \wr G}_{\prod'(\Sigma_{a_j} \wr' \Sigma_j) \times' (\Sigma_{a_j} \wr' \Sigma_{m-j})} \left(\sigma(a_0, \dots, a_m)_\star \left(\mathrm{res}^{(\Sigma_i \wr G) \times (\Sigma_{km-i} \wr G)}_{\left(\prod' \Sigma_{a_j} \wr' \Sigma_j\right) \times' \left(\prod' \Sigma_{a_j} \wr' \Sigma_{m-j}\right)} (x_i \times y_{km-i}) \right) \right) \end{aligned}$$

where the sum ranges over all tuples $(a_0, \dots, a_m)$ that satisfy (5.2.12). If x_i and y_{km-i} are the respective components of two exponential sequences $x, y \in \exp(R; G)$, then

$$\begin{aligned}
\sigma(a_0, \dots, a_m)_\star & \left(\operatorname{res}^{(\Sigma_i \wr G)\times(\Sigma_{km-i}\wr G)}_{\left(\prod' \Sigma_{a_j} \wr' \Sigma_j\right)\times'\left(\prod' \Sigma_{a_j}\wr'\Sigma_{m-j}\right)}(x_i \times y_{km-i}) \right) \\
&= \sigma(a_0, \dots, a_m)_\star \left(\operatorname{res}^{\Sigma_i \wr G}_{\prod' \Sigma_{a_j}\wr' \Sigma_j}(x_i) \times \operatorname{res}^{\Sigma_{km-i}\wr G}_{\prod' \Sigma_{a_j}\wr'\Sigma_{m-j}}(y_{km-i}) \right) \\
&= \sigma(a_0, \dots, a_m)_\star \left(\prod\nolimits_{j=0}^m \Psi^*_{a_j, j}(x_{a_j j}) \times \prod\nolimits_{j=0}^m \Psi^*_{a_j, m-j}(y_{a_j(m-j)}) \right) \\
&= \prod\nolimits_{j=0}^m \left(\Psi^*_{a_j,j}(x_{a_j j}) \times \Psi^*_{a_j,m-j}(y_{a_j(m-j)}) \right) .
\end{aligned}$$

Given $x, y \in \exp(R; G)$ we have

$$(\kappa(x) \oplus \kappa(y))_m = \oplus_{j=0}^m \operatorname{tr}_{j,m-j}(\kappa(x)_j \times \kappa(y)_{m-j}) ,$$

where the sum on the right is taken in the group $\exp(R; \Sigma_m \wr G)$ under $\oplus$. Expanding this further we arrive at the expression

$$\begin{aligned}
((\kappa(x) \oplus \kappa(y))_m)_k &= \sum_{a_0+\dots+a_m=k} \operatorname{tr}_{a_0,\dots,a_m}\left(\prod\nolimits_{j=0}^m \operatorname{tr}_{j,m-j}(\kappa(x)_j \times \kappa(y)_{m-j})_{a_j} \right) \\
&= \sum_{a_0+\dots+a_m=k} \operatorname{tr}_{a_0,\dots,a_m}\left(\prod\nolimits_{j=0}^m \operatorname{tr}^{\Sigma_{a_j}\wr\Sigma_m}_{\Sigma_{a_j}\wr\Sigma_{j,m-j}}(\Psi^*_{a_j,j}(x_{a_j j}) \times \Psi^*_{a_j,m-j}(y_{a_j(m-j)})) \right) \\
&= \sum_{a_0+\dots+a_m=k} \operatorname{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\prod'(\Sigma_{a_j}\wr'\Sigma_j)\times'(\Sigma_{a_j}\wr'\Sigma_{m-j})}\left(\prod\nolimits_{j=0}^m \Psi^*_{a_j,j}(x_{a_j j}) \times \Psi^*_{a_j,m-j}(y_{a_j(m-j)}) \right) \\
&= \sum_{a_0+\dots+a_m=k} \operatorname{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\prod'(\Sigma_{a_j}\wr'\Sigma_j)\times'(\Sigma_{a_j}\wr'\Sigma_{m-j})}\left(\prod\nolimits_{j=0}^m \Psi^*_{a_j,j}(x_{a_j j}) \times \Psi^*_{a_j,m-j}(y_{a_j(m-j)}) \right) \\
&= \sum_{i=0}^{km} \Psi^*_{k,m}(\operatorname{tr}_{i,km-i}(x_i \times y_{km-i})) = (\kappa(x \oplus y)_m)_k
\end{aligned}$$

in the group $R(\Sigma_k \wr \Sigma_m \wr G)$. Here $\operatorname{tr}_{a_0,\dots,a_m}$ is shorthand notation for the transfer along the monomorphism

$$\Phi_{a_0,\dots,a_m} : (\Sigma_{a_0} \wr \Sigma_m \wr G) \times \dots \times (\Sigma_{a_j} \wr \Sigma_m \wr G) \times \dots \times (\Sigma_{a_m} \wr \Sigma_m \wr G) \longrightarrow \Sigma_k \wr \Sigma_m \wr G ,$$

the analog of the monomorphism (5.1.4) with multiple inputs (and for the group $\Sigma_m \wr G$ instead of G). So the map $\kappa_R(G) : \exp(R; G) \longrightarrow \exp(\exp(R); G)$ is additive, and hence a ring homomorphism.

It remains to check that the maps κ_R commute with restrictions and transfers. For every continuous homomorphism $\alpha : K \longrightarrow G$ the relations

$$\begin{aligned}
(\kappa(\alpha^*(x))_m)_k &= \Psi^*_{k,m}((\Sigma_{km} \wr \alpha)^*(x_{km})) = (\Sigma_k \wr \Sigma_m \wr \alpha)^*(\Psi^*_{k,m}(x_{km})) \\
&= (\Sigma_k \wr \Sigma_m \wr \alpha)^*((\kappa(x)_m)_k) = ((\alpha^*(\kappa(x)))_m)_k
\end{aligned}$$

hold in $R(\Sigma_k \wr \Sigma_m \wr K)$. So $\kappa_R(K) \circ \exp(R; \alpha^*) = \exp(\exp(R); \alpha^*) \circ \kappa_R(G)$.

The compatibility of κ with transfers needs another application of a double coset formula. Indeed, for every closed subgroup H of G, the group $\Sigma_{km} \wr G$ consists of a single double coset for the left $(\Sigma_k \wr \Sigma_m \wr G)$-action and right $(\Sigma_{km} \wr H)$-action, and

$$(\Sigma_k \wr \Sigma_m \wr G) \ \cap \ (\Sigma_{km} \wr H) \ = \ \Sigma_k \wr \Sigma_m \wr H \ .$$

So for every $x \in \exp(R; H)$, the relations

$$\begin{aligned}(\kappa(\mathrm{tr}_H^G(x))_m)_k \ &= \ \Psi_{k,m}^* \left(\mathrm{tr}_{\Sigma_{km}\wr H}^{\Sigma_{km}\wr G}(x_{km})\right) \ = \ \mathrm{tr}_{\Sigma_k\wr\Sigma_m\wr H}^{\Sigma_k\wr\Sigma_m\wr G}(\Psi_{k,m}^*(x_{km})) \\ &= \ \mathrm{tr}_{\Sigma_k\wr\Sigma_m\wr H}^{\Sigma_k\wr\Sigma_m\wr G}((\kappa(x)_m)_k) \ = \ \left((\mathrm{tr}_H^G(\kappa(x)))_m\right)_k\end{aligned}$$

hold in $R(\Sigma_k \wr \Sigma_m \wr G)$. Hence $\kappa \circ \mathrm{tr}_H^G = \mathrm{tr}_H^G \circ \kappa$. Altogether this shows that the maps $\kappa_R(G)$ are ring homomorphisms and compatible with restrictions and transfers, so they form a morphism of global Green functors.

(iii) We have to show that the transformation κ is coassociative, and counital with respect to η, and these are all straightforward from the definitions. The counitality relations

$$\eta_{\exp(M)} \circ \kappa_M \ = \ \mathrm{Id}_{\exp(M)} \ = \ \exp(\eta_M) \circ \kappa_M$$

come down to the facts that the homomorphism $\Psi_{k,1}$ is the result of applying $\Sigma_k \wr -$ to the preferred isomorphism $\Sigma_1 \wr G \cong G$, and that the homomorphism $\Psi_{1,n}$ is the preferred isomorphism $\Sigma_1 \wr \Sigma_n \wr G \ \cong \ \Sigma_n \wr G$. The coassociativity relation

$$\exp(\kappa_M) \circ \kappa_M \ = \ \kappa_{\exp(M)} \circ \kappa_M$$

ultimately boils down to the observation that the following square of monomorphisms commutes:

$$\begin{array}{ccc} \Sigma_k \wr \Sigma_m \wr \Sigma_n \wr G & \xrightarrow{\Sigma_k \wr \Psi_{m,n}} & \Sigma_k \wr \Sigma_{mn} \wr G \\ {\scriptstyle \Psi_{k,m}} \downarrow & & \downarrow {\scriptstyle \Psi_{k,mn}} \\ \Sigma_{km} \wr \Sigma_n \wr G & \xrightarrow[\Psi_{km,n}]{} & \Sigma_{kmn} \wr G \end{array}$$

□

Now we can finally get to the main result of this section, identifying global power functors with coalgebras over the comonad of exponential sequences. We suppose that R is a global Green functor and $P : R \longrightarrow \exp(R)$ a morphism of global Green functors. For every compact Lie group G, a sequence of operations $P^m : R(G) \longrightarrow R(\Sigma_m \wr G)$ is then defined by

$$P^m(x) \ = \ (P(x))_m \ ,$$

i.e., $P^m(x)$ is the mth component of the exponential sequence $P(x)$.

Theorem 5.2.13 (Comonadic description of global power functors)**.**

(i) *Let R be a global Green functor and $P : R \longrightarrow \exp(R)$ a morphism of global Green functors that makes R into a coalgebra over the comonad* $(\exp, \eta, \kappa)$. *Then the operations $P^m : R(G) \longrightarrow R(\Sigma_m \wr G)$ make R a global power functor.*

(ii) *The functor*

$$(\exp\text{-coalgebras}) \longrightarrow \mathcal{G}l\mathcal{P}ow , \quad (R, P) \longmapsto (R, \{P^m\}_{m\geq 0})$$

is an isomorphism of categories.

Proof (i) The fact that $P : R \longrightarrow \exp(R)$ takes values in exponential sequences is equivalent to the restriction condition of the power operations. The fact that $P : R \longrightarrow \exp(R)$ is a morphism of global Green functors encodes simultaneously the unit, contravariant naturality, transfer, multiplicativity and additivity relations of a global power functor. The identity relation $P^1 = \mathrm{Id}$ is equivalent to the counit condition of a coalgebra, i.e., that the composite

$$R \xrightarrow{P} \exp(R) \xrightarrow{\eta_R} R$$

is the identity. The transitivity relation is equivalent to

$$\exp(P) \circ P = \kappa_R \circ P ,$$

the coassociativity condition of a coalgebra.

Part (ii) is essentially reading part (i) backwards, and we omit the details. □

The interpretation of global power functors as coalgebras over a comonad has some useful consequences. In general, the forgetful functor from any category of coalgebras to the underlying category has a right adjoint 'cofree' functor. In particular, colimits in a category of coalgebras are created in the underlying category. In our situation that means:

Corollary 5.2.14. (i) *Colimits in the category of global power functors exist and are created in the underlying category of global Green functors.*

(ii) *For every global Green functor R, the maps*

$$P^m : \exp(R; G) \longrightarrow \exp(R; \Sigma_m \wr G) , \; P^m(x) = \kappa(x)_m = (\Psi^*_{k,m}(x_{km}))_{k\geq 0}$$

make the global Green functor $\exp(R)$ *a global power functor.*

(iii) *When viewed as a functor to the category of global power functors as in* (ii)*, the functor* exp *is right adjoint to the forgetful functor.*

Example 5.2.15 (Coproducts)**.** We let R and S be two global Green functors. Global Green functors are the commutative monoids, with respect to the box product, in the category of global functors. So the box product $R\Box S$ is the

coproduct in the category of global Green functors, with multiplication defined as the composite

$$R\Box S\,\Box R\Box S \xrightarrow{R\Box\tau_{S,R}\Box S} R\Box R\Box S\,\Box S \xrightarrow{\mu_R\Box\mu_S} R\Box S\ .$$

If $P : R \longrightarrow \exp(R)$ and $P' : S \longrightarrow \exp(S)$ are global power structures on R and S, then $R\Box S$ has preferred power operations specified by the morphism of global Green functors

$$R\Box S \xrightarrow{P\Box P'} \exp(R)\Box\exp(S) \longrightarrow \exp(R\Box S)\ ,$$

where the second morphism is the canonical one from the coproduct of exp to the values of exp at a coproduct. With these power operations, $R\Box S$ becomes a coproduct of R and S in the category of global power functors, by Corollary 5.2.14 (i).

This abstract definition of the power operations on $R\Box S$ can be made more explicit. Indeed, the power operations on $R\Box S$ are determined by the formula

$$P^m(x\times y) \;=\; \Delta^*(P^m(x)\times P^m(y))$$

for all compact Lie groups G and K and classes $x\in R(G)$ and $y\in S(K)$, and by the relations of the power operations. Here $\Delta : \Sigma_m\wr(G\times K) \longrightarrow (\Sigma_m\wr G)\times(\Sigma_m\wr K)$ is the diagonal monomorphism (2.2.11).

The coproduct of global power functors is realized by the coproduct of ultra-commutative ring spectra in the following sense. If E and F are ultra-commutative ring spectra, then the ring spectra morphisms $E \longrightarrow E\wedge F$ and $F \longrightarrow E\wedge F$ induce morphisms of global power functors $\underline{\pi}_0(E) \longrightarrow \underline{\pi}_0(E\wedge F)$ and $\underline{\pi}_0(F) \longrightarrow \underline{\pi}_0(E\wedge F)$, and together they define a morphism from the coproduct of global power functors

$$\underline{\pi}_0(E)\,\Box\,\underline{\pi}_0(F) \longrightarrow \underline{\pi}_0(E\wedge F)\ .$$

If E and F are globally connective and at least one of them is flat as an orthogonal spectrum, then this is an isomorphism of global functors by Proposition 4.4.15, hence an isomorphism of global power functors.

Example 5.2.16 (Localization of global power functors)**.** Now we discuss localizations of global power functors. We first consider a global Green functor R and a multiplicative subset $S\subseteq R(e)$ in the 'underlying ring', i.e., the value at the trivial group. We define a global Green functor $R[S^{-1}]$ and a morphism of global Green functors $i : R \longrightarrow R[S^{-1}]$. The value at a compact Lie group G is the ring

$$R[S^{-1}](G) \;=\; R(G)[p_G^*(S)^{-1}]\ ,$$

the localization of the ring $R(G)$ at the multiplicative subset obtained as the

image of S under the inflation homomorphism $p_G^* : R(e) \longrightarrow R(G)$. The value of the morphism i at G is the localization map $R(G) \longrightarrow R(G)[p_G^*(S)^{-1}]$. If $\alpha : K \longrightarrow G$ is a continuous homomorphism, the relation $p_G \circ \alpha = p_K$ implies that the ring homomorphism

$$\alpha^* \ : \ R(G) \ \longrightarrow \ R(K)$$

takes the set $p_G^*(S)$ to the set $p_K^*(S)$. So the universal property of localization provides a unique ring homomorphism

$$\alpha[S^{-1}]^* \ : \ R[S^{-1}](G) \ = \ R(G)[p_G^*(S)^{-1}] \ \longrightarrow \ R(K)[p_K^*(S)^{-1}] \ = \ R[S^{-1}](K)$$

such that $\alpha[S^{-1}]^* \circ i(G) = i(K) \circ \alpha^*$. Again by the universal property of localizations, this data produces a contravariant functor from the category Rep to the category of commutative rings.

Now we let H be a closed subgroup of G. We consider $R(H)$ as a module over $R(G)$ via the restriction homomorphism $\mathrm{res}^G_H : R(G) \longrightarrow R(H)$. Because $\mathrm{res}^G_H(p_G^*(S)) = p_H^*(S)$, the localization $R(H)[p_H^*(S)^{-1}] = R[S^{-1}](H)$ is also a localization of $R(H)$ at $p_G^*(S)$ as an $R(G)$-module. The reciprocity formula means that the transfer map $\mathrm{tr}^G_H : R(H) \longrightarrow R(G)$ is a homomorphism of $R(G)$-modules. The composite $R(G)$-linear map

$$R(H) \ \xrightarrow{\mathrm{tr}^G_H} \ R(G) \ \xrightarrow{i(G)} \ R(G)[p_G^*(S)^{-1}] \ = \ R[S^{-1}](G)$$

thus extends over a unique $R[S^{-1}](G)$-linear map

$$\mathrm{tr}[S^{-1}]^G_H \ : \ R[S^{-1}](H) \ = \ R(H)[p_G^*(S)^{-1}] \ \longrightarrow \ R[S^{-1}](G)$$

such that $\mathrm{tr}[S^{-1}]^G_H \circ i(H) = i(G) \circ \mathrm{tr}^G_H$. Reciprocity for $\mathrm{tr}[S^{-1}]^G_H$ is equivalent to $R[S^{-1}](G)$-linearity. The other necessary properties of transfers, such as transitivity, compatibility with inflations, vanishing for infinite Weyl groups, and the double coset formula all follow from corresponding properties in the global Green functor R and the universal property of localization. So altogether this shows that the objectwise localizations assemble into a new global Green functor $R[S^{-1}]$; the homomorphisms $i(G)$ altogether form a morphism of global Green functors $i : R \longrightarrow R[S^{-1}]$ by construction. The following universal property is also straightforward from the universal property of localizations of commutative rings and modules.

Proposition 5.2.17. *Let R be a global Green functor and S a multiplicative subset of the underlying ring $R(e)$. Let $f : R \longrightarrow R'$ be a morphism of global Green functors such that all elements of the set $f(e)(S)$ are invertible in the ring $R'(e)$. Then there is a unique homomorphism of global Green functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f}i = f$.*

Now we let R be a global power functor. If we want the localization $R[S^{-1}]$ to inherit power operations, then we need an extra hypothesis on the multiplicative subset S.

Theorem 5.2.18. *Let R be a global power functor and S a multiplicative subset of the underlying ring $R(e)$. Suppose that the multiplicative subset*

$$P^m(S) \subset R(\Sigma_m)$$

becomes invertible in the ring $R[S^{-1}](\Sigma_m)$, for every $m \geq 1$.

(i) *There is a unique extension of the global Green functor $R[S^{-1}]$ to a global power functor such that the morphism $i : R \longrightarrow R[S^{-1}]$ is a morphism of global power functors.*

(ii) *Let $f : R \longrightarrow R'$ be a morphism of global power functors such that all elements of the set $f(e)(S)$ are invertible in the ring $R'(e)$. Then there is a unique homomorphism of global power functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f}i = f$.*

Proof (i) We use the comonadic description of global power functors given in Theorem 5.2.13. This exhibits the power operations of R as a morphism of global Green functors $P : R \longrightarrow \exp(R)$.

The multiplication in the ring $\exp(R; e)$ is componentwise, and for every $s \in S$ the element $P^m(s)$ has a multiplicative inverse $t_m \in R[S^{-1}](\Sigma_m)$ by hypothesis. We claim that the sequence $t = (t_m)_{m\geq 0}$ is again exponential, and hence an element of the ring $\exp(R[S^{-1}], e)$. Indeed, for $0 < i < m$ we have

$$\begin{aligned} \Phi^*_{i,m-i}(t_m) \cdot (P^i(s) \times P^{m-i}(s)) \ &= \ \Phi^*_{i,m-i}(t_m) \cdot \Phi^*_{i,m-i}(P^m(s)) \\ &= \ \Phi^*_{i,m-i}(t_m \cdot P^m(s)) \ = \ \Phi^*_{i,m-i}(1) \ = \ 1 \ . \end{aligned}$$

Since $t_i \times t_{m-i}$ is also inverse to $P^i(s) \times P^{m-i}(s)$ and inverse are unique, we conclude that $\Phi^*_{i,m-i}(t_m) = t_i \times t_{m-i}$. This shows that the inverses form another exponential sequence.

The relation

$$P(s) \cdot t \ = \ (P^m(s) \cdot t_m)_m \ = \ 1$$

holds in the ring $\exp(R[S^{-1}]; e)$, by construction. So the composite morphism

$$R \xrightarrow{P} \exp(R) \xrightarrow{\exp(i)} \exp(R[S^{-1}])$$

takes S to invertible elements. The universal property of Proposition 5.2.17 thus provides a unique morphism of global Green functors

$$P[S^{-1}] \ : \ R[S^{-1}] \ \longrightarrow \ \exp(R[S^{-1}])$$

such that $P[S^{-1}] \circ i = \exp(i) \circ P$. Now we observe that

$$\eta_{R[S^{-1}]} \circ P[S^{-1}] \circ i \ = \ \eta_{R[S^{-1}]} \circ \exp(i) \circ P \ = \ i \circ \eta_R \circ P \ = \ i \ ;$$

the uniqueness clause in the universal property of Proposition 5.2.17 then implies that $\eta_{R[S^{-1}]} \circ P[S^{-1}]$ is the identity of $R[S^{-1}]$. Similarly,

$$\begin{aligned} \exp(P[S^{-1}]) \circ P[S^{-1}] \circ i \ &= \ \exp(P[S^{-1}]) \circ \exp(i) \circ P \ = \ \exp(P[S^{-1}] \circ i) \circ P \\ &= \ \exp(\exp(i) \circ P) \circ P \ = \ \exp(\exp(i)) \circ \exp(P) \circ P \\ &= \ \exp(\exp(i)) \circ \kappa_R \circ P \ = \ \kappa_{R[S^{-1}]} \circ \exp(i) \circ P \\ &= \ \kappa_{R[S^{-1}]} \circ P[S^{-1}] \circ i \ . \end{aligned}$$

The uniqueness clause then implies that

$$\exp(P[S^{-1}]) \circ P[S^{-1}] \ = \ \kappa_{R[S^{-1}]} \circ P[S^{-1}] \ .$$

So the morphism $P[S^{-1}]$ is a coalgebra structure over the exp comonad. The relation $P[S^{-1}] \circ i = \exp(i) \circ P$ then says that i is a morphism of exp-coalgebras, so this completes the proof.

(ii) Since morphisms of global power functors are in particular morphisms of global Green functors, the uniqueness clause follows from Proposition 5.2.17. If $f : R \longrightarrow R'$ is a morphism of global power functors such that all elements of $f(e)(S)$ are invertible in the ring $R'(e)$, then Proposition 5.2.17 provides a homomorphism of global Green functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f} i = f$. We need to show that $\bar{f}$ is also compatible with the power operations. We let

$$P[S^{-1}] \ : \ R[S^{-1}] \ \longrightarrow \ \exp(R[S^{-1}]) \qquad \text{and} \qquad P' \ : \ R' \ \longrightarrow \ \exp(R')$$

be the morphisms of global Green functors that encode the exp-coalgebra structures. We observe that

$$\begin{aligned} \exp(\bar{f}) \circ P[S^{-1}] \circ i \ &= \ \exp(\bar{f}) \circ \exp(i) \circ P \\ &= \ \exp(f) \circ P \ = \ P' \circ f \ = \ P' \circ \bar{f} \circ i \ ; \end{aligned}$$

the third equation is the hypothesis that f is a morphism of global power functors. This is an equality between morphisms of global Green functors, so the uniqueness of Proposition 5.2.17 shows that $\exp(\bar{f}) \circ P[S^{-1}] = P' \circ \bar{f}$, i.e., $\bar{f}$ is a morphism of exp-coalgebras. □

Example 5.2.19 (Localization at a subring of $\mathbb{Q}$)**.** We use Theorem 5.2.18 to show that global power functors can always be rationalized; more generally, power operations 'survive' localization at any subring of the ring $\mathbb{Q}$ of rational numbers. We consider a global power functor R and a natural number $n \geq 2$. We claim that for every $m \geq 1$ the element $P^m(n)$ of $R(\Sigma_m)$ becomes invertible in the ring $R(\Sigma_m)[1/n]$. We argue by induction over m; we start with $m = 1$,

where the relation $P^1(n) = n$ shows the claim. Now we consider $m \geq 2$ and assume that for all $0 \leq i < m$ the element $P^i(n)$ has an inverse t_i in the ring $R(\Sigma_i)[1/n]$. Then for every n-tuple $(i_1, \dots, i_n)$ that satisfies $0 \leq i_j < m$ and $i_1 + \cdots + i_n = m$ we get the relation

$$\begin{aligned}\operatorname{res}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(P^m(n)) \cdot (t_{i_1} \times \cdots \times t_{i_n}) \ &= \ (P^{i_1}(n) \times \cdots \times P^{i_n}(n)) \cdot (t_{i_1} \times \cdots \times t_{i_n}) \\ &= \ (P^{i_1}(n) \cdot t_{i_1}) \times \cdots \times (P^{i_n}(n) \cdot t_{i_n}) \ = \ 1 \times \cdots \times 1\end{aligned}$$

in the ring $R(\Sigma_{i_1} \times \cdots \times \Sigma_{i_n})[1/n]$. Thus

$$\begin{aligned}P^m(n) \ &= \ P^m(\underbrace{1 + \cdots + 1}_{n}) \ = \ \sum_{i_1+\cdots+i_n=m} \operatorname{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(P^{i_1}(1) \times \cdots \times P^{i_n}(1)) \\ &= \ n \cdot P^m(1) \ + \sum_{i_1+\cdots+i_n=m,\ i_j<m} \operatorname{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(1 \times \cdots \times 1) \\ &= \ n \ + \sum_{i_1+\cdots+i_n=m,\ i_j<m} \operatorname{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(\operatorname{res}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(P^m(n)) \cdot (t_{i_1} \times \cdots \times t_{i_n})) \\ &= \ n \ + \sum_{i_1+\cdots+i_n=m,\ i_j<m} P^m(n) \cdot \operatorname{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(t_{i_1} \times \cdots \times t_{i_n})\end{aligned}$$

Rearranging the terms gives

$$P^m(n) \cdot \left(1 - \sum_{i_1+\cdots+i_n=m,\ i_j<m} \operatorname{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(t_{i_1} \times \cdots \times t_{i_n}) \right) \ = \ n \ .$$

So $P^m(n)$ has an inverse in the ring $R(\Sigma_m)[1/n]$, and this completes the inductive step.

Now we let S be any multiplicative subset of the ring of integers. By the previous paragraph, Theorem 5.2.18 applies and provides a unique structure of global power functor on the global Green functor $R[S^{-1}]$ such that the morphism $i : R \longrightarrow R[S^{-1}]$ is a morphism of global power functors. In particular, if we let S be the set of all positive integers, we can conclude that the rationalization $\mathbb{Q} \otimes R$ of R has a unique structure of global power functor such that the localization map $R \longrightarrow \mathbb{Q} \otimes R$ is a morphism of global power functors.

Remark 5.2.20 (Monadic description of global power functors)**.** Now we explain that the category of global power functors in not only comonadic, but also monadic over the category of global Green functors. In fact, both categories are examples of algebras over *multisorted algebraic theories* (also called *colored theories*). The 'sorts' (or 'colors') are the compact Lie groups and the content of this claim is that the structure of global Green functors and global power functors can be specified by giving the values $R(G)$ at every compact Lie group, together with n-ary operations for different $n \geq 0$ and varying inputs and output, and relations between composites of those operations.

In the case of global Green functors, the operations to be specified are

- the constants given by the additive and multiplicative units in the rings $R(G)$,
- the unary operations given by the additive inverse map in $R(G)$, the restriction maps and transfers, and
- the binary operations specifying the addition and multiplication in the rings $R(G)$.

The relations include, among others, the neutrality, associativity and commutativity of addition and multiplication; the distributivity in the rings $R(G)$; the additivity of restriction and transfers; the functoriality of restrictions and transitivity of transfers; and the double coset and reciprocity formulas.

Global power functors have additional unary operations, the power operations, and additional relations as listed in Definition 5.1.6.

Proposition 5.2.21. *The forgetful functor from the category of global power functors to the category of global Green functors has a left adjoint. The category of global power functors is isomorphic to the category of algebras over the monad of this adjunction.*

Proof For the existence of the left adjoint we have to show that for every global Green functor R the functor

$$\mathcal{G}l\mathcal{P}ow \longrightarrow (\text{sets})\ , \quad S \longmapsto \mathcal{G}l\mathcal{G}re(R, S)$$

is representable. This is a formal consequence of the existence of free global power functors, colimits of global power functors and the fact that global power functors are a multi-sorted theory. We explain this in more detail, without completely formalizing the argument.

We choose a set of compact Lie groups $\{K_i\}_{i\in I}$ that contains one compact Lie group from every isomorphism class. We form the global power functor

$$L \;=\; \Box_{i\in I, x\in R(K_i)} C_{i,x}\ ,$$

a coproduct, indexed by all pairs (i, x) consisting of an index $i \in I$ and an element $x \in R(K_i)$, of free global power functors

$$C_{i,x} \;=\; C_{K_i}$$

generated by the compact Lie groups K_i. On this free global power functor we impose the minimal amount of relations so that the maps $R(K_i) \longrightarrow L(K_i)$ that send $x \in R(K_i)$ to the generator indexed by (i, x) becomes a morphism of global Green functors. Here ‘imposing relations’ means that we form another box product L' of free global power functors, with one box factor for each relation between elements in the various sets $R(K_i)$. For example, we include one factor for the sum of each pair of elements in the same set $R(K_i)$, another

factor for the product of each pair of elements in the same set $R(K_i)$, and more factors for zero elements, multiplicative units, all restriction relations and all transfers relations. Then we form a coequalizer, in the category of global power functors

$$L' \rightrightarrows L \longrightarrow F$$

where the two morphisms from L' to L restrict, on each box factor, to the morphism that represents the respective relation. The resulting global power functor then represents the functor $\mathcal{G}l\mathcal{G}re(R,-)$, so we can take F as the value of the left adjoint on R.

Since global power functors are equivalent to the coalgebras over the exp-comonad, the forgetful functor creates all colimits, in particular coequalizers. So by Beck's monadicity theorem (see for example [105, VI.7 Thm. 1]), the tautological functor from global power functors to algebras over the adjunction monad is an isomorphism of categories. □

Example 5.2.22 (Limits)**.** The category of global power functors has limits, and they are defined 'groupwise'. A special case of a limit is the product of global power functors, which is realized by the product of ultra-commutative ring spectra. Indeed, if E and F are ultra-commutative ring spectra, then so is the product $E \times F$ of the underlying orthogonal spectra, and the canonical map

$$\underline{\pi}_0(E \times F) \longrightarrow \underline{\pi}_0(E) \times \underline{\pi}_0(F)$$

is an isomorphism of global power functors (by Corollary 3.1.37 (ii)).

5.3 Examples

In this section we discuss various examples of and constructions with global power functors, and how these are realized topologically by ultra-commutative ring spectra. These examples include the Burnside ring global power functor (Example 5.3.1), the global functor represented by an abelian compact Lie group (Proposition 5.3.3), and constant global power functors (Example 5.3.7). The orthogonal spectrum $\mathcal{H}A$ consisting of the A-linearizations of spheres (Construction 5.3.8) tries to be an Eilenberg-Mac Lane spectrum for the constant global functor, and it is so on finite groups. Closely related to $\mathcal{H}\mathbb{Z}$ is the infinite symmetric product spectrum Sp^∞, see Example 5.3.10; this ultra-commutative ring spectrum comes with a filtration by finite symmetric products, realizing an interesting filtration of the Burnside ring global functor on $\underline{\pi}_0$, see (5.3.13). We close this section with a discussion of the complex representation ring global functor (Example 5.3.18), and a global view on 'explicit Brauer induction' (Remark 5.3.19).

Example 5.3.1 (Burnside ring global functor)**.** The Burnside ring global functor $\mathbb{A} = \mathbf{A}(e, -)$ is the unit object for the box product of global functors, and hence an initial object in the category of global Green functors. Initial objects are examples of colimits, so Corollary 5.2.14 (i) implies that $\mathbb{A}$ has a unique structure of global power functor. Indeed, there is a unique morphism $P : \mathbb{A} \longrightarrow \exp(\mathbb{A})$ of global Green functors (since $\mathbb{A}$ is initial), and the coalgebra diagrams commute (again since $\mathbb{A}$ is initial). With these power operations, $\mathbb{A}$ is also an initial global power functor.

We can make the power operations in the Burnside ring global functor more explicit. Indeed, the group $\mathbb{A}(G)$ is free abelian with a basis given by the elements $t_H = \mathrm{tr}_H^G(p_H^*(1))$ for every conjugacy class of closed subgroups $H \leq G$ with finite Weyl group, where $p_H : H \longrightarrow e$ is the unique homomorphism. On these generators, the naturality properties of a global power functor force the power operations to be

$$\begin{aligned} P^m(t_H) \ &= \ P^m(\mathrm{tr}_H^G(p_H^*(1))) \ = \ \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}((\Sigma_m \wr p_H)^*(P^m(1))) \\ &= \ \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}((\Sigma_m \wr p_H)^*(p_{\Sigma_m}^*(1))) \ = \ \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}(p_{\Sigma_m \wr H}^*(1)) \ = \ t_{\Sigma_m \wr H} \ . \end{aligned} \tag{5.3.2}$$

This determines the power operations in general by the additivity property, and also shows the uniqueness.

When restricted to *finite* groups, the ring $\mathbb{A}(G)$ is isomorphic to the Grothendieck group of finite G-sets, and in this description the power operations are given by raising a finite G-set to a power, i.e., the power map

$$P^m \ : \ \mathbb{A}(G) \ \longrightarrow \ \mathbb{A}(\Sigma_m \wr G)$$

takes the class of a finite G-set S to the class of the $(\Sigma_m \wr G)$-set S^m. Indeed, for the additive generator $[G/H] = t_H$ of $\mathbb{A}(G)$ this is the relation (5.3.2), and for general finite G-sets it follows from the additivity formula for power operations and the fact that for two finite G-sets S and T the power $(S \amalg T)^m$ is $(\Sigma_m \wr G)$-equivariantly isomorphic to the coproduct

$$\coprod_{k=0}^m (\Sigma_m \wr G) \times_{(\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)} (S^k \times T^{m-k}) \ .$$

The canonical power operations in the Burnside ring global functor correspond to the homotopy theoretic power operations for the global sphere spectrum. Indeed, since $\mathbb{A}$ is initial in both the category of global Green functors and in the category of global power functors, any isomorphism of global Green functors is automatically compatible with power operations. In other words,

we can conclude that the square

$$\begin{tikzcd} \mathbb{A}(G) \arrow[r, "P^m"] \arrow[d, "\cong"'] & \mathbb{A}(\Sigma_m \wr G) \arrow[d, "\cong"] \\ \pi_0^G(\mathbb{S}) \arrow[r, "P^m"'] & \pi_0^{\Sigma_m \wr G}(\mathbb{S}) \end{tikzcd}$$

commutes for all G and m without having to go back to the definition of the operations in $\underline{\pi}_0(\mathbb{S})$; the vertical maps are the action on the multiplicative units.

The previous example generalizes to the represented global functor $\mathbf{A}_A = \mathbf{A}(A, -)$ for every *abelian* compact Lie group, which has a preferred structure of a global power functor. The following proposition is a stable analog of Proposition 2.2.23, which describes a structure of global power monoid on the Rep-functor represented by A.

Proposition 5.3.3. *Let A be an abelian compact Lie group. The represented global functor $\mathbf{A}_A$ has a unique structure of global power functor subject to the following two conditions:*

- *The multiplication is the composite*

$$\mathbf{A}_A \Box \mathbf{A}_A \;\cong\; \mathbf{A}_{A\times A} \xrightarrow{\mathbf{A}(\mu^*,-)} \mathbf{A}_A \;,$$

  where $\mu : A \times A \longrightarrow A$ is the multiplication and $\mu^* \in \mathbf{A}(A, A \times A)$ is the associated restriction morphism.
- *The power operations satisfy*

$$P^m(1_A) \;=\; p_m^* \quad in \quad \mathbf{A}(A, \Sigma_m \wr A) \;,$$

  the inflation operation of the continuous homomorphism

$$p_m \;:\; \Sigma_m \wr A \longrightarrow A \;, \quad (\sigma; a_1, \dots, a_m) \longmapsto a_1 \cdot \ldots \cdot a_m \;.$$

Moreover, for every global power functor R the evaluation map

$$\mathcal{G}l\mathcal{P}ow(\mathbf{A}_A, R) \longrightarrow R(A) \;, \quad f \longmapsto f_A(1_A)$$

is injective with image the set of those $x \in R(A)$ such that $P^m(x) = p_m^*(x)$ for all $m \geq 1$.

Proof Since the convolution product of two represented functors is again represented, morphisms of global functors $\mathbf{A}_A \Box \mathbf{A}_A \longrightarrow \mathbf{A}_A$ are determined by their effect on the element $1_A \Box 1_A \in (\mathbf{A}_A \Box \mathbf{A}_A)(A \times A)$. The first condition fixes this to be the operation μ^*. So there is at most one structure of global Green functor on $\mathbf{A}_A$ that satisfies the first condition. Since this representable global functor is freely generated by the identity 1_A in $\mathbf{A}(A, A)$, the power operations

are all determined by naturality from the effect on this generator. So together we have shown that there is at most one structure of global power functor that satisfies the two conditions.

Now we construct the desired structure. The associativity, commutativity and unitality of the multiplication $\mu : A \times A \longrightarrow A$ readily imply that the associated multiplication of $\mathbf{A}_A$ is associative, commutative and unital. So this multiplication makes $\mathbf{A}_A$ a global Green functor. The relations

$$p_m \circ \Phi_{k,m-k} = p_k \cdot p_{m-k} : (\Sigma_k \wr A) \times (\Sigma_{m-k} \wr A) \longrightarrow A$$

imply that the sequence $(p_m^*)_{m\geq 0}$ is exponential. So the Yoneda lemma provides a unique morphism of global functors

$$P = (P^m)_{m\geq 0} : \mathbf{A}_A \longrightarrow \exp(\mathbf{A}_A) \qquad \text{such that} \qquad P^m(1_A) = p_m^* .$$

We claim that P is a morphism of global Green functors. The unitality follows from the fact that the restriction of p_m to $\Sigma_m \wr e$ is the trivial homomorphism. Since the global functor $\mathbf{A}_A \Box \mathbf{A}_A$ is representable by $A \times A$, the Yoneda lemma reduces the multiplicativity property to the relation

$$\exp(\mathbf{A}_A, \mu^*)((p_m^*)_m) = \exp(\mathbf{A}_A, q_1^*)((p_m^*)_m) \cdot \exp(\mathbf{A}_A, q_2^*)((p_m^*)_m)$$

in the ring $\exp(\mathbf{A}_A, A \times A)$, where $q_1, q_2 : A \times A \longrightarrow A$ are the two projections. This relation, in turn, is a consequence of the commutativity of the following diagram of group homomorphisms:

$$\begin{tikzcd} \Sigma_m \wr (A \times A) \arrow[r, "\Sigma_m \wr \mu"] \arrow[d, "\Delta"'] & \Sigma_m \wr A \arrow[d, "p_m"] \\ (\Sigma_m \wr A) \times (\Sigma_m \wr A) \arrow[r, "p_m \cdot p_m"'] & A \end{tikzcd}$$

Here the left vertical map is a diagonal, given by

$$\Delta(\sigma;\, (a_1, \bar{a}_1), \dots, (a_m, \bar{a}_m)) = ((\sigma;\, a_1, \dots, a_m), (\sigma;\, \bar{a}_1, \dots, \bar{a}_m)) .$$

We conclude the construction of the global power structure by showing that the morphism $P : \mathbf{A}_A \longrightarrow \exp(\mathbf{A}_A)$ provides a coalgebra structure over the exp-comonad. It suffices to show the relations $\eta_{\mathbf{A}_A} \circ P = \mathrm{Id}$ and $\exp(P) \circ P = \kappa_{\mathbf{A}_A} \circ P$ on the universal element 1_A in $\mathbf{A}_A(A)$, one more time by the Yoneda lemma. For the first one, this boils down to the fact that $p_1 : \Sigma_1 \wr A \longrightarrow A$ is the preferred isomorphism. The second one follows from the commutativity of the

following square:

$$\begin{tikzcd}
\Sigma_k \wr \Sigma_m \wr A \arrow[r, "\Psi_{k,m}"] \arrow[d, "\Sigma_k \wr p_m"'] & \Sigma_{km} \wr A \arrow[d, "p_{km}"] \\
\Sigma_k \wr A \arrow[r, "p_k"'] & A
\end{tikzcd}$$

□

The global power functor $\mathbf{A}_A$ described in Proposition 5.3.3 is realized by an ultra-commutative ring spectrum. Indeed, Construction 2.3.23 provides a global classifying space $R(BA)$ for the abelian compact Lie group A with an ultra-commutative multiplication. The associated unreduced suspension spectrum $\Sigma^\infty_+ R(BA)$ is then an ultra-commutative ring spectrum. Theorem 4.2.5 shows that $\underline{\pi}_0(\Sigma^\infty_+ R(BA))$ is freely generated, as a global functor, by the stable tautological class $e_A \in \pi_0^A(\Sigma^\infty_+ R(BA))$, the stabilization of the unstable tautological class u_A. The characterization of the multiplication and power operations on $\mathbf{A}_A$ only involve restriction maps (but no transfers); so the fact that $\underline{\pi}_0(\Sigma^\infty_+ R(BA))$ realizes the global power structure of Proposition 5.3.3 follows from the corresponding unstable relations in the global power monoid $\underline{\pi}_0(R(BA))$, as explained in Example 2.3.23.

For a compact Lie group G we define

$$u_G^{ucom} \in \pi_0^G(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}} G))$$

as the image of the unstable tautological class $u_G \in \pi_0^G(B_{\mathrm{gl}} B)$, defined in (1.5.11), under the two composites in the commutative square

$$\begin{tikzcd}
\pi_0^G(B_{\mathrm{gl}} B) \arrow[r, "\pi_0^G(\eta)"] \arrow[d, "\sigma^G"'] & \pi_0^G(\mathbb{P}(B_{\mathrm{gl}} B)) \arrow[d, "\sigma^G"] \\
\pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} B) \arrow[r, "\pi_0^G(\Sigma^\infty_+ \eta)"'] & \pi_0^G(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}} B))
\end{tikzcd}$$

where $\eta : B_{\mathrm{gl}} G \longrightarrow \mathbb{P}(B_{\mathrm{gl}} G)$ is the adjunction unit, i.e., the inclusion of the homogeneous summand for $m = 1$. The stabilization map σ^G was defined in (4.1.10). Here, as usual, we have made an implicit choice of non-zero faithful G-representation V, and the global classifying space is $B_{\mathrm{gl}} G = \mathbf{L}_{G,V}$. We already know that

- the class u_G freely generates $\underline{\pi}_0(B_{\mathrm{gl}} G)$ as a Rep-functor (Proposition 1.5.12 (ii)),
- the class $u_G^{umon} = \pi_0^G(\eta)(u_G)$ freely generates $\underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}} G))$ as a global power monoid (Theorem 2.2.24 (ii)), and
- the class $e_G = \sigma^G(u_G)$ freely generates $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} G)$ as a global functor (Proposition 4.2.5).

The next proposition complements these results and shows that the global power functor $\underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}B))$ is freely generated by the class u_G^{ucom}. The free global power functor C_G generated by the compact Lie group G was introduced in Example 5.1.19. The underlying global functor is

$$C_G \ = \ \bigoplus\nolimits_{m\geq 0} \mathbf{A}(\Sigma_m \wr G, -) \ ,$$

and there is a preferred element $1_G \in C_G(G)$, the identity operation of π_0^G in the summand indexed by $m = 1$. The adjective 'free' is justified by Proposition 5.2.6 (ii), which says that for every global power functor R and every element $x \in R(G)$ there is a unique morphism of global power functors $f : C_G \longrightarrow R$ such that $f(1_G) = x$.

Proposition 5.3.4. *Let G be a compact Lie group.*

(i) *The unique morphism of global power functors*

$$\varphi \ : \ C_G \ \longrightarrow \ \underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}G))$$

satisfying $\varphi(1_G) = u_G^{ucom}$ *is an isomorphism.*

(ii) *For every global power functor R and every element* $x \in R(G)$ *there is a unique morphism of global power functors* $f : \underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow R$ *such that* $f(u_G^{ucom}) = x$.

Proof (i) The ultra-commutative monoid $\mathbb{P}(B_{\mathrm{gl}}G)$ is the disjoint union of the orthogonal spaces $\mathbb{P}^m(B_{\mathrm{gl}}G)$ for $m \geq 0$. Moreover, $\mathbb{P}^m(B_{\mathrm{gl}}G)$ is a global classifying space for the group $\Sigma_m \wr G$, in such a way that the class $[m](u_G) \in \pi_0^G(\mathbb{P}^m(B_{\mathrm{gl}}G))$ matches the unstable tautological class $u_{\Sigma_m \wr G}$, by Example 2.2.19. So by Proposition 4.2.5 the morphism of global functors

$$\psi_m \ : \ \mathbf{A}(\Sigma_m \wr G, -) \ \longrightarrow \ \underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}^m(B_{\mathrm{gl}}G))$$

classified by $\sigma^{\Sigma_m \wr G}([m](u_G)) \ = \ P^m(\sigma^G(u_G))$ in $\pi_0^{\Sigma_m \wr G}(\Sigma^\infty_+ \mathbb{P}^m(B_{\mathrm{gl}}G))$ is an isomorphism. The functor Σ^∞_+ takes a disjoint union of orthogonal spaces to a wedge of orthogonal spectra, and equivariant stable homotopy groups take wedges to sums. Hence the morphism

$$\begin{aligned}\psi \ = \ \bigoplus\nolimits_{m\geq 0} \psi_m \ : \ C_G \ &= \bigoplus\nolimits_{m\geq 0} \mathbf{A}(\Sigma_m \wr G, -) \\ &\longrightarrow \bigoplus\nolimits_{m\geq 0} \underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}^m(B_{\mathrm{gl}}G)) \ = \ \underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}G))\end{aligned}$$

is an isomorphism of global functors.

The power operations in C_G satisfy $[m](1_G) = 1_{\Sigma_m \wr G}$; since φ is a morphism of global power functors, it also satisfies

$$\varphi(1_{\Sigma_m \wr G}) \ = \ \varphi([m](1_G)) \ = \ [m](\varphi(1_G)) \ = \ [m](u_G^{umon}) \ = \ \sigma^G([m](u_G)) \, .$$

So the morphisms φ and ψ coincide on the classes $1_{\Sigma_m \wr G}$ for all $m \geq 0$. Since

these classes generated C_G as a global functor, we conclude that $\varphi = \psi$. Since ψ is an isomorphism, this proves the claim.

Part (ii) is the combination of (i) and the freeness property of C_G from Proposition 5.2.6 (ii). □

Example 5.3.5 (Monoid rings)**.** We let R be a global Green functor and M a commutative monoid. We denote by $R[M]$ the *monoid ring functor*; its value at a compact Lie group G is given by

$$(R[M])(G) \ = \ R(G)[M] \ ,$$

the monoid ring of M over $R(G)$. The structure as global functor is induced from the structure of R and constant in M. The multiplication and unit are induced from the multiplication and units of R and M. The global Green functor $R[M]$ can be characterized as follows by the functor that it represents: for every global Green functor S, morphisms $R[M] \longrightarrow S$ biject with pairs consisting of a morphism of global Green functors $R \longrightarrow S$ and a monoid homomorphism $M \longrightarrow (S(e), \cdot)$ to the multiplicative monoid of the underlying ring $S(e)$.

Now suppose that R is even a global power functor. Then $R[M]$ inherits a natural structure as global power functor: we define the power operation

$$P^m \ : \ R(G)[M] \ \longrightarrow \ R(\Sigma_m \wr G)[M]$$

by $P^m(r \cdot x) = P^m(r) \cdot x^m$ for $r \in R(G)$ and $x \in M$, and then we extend this by additivity to general elements in $R(G)[M]$. In this upgraded setting, the global power functor $R[M]$ has a similar characterization as in the previous paragraph: for a global power functor S, we let

$$\mathrm{Mon}(S) \ = \ \{x \in S(e) \mid P^m(x) = (p_{\Sigma_m})^*(x) \text{ for all } m \geq 1\}$$

be the set of *monoid-like elements* of S; here $p_{\Sigma_m} : \Sigma_m \longrightarrow e$ is the unique homomorphism. The set $\mathrm{Mon}(S)$ is a submonoid of the multiplicative monoid of $S(e)$. Then morphisms $R[M] \longrightarrow S$ biject with pairs consisting of a morphism of global power functors $R \longrightarrow S$ and a monoid homomorphism $M \longrightarrow \mathrm{Mon}(S)$ to the monoid-like elements of S.

Now we let E be an ultra-commutative ring spectrum. Then the monoid ring spectrum $E[M] = E \wedge M_+$ is another ultra-commutative ring spectrum, see Construction 4.1.19. The unit of M induces a morphism of ultra-commutative ring spectra $E \longrightarrow E[M]$ that induces a morphism of global power functors $\underline{\pi}_0(E) \longrightarrow \underline{\pi}_0(E[M])$. Similarly, the unit of E induces a monoid homomorphism $M \longrightarrow E(0) \wedge M_+$, and this induces a monoid homomorphism $M \longrightarrow \mathrm{Mon}(\underline{\pi}_0(E[M]))$ to the monoid-like elements in the multiplicative monoid of the ring $\pi_0^e(E[M])$. The universal property of the algebraic monoid ring com-

bines these two pieces of data into a morphism of global power functors

$$(\underline{\pi}_0 E)[M] \longrightarrow \underline{\pi}_0(E[M])\ . \tag{5.3.6}$$

Additively, the left-hand side is a direct sum of copies of $\underline{\pi}_0(E)$, indexed by the elements of M. Similarly, the orthogonal spectrum $E[M]$ is a wedge of copies of E, indexed by the elements of M. So the right-hand side $\underline{\pi}_0(E[M])$ is also a direct sum of copies of $\underline{\pi}_0(E)$ (by Corollary 3.1.37 (i)). We conclude that the morphism of global power functors (5.3.6) is an isomorphism.

Example 5.3.7 (Constant global power functors and Eilenberg-Mac Lane spectra)**.** We let B be a commutative ring. Then the constant global functor $\underline{B}$ (see Example 4.2.8 (iii)) is naturally a global Green functor, via the ring structure of B. For every compact Lie group G, every exponential sequence $x \in \exp(\underline{B}; G)$ satisfies $x_n = (x_1)^n$; so an exponential sequence is completely determined by the element x_1, which can be chosen arbitrarily. Hence the morphism

$$\eta_{\underline{B}} \ :\ \exp(\underline{B}) \longrightarrow \underline{B}$$

is an isomorphism of global Green functors. So when restricted to constant global Green functors, the exp comonad is isomorphic to the identity. Thus $\underline{B}$ has a unique structure of coalgebra over the comonad exp, with structure morphism the inverse of $\eta_{\underline{B}}$.

The preferred global power structure on the constant global functor $\underline{B}$ can also be derived more directly. In fact, since the power operation P^m has to be an equivariant refinement of the mth power map in $R(G)$ (compare Remark 2.2.9), the only possibility to define P^m is as

$$P^m \ :\ \underline{B}(G) = B \longrightarrow B = \underline{B}(\Sigma_m \wr G)\ , \quad b \longmapsto b^m\ ,$$

the mth power in the ring B.

Since $\underline{B}$ is constant, every morphism $R \longrightarrow \underline{B}$ of global functors is determined by the map $R(e) \longrightarrow \underline{B}(e) = B$. Moreover, every ring homomorphism $\psi : R(e) \longrightarrow B$ extends uniquely to a morphism of global power functors $\hat{\psi} : R \longrightarrow \underline{B}$ by defining its value at G as the composite

$$R(G) \xrightarrow{\mathrm{res}^G_e} R(e) \xrightarrow{\psi} B\ .$$

In other words, the functor

$$(\text{commutative rings}) \longrightarrow \mathcal{G}l\mathcal{P}ow\ , \quad B \longmapsto \underline{B}$$

is right adjoint to the functor that takes a global power functor R to the underlying ring $R(e)$.

The author does not know an explicit point-set level model for an ultra-commutative ring spectrum that realizes the constant global power functor $\underline{B}$ of a commutative ring B. The Eilenberg-Mac Lane spectrum $\mathcal{H}B$, discussed in the next construction, is an ultra-commutative ring spectrum and tries to realize $\underline{B}$: the global power functor $\underline{\pi}_0(\mathcal{H}B)$ is indeed constant *on finite groups*, but the restriction maps are not generally isomorphisms, compare Example 5.3.14. The morphism of global power functors $\underline{\pi}_0(\mathcal{H}B) \longrightarrow \underline{B}$ adjoint to the identification $\pi_0^e(\mathcal{H}B) \cong B$ is thus an isomorphism at finite groups (and some other compact Lie groups), but *not* an isomorphism in general.

Construction 5.3.8. Let A be an abelian group. The *Eilenberg-Mac Lane spectrum* $\mathcal{H}A$ is defined at an inner product space V by

$$(\mathcal{H}A)(V) \;=\; A[S^V]\;,$$

the reduced A-linearization of the V-sphere. The orthogonal group $O(V)$ acts through the action on S^V and the structure map $\sigma_{V,W} : S^V \wedge (\mathcal{H}A)(W) \longrightarrow (\mathcal{H}A)(V \oplus W)$ is given by

$$S^V \wedge A[S^W] \;\longrightarrow\; A[S^{V\oplus W}]\;, \quad v \wedge \left(\sum\nolimits_i m_i \cdot w_i\right) \;\longmapsto\; \sum\nolimits_i m_i \cdot (v \wedge w_i)\;.$$

Non-equivariantly, $A[S^V]$ is an Eilenberg-Mac Lane space of type (A, n), where $n = \dim(V)$. Hence the underlying non-equivariant homotopy type of $\mathcal{H}A$ is that of an Eilenberg-Mac Lane spectrum for A.

As we shall now discuss, the equivariant homotopy groups of $\mathcal{H}A$ are in general *not* concentrated in dimension 0, and hence $\mathcal{H}A$ is not the Eilenberg-Mac Lane spectrum of a global functor. However, on *finite* groups, the equivariant behavior of $\mathcal{H}A$ is as expected. We recall from Definition 4.5.6 that an orthogonal spectrum is *left induced* from the global family $\mathcal{F}in$ of finite groups if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}in} \longrightarrow \mathcal{GH}$ from the $\mathcal{F}in$-global homotopy category. The inclusion of generators is a homeomorphism $A \cong A[S^0] = (\mathcal{H}A)(0)$ and this induces a bijection

$$A \;\cong\; \pi_0(A) \;=\; [S^0, (\mathcal{H}A)(0)]\;.$$

Since $\mathcal{H}A$ is in particular a non-equivariant Ω-spectrum (for example by Proposition 5.3.9 below), the composite of this bijection with the stabilization map is an isomorphism of abelian groups

$$A \;\cong\; \pi_0^e(\mathcal{H}A)\;.$$

The restriction maps

$$\mathrm{res}^G_e \;:\; \pi_0^G(\mathcal{H}A) \;\longrightarrow\; \pi_0^e(\mathcal{H}A) \;\cong\; A \;=\; \underline{A}(G)$$

form a morphism of global functors $\underline{\pi}_0(\mathcal{H}A) \longrightarrow \underline{A}$ to the constant global functor with value A. Since $\mathcal{H}A$ is globally connective (by the next proposition), there is a unique morphism

$$\rho \ : \ \mathcal{H}A \ \longrightarrow \ H\underline{A}$$

in the global stable homotopy category that realizes the morphism on π_0^e.

Proposition 5.3.9. *For every abelian group A the Eilenberg-Mac Lane spectrum $\mathcal{H}A$ is globally connective, left induced from the global family $\mathcal{F}in$ of finite groups and a $\mathcal{F}in$-Ω-spectrum. The morphism $\rho : \mathcal{H}A \longrightarrow H\underline{A}$ is a $\mathcal{F}in$-global equivalence.*

Proof The orthogonal spectrum $\mathcal{H}A$ is obtained by evaluation of a $\mathbf{\Gamma}$-space $\underline{A}$ on spheres, where $\underline{A}(T) = A[T]$ is the reduced A-linearization of a finite based set T. So $\mathcal{H}A$ is left induced from the global family $\mathcal{F}in$ by Proposition 4.5.15 (ii).

The $\mathbf{\Gamma}$-space $\underline{A}$ is actually a $\mathbf{\Gamma}$-*set*, i.e., all its values have the discrete topology. The latching map $l_n : \operatorname{colim}_{U\subsetneq\{1,\dots,n\}} A[U] \longrightarrow A[\{1,\dots,n\}]$ defined in (B.31) is thus an injective map (by Proposition B.32 (ii)) between discrete Σ_n-spaces, hence a Σ_n-cofibration. It is then a $(G\times\Sigma_n)$-cofibration if we let a compact Lie group G act trivially. Proposition B.43 (ii) then shows that the orthogonal G-spectrum $\underline{A}(\mathbb{S}) = \mathcal{H}A$ (with trivial G-action) is equivariantly connective. Since G was arbitrary, this proves that $\mathcal{H}A$ is globally connective.

Dos Santos shows in [142] that for every representation V of a finite group G, the G-space $\mathcal{H}A(V) = A[S^V]$ is an equivariant Eilenberg-Mac Lane space of type (A, V), i.e., the G-space $\operatorname{map}_*(S^V, A[S^V])$ has homotopically discrete fixed-points for all subgroups H of G and the natural map

$$A \ \longrightarrow \ [S^V, A[S^V]]^H \ = \ \pi_0\left(\operatorname{map}_*^H(S^V, A[S^V])\right)$$

sending $m \in A$ to the homotopy class of $m\cdot - : S^V \longrightarrow A[S^V]$ is an isomorphism. This shows that $\mathcal{H}A$ is a $\mathcal{F}in$-Ω-spectrum for the constant global functor $\underline{A}$.

We offer an independent proof of the $\mathcal{F}in$-Ω-property via the $\mathbf{\Gamma}$-G-space techniques of Segal and Shimakawa [155, 157], in our adaptation of Theorem B.61. We let G act trivially on the $\mathbf{\Gamma}$-space $\underline{A}$. The G-maps

$$P_S \ : \ \underline{A}(S_+) \ \longrightarrow \ \operatorname{map}(S, \underline{A}(1_+)) \ = \ A^S \ , \quad P_S(x)(s) \ = \ \underline{A}(p_s)(x) \ ,$$

are homeomorphisms, so in particular G-weak equivalences. So $\underline{A}$ is a special $\mathbf{\Gamma}$-G-space in the sense of Shimakawa [157, Def. 1.3], see also Definition B.49 below. Since $\pi_0(\underline{A}(1_+))$ is isomorphic to A, and hence a group (as opposed to a monoid only), the $\mathbf{\Gamma}$-G-space $\underline{A}$ is very special. Since $\underline{A}$ is also G-cofibrant, Theorem B.61 shows that for every finite group G and every pair

of G-representations V and W, the adjoint structure map $\tilde{\sigma}_{V,W} : A[S^W] \longrightarrow \mathrm{map}_*(S^V, A[S^{V\oplus W}])$ is a G-weak equivalence. □

The properties mentioned in the previous proposition do *not* generalize to compact Lie groups of positive dimension, i.e., contrary to what one may expect at first, $(\mathcal{H}A)_G$ is not generally a G-Ω-spectrum, not all restriction maps in dimension 0 are isomorphisms (see Example 5.3.14), and the groups $\pi_*^G(\mathcal{H}A)$ need not be concentrated in dimension 0 (see Theorem 5.3.16).

We now consider the important special case $A = \mathbb{Z}$. The equivariant homotopy group $\pi_0^G(\mathcal{H}\mathbb{Z})$ may be larger than a single copy of the integers, and we are now going to give a presentation of $\pi_0^G(\mathcal{H}\mathbb{Z})$. Before we do so, we compare $\mathcal{H}\mathbb{Z}$ to the 'infinite symmetric product of the sphere spectrum'.

Example 5.3.10 (Infinite symmetric product)**.** There is no essential difference if we consider the infinite symmetric product Sp^∞ (i.e., the reduced free abelian monoid) instead of the reduced free abelian group $\mathbb{Z}[-]$ generated by representation spheres: the level-wise inclusions of the free abelian monoids into the free abelian groups provide a morphism of ultra-commutative ring spectra

$$Sp^\infty = \{Sp^\infty(S^V)\}_V \longrightarrow \{\mathbb{Z}[S^V]\}_V = \mathcal{H}\mathbb{Z}\,, \tag{5.3.11}$$

and this morphism is a global equivalence by the following proposition.

In his unpublished preprint [155], Segal argues that for every finite group G and every G-representation V with $V^G \neq 0$ the map

$$Sp^\infty(S^V) \longrightarrow \mathbb{Z}[S^V]$$

is a G-weak equivalence. A published proof of this fact appears as Proposition A.6 in Dugger's paper [45]. This generalizes to compact Lie groups:

Proposition 5.3.12. *Let G be a compact Lie group.*

(i) *For every G-representation V such that $V^G \neq 0$, the natural map $Sp^\infty(S^V) \longrightarrow \mathbb{Z}[S^V]$ is a G-weak equivalence.*
(ii) *The morphism* (5.3.11) *is a global equivalence of orthogonal spectra from Sp^∞ to $\mathcal{H}\mathbb{Z}$.*

Proof (i) By application to all closed subgroups of G, it suffices to show that the fixed-point map $(Sp^\infty(S^V))^G \longrightarrow (\mathbb{Z}[S^V])^G$ is a non-equivariant weak equivalence. We let $G^\circ \leq G$ denote the connected component of the identity element and $\bar{G} = G/G^\circ$ the finite group of components of G. To calculate G-fixed-points we can first take G°-fixed-points and then $\bar{G}$-fixed-points. Proposition B.42 provides homeomorphisms

$$(Sp^\infty(S^V))^G \cong (Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \qquad \text{and} \qquad (\mathbb{Z}[S^V])^G \cong (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}}\,.$$

Since V^{G° is an orthogonal representation of the finite group $\bar{G}$ the map

$$(Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \longrightarrow (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}} ,$$

is a weak equivalence by [45, Prop. A.6]. Strictly speaking, Dugger's proposition is stated only for geometric realizations of $\bar{G}$-simplicial sets; since the representation sphere $S^{V^{G^\circ}}$ admits the structure of a finite $\bar{G}$-CW-complex, it is $\bar{G}$-homotopy equivalent to the realization of a $\bar{G}$-simplicial set, compare Proposition B.46 (ii). So we can also apply Dugger's result in our situation. Part (ii) is then immediate from (i). □

The infinite symmetric product of any based space X has a natural filtration

$$X = Sp^1(X) \subseteq Sp^2(X) \subseteq \ldots \subseteq Sp^n(X) \subseteq \ldots$$

by the finite symmetric products. Evaluation on spheres provides orthogonal spectra $Sp^n = Sp^n(\mathbb{S})$ and a filtration

$$\mathbb{S} = Sp^1 \subseteq Sp^2 \subseteq \ldots \subseteq Sp^n \subseteq \ldots$$

of Sp^∞ by orthogonal subspectra. We study the equivariant and global properties of this filtration in [144]. The 0th homotopy group global functor of the orthogonal spectrum Sp^n has a very compact description as follows. We recall that the Burnside ring $\mathbb{A}(G)$ of a compact Lie group G is freely generated by the classes $t_H^G = \mathrm{tr}_H^G(p_H^*(1))$ where H runs through a set of representatives of the conjugacy classes of subgroups of G with finite Weyl group. Theorem 3.12 of [144] shows that the global functor $\underline{\pi}_0(Sp^n)$ is the quotient of the Burnside ring global functor by the global subfunctor generated by the element $n \cdot 1 - t_{\Sigma_{n-1}}^{\Sigma_n}$ in $\mathbb{A}(\Sigma_n)$,

$$\underline{\pi}_0(Sp^n) \cong \mathbb{A}/\langle n \cdot 1 - t_{\Sigma_{n-1}}^{\Sigma_n} \rangle . \tag{5.3.13}$$

Letting n go to infinity and combining this with part (ii) of Proposition 5.3.12 gives the following calculation of the global functor $\underline{\pi}_0(\mathcal{H}\mathbb{Z})$. We let I_∞ denote the global subfunctor of the Burnside ring global functor $\mathbb{A}$ generated by the classes $n \cdot 1 - t_{\Sigma_{n-1}}^{\Sigma_n}$ for all $n \geq 1$. We show in [144, Thm. 3.12] that the unit map $\mathbb{S} \longrightarrow Sp^\infty$ induces an isomorphism of global functors

$$\mathbb{A}/I_\infty \cong \underline{\pi}_0(Sp^\infty) \cong \underline{\pi}_0(\mathcal{H}\mathbb{Z}) .$$

This calculation can be made even more explicit. Elementary algebra (see [144, Prop. 4.1]) identifies the value $I_\infty(G)$ at a compact Lie group G as the subgroup of $\mathbb{A}(G)$ generated by the classes

$$[H : K] \cdot t_H^G - t_K^G$$

for all nested sequences of closed subgroup $K \leq H \leq G$ such that $W_G H$ is finite

and K has finite index in H. The possibility that K has infinite Weyl group in G is allowed, in which case $t_K^G = 0$. This presents the equivariant homotopy group as an explicit quotient of the Burnside ring of G:

$$\pi_0^G(\mathcal{H}\mathbb{Z}) \cong \pi_0^G(Sp^\infty) \cong \mathbb{A}(G)/I_\infty(G)$$

Example 5.3.14. For every finite group G, the group $\pi_0^G(\mathcal{H}\mathbb{Z})$ is free of rank 1, generated by the multiplicative unit, and the inflation map $p_G^* : \pi_0^e(\mathcal{H}\mathbb{Z}) \longrightarrow \pi_0^G(\mathcal{H}\mathbb{Z})$ is an isomorphism. This does *not* persist to general compact Lie groups of positive dimension. An explicit example for which $\pi_0^G(\mathcal{H}\mathbb{Z})$ has rank bigger than one is $G = SU(2)$. Then the classes 1 and $\mathrm{tr}_N^{SU(2)}(1)$ are a $\mathbb{Z}$-basis for $\pi_0^{SU(2)}(\mathcal{H}\mathbb{Z})$ modulo torsion, see [144, Ex. 4.16], where N is a maximal torus normalizer.

To illustrate that the orthogonal spectrum $\mathcal{H}A$ is typically *not* an Eilenberg-Mac Lane spectrum of any global functor, we calculate the $U(1)$-equivariant homotopy group $\pi_1^{U(1)}(\mathcal{H}A)$.

Construction 5.3.15. As we shall now see, all classes in the group $\pi_1^{U(1)}(\mathcal{H}A)$ arise in a systematic way via dimension shifting transfers from finite subgroups of $U(1)$. We organize these transfers into a natural isomorphism from $A \otimes \mathbb{Q}$ to $\pi_1^{U(1)}(\mathcal{H}A)$.

We let C be any finite subgroup of $U(1)$. We identify $\mathbb{R}$ with the tangent space of the distinguished coset eC in $U(1)/C$ via the differential of the smooth curve

$$\mathbb{R} \longrightarrow U(1)/C\,, \quad t \longmapsto e^{2\pi i t}\cdot C\,.$$

The upshot of this identification $\mathbb{R} \cong T_{eC}(U(1)/C)$ is that we can view the dimension shifting transfer (3.2.23) as a homomorphism

$$\mathrm{Tr}_C^{U(1)} \;:\; \pi_1^C(E \wedge S^1) \longrightarrow \pi_1^{U(1)}(E)\,.$$

We can then define an additive map $\psi_C : A \longrightarrow \pi_1^{U(1)}(\mathcal{H}A)$ as the composite

$$A \cong \pi_0^e(\mathcal{H}A) \xrightarrow[\cong]{p_C^*} \pi_0^C(\mathcal{H}A) \xrightarrow[\cong]{-\wedge S^1} \pi_1^C(\mathcal{H}A \wedge S^1) \xrightarrow{\mathrm{Tr}_C^{U(1)}} \pi_1^{U(1)}(\mathcal{H}A)$$

of inflation, the suspension isomorphism and the dimension shifting transfer.

Now we consider a subgroup C' of C of index m. Since the restriction of $\underline{\pi}_0(\mathcal{H}A)$ to finite groups is a constant global functor, the relation

$$\begin{aligned}\mathrm{Tr}_{C'}^{U(1)} \circ(- \wedge S^1) \circ p_{C'}^* \;&=\; \mathrm{Tr}_C^{U(1)} \circ \mathrm{tr}_{C'}^C \circ(- \wedge S^1) \circ p_{C'}^* \\ &=\; \mathrm{Tr}_C^{U(1)} \circ(- \wedge S^1) \circ \mathrm{tr}_{C'}^C \circ p_{C'}^* \;=\; m\cdot \mathrm{Tr}_C^{U(1)} \circ(- \wedge S^1) \circ p_C^*\end{aligned}$$

holds as homomorphisms $\pi_0^e(\mathcal{H}A) \longrightarrow \pi_1^{U(1)}(\mathcal{H}A)$, by transitivity of transfers

(Proposition 3.2.29). Hence $\psi_{C'} = m \cdot \psi_C$ and we obtain a well-defined homomorphism

$$\psi \ : \ A \otimes \mathbb{Q} \longrightarrow \pi_1^{U(1)}(\mathcal{H}A) \qquad \text{by} \qquad \psi(a \otimes r/s) \ = \ r \cdot \psi_{C_s}(a) \ ,$$

where $C_s \subset U(1)$ is group of sth roots of unity.

Theorem 5.3.16. *The homomorphism $\psi : A \otimes \mathbb{Q} \longrightarrow \pi_1^{U(1)}(\mathcal{H}A)$ is an isomorphism for every abelian group A.*

Proof Since the orthogonal spectrum $\mathcal{H}A$ is obtained from a $\mathbf{\Gamma}$-space by evaluation on spheres, the inflation homomorphism

$$p^* \ : \ \pi_*^e(\mathcal{H}A) \ = \ \Phi_*^e(\mathcal{H}A) \ \longrightarrow \ \Phi_*^{U(1)}(\mathcal{H}A)$$

is an isomorphism of geometric fixed-point homotopy groups, by Proposition 4.5.15 (ii). The non-equivariant homotopy groups of $\mathcal{H}A$ are concentrated in dimension 0, so the $U(1)$-equivariant geometric fixed-point homotopy groups $\Phi_k^{U(1)}(\mathcal{H}A)$ vanish for all $k \neq 0$. The isotropy separation sequence (3.3.9) thus shows that the map $E\mathcal{P} \longrightarrow *$ induces an isomorphism

$$\pi_1^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}_+) \ \cong \ \pi_1^{U(1)}(\mathcal{H}A) \ ,$$

where $E\mathcal{P}$ is a universal $U(1)$-space for the family of proper closed (i.e., finite) subgroups of $U(1)$.

We let X be mapping telescope of the sequence of projections

$$U(1) \ \longrightarrow \ U(1)/C_2 \ \longrightarrow \ U(1)/C_6 \ \longrightarrow \ \dots \ \longrightarrow \ U(1)/C_{n!} \ \longrightarrow \ \dots \ .$$

This is a $U(1)$-CW-complex without $U(1)$-fixed-points and such that the fixed-point space X^C is path connected for every finite subgroup C of $U(1)$. So we can build a universal space $E\mathcal{P}$ by attaching equivariant cells with finite isotropy groups of dimension 2 and larger to X. The Wirthmüller isomorphisms

$$\pi_k^{U(1)}(\mathcal{H}A \wedge (U(1)/C)_+ \wedge S^n) \ \cong \ \pi_k^C(\mathcal{H}A \wedge S^1 \wedge S^n) \ \cong \ \pi_{k-1-n}^C(\mathcal{H}A)$$

and the fact that the C-equivariant homotopy groups of $\mathcal{H}A$ are concentrated in dimension 0 mean that the group $\pi_k^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}/X)$ is trivial for all $k \leq 2$. Hence the inclusion $X \longrightarrow E\mathcal{P}$ induces an isomorphism

$$\pi_1^{U(1)}(\mathcal{H}A \wedge X_+) \ \xrightarrow{\cong} \ \pi_1^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}_+) \ .$$

We let $f^n : U(1)/C_{n!} \longrightarrow X$ be the inclusion. The dimension shifting transfer map $\mathrm{Tr}_C^{U(1)}$ factors as the composite

$$\begin{aligned} \pi_1^C(\mathcal{H}A \wedge S^1) \ &\xrightarrow{U(1)\ltimes_C -} \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{n!})_+) \\ &\xrightarrow{\ f_*^n\ } \ \pi_1^{U(1)}(\mathcal{H}A \wedge X_+) \ \longrightarrow \ \pi_1^{U(1)}(\mathcal{H}A) \ . \end{aligned}$$

So the map ψ factors through $\pi_1^{U(1)}(\mathcal{H}A \wedge X_+)$. Since $\mathcal{H}A \wedge X_+$ is a mapping telescope, its equivariant homotopy groups can be calculated as the colimit of the sequence:

$$\pi_1^{U(1)}(\mathcal{H}A \wedge U(1)_+) \longrightarrow \dots \longrightarrow \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{n!})_+) \longrightarrow \dots$$

We rewrite this using the Wirthmüller and suspension isomorphisms as

$$\pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_k)_+) \cong \pi_1^{C_k}(\mathcal{H}A \wedge S^1) \cong \pi_0^{C_k}(\mathcal{H}A) \cong A .$$

Example 3.2.31 shows that for all $k, m \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{C_k}(\mathcal{H}A \wedge S^1) \arrow[r, "\mathrm{tr}_{C_k}^{C_{km}}"] \arrow[d, "U(1)\ltimes_{C_k} -"', "\cong"] & \pi_1^{C_{km}}(\mathcal{H}A \wedge S^1) \arrow[d, "\cong"', "U(1)\ltimes_{C_{km}} -"] \\
\pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_k)_+) \arrow[r, "(\mathcal{H}A\wedge \mathrm{proj})_*"'] & \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{km})_+)
\end{tikzcd}$$

Transfers commute with the suspension isomorphism and in the global functor $\underline{\pi}_0(\mathcal{H}A)$, the transfer $\mathrm{tr}_{C_k}^{C_{km}}$ is multiplication by the index $[C_{km} : C_k] = m$; so the above homotopy group sequence becomes the sequence

$$A \xrightarrow{\cdot 2} A \xrightarrow{\cdot 3} \cdots \longrightarrow A \xrightarrow{\cdot n} A \longrightarrow \cdots .$$

The colimit of this sequence is isomorphic to $A \otimes \mathbb{Q}$, compatible with the map $\psi : A \otimes \mathbb{Q} \longrightarrow \pi_1^{U(1)}(\mathcal{H}A \wedge X_+)$. This proves the claim. □

Example 5.3.17. The right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor from the global to the non-equivariant stable homotopy category is modeled on the point-set level by the functor $b : \mathcal{S}p \longrightarrow \mathcal{S}p$ discussed in Construction 4.5.21. The global homotopy type of bE is that of a Borel cohomology theory, and in particular,

$$\pi_0^G(bE) \cong E^0(BG) ,$$

natural in G for transfers and restriction maps. The functor b is lax symmetric monoidal, so it takes an ultra-commutative ring spectrum R to an ultra-commutative ring spectrum bR; the power operations

$$P^m : \pi_0^e(bE) \longrightarrow \pi_0^{\Sigma_m}(bE)$$

correspond to the classical power operations

$$\mathcal{P}_m : E^0(*) \longrightarrow E^0(B\Sigma_m) ,$$

compare the more general Remark 5.1.14.

Example 5.3.18 (Representation ring global power functor)**.** As G varies over all compact Lie groups, the unitary representation rings $\mathbf{RU}(G)$ form the *unitary representation ring global functor* **RU**, compare Example 4.2.8 (iv). This is classical in the restricted realm of finite groups, but somewhat less familiar for compact Lie groups in general. The restriction maps $\alpha^* : \mathbf{RU}(G) \longrightarrow \mathbf{RU}(K)$ are induced by restriction of representations along a homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [150, § 2]. If H is a subgroup of finite index of G, then this induction sends the class of an H-representation V to the induced G-representation $\mathrm{map}^H(G, V)$; in general, induction sends actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for **RU** was proven by Snaith [160, Thm. 2.4].

The representation rings also have well-known power operations

$$P^m \ : \ \mathbf{RU}(G) \ \longrightarrow \ \mathbf{RU}(\Sigma_m \wr G) \ ;$$

on the class of a G-representation V, the power operation is represented by the tensor power

$$P^m[V] \ = \ [V^{\otimes m}] \ ,$$

using the canonical action of $\Sigma_m \wr G$ on $V^{\otimes m}$. Since power operations are not additive, one has to argue why this assignment extends to virtual representations. The standard way is to assemble all power operations on representations into a map

$$P \ : \ \mathbf{RU}^+(G) \ \longrightarrow \ \exp(\mathbf{RU}; G) \ , \quad P([V]) \ = \ \left([V^{\otimes m}]\right)_{m \geq 0}$$

from the monoid of isomorphism classes of G-representations. If W is another G-representation, then

$$(V \oplus W)^{\otimes m} \qquad \text{and} \qquad \bigoplus\nolimits_{k=0}^{m} \mathrm{tr}_{(\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)}^{\Sigma_m \wr G} (V^{\otimes k} \otimes W^{\otimes (m-k)})$$

are isomorphic as $(\Sigma_m \wr G)$-representations, because tensor product distributes over direct sum. This means that the total power map P is a monoid homomorphism from $\mathbf{RU}^+(G)$ to the group $\exp(\mathbf{RU}; G)$ under $\oplus$. So the total power operation extends uniquely to a group homomorphism

$$P \ : \ \mathbf{RU}(G) \ \longrightarrow \ \exp(\mathbf{RU}; G)$$

on the representation ring. The operation P^m is then the mth factor of this extension. The representation ring global power functor **RU** is realized by the ultra-commutative ring spectrum **KU**, the unitary global K-theory spectrum, see Theorem 6.4.24 below.

Remark 5.3.19 (Brauer induction)**.** By Brauer's theorem [24, Thm. I] the complex representation ring of a finite group is generated, as an abelian group, by representations that are induced from 1-dimensional representations of subgroups. Segal generalized this result to compact Lie groups in [150, Prop. 3.11 (ii)], where 'induction' refers to the smooth induction. In fact, in the world of compact Lie groups, Segal's smooth induction for not necessarily finite index subgroups makes the proof quite transparent, as we shall now recall. In our language the statement can be expressed by saying that the representation ring global functor **RU** is 'cyclic' in the sense that it is generated by a single element, the class $x \in \mathbf{RU}(U(1))$ of the tautological 1-dimensional representation of the circle group $U(1)$. Equivalently, the morphism of global functors

$$\mathrm{ev}_x \ : \ \mathbf{A}(U(1), -) \ \longrightarrow \ \mathbf{RU}$$

classified by the element x is an epimorphism. We recall the argument: we let $i : U(1) \times U(n-1) \longrightarrow U(n)$ be the block sum embedding and $q : U(1) \times U(n-1) \longrightarrow U(1)$ the projection to the first factor. The character formula [150, p. 119] for induced representations shows that the smooth transfer

$$i_!(q^*(x)) \ \in \ \mathbf{RU}(U(n)) \tag{5.3.20}$$

has the same character as the tautological n-dimensional representation of $U(n)$. Since characters determine unitary representations of compact Lie groups, $i_!(q^*(x))$ equals the class of the tautological representation τ_n of $U(n)$. Any unitary representation of a compact Lie group G of dimension n is isomorphic to $\alpha^*(\tau_n)$ for a continuous homomorphism $\alpha : G \longrightarrow U(n)$; so the class of such a representation equals

$$\alpha^*(i_!(q^*(x))) \ \in \ \mathbf{RU}(G) .$$

So the global functor **RU** is generated by the single class $x = \tau_1$.

An interesting line of investigation, dubbed *explicit Brauer induction*, started with Snaith's paper [160]. Informally speaking, an 'explicit Brauer induction' is a section to the map

$$\mathbf{A}(U(1), G) \ \longrightarrow \ \mathbf{RU}(G)$$

that is specified by a direct recipe, for example an explicit formula, and has naturality properties as the group G varies. So such a map is an 'explicit and natural' way to write a (virtual) representation as a sum of induced representations of 1-dimensional representations. The first explicit Brauer induction was Snaith's formula [160, Thm. (2.16)]; however, Snaith's maps are not additive and not compatible with restriction to subgroups. Later Boltje [20] specified a different explicit Brauer induction formula by purely algebraic means; Symonds [170] gave a topological interpretation of Boltje's construction. The

Boltje-Symonds maps are additive and natural for restriction along group homomorphisms; the maps are not (and in fact cannot be) in general compatible with transfers. In our present language, the Boltje-Symonds maps form a natural transformation of Rep-abelian groups from $\mathbf{RU}$ to $\mathbf{A}(U(1),-)$. We recall the construction of these maps; we follow Symonds' approach, as his reasoning is very much in the spirit of global homotopy theory.

The starting point is the formula (5.3.20) that expresses the class of the tautological $U(n)$-representation in $\mathbf{RU}(U(n))$ as a smooth induction of a specific 1-dimensional representation of the subgroup $U(1)\times U(n-1)$, namely the one whose character is the projection $q : U(1)\times U(n-1) \longrightarrow U(1)$ to the first factor. Formula (5.3.20) suggest a class in $\mathbf{A}(U(1),U(n))$ as the image of the tautological $U(n)$-representation; if we also want naturality and additivity, then this fixes things completely:

Theorem 5.3.21 (Boltje [20], Symonds [170])**.** *There is a unique natural transformation of* Rep-*abelian groups*

$$b \;:\; \mathbf{RU} \;\longrightarrow\; \mathbf{A}(U(1),-)$$

that satisfies

$$b_{U(n)}(\tau_n) \;=\; \mathrm{tr}_{U(1)\times U(n-1)}^{U(n)} \circ q^* \qquad \textit{in} \quad \mathbf{A}(U(1),U(n))\,.$$

Moreover:

(i) *The transformation b is a section to the evaluation morphism of global power functors* $\mathrm{ev}_x : \mathbf{A}(U(1),-) \longrightarrow \mathbf{RU}$ *at the class $x \in \mathbf{RU}(U(1))$.*

(ii) *The value of b_G at the 1-dimensional representation with character $\chi : G \longrightarrow U(1)$ is given by*

$$b_G[\chi^*(\tau_1)] \;=\; \chi^* \;\in\; \mathbf{A}(U(1),G)\,.$$

Proof Every class in $\mathbf{RU}(G)$ is a formal difference of classes of actual representations, and every n-dimensional representation is the restriction of τ_n along some continuous homomorphism $G \longrightarrow U(n)$. So uniqueness is a consequence of naturality and additivity.

Conversely, this also suggests how to define the transformation. If V is any n-dimensional unitary G-representation, then V is isomorphic to $\alpha^*(\mathbb{C}^n)$ for some continuous homomorphism $\alpha : G \longrightarrow U(n)$, unique up to conjugacy. So we set

$$b_G[V] \;=\; b_G(\alpha^*(\tau_n)) \;=\; \alpha^* \circ \mathrm{tr}_{U(1)\times U(n-1)}^{U(n)} \circ q^* \;\in\; \mathbf{A}(U(1),G)\,.$$

This defines a set theoretic map

$$b_G \;:\; \mathbf{RU}^+(G) \;\longrightarrow\; \mathbf{A}(U(1),G)$$

from the monoid of isomorphism classes of unitary G-representations, and these maps are automatically compatible with restriction along group homomorphisms. The double coset formula (3.4.14) for $\mathrm{res}^{U(n+m)}_{U(n)\times U(m)} \circ \mathrm{tr}^{U(n+m)}_{U(1)\times U(n+m-1)}$ implies the relation

$$\mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(b_{U(n+m)}(\tau_{n+m})) \;=\; b_{U(n)}(\tau_n) \;+\; b_{U(m)}(\tau_m)$$

in the group $\mathbf{A}(U(1), U(n) \times U(m))$. Hence the maps b_G are additive, and so they extend uniquely to a group homomorphism

$$b_G \;:\; \mathbf{RU}(G) \;\longrightarrow\; \mathbf{A}(U(1), G)$$

on the Grothendieck group, for which we use the same name. These homomorphisms are still compatible with restriction along continuous group homomorphisms.

It remains to show the additional properties. The transformation $\mathrm{ev}_x \circ b : \mathbf{RU} \longrightarrow \mathbf{RU}$ is additive and natural for restriction along continuous homomorphisms, so for property (i) it suffices to show the relation $\mathrm{ev}_x \circ b = \mathrm{Id}$ in the universal examples, i.e., for the tautological representations of the unitary groups $U(n)$. This universal example is taken care of by the formula (5.3.20).

Property (ii) holds because for $n = 1$ the map q is the identity of $U(1)$. □

While the above construction of the explicit Brauer map $b : \mathbf{RU} \longrightarrow \mathbf{A}(U(1), -)$ is slick, it is not yet particularly explicit. To write the class $b_G[V]$ as a $\mathbb{Z}$-linear combination of transfers of 1-dimensional representations of subgroups of G, one would now have to write the classifying homomorphism $\alpha : G \longrightarrow U(n)$ for V as the composite of an epimorphism and a subgroup inclusion and then expand the term $\alpha^* \circ \mathrm{tr}^{U(n)}_{U(1)\times U(n-1)}$ using the compatibility of transfers with inflation, and the double coset formula for the restriction of a transfer. Readers desperate for a truly explicit formula can find one in [20, Thm. (2.1)] or [21, Thm. 2.24 (e)].

5.4 Global model structure

In this section we construct a model structure on the category of ultra-commutative ring spectra with global equivalences as the weak equivalences, see Theorem 5.4.3. The strategy is the same is in the unstable situation in Section 2.1: we lift the positive version of the global model structure to commutative monoid objects with the help of the general lifting theorem [188, Thm. 3.2].

We also calculate the algebra of natural operations on the homotopy groups of ultra-commutative ring spectra: we show that these operations are freely generated by restrictions, transfers and power operations, see Theorem 5.4.4

for the precise statement. Then we show in Theorem 5.4.14 that every global power functor can be realized by an ultra-commutative ring spectrum; the realization can moreover be chosen as a global Eilenberg-Mac Lane spectrum, i.e., such that all equivariant homotopy groups in non-zero dimensions vanish.

The construction of the global model structure on the category of ultra-commutative ring spectra is largely parallel to the unstable precursor, the global model structure for ultra-commutative monoids in Theorem 2.1.15. The analogous arguments as in the unstable case in Corollary 2.1.4 show that the category of ultra-commutative ring spectra is complete and cocomplete and that the forgetful functor to the category of orthogonal spectra creates all limits, all filtered colimits and those coequalizers that are reflexive in the category of orthogonal spectra. The category of ultra-commutative ring spectra is tensored and cotensored over the category $\mathbf{T}$ of spaces, by the same arguments in Construction 2.1.6 for ultra-commutative monoids.

We recall from [64, Def. 3] the notion of a *symmetrizable cofibration* and *symmetrizable acyclic cofibration*, see also Definition 2.1.11. For a morphism $i : A \longrightarrow B$ of orthogonal spectra we let $K^n(i)$ denote the n-cube of orthogonal spectra whose value at a subset $S \subseteq \{1, 2, \dots, n\}$ is

$$K^n(i)(S) \ = C_1 \wedge C_2 \wedge \ldots \wedge C_n \qquad \text{with} \qquad C_j \ = \begin{cases} A & \text{if } j \notin S \\ B & \text{if } j \in S. \end{cases}$$

All morphisms in the cube $K^n(i)$ are smash products of identities and copies of the morphism $i : A \longrightarrow B$. We let $Q^n(i)$ denote the colimit of the punctured n-cube, i.e., the cube $K^n(i)$ with the terminal vertex removed, and $i^{\Box n} : Q^n(i) \longrightarrow K^n(i)(\{1, \dots, n\}) = B^{\wedge n}$ the canonical map. So the morphism $i^{\Box n}$ is an iterated pushout product morphism. The symmetric group Σ_n acts on $Q^n(i)$ and $B^{\wedge n}$ by permuting the smash factors, and the iterated pushout product morphism $i^{\Box n} : Q^n(i) \longrightarrow B^{\wedge n}$ is Σ_n-equivariant. The morphism $i : A \longrightarrow B$ of orthogonal spectra is a *symmetrizable cofibration* (or *symmetrizable acyclic cofibration*) if the morphism

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (or acyclic cofibration) for every $n \geq 1$. Since the morphism $i^{\Box 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration, and similarly for acyclic cofibrations.

The next theorem says that in the category of orthogonal spectra, all cofibrations and acyclic cofibrations in the positive global model structure are symmetrizable with respect to the monoidal structure given by the smash product.

Theorem 5.4.1. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spectra. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \;:\; Q^n(i)/\Sigma_n \;\longrightarrow\; B^{\wedge n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spectra are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spectra that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \;:\; Q^n(i)/\Sigma_n \;\longrightarrow\; B^{\wedge n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spectra are symmetrizable.

Proof (i) We recall from Theorem 4.3.17 (iii) the set

$$I_{\mathcal{A}ll} \;=\; \{\, G_m((O(m)/H \times i_k)_+) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spectra, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. The set $I_{\mathcal{A}ll}$ detects the acyclic fibrations in the strong level model structure of orthogonal spectra. In particular, every flat cofibration is a retract of an $I_{\mathcal{A}ll}$-cell complex. By [64, Cor. 9] or [188, Lemma A.1] it suffices to show that the generating flat cofibrations in $I_{\mathcal{A}ll}$ are symmetrizable.

For a space A, the orthogonal spectrum $G_m((O(m)/H \times A)_+)$ is isomorphic to $F_{H,\mathbb{R}^m} \wedge A_+$; so we show more generally that every morphism of the form

$$F_{G,V} \wedge (i_k)_+ \;:\; F_{G,V} \wedge \partial D^k_+ \;\longrightarrow\; F_{G,V} \wedge D^k_+$$

is a symmetrizable cofibration, where V is any representation of a compact Lie group G. The symmetrized iterated pushout product

$$(F_{G,V} \wedge (i_k)_+)^{\square n}/\Sigma_n \;:\; Q^n(F_{G,V} \wedge (i_k)_+)/\Sigma_n \;\longrightarrow\; (F_{G,V} \wedge D^k_+)^{\wedge n}/\Sigma_n \quad (5.4.2)$$

is isomorphic to

$$F_{\Sigma_n \wr G, V^n}((i_k^{\square n})_+) \;:\; F_{\Sigma_n \wr G, V^n}(Q^n(i_k)_+) \;\longrightarrow\; F_{\Sigma_n \wr G, V^n}((D^k)^n_+)\,,$$

where

$$i_k^{\square n} \;:\; Q^n(i_k) \;\longrightarrow\; (D^k)^n$$

is the n-fold pushout product of the inclusion $i_k : \partial D^k \longrightarrow D^k$, with respect to the cartesian product of spaces. The map $i_k^{\square n}$ is Σ_n-equivariant, and we showed in the proof of the analogous unstable result in Theorem 2.1.13 (i) that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. Proposition B.14 (i) then shows that $i_k^{\square n}$ is also a cofibration of $(\Sigma_n \wr G)$-spaces, with respect to the action by restriction along the projection $(\Sigma_n \wr G) \longrightarrow \Sigma_n$. So the morphism (5.4.2) is a flat cofibration.

(ii) Theorem 4.3.17 (iii) describes a set $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ of generating acyclic

cofibrations for the global model structure on the category of orthogonal spectra. From this we obtain a set $J^+ \cup K^+$ of generating acyclic cofibration for the *positive* global model structure of Proposition 4.3.33 by restricting to those morphisms in $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ that are positive cofibrations, i.e., homeomorphisms in level 0; so explicitly, we set

$$J^+ \ = \ \{\, G_m((O(m)/H \times j_k)_+) \mid m \geq 1, k \geq 0, H \leq O(m)\}\ ,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+ \ = \ \bigcup_{G,V,W\ :\ V \neq 0} \mathcal{Z}(\lambda_{G,V,W})\ ,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\lambda_{G,V,W} : F_{G,V\oplus W} S^V \longrightarrow F_{G,W}$. Here (G, V, W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a G-representation V and non-zero faithful G-representation W. By [64, Cor. 9] or [188, Lemma A.1] it suffices to show that all morphisms in $J^+ \cup K^+$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m((O(m)/H \times j_k)_+)$ in J^+. For every $n \geq 1$, the morphism

$$(G_m((O(m)/H \times j_k)_+))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the morphism j_k is a homotopy equivalence of spaces, so the morphism $G_m((O(m)/H \times j_k)_+)$ is a homotopy equivalence of orthogonal spectra; the morphism $\mathbb{P}^n(G_m((O(m)/H \times j_k)_+))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition 2.1.12 (i). Then [64, Cor. 23] shows that $G_m((O(m)/H \times j_k)_+)$ is a symmetrizable acyclic cofibration. This takes care of the set J^+.

Now we consider the morphisms in the set K^+. Since G acts faithfully on the non-zero inner product space W, the action of the wreath product $\Sigma_n \wr G$ on W^n is again faithful. So the morphism

$$\lambda_{\Sigma_n \wr G, V^n, W^n} \ : \ F_{\Sigma_n \wr G, V^n \oplus W^n} S^{V^n} \ \longrightarrow \ F_{\Sigma_n \wr G, W^n}$$

is a global equivalence by Theorem 4.1.29. The vertical morphisms in the commutative square

$$\begin{tikzcd}
F_{\Sigma_n \wr G, V^n \oplus W^n} S^{V^n} \arrow[r, "\lambda_{\Sigma_n \wr G, V^n, W^n}"] \arrow[d, "\cong"'] & F_{\Sigma_n \wr G, W^n} \arrow[d, "\cong"] \\
\mathbb{P}^n(F_{G,V\oplus W} S^V) \arrow[r, "\mathbb{P}^n(\lambda_{G,V,W})"'] & \mathbb{P}^n(F_{G,W})
\end{tikzcd}$$

are isomorphisms; so the morphism $\mathbb{P}^n(\lambda_{G,V,W})$ is a global equivalence. Proposition 2.1.12 (iii) then shows that all morphisms in $\mathcal{Z}(\lambda_{G,V,W})$ are symmetrizable acyclic cofibrations. □

Now we put all the pieces together. We call a morphism of ultra-commutative ring spectra a *global equivalence* (or *positive global fibration*) if the underlying morphism of orthogonal spectra is a global equivalence (or fibration in the positive global model structure of Proposition 4.3.33).

Theorem 5.4.3 (Global model structure for ultra-commutative ring spectra)**.**

(i) *The global equivalences and positive global fibrations are part of a cofibrantly generated, proper, topological model structure on the category of ultra-commutative ring spectra, the* global model structure*.*

(ii) *Let $j : R \longrightarrow S$ be a cofibration in the global model structure of ultra-commutative ring spectra.*

  (a) *The morphism of R-modules underlying j is a cofibration in the global model structure of R-modules of Corollary* 4.3.29 (i).

  (b) *The morphism of orthogonal spectra underlying j is an h-cofibration.*

  (c) *If the underlying orthogonal spectrum of R is flat, then j is a flat cofibration of orthogonal spectra.*

Proof The proof is completely parallel to the proof of the analogous unstable theorem, Theorem 2.1.15.

(i) The positive global model structure of orthogonal spectra (Proposition 4.3.33) is monoidal and cofibrantly generated. The 'unit axiom' also holds: we let $f : \mathbb{S}^c \longrightarrow \mathbb{S}$ be any positive flat replacement of the orthogonal sphere spectrum. Then for every orthogonal spectrum X the induced morphism $f \wedge X : \mathbb{S}^c \wedge X \longrightarrow \mathbb{S} \wedge X$ is a global equivalence by Theorem 4.3.27 (ii). The monoid axiom holds by Proposition 4.3.28. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 5.4.1, so the model structure satisfies the 'commutative monoid axiom' of [188, Def. 3.1]. The symmetric algebra functor $\mathbb{P}$ commutes with filtered colimits by the analog of Corollary 2.1.4 for ultra-commutative ring spectra. Theorem 3.2 of [188] thus shows that the positive global model structure of orthogonal spectra lifts to a cofibrantly generated model structure on the category of ultra-commutative ring spectra. The global model structure is topological by Proposition B.5; here we take $\mathcal{G}$ as the set of free ultra-commutative ring spectra $\mathbb{P}(F_{H,\mathbb{R}^m})$ for all $m \geq 1$ and all closed subgroups H of $O(m)$, and we take $\mathcal{Z}$ as the set of acyclic cofibrations $\mathbb{P}(c(\lambda_{G,V,W}))$ for the mapping cone inclusions $c(\lambda_{G,V,W})$ of the global equivalences $\lambda_{G,V,W} : F_{G,V\oplus W}S^W \longrightarrow F_{G,W}$, indexed by representatives as in the definition of the set K^+. Since weak equivalences, fibrations and pullbacks of

ultra-commutative ring spectra are are created on underlying orthogonal spectra, right properness is inherited from the positive global model structure of orthogonal spectra (Proposition 4.3.33). We defer the proof of left properness until after the proof of part (ii).

Part (ii) is proved in exactly the same way as in the unstable case in Theorem 2.1.15; whenever the unstable proof refers to the model category of R-modules in Corollary 1.4.15, the stable proof instead uses Corollary 4.3.29. For the symmetrizability of the cofibrations, the stable proof uses Theorem 5.4.1 (i) instead of Theorem 2.1.13 (i). We refrain from repeating the remaining details.

It remains to show that the model structure is left proper. A pushout square of ultra-commutative ring spectra has the form

$$\begin{tikzcd} R \arrow[r, "f", "\simeq"'] \arrow[d, "j"'] & T \arrow[d, "j \wedge_R T"] \\ S \arrow[r, "S \wedge_R f"'] & S \wedge_R T \end{tikzcd}$$

where S and T are considered as R-modules by restriction along j and f, respectively. For left properness we now suppose that j is a cofibration and f is a global equivalence. By part (a) of (ii), the morphism j is then a cofibration of R-modules in the global model structure of Corollary 4.3.29 (i). Since R is cofibrant in that model structure, also S is cofibrant as an R-module. Proposition 4.3.30 then shows that the functor $S \wedge_R -$ preserves global equivalences. So the cobase change $S \wedge_R f$ of f is a global equivalence. This shows that the global model structure of ultra-commutative ring spectra is left proper. □

Now we can show that the restriction maps, (additive) transfer maps and (multiplicative) power operations generate all natural operations between the 0th equivariant homotopy groups of ultra-commutative ring spectra. The strategy is the one that we have employed several times before: the functor π_0^G from ultra-commutative ring spectra to sets is representable, namely by $\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}G)$, the unreduced suspension spectrum of the free ultra-commutative monoid generated by $B_{\mathrm{gl}}G$. So we have to determine the equivariant homotopy groups $\pi_0^K(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}G))$, which just means assembling various results already proved.

Theorem 5.4.4. *Let G and K be compact Lie groups. The group of natural transformations $\pi_0^G \longrightarrow \pi_0^K$ of set-valued functors on the category of ultra-commutative ring spectra is a free abelian group with basis the operations*

$$\mathrm{tr}_L^K \circ \alpha^* \circ P^m \ : \ \pi_0^G \longrightarrow \pi_0^K$$

for all $m \geq 0$ and all $(K \times (\Sigma_m \wr G))$-conjugacy classes of pairs (L, α) con-

sisting of a closed subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow \Sigma_m \wr G$.

Proof We let W be any non-zero faithful G-representation and write $B_{\mathrm{gl}}G = \mathbf{L}_{G,W}$ for the global classifying space of G based on W. We denote by u_G^{ucom} the class in $\pi_0^G(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ obtained by pushing forward the tautological class $u_{G,W}$ along the adjunction unit $\eta : B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ and then applying the stabilization map σ^G of (4.1.10).

We apply the representability result of Proposition 1.5.13 to the category of ultra-commutative ring spectra and the adjoint functor pair:

$$\Sigma_+^\infty \circ \mathbb{P} \ : \ spc \rightleftarrows ucom \ : \ U \circ \Omega^\bullet$$

If V is any G-representation, then the restriction morphism $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,W}$ is a global equivalence between positive flat orthogonal spaces. We showed in the proof of Theorem 2.1.13 (ii) that the induced morphism of free ultra-commutative monoids $\mathbb{P}(\rho_{G,V,W}) : \mathbb{P}(\mathbf{L}_{G,V\oplus W}) \longrightarrow \mathbb{P}(\mathbf{L}_{G,W})$ is a global equivalence. So the induced morphism of unreduced suspension spectra $\Sigma_+^\infty \mathbb{P}(\rho_{G,V,W})$ is a global equivalence by Corollary 4.1.9. In particular, the morphism of Rep-functors $\underline{\pi}_0(\Sigma_+^\infty \mathbb{P}(\rho_{G,V,W}))$ is an isomorphism. So Proposition 1.5.13 applies and shows that evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{ucom}(\pi_0^G, \pi_0^K) \ \longrightarrow \ \pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))\ , \quad \tau \ \longmapsto \ \tau(u_G^{ucom})$$

to the 0th K-equivariant homotopy group of the ultra-commutative ring spectrum $\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)$.

The identification (2.2.20) and Proposition 1.5.12 (ii) show that the homotopy set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$ bijects with the set

$$\coprod\nolimits_{m\geq 0} \mathrm{Rep}(L, \Sigma_m \wr G)$$

by sending the conjugacy class of $\alpha : L \longrightarrow \Sigma_m \wr G$ to $\alpha^*([m](u_G)) = \alpha^*(u_{\Sigma_m \wr G})$. Proposition 4.1.11 then implies that the group $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group, and it specifies a basis consisting of the elements

$$\mathrm{tr}_L^K(\sigma^L(x))$$

where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$. So together this shows that $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group with basis the classes

$$\begin{aligned} \mathrm{tr}_L^K(\sigma^L(\alpha^*([m](u_G)))) \ &= \ \mathrm{tr}_L^K(\alpha^*(\sigma^{\Sigma_m \wr G}([m](u_G)))) \\ &= \ \mathrm{tr}_L^K(\alpha^*(P^m(\sigma^G(u_G)))) \ = \ \mathrm{tr}_L^K(\alpha^*(P^m(u_G^{ucom}))) \end{aligned}$$

for all (m, L, α) as in the statement of the theorem. □

Our last major goal in this chapter is to show that every global power functor is realized by an ultra-commutative ring spectrum that can moreover be chosen to have all its equivariant homotopy groups concentrated in dimension 0, see Theorem 5.4.14 below. Towards this aim we will show various auxiliary results, which may be of independent interest. We start with a proposition that describes the effect on the homotopy group global power functors of 'coning off' a free ultra-commutative ring spectrum. We can only analyze this process algebraically if all spectra involved are globally connective, for otherwise the symmetric powers $\mathbb{P}^m(A)$ generate non-trivial equivariant homotopy in arbitrarily low negative dimensions.

We recall from Example 5.2.15 that the coproduct in the category of global power functors is given by the box product of underlying global functors. So if $f : R \longrightarrow F$ and $g : S \longrightarrow F$ are morphisms of global power functors, we let $f \diamond g : R \Box S \longrightarrow F$ denote their categorical sum, i.e., the composite

$$R \Box S \xrightarrow{f \Box g} F \Box F \xrightarrow{\text{multiplication}} F .$$

If A is an orthogonal spectrum and R an ultra-commutative ring spectrum, then the zero morphism of orthogonal spectra from A to R freely generates a morphism of ultra-commutative ring spectra $\tau_A : \mathbb{P}A \longrightarrow R$ that we call the 'trivial morphism'. In terms of the wedge decomposition $\mathbb{P}A = \bigvee_{m \geq 0} \mathbb{P}^m(A)$, this trivial morphism is the unit morphism of R on $\mathbb{P}^0(A) = \mathbb{S}$, and trivial on all other wedge summands. We recall that $CA = A \wedge [0, 1]$ is the cone of an orthogonal spectrum A, and $i = (- \wedge 1) : A \longrightarrow CA$ is the cone inclusion.

Proposition 5.4.5. *Let A be a positively flat orthogonal spectrum such that the free ultra-commutative ring spectrum $\mathbb{P}A$ is globally connective. Let R be a globally connective ultra-commutative ring spectrum and $\rho : \mathbb{P}A \longrightarrow R$ a morphism of ultra-commutative ring spectra. We define T as the following pushout, in the category of ultra-commutative ring spectra:*

$$\begin{tikzcd} \mathbb{P}A \arrow[r, "\mathbb{P}i"] \arrow[d, "\rho"'] & \mathbb{P}(CA) \arrow[d] \\ R \arrow[r, "\psi"'] & T \end{tikzcd}$$

Then T is globally connective, the morphism of global power functors $\underline{\pi}_0(\psi) : \underline{\pi}_0(R) \longrightarrow \underline{\pi}_0(T)$ is surjective and a coequalizer, in the category of global power functors, of the two morphisms

$$\underline{\pi}_0(R) \diamond \underline{\pi}_0(\rho) \, , \; \underline{\pi}_0(R) \diamond \underline{\pi}_0(\tau_A) \; : \; \underline{\pi}_0(R) \Box \underline{\pi}_0(\mathbb{P}A) \; \longrightarrow \; \underline{\pi}_0(R) \, .$$

Proof The cone $CA = A \wedge [0, 1]$ is isomorphic to the realization of the simplicial orthogonal spectrum $B_\bullet(A, A, *)$, the bar construction with respect to

wedge. Explicitly,

$$B_m(A, A, *) \;=\; A \wedge \Delta[1]_m \;,$$

where $\Delta[1]$ is the simplicial 1-simplex, pointed by the vertex 0; the simplicial structure is induced by that of $\Delta[1]$. Since $\Delta[1]$ has $(m + 1)$ non-basepoint simplices of dimension m, the spectrum $B_m(A, A, *)$ is a wedge of $m + 1$ copies of A.

Since the free functor $\mathbb{P} : \mathcal{S}p \longrightarrow ucom$ is an enriched left adjoint, it preserves coends and cotensors with spaces. So $\mathbb{P}(CA)$ is isomorphic to the realization, internal to ultra-commutative ring spectra, of the simplicial ultra-commutative ring spectrum $\mathbb{P}(B_\bullet(A, A, *))$. Under this isomorphism, the morphism $\mathbb{P}i : \mathbb{P}A \longrightarrow \mathbb{P}(CA)$ identifies $\mathbb{P}A$ with the degeneracies of the object of 0-simplices. As a left adjoint, the free functor preserves coproducts, i.e., it takes wedges of orthogonal spectra to smash products of ultra-commutative ring spectra. So $\mathbb{P}(B_\bullet(A, A, *))$ is isomorphic, as a simplicial ultra-commutative ring spectrum, to $B^\wedge_\bullet(\mathbb{P}A, \mathbb{P}A, \mathbb{S})$, the bar construction of $\mathbb{P}A$ with respect to smash product.

Since colimits commute among themselves and with cotensors, we can describe the pushout $T = R \wedge_{\mathbb{P}A} \mathbb{P}(CA)$ as the realization of the simplicial ultra-commutative ring spectrum $R \wedge_{\mathbb{P}A} B^\wedge_\bullet(\mathbb{P}A, \mathbb{P}A, \mathbb{S})$ where we take pushouts in every simplicial dimension. The smash product over $\mathbb{P}A$ 'cancels' with the 0th smash factor in the bar construction, so we can take T as the internal realization of the bar construction $B^\wedge_\bullet(R, \mathbb{P}A, \mathbb{S})$.

By the stable analog of Proposition 2.1.7, the realization internal to the category of ultra-commutative ring spectra can equivalently be taken in the underlying category of orthogonal spectra. The underlying realization is the sequential colimit of partial realizations $B^{[n]}$, i.e., 'skeleta' in the simplicial direction, defined as the coend

$$B^{[n]} \;=\; \int^{[m]\in\mathbf{\Delta}_{\le n}} B^\wedge_m(R, \mathbb{P}A, \mathbb{S}) \wedge \Delta^m_+$$

of the restriction to the full subcategory $\mathbf{\Delta}_{\le n}$ of $\mathbf{\Delta}$ with objects all $[m]$ with $m \le n$. We refer to [63, VII Sec. 1-3] for background on realization of simplicial objects and the skeleton filtration. The realization $|B^\wedge_\bullet(R, \mathbb{P}A, \mathbb{S})|$ is the colimit of the sequence of orthogonal spectra

$$R \;=\; B^{[0]} \;\longrightarrow\; B^{[1]} \;\longrightarrow\; \ldots \;\longrightarrow\; B^{[n]} \;\longrightarrow\; \ldots \,. \tag{5.4.6}$$

Moreover, the n-skeleton $B^{[n]}$ is obtained from the previous stage as a pushout

of orthogonal spectra:

$$\begin{tikzcd} L_n^{\Delta} \wedge \Delta_+^n \cup_{L_n^{\Delta} \wedge \partial \Delta_+^n} B_n^{\wedge}(R, \mathbb{P}A, \mathbb{S}) \wedge \partial \Delta_+^n \arrow[r, "\mathrm{incl}"] \arrow[d] & B_n^{\wedge}(R, \mathbb{P}A, \mathbb{S}) \wedge \Delta_+^n \arrow[d] \\ B^{[n-1]} \arrow[r] & B^{[n]} \end{tikzcd} \tag{5.4.7}$$

see (3.8) of [63, VII Sec. 3]. Here the orthogonal spectrum L_n^{Δ} is the nth latching object in the simplicial direction, compare (1.5) of [63, VII Sec. 1], or (1.2.35). In our situation, the latching morphism $L_n^{\Delta} \longrightarrow B_n^{\wedge}(R, \mathbb{P}A, \mathbb{S})$ is the morphism

$$R \wedge i^{\square n} \ : \ R \wedge Q^n(i) \ \longrightarrow \ R \wedge (\mathbb{P}A)^{\wedge n} \ = \ B_n^{\wedge}(R, \mathbb{P}A, \mathbb{S}) \ ,$$

where $i^{\square n}$ is the iterated pushout product, with respect to smash product of orthogonal spectra, of the unit morphism $\mathbb{S} \longrightarrow \mathbb{P}A$. Since flat cofibrations are symmetrizable (Theorem 5.4.1 (i)), this unit morphism is a flat cofibration. The pushout product property of the flat cofibrations (Proposition 4.3.24) thus shows that $i^{\square n}$, and hence the latching morphism, is a flat cofibration.

So both horizontal morphisms in the pushout square (5.4.7) are h-cofibrations, and the cokernel of the inclusion $B^{[n-1]} \longrightarrow B^{[n]}$ is isomorphic to

$$(B_n^{\wedge}(R, \mathbb{P}A, \mathbb{S})/L_n^{\Delta}) \wedge S^n \ \cong \ R \wedge (\bar{\mathbb{P}}A)^{\wedge n} \wedge S^n$$

where $\bar{\mathbb{P}}A = (\mathbb{P}A)/\mathbb{S} = \bigvee_{m \geq 1} \mathbb{P}^m(A)$. Since A is flat and flat cofibrations are symmetrizable, the underlying orthogonal spectrum of $\bar{\mathbb{P}}A$ is flat. Since $\mathbb{P}A$ is globally connective, so is $\bar{\mathbb{P}}A$. Since R is also globally connective, so is $R \wedge (\bar{\mathbb{P}}A)^{\wedge n}$, by Proposition 4.4.15. This shows that the cokernel of the inclusion $B^{[n-1]} \longrightarrow B^{[n]}$ is globally $(n-1)$-connected. Since the inclusion is also an h-cofibration, Corollary 3.1.38 (i) shows that all the morphisms in the sequence (5.4.6) induce isomorphisms on the global functors $\underline{\pi}_k$ for all $k < 0$ and an epimorphism on $\underline{\pi}_0$. Moreover, starting from $B^{[1]}$ onward, the morphisms even induce an isomorphism on $\underline{\pi}_0$. Each morphism in the sequence is an h-cofibration, and so level-wise a closed embedding. Since equivariant homotopy groups commute with sequential colimits over closed embeddings (Proposition 3.1.41 (i)), we conclude that also the morphism $\underline{\pi}_k(\psi) : \underline{\pi}_k(R) \longrightarrow \underline{\pi}_k(T)$ is an isomorphism of global functors for all $k < 0$ and an epimorphism for $k = 0$. Since R is globally connective, this shows in particular that the colimit T is globally connective. Moreover, the inclusion $B^{[1]} \longrightarrow |B_\bullet^{\wedge}(R, \mathbb{P}A, \mathbb{S})| = T$ induces an isomorphism on $\underline{\pi}_0$.

It remains to identify $\underline{\pi}_0(\psi) : \underline{\pi}_0(R) \longrightarrow \underline{\pi}_0(T)$ as a coequalizer. The latching object L_1^{Δ} is simply the spectrum $B_0^{\wedge}(R, \mathbb{P}A, \mathbb{S}) = R$, and for $n = 1$ the pushout

(5.4.7) specializes to a pushout of orthogonal spectra

$$\begin{tikzcd} R \wedge \mathbb{P}A \wedge \{0,1\}_+ \arrow[r, "\mathrm{incl}"] \arrow[d, "d_0^* + d_1^*"'] & R \wedge \mathbb{P}A \wedge [0,1]_+ \arrow[d] \\ R \arrow[r] & B^{[1]} \end{tikzcd} \tag{5.4.8}$$

The orthogonal spectrum $R \wedge \mathbb{P}A \wedge \{0,1\}_+$ is the wedge of two copies of $R \wedge \mathbb{P}A$. On one copy the left vertical morphism is the simplicial face morphism $d_0^* : R \wedge \mathbb{P}A \longrightarrow R$, which is the product, with respect to the multiplication of R, of the identity and the morphism $\rho : \mathbb{P}A \longrightarrow R$. On the other copy the left vertical morphism is the simplicial face morphism $d_1^* : R \wedge \mathbb{P}A \longrightarrow R$, which is the product of the identity of R and the trivial morphism $\tau_A : \mathbb{P}A \longrightarrow R$, adjoint to the zero morphism from A to R.

The orthogonal spectra underlying R and $\mathbb{P}A$ are globally connective by hypothesis. Moreover, $\mathbb{P}A$ is flat because A is and because flat cofibrations are symmetrizable. So by Proposition 4.4.15, the orthogonal spectrum $R \wedge \mathbb{P}A$ is again globally connective. Since the two horizontal morphisms in the pushout square (5.4.8) are h-cofibrations, the equivariant homotopy groups of the pushout $B^{[1]}$ participate in an exact Mayer-Vietoris sequence

$$\underline{\pi}_0(R \wedge \mathbb{P}A) \oplus \underline{\pi}_0(R \wedge \mathbb{P}A) \longrightarrow \underline{\pi}_0(R) \oplus \underline{\pi}_0(R \wedge \mathbb{P}A) \longrightarrow \underline{\pi}_0(B^{[1]}) \longrightarrow 0\,,$$

where the first map is given by

$$(x, y) \longmapsto (\underline{\pi}_0(d_0^*)(x) + \underline{\pi}_0(d_1^*)(y),\ x + y)\,.$$

Using the identification $\underline{\pi}_0(B^{[1]}) \cong \underline{\pi}_0(T)$ induced by the inclusion $B^{[1]} \longrightarrow T$, we can rewrite this as an exact sequence of global functors

$$\underline{\pi}_0(R \wedge \mathbb{P}A) \xrightarrow{\underline{\pi}_0(d_0^*) - \underline{\pi}_0(d_1^*)} \underline{\pi}_0(R) \xrightarrow{\underline{\pi}_0(\psi)} \underline{\pi}_0(T) \longrightarrow 0\,. \tag{5.4.9}$$

Since R and $\mathbb{P}A$ are globally connective and $\mathbb{P}A$ is flat, the canonical morphism

$$\underline{\pi}_0(R) \Box \underline{\pi}_0(\mathbb{P}A) \longrightarrow \underline{\pi}_0(R \wedge \mathbb{P}A)$$

is an isomorphism of global power functors by Example 5.2.15. Under this identification, the effects of the two face morphisms $d_0^*, d_1^* : R \wedge \mathbb{P}A \longrightarrow R$ on $\underline{\pi}_0$ correspond to the morphisms of global power functors

$$\underline{\pi}_0(R) \diamond \underline{\pi}_0(\rho) \qquad \text{and} \qquad \underline{\pi}_0(R) \diamond \underline{\pi}_0(\tau_A)\,.$$

So we may show that $\underline{\pi}_0(\psi)$ is a coequalizer of the two morphisms $\underline{\pi}_0(d_0^*)$ and $\underline{\pi}_0(d_1^*)$. To this end we consider a morphism of global power functors $\varphi : \underline{\pi}_0(R) \longrightarrow F$ satisfying $\varphi \circ \underline{\pi}_0(d_0^*) = \varphi \circ \underline{\pi}_0(d_1^*)$. By exactness of the sequence (5.4.9) there is a unique morphism of global functors $\bar{\varphi} : \underline{\pi}_0(T) \longrightarrow F$

such that $\bar{\varphi} \circ \underline{\pi}_0(\psi) = \varphi$. Since $\underline{\pi}_0(\psi)$ is a surjective homomorphism of global power functors and the composite $\bar{\varphi}\circ\underline{\pi}_0(\psi)$ is a homomorphism of global power functors, the morphism $\bar{\varphi}$ is also compatible with multiplication and power operations, hence a morphism of global power functors. This establishes the universal property of a coequalizer, and hence completes the proof. □

Now we have all the necessary tools to show that the topological construction of free ultra-commutative ring spectra realizes the algebraic construction of free global power functors, at least for sufficiently cofibrant and globally connective spectra.

Definition 5.4.10. A *free global power functor* generated by a global functor F is a pair (R, i) consisting of a global power functor R and a morphism of global functors $i : F \longrightarrow R$ with the following property: for every global power functor S and every morphism of global functors $j : F \longrightarrow S$ there is a unique morphism of global power functors $\varphi : R \longrightarrow S$ such that $\varphi \circ i = j$.

In other words, (R, i) is a free global power functor for F if and only if it represents the functor

$$\mathcal{G}l\mathcal{P}ow \longrightarrow (\text{sets}) , \quad S \longmapsto \mathcal{GF}(F, S) .$$

For every orthogonal spectrum X, the free ultra-commutative ring spectrum $\mathbb{P}X$ gives rise to a global power functor $\underline{\pi}_0(\mathbb{P}X)$. The adjunction unit $\eta : X \longrightarrow \mathbb{P}X$ (i.e., the inclusion of the 'linear' wedge summand) is a morphism of orthogonal spectra, and thus induces a morphism of global functors

$$\underline{\pi}_0(\eta) : \underline{\pi}_0(X) \longrightarrow \underline{\pi}_0(\mathbb{P}X) .$$

Theorem 5.4.11. *Let X be a positively flat globally connective orthogonal spectrum. Then the free ultra-commutative ring spectrum $\mathbb{P}X$ is globally connective and the pair $(\underline{\pi}_0(\mathbb{P}X), \underline{\pi}_0(\eta))$ is a free global power functor generated by the global functor $\underline{\pi}_0(X)$.*

Proof We let $\mathcal{Y}$ be the class of globally connective orthogonal spectra Y such that for some (hence any) global equivalence $Y^c \longrightarrow Y$ from a positively flat orthogonal spectrum the claim of the theorem holds for Y^c. We let G be a compact Lie group and V a non-zero faithful G-representation. Then $B_{\text{gl}}G = \mathbf{L}_{G,V}$ is a global classifying space for G. The free ultra-commutative ring spectrum $\mathbb{P}(\Sigma^\infty_+ B_{\text{gl}}G)$ is isomorphic to the unreduced suspension spectrum of the free ultra-commutative monoid $\mathbb{P}(B_{\text{gl}}G)$; so $\mathbb{P}(\Sigma^\infty_+ B_{\text{gl}}G)$ is globally connective by Proposition 4.1.11. By Proposition 5.3.4 the global power functor

$$\underline{\pi}_0(\mathbb{P}(\Sigma^\infty_+ B_{\text{gl}}G))$$

is freely generated by the class u_G^{ucom} in $\pi_0^G(\mathbb{P}(\Sigma^\infty_+ B_{\text{gl}}G))$. Moreover, the global

functor $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ is freely generated by the stable tautological class e_G, by Proposition 4.2.5. So altogether this shows that the orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ belongs to the class $\mathcal{Y}$.

Now we establish two closure properties for the class $\mathcal{Y}$. We let $\{X_i\}_{i\in I}$ be a family of orthogonal spectra in $\mathcal{Y}$; we claim that the wedge $\bigvee_{i\in I} X_i$ also belongs to $\mathcal{Y}$. Since wedges of orthogonal spectra preserve global equivalences (by Corollary 3.1.37 (i)), we can assume that each X_i is positively flat. Then the wedge $\bigvee_{i\in I} X_i$ is also positively flat. The underlying orthogonal spectrum of $\mathbb{P}(\bigvee_{i\in I} X_i)$ is the wedge, indexed by functions $f : I \longrightarrow \mathbb{N}$ with finite support (i.e., $f(i) = 0$ for almost all $i \in I$) of smash products of the finite symmetric powers:

$$\mathbb{P}\Big(\bigvee\nolimits_{i\in I} X_i\Big) \cong \bigvee_{f:I\longrightarrow\mathbb{N},\text{finite supp}} \ \bigwedge_{f(i)\neq 0} \mathbb{P}^{f(i)}(X_i)\ .$$

Since flat cofibrations are symmetrizable and X_i is flat, the symmetric power $\mathbb{P}^{f(i)}(X_i)$ is flat. Since $\mathbb{P}(X_i)$ is globally connective, so is its retract $\mathbb{P}^{f(i)}(X_i)$. So for every finitely supported function $f : I \longrightarrow \mathbb{N}$, the morphism

$$\Box_{f(i)\neq 0}\, \underline{\pi}_0\left(\mathbb{P}^{f(i)}(X_i)\right) \longrightarrow \underline{\pi}_0\left(\bigwedge\nolimits_{f(i)\neq 0} \mathbb{P}^{f(i)}(X_i)\right)$$

from the iterated box product is an isomorphism of global functors by Proposition 4.4.15. Since equivariant homotopy groups take wedges to sums by Corollary 3.1.37 (i), this establishes a sequence of isomorphisms of global functors

$$\begin{aligned}
\operatorname{colim}_{J\subset I,\ J \text{ finite}} \Box_{j\in J} \underline{\pi}_0(\mathbb{P}(X_j)) &\cong \bigoplus_{f:I\longrightarrow\mathbb{N},\text{finite supp}} \ \Box_{f(i)\neq 0} \underline{\pi}_0(\mathbb{P}^{f(i)}(X_i)) \\
&\cong \bigoplus_{f:I\longrightarrow\mathbb{N},\text{finite supp}} \underline{\pi}_0\left(\bigwedge\nolimits_{f(i)\neq 0} \mathbb{P}^{f(i)}(X_i)\right) \\
&\xrightarrow{\cong} \underline{\pi}_0\left(\mathbb{P}\left(\bigvee\nolimits_{i\in I} X_i\right)\right)
\end{aligned}$$

We claim that the source of this isomorphism is the coproduct, in the category of global power functors, of the global power functors $\underline{\pi}_0(\mathbb{P}(X_i))$. Indeed, the coproduct of two global power functors is given by the box product of underlying global functors, see Example 5.2.15. As in any category, a general (possibly infinite) I-indexed coproduct can be obtained as the filtered colimit, formed over the poset of finite subsets of I, of the finite coproducts. Moreover, filtered colimits in the category of global Green functors, and hence also of global power functors, are created in the underlying category of global functors, see Proposition 2.1.3. This proves the claim.

Since $\underline{\pi}_0(\mathbb{P}(X_i))$ is freely generated by the global functor $\underline{\pi}_0(X_i)$ and coproducts of free objects are free, the right-hand side of the isomorphism is a free global power functor generated by the global functor $\bigoplus_{i\in I} \underline{\pi}_0(X_i)$. Equivariant

homotopy groups take wedges to direct sums, so the canonical morphism

$$\bigoplus_{i\in I} \underline{\pi}_0(X_i) \longrightarrow \underline{\pi}_0\left(\bigvee_{i\in I} X_i\right)$$

is an isomorphism of global functors. So altogether we obtain that $\underline{\pi}_0(\mathbb{P}(\bigvee_{i\in I} X_i))$ is a free global power functor generated by the global functor $\underline{\pi}_0(\bigvee_{i\in I} X_i)$. This concludes the proof that the class $\mathcal{Y}$ is closed under wedges.

Now we claim that the class $\mathcal{Y}$ is also closed under cones. In other words, we let

$$A \longrightarrow B \longrightarrow C \longrightarrow A \wedge S^1$$

be a distinguished triangle in the global stable homotopy category such that A and B belong to $\mathcal{Y}$; we must show that C also belongs to $\mathcal{Y}$. After replacing the distinguished triangle by an isomorphic one, we can assume that A and B are positively flat and $C = Cf$ is the mapping cone of a morphism $f : A \longrightarrow B$ of orthogonal spectra, i.e., the pushout of the diagram of orthogonal spectra

$$B \xleftarrow{f} A \xrightarrow{i} CA ,$$

where the right morphism is the cone inclusion. As a left adjoint, the free functor $\mathbb{P}$ preserves pushouts, so $\mathbb{P}(Cf)$ is a pushout of the diagram of ultra-commutative ring spectra

$$\mathbb{P}B \xleftarrow{\mathbb{P}f} \mathbb{P}A \xrightarrow{\mathbb{P}i} \mathbb{P}(CA) .$$

Since A and B belong to $\mathcal{Y}$, both $\mathbb{P}B$ and $\mathbb{P}A$ are globally connective. So Proposition 5.4.5 applies and shows that the morphism

$$\mathbb{P}j : \mathbb{P}B \longrightarrow \mathbb{P}(Cf)$$

is a coequalizer, in the category of global power functors, of the two morphisms

$$\underline{\pi}_0(\mathbb{P}B) \diamond \underline{\pi}_0(\mathbb{P}f) ,\ \underline{\pi}_0(\mathbb{P}B) \diamond \underline{\pi}_0(\tau_A) \ :\ \underline{\pi}_0(\mathbb{P}B)\Box\underline{\pi}_0(\mathbb{P}A) \longrightarrow \underline{\pi}_0(\mathbb{P}B) .$$

Since A and B belong to the class $\mathcal{Y}$, the global power functors $\underline{\pi}_0(\mathbb{P}A)$ and $\underline{\pi}_0(\mathbb{P}B)$ are freely generated by the global functors $\underline{\pi}_0(A)$ and $\underline{\pi}_0(B)$, respectively. So the coequalizer property witnesses that the global power functor $\underline{\pi}_0(\mathbb{P}(Cf))$ is freely generated by the cokernel of the morphism of global functors $\underline{\pi}_0(f) : \underline{\pi}_0(A) \longrightarrow \underline{\pi}_0(B)$. Since A is globally connective,

$$\underline{\pi}_0(A) \xrightarrow{\underline{\pi}_0(f)} \underline{\pi}_0(B) \longrightarrow \underline{\pi}_0(Cf) \longrightarrow 0$$

is an exact sequence of global functors by Proposition 3.1.36. This concludes the proof that the pair $(\underline{\pi}_0(\mathbb{P}(Cf)), \underline{\pi}_0(\eta))$ is a free global power functor generated by the global functor $\underline{\pi}_0(Cf)$. So we have also shown that the class $\mathcal{Y}$ is closed under cones.

Altogether this shows that the class of orthogonal spectra $\mathcal{Y}$ is closed under wedges and cones and contains the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for all compact Lie groups G. Proposition 4.4.13 then shows that $\mathcal{Y}$ is the class of all globally connective orthogonal spectra. This proves the theorem. □

Theorem 5.4.12. *Let X be a positively flat globally $(n-1)$-connected orthogonal spectrum, for some $n \geq 0$. Then for every $m \geq 1$ the orthogonal spectrum $\mathbb{P}^m(X)$ is globally $(n-1)$-connected.*

Proof We choose a cofibrant replacement $q : Y \longrightarrow \Omega^n X$ of $\Omega^n X$ in the positive global model structure of Proposition 4.3.33. Then Y is globally connective (i.e., globally (-1)-connected). The adjoint $q^\flat$ of q factors as the composite

$$Y \wedge S^n \xrightarrow{q \wedge S^n} (\Omega^n X) \wedge S^n \xrightarrow{\epsilon_X} X \ .$$

The first morphism is a global equivalence since suspension is fully homotopical, and the second morphism is a global equivalence by Proposition 3.1.25 (ii). So the morphism $q^\flat$ is a global equivalence.

We show that $\mathbb{P}^m(Y \wedge S^n)$ is globally $(n-1)$-connected. We let

$$\nu_m \ = \ \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1 + \cdots + x_m = 0\}$$

denote the reduced natural Σ_m-representation, where Σ_m acts by permutation of coordinates. The smash power $(S^n)^{\wedge m}$ is Σ_m-equivariantly homeomorphic to the smash product of S^n (with trivial Σ_m-action) and the representation sphere $S^{\mathbb{R}^n \otimes \nu_m}$. This decomposition induces an isomorphism of orthogonal spectra

$$\mathbb{P}^m(Y \wedge S^n) \ \cong \ (Y^{\wedge m} \wedge (S^n)^{\wedge m})/\Sigma_m \ \cong \ (Y^{\wedge m} \wedge S^{\mathbb{R}^n \otimes \nu_m})/\Sigma_m \wedge S^n \ .$$

The sphere $S^{\mathbb{R}^n \otimes \nu_m}$ admits a finite Σ_m-CW-structure. Moreover, all isotropy groups are conjugate to subgroups of the form $\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}$ with $i_1 + \cdots + i_k = m$. So the orthogonal spectrum $\mathbb{P}^m(Y \wedge S^n)$ admits a finite filtration, along h-cofibrations, with subquotients isomorphic to

$$\begin{aligned}(Y^{\wedge m} \wedge \Sigma_m/(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})_+ \wedge S^k)/\Sigma_m \wedge S^n \ &\cong \ Y^{\wedge m}/(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}) \wedge S^{k+n} \\ &\cong \ \mathbb{P}^{i_1}(Y) \wedge \ldots \wedge \mathbb{P}^{i_k}(Y) \wedge S^{k+n} \ .\end{aligned}$$

Since Y is positively flat and globally connective, Theorem 5.4.11 says that the free ultra-commutative ring spectrum $\mathbb{P}Y$ is globally connective. Since $\mathbb{P}^{i_k}(Y)$ is a retract of $\mathbb{P}Y$, it is also globally connective. Since flat cofibrations are symmetrizable (Theorem 5.4.1 (i)), each of the orthogonal spectra $\mathbb{P}^{i_j}(Y)$ is again flat. So the smash product $\mathbb{P}^{i_1}(Y) \wedge \ldots \wedge \mathbb{P}^{i_k}(Y)$ is globally connective by Proposition 4.4.15. Altogether this shows that each of the subquotients of $\mathbb{P}^m(Y \wedge S^n)$ is globally $(n-1)$-connected. So the spectrum $\mathbb{P}^m(Y \wedge S^n)$ itself is globally $(n-1)$-connected.

The free functor $\mathbb{P} : \mathcal{S}p \longrightarrow ucom$ is a left Quillen functor from the positive global model structure on orthogonal spectra to the global model structure of ultra-commutative ring spectra. So $\mathbb{P}$ preserves global equivalences between positively flat orthogonal spectra, hence so does its retract $\mathbb{P}^m$. So the morphism $\mathbb{P}^m(q^\flat) : \mathbb{P}^m(Y \wedge S^n) \longrightarrow \mathbb{P}^m(X)$ is a global equivalence, and $\mathbb{P}^m(X)$ is globally $(n-1)$-connected. □

The next theorem shows that we can 'kill higher homotopy groups' in the world of ultra-commutative ring spectra. Again, the construction needs the hypothesis of global connectivity.

Theorem 5.4.13. *Let R be a globally connective ultra-commutative ring spectrum and $n \geq 1$. Then there is a cofibration of ultra-commutative ring spectra $\psi : R \longrightarrow T$ with the following properties:*

- *the induced morphism of global functors $\underline{\pi}_k(\psi) : \underline{\pi}_k(R) \longrightarrow \underline{\pi}_k(T)$ is bijective for $k < n$, and*
- *the global functor $\underline{\pi}_n(T)$ is trivial.*

Proof We choose an index set J, compact Lie groups G_j and elements $y_j \in \pi_n^{G_j}(R)$, for $j \in J$, that altogether generate the global functor $\underline{\pi}_n(R)$. We represent each class y_j as a morphism of orthogonal spectra

$$f_j \ : \ \Sigma^\infty_+ B_{\mathrm{gl}} G_j \wedge S^n \ \longrightarrow \ R$$

that sends the n-fold suspension of the stable tautological class e_{G_j} to y_j; this involves an implicit choice of non-zero faithful G_j-representation. We form the wedge of all these morphisms

$$F \ : \ X \ = \ \bigvee_{j \in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j \wedge S^n \ \longrightarrow \ R \ .$$

All we need to remember about F is that its source X is globally $(n-1)$-connected and positively flat, and that the morphism of global functors

$$\underline{\pi}_n(F) \ : \ \underline{\pi}_n(X) \ \longrightarrow \ \underline{\pi}_n(R)$$

is surjective. We extend F freely to a morphism of ultra-commutative ring spectra $\tilde{F} : \mathbb{P}X \longrightarrow R$. We let T be the pushout, in the category of ultra-commutative ring spectra, of the diagram

$$R \ \xleftarrow{\tilde{F}} \ \mathbb{P}X \ \xrightarrow{\mathbb{P}i} \ \mathbb{P}(CX) \ ,$$

where CX is the cone of the orthogonal spectrum X, and $i : X \longrightarrow CX$ is the cone inclusion. Since X is positively flat, the cone inclusion is a positive cofibration of orthogonal spectra, and so $\mathbb{P}i$ is a cofibration of ultra-commutative

ring spectra. The cobase change $\psi : R \longrightarrow T$ of $\mathbb{P}i$ is then also a cofibration of ultra-commutative ring spectra.

We recall that the cobase change of a free morphism $\mathbb{P}i : \mathbb{P}X \longrightarrow \mathbb{P}(CX)$ comes with a filtration by 'number of factors from $\mathbb{P}(CX)$', see for example [188, Prop. B.2]. More precisely, the morphism $\psi : R \longrightarrow T$ is the sequential composite of morphisms of R-module spectra

$$R = P_0 \xrightarrow{\psi_1} P_1 \xrightarrow{\psi_2} P_2 \xrightarrow{\psi_3} \cdots ,$$

such that each step in the filtration can be obtained from the previous one as a pushout of R-modules:

$$\begin{tikzcd} R \wedge Q^m(i)/\Sigma_m \arrow[r, "i^{\square m}/\Sigma_m"] \arrow[d] & R \wedge (CX)^{\wedge m}/\Sigma_m \arrow[d] \\ P_{m-1} \arrow[r, "\psi_m"'] & P_m \end{tikzcd}$$

Since $i : X \longrightarrow CX$ is a flat cofibration, and flat cofibrations are symmetrizable (Theorem 5.4.1 (i)), the upper horizontal morphism is a cofibration of R-modules, hence so is its cobase change $\psi_m : P_{m-1} \longrightarrow P_m$. Moreover, the pushout square witnesses that the cokernel of ψ_m is isomorphic to

$$R \wedge \mathrm{coker}(i^{\square m})/\Sigma_m \cong R \wedge (CX/X)^{\wedge m}/\Sigma_m \cong R \wedge \mathbb{P}^m(X \wedge S^1) .$$

Since X is globally $(n-1)$-connected, its suspension $X \wedge S^1$ is globally n-connected, and so the mth symmetric power $\mathbb{P}^m(X \wedge S^1)$ is also globally n-connected for all $m \geq 1$, by Theorem 5.4.12. Since R is connective and $\mathbb{P}^m(X \wedge S^1)$ is flat, the smash product $R \wedge \mathbb{P}^m(X \wedge S^1)$ is again globally n-connected by Proposition 4.4.15. In particular, the morphism $\underline{\pi}_k(\psi_m) : \underline{\pi}_k(P_{m-1}) \longrightarrow \underline{\pi}_k(P_m)$ is bijective for $k < n$ and surjective for $k = n$. Since all morphisms $\psi_m : P_{m-1} \longrightarrow P_m$ are h-cofibrations of orthogonal spectra, they are level-wise closed embeddings, and so that canonical map

$$\mathrm{colim}_m\, \underline{\pi}_k(P_m) \longrightarrow \underline{\pi}_k(\mathrm{colim}_m\, P_m) = \underline{\pi}_k(T)$$

is an isomorphism by Proposition 3.1.41 (i). Altogether this shows that the morphism $\underline{\pi}_k(\psi) : \underline{\pi}_k(R) \longrightarrow \underline{\pi}_k(T)$ is bijective for $k < n$ and surjective for $k = n$.

The diagram

$$\begin{tikzcd} \underline{\pi}_n(X) \arrow[r, "\underline{\pi}_n(\eta_X)"] \arrow[dr, "\underline{\pi}_n(F)"'] & \underline{\pi}_n(\mathbb{P}X) \arrow[r, "\underline{\pi}_n(\mathbb{P}(i))"] \arrow[d, "\underline{\pi}_n(\tilde{F})"] & \underline{\pi}_n(\mathbb{P}(CX)) \arrow[d] \\ & \underline{\pi}_n(R) \arrow[r, "\underline{\pi}_n(\psi)"'] & \underline{\pi}_n(T) \end{tikzcd}$$

commutes and the composite through the upper right corner is the zero map because CX is contractible. Since $\underline{\pi}_n(F)$ and $\underline{\pi}_n(\psi)$ are surjective, we conclude that the global functor $\underline{\pi}_n(T)$ is trivial. $\square$

Now we can finally show that every global power functor is realized by an ultra-commutative ring spectrum. The analogous result in equivariant stable homotopy theory for a fixed finite group has been obtained by Ullman [181]. More is true: the next theorem effectively constructs a right adjoint functor

$$H \;:\; \mathcal{G}l\mathcal{P}ow \;\longrightarrow\; \mathrm{Ho}^{\mathrm{gl.\,connective}}(ucom)$$

to the functor $\underline{\pi}_0$ such that the adjunction counit is an isomorphism $\underline{\pi}_0(HF) \cong F$ of global power functors. In other words, for every globally connective ultra-commutative ring spectrum T, the functor $\underline{\pi}_0$ restricts to a bijection

$$\underline{\pi}_0 \;:\; \mathrm{Ho}(ucom)(T, HF) \;\cong\; \mathcal{G}l\mathcal{P}ow(\underline{\pi}_0(T), F) \;.$$

However, we won't prove these more general facts.

Theorem 5.4.14. *Let F be a global power functor. There is an ultra-commutative ring spectrum HF such that $\underline{\pi}_k(HF) = 0$ for all $k \neq 0$ and an isomorphism of global power functors*

$$\underline{\pi}_0(HF) \;\cong\; F \;.$$

Proof The t-structure on the global stable homotopy category of Theorem 4.4.9 (for the global family of all compact Lie groups) lets us choose an Eilenberg-Mac Lane spectrum for the underlying global functor of F, see Remark 4.4.12. Cofibrant replacement in the positive global model structure of Proposition 4.3.33 then provides a globally connective positive flat orthogonal spectrum X and an isomorphism of global functors $\underline{\pi}_0(X) \cong F$.

We form the free ultra-commutative ring spectrum $\mathbb{P}X$. Theorem 5.4.11 then shows that $\mathbb{P}X$ is globally connective and $\underline{\pi}_0(\mathbb{P}X)$ is a free global power functor generated by the global functor $\underline{\pi}_0(X)$, with respect to the morphism $\underline{\pi}_0(\eta)$: $\underline{\pi}_0(X) \longrightarrow \underline{\pi}_0(\mathbb{P}X)$. In particular, there is a unique morphism of global power functors

$$\epsilon \;:\; \underline{\pi}_0(\mathbb{P}X) \;\longrightarrow\; F$$

such that $\epsilon \circ \underline{\pi}_0(\eta)$ is the previous isomorphism. The morphism ϵ is then surjective.

We choose a positively cofibrant globally connective orthogonal spectrum A together with a morphism of orthogonal spectra $\rho' : A \longrightarrow \mathbb{P}X$ such that the image of the morphism $\underline{\pi}_0(\rho') : \underline{\pi}_0(A) \longrightarrow \underline{\pi}_0(\mathbb{P}X)$ coincides with the kernel of ϵ. For example, we can choose compact Lie groups G_j and elements

$y_j \in \pi_0^{G_j}(\mathbb{P}X)$ that altogether generate this kernel as a global functor. Then we represent each class y_j as a morphism of orthogonal spectra

$$f_j \ : \ \Sigma^\infty_+ B_{\mathrm{gl}} G_j \ \longrightarrow \ \mathbb{P}X$$

that sends the stable tautological class e_{G_j} to y_j, and let $\rho' : A \longrightarrow \mathbb{P}X$ be the wedge of all these morphisms.

We let $\rho : \mathbb{P}A \longrightarrow \mathbb{P}X$ be the free extension of ρ' to a morphism of ultra-commutative ring spectra. Then we define T as the following pushout, in the category of ultra-commutative ring spectra:

$$\begin{tikzcd} \mathbb{P}A \arrow[r, "\mathbb{P}i"] \arrow[d, "\rho"'] & \mathbb{P}(CA) \arrow[d] \\ \mathbb{P}X \arrow[r, "\psi"'] & T \end{tikzcd}$$

Since $\mathbb{P}A$ and $\mathbb{P}X$ are both globally connective by Theorem 5.4.11, we can apply Proposition 5.4.5 and conclude that T is globally connective, the morphism of global power functors $\underline{\pi}_0(\psi) : \underline{\pi}_0(\mathbb{P}X) \longrightarrow \underline{\pi}_0(T)$ is surjective, and a coequalizer, in the category of global power functors, of the two morphisms

$$\underline{\pi}_0(\mathbb{P}X) \diamond \underline{\pi}_0(\rho) \ , \ \underline{\pi}_0(\mathbb{P}X) \diamond \underline{\pi}_0(\tau_A) \ : \ \underline{\pi}_0(\mathbb{P}X) \Box \underline{\pi}_0(\mathbb{P}A) \ \longrightarrow \ \underline{\pi}_0(\mathbb{P}X) \ .$$

Now $\underline{\pi}_0(\mathbb{P}A)$ is the free global power functor generated by the global functor $\underline{\pi}_0(A)$, by Theorem 5.4.11. So the effect of the coequalizer is to annihilate the global power ideal generated by the image of $\underline{\pi}_0(\rho') : \underline{\pi}_0(A) \longrightarrow \underline{\pi}_0(\mathbb{P}X)$. This global power ideal is, by construction, the kernel of $\epsilon : \underline{\pi}_0(\mathbb{P}X) \longrightarrow F$. So ϵ descends to an isomorphism of global power functors between $\underline{\pi}_0(T)$ and F.

Now we use Theorem 5.4.13 to kill the higher homotopy groups of T. More precisely, we construct a sequence of cofibrations of ultra-commutative ring spectra

$$T \ = \ T_0 \ \longrightarrow \ T_1 \ \longrightarrow \ \ldots \ \longrightarrow \ T_n \ \longrightarrow \ \ldots \qquad (5.4.15)$$

by induction on n. We obtain T_n by applying Theorem 5.4.13 in dimension n to $R = T_{n-1}$. Then T_n is globally connective, $\underline{\pi}_0(T_n)$ is isomorphic to $\underline{\pi}_0(T)$, and hence to F, and the global functors $\underline{\pi}_k(T_n)$ are trivial for all $1 \leq k \leq n$.

Finally, we define HF as the colimit of the sequence (5.4.15) of ultra-commutative ring spectra. Each morphism in the sequence is a cofibration of ultra-commutative ring spectra, hence an h-cofibration of underlying orthogonal spectra (see Theorem 5.4.3 (ii)), and so level-wise a closed embedding. Since equivariant homotopy groups commute with sequential colimits over closed embeddings (see Proposition 3.1.41 (i)), the colimit has the desired properties of an ultra-commutative Eilenberg-Mac Lane spectrum for F. □

6

Global Thom and K-theory spectra

The final chapter of this book is devoted to an in-depth study of interesting examples of ultra-commutative ring spectra, in particular global Thom spectra and ultra-commutative global models for various flavors of equivariant K-theory spectra. In Section 6.1 we discuss global refinements of the classical Thom spectrum MO, as well as Thom spectra for the other families of classical Lie groups, such as MSO and MU. We carefully analyze two different global forms of the Thom spectrum MO that represents unoriented bordism, namely the ultra-commutative Thom ring spectrum **MO**, and an E_∞-orthogonal ring spectrum **mO**. Both Thom spectra are the homogeneous degree 0 summands in certain $\mathbb{Z}$-graded periodic extensions **MOP** and **mOP**, respectively. The Thom spectrum **mO** is the natural target for the Thom-Pontryagin map from geometric equivariant bordism. The ultra-commutative ring spectrum **MO** is the $\mathbb{R}$-analog of tom Dieck's equivariant homotopical bordism. The equivariant homology theory represented by **MO** is a localization of the one represented by **mO**, formed by inverting certain 'inverse Thom classes', see Corollary 6.1.35. The Thom spectrum **mO** comes with a rank filtration that we use to show that **mO** is globally connective, and to calculate the global functor $\underline{\pi}_0(\mathbf{mO})$, see Theorem 6.1.44.

Section 6.2 is devoted to equivariant bordism and its relation to the equivariant homology theories represented by the global Thom spectra of Section 6.1. We recall various facts about equivariant bordism groups of smooth compact G-manifolds in some detail, highlighting the global perspective. The main result of this section is Theorem 6.2.33 which says that when G is isomorphic to a product of a finite group and a torus, then the Thom-Pontryagin map is an isomorphism from G-equivariant bordism to G-equivariant **mO**-homology. This result is usually credited to Wasserman, because it can be derived from his equivariant transversality theorem [184, Thm. 3.11]. Theorem 6.2.37 gives a localized version of this result: the Thom-Pontryagin map is an isomorphism from *stable* equivariant bordism to $\mathbf{mO}[1/\tau]$-theory, without any restriction

on the compact Lie group. Given that $\mathbf{mO}[1/\tau]$-theory is isomorphic to **MO**-theory (by Corollary 6.1.35), this is equivalent to a result of Bröcker and Hook [27, Thm. 4.1] that identifies stable equivariant bordism with equivariant **MO**-homology.

Section 6.3 discusses the ultra-commutative ring spectrum **ku**, the *connective global K-theory spectrum*, see Construction 6.3.9. The degree zero equivariant cohomology theory represented by **ku** tries hard to be equivariant K-theory (see Theorem 6.3.31 for the precise statement), and there is a ring homomorphism $\mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ from the complex representation ring that is an isomorphism whenever G is finite, and almost a morphism of global power functors, compare Theorem 6.3.33. We also introduce the Bott class in the group $\pi_2^e(\mathbf{ku})$ (see Construction 6.3.42) and the more general equivariant Bott classes associated with G-Spin^c-representations (see Construction 6.3.46).

The final Section 6.4 reviews **KU**, the *periodic global K-theory* spectrum, see Construction 6.4.9. This is an ultra-commutative ring spectrum whose G-homotopy type realizes G-equivariant periodic K-theory, in the sense that the equivariant cohomology represented by **KU** on finite G-CW-complexes is isomorphic to equivariant K-theory, see Corollary 6.4.23. Moreover, $\underline{\pi}_0(\mathbf{KU})$ is isomorphic, as a global power functor, to the complex representation ring functor **RU**, by Theorem 6.4.24. Periodic global K-theory receives a morphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ from connective global K-theory; the image of the Bott class under j becomes a unit (Theorem 6.4.29), justifying the adjective 'periodic' for **KU**. Our final example of an ultra-commutative ring spectrum is $\mathbf{ku}^c$, the *global connective K-theory* (see Construction 6.4.32), a certain global refinement of Greenlees' equivariant connective K-theory [66].

6.1 Global Thom spectra

In this section we discuss two different global forms of the Thom spectrum MO that represents unoriented bordism, namely the ultra-commutative Thom ring spectrum **MO**, and a variation **mO** that is only E_∞-commutative. Both Thom spectra are the homogeneous degree 0 summands in certain $\mathbb{Z}$-graded periodic extensions **MOP** and **mOP**, respectively. All four orthogonal spectra are Thom spectra over certain orthogonal spaces defined in Section 2.4. The partners are easy to identify from the notation: the relevant orthogonal spaces either have a **B** or **b** in their name, and in the corresponding Thom spectrum this letter is replaced by an **M** or **m**, respectively.

While the spectrum **mO** has less structure than **MO** (it is 'only' E_∞, not ultra-commutative), it is closely related to geometry. Indeed, the Thom spec-

trum **mO** is the natural target for the Thom-Pontryagin map from geometric equivariant bordism. The ultra-commutative ring spectrum **MO** is the $\mathbb{R}$-analog of equivariant homotopical bordism, due to tom Dieck [175]; the unoriented version **MO** is studied in detail in [27]. The following commutative diagram gives a schematic overview of the relevant equivariant homology theories:

$$\begin{tikzcd}
\mathcal{N}_*^G \arrow[r, "\Theta^G"] \arrow[d] & \mathbf{mO}_*^G \arrow[r] \arrow[d] & \mathbf{mOP}_*^G \arrow[d] \\
\mathfrak{N}_*^{G:S} \arrow[r, "\cong", "\Theta^G"'] & \mathbf{MO}_*^G \arrow[r] & \mathbf{MOP}_*^G
\end{tikzcd}$$

The two theories in the middle column are the equivariant homology theories represented by the orthogonal Thom spectra **mO** and **MO**. The vertical transformations are isomorphisms for $G = e$, but not in general. In fact, when G has more than one element, then $\pi_*^G(\mathbf{MO})$ has non-trivial elements in negative degrees, while **mO** is globally connective. The vertical transformations are localizations, i.e., they become isomorphisms after inverting all inverse Thom classes, compare Corollary 6.1.35. The two theories **mOP** and **MOP** in the last column are the periodic versions of **mO** and **MO**; each is a wedge of all suspensions and desuspensions of the non-periodic version.

In the left column, $\mathcal{N}_*^G$ is geometrically defined equivariant bordism, and $\mathfrak{N}_*^{G:S}$ is *stable* equivariant bordism, a certain localization of $\mathcal{N}_*^G$. So the two theories in the left column are not represented by orthogonal spectra, but they are defined from bordism classes of G-manifolds; we will recall these geometric theories in Section 6.2. The transformations labeled Θ^G are the equivariant Thom-Pontryagin construction and its 'stabilization'. The upper Thom-Pontryagin map $\Theta^G : \mathcal{N}_*^G \longrightarrow \pi_*^G(\mathbf{mO})$ is an isomorphism whenever G is isomorphic to a product of a finite group and a torus; this result seems to have been folklore at some point, and we give a proof in Theorem 6.2.33. The upper Thom-Pontryagin map is not an isomorphism for more general compact Lie groups; in fact the geometric bordism theory $\mathcal{N}_*^G$ cannot in general be represented by an orthogonal G-spectrum because the Wirthmüller map fails to be an isomorphism for those closed subgroups H of G that act non-trivially on the tangent space $T_{eH}(G/H)$, compare Remark 6.2.13. The stabilized Thom-Pontryagin map $\Theta^G : \mathfrak{N}_*^{G:S} \longrightarrow \pi_*^G(\mathbf{MO})$ is an isomorphism in complete generality, by a theorem of Bröcker and Hook [27, Thm. 4.1]; we derive this fact in a different way, see Remark 6.2.38.

The Thom spectrum **mO** comes with a rank filtration, the subquotients of which we identify in Theorem 6.1.42. We use the rank filtration to show that **mO** is globally connective, and we calculate the global functor $\underline{\pi}_0(\mathbf{mO})$ in Theorem 6.1.44.

Example 6.1.1. We start with the ultra-commutative ring spectrum **MGr**, the Thom spectrum over the additive Grassmannian **Gr**, discussed in Example 2.3.12. We recall that the value of **Gr** at an inner product space V is

$$\mathbf{Gr}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr_n(V) \ ,$$

the disjoint union of all Grassmannians in V. Over the space $\mathbf{Gr}(V)$ sits a tautological euclidean vector bundle (of non-constant rank) whose total space consists of pairs $(x, U) \in V \times \mathbf{Gr}(V)$ such that $x \in U$. We define $\mathbf{MGr}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. The structure maps are given by

$$\mathbf{O}(V, W) \wedge \mathbf{MGr}(V) \ \longrightarrow \ \mathbf{MGr}(W) \ , \ (w, \varphi) \wedge (x, U) \ \longmapsto \ (w+\varphi(x), \ \varphi^\perp \oplus \varphi(U)) \ ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of $\varphi : V \longrightarrow W$. Multiplication maps are defined by direct sum, i.e.,

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{MGr}(V) \wedge \mathbf{MGr}(W) \ &\longrightarrow \ \ \mathbf{MGr}(V \oplus W) \\ (x, U) \wedge (x', U') \ \ &\longmapsto \ ((x, x'), \ U \oplus U') \ . \end{aligned}$$

Unit maps are defined by

$$\eta(V) \ : \ S^V \ \longrightarrow \ \mathbf{MGr}(V) \ , \quad v \ \longmapsto \ (v, V) \ .$$

The multiplication maps are binatural, associative, commutative and unital, and all this structure makes **MGr** an ultra-commutative ring spectrum. The orthogonal spectrum **MGr** is graded, with kth homogeneous summand given by

$$\mathbf{MGr}^{[k]}(V) \ = \ Th(Gr_{|V|+k}(V)) \ .$$

So **MGr** is concentrated in non-positive gradings, i.e., $\mathbf{MGr}^{[k]}$ is trivial for $k > 0$. The unit morphism $\eta : \mathbb{S} \longrightarrow \mathbf{MGr}$ is an isomorphism onto the summand $\mathbf{MGr}^{[0]}$.

Now we let V be a representation of a compact Lie group G. We define the *inverse Thom class*

$$\tau_{G,V} \ \in \ \mathbf{MGr}_0^G(S^V) \tag{6.1.2}$$

as the class represented by the G-map

$$t_{G,V} \ : \ S^V \ \longrightarrow \ Th(Gr(V)) \wedge S^V \ = \ \mathbf{MGr}(V) \wedge S^V \ , \quad v \ \longmapsto \ (0, \{0\}) \wedge (-v) \ .$$

If V has dimension m, then the class $\tau_{G,V}$ has internal degree $-m$, i.e., it lies in the homogeneous summand $\mathbf{MGr}^{[-m]}$. The justification for the name 'inverse Thom class' is that in the theory **MOP**, the image of the class $\tau_{G,V}$ becomes invertible, and its inverse is the Thom class of V (considered as G-vector bundle over a point). We explain this in more detail in Theorem 6.1.17 below.

So while the theory **MGr** is not globally orientable, and does not have Thom isomorphisms for equivariant bundles, informally speaking the inverses of the prospective Thom classes are already present in **MGr**.

Remark 6.1.3 (**MGr** as a wedge of semifree spectra)**.** We recall from Construction 4.1.23 that the semifree orthogonal spectrum generated by the tautological $O(m)$-representation ν_m on $\mathbb{R}^m$ is given by

$$F_{O(m),\nu_m} \;=\; \mathbf{O}(\nu_m, -)/O(m)\,.$$

We claim that the spectrum **MGr** is isomorphic to the wedge of these semifree orthogonal spectra. Indeed, the maps

$$\mathbf{O}(\nu_m, V)/O(m) \;\longrightarrow\; \mathbf{MGr}^{[-m]}(V)\,, \quad (v,\varphi)\cdot O(m) \;\longmapsto\; (v, \varphi^\perp)$$

define an isomorphism of orthogonal spectra from $F_{O(m),\nu_m}$ to the homogeneous summand $\mathbf{MGr}^{[-m]}$. This isomorphism takes the equivariant homotopy class $a_{O(m),\nu_m}$ in $\pi_0^G(F_{O(m),\nu_m} \wedge S^{\nu_m})$ defined in (4.4.16) to the inverse Thom class $\tau_{O(m),\nu_m}$ of the tautological $O(m)$-representation, by direct inspection of the definitions.

Now we suppose that $\varphi : V \longrightarrow W$ is an isomorphism of orthogonal G-representations. Then φ compactifies to a G-equivariant homeomorphism $S^\varphi : S^V \longrightarrow S^W$ and hence induces an isomorphism

$$(S^\varphi)_* \;:\; \mathbf{MGr}_0^G(S^V) \;\longrightarrow\; \mathbf{MGr}_0^G(S^W)\,.$$

The following properties of the inverse Thom classes $\tau_{G,V}$ are all straightforward from the definition. The power operations that show up in part (v) are defined in (5.1.2); the norm maps of part (vi) are introduced in Remark 5.1.7.

Proposition 6.1.4. *The inverse Thom classes $\tau_{G,V}$ have the following properties.*

(i) *For every isomorphism $\varphi : V \longrightarrow W$ of orthogonal G-representations, the induced isomorphism $(S^\varphi)_*$ takes the class $\tau_{G,V}$ to the class $\tau_{G,W}$.*

(ii) *The class $\tau_{G,0}$ of the trivial 0-dimensional G-representation is the multiplicative unit* 1 *in* $\mathbf{MGr}_0^G(S^0) = \pi_0^G(\mathbf{MGr})$.

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^*(\tau_{G,V}) \;=\; \tau_{K,\alpha^* V}$$

holds in $\mathbf{MGr}_0^K(S^{\alpha^*(V)})$.

(iv) *For all orthogonal G-representations V and W the relation*

$$\tau_{G,V} \cdot \tau_{G,W} \;=\; \tau_{G,V\oplus W}$$

holds in $\mathbf{MGr}_0^G(S^{V\oplus W})$.

(v) *For all orthogonal G-representations V and all $k \geq 1$ the relation*

$$P^k(\tau_{G,V}) \;=\; \tau_{\Sigma_k \wr G, V^k}$$

holds in $\mathbf{MGr}_0^{\Sigma_k \wr G}(S^{V^k})$.

(vi) *For every closed subgroup H of G of finite index and all orthogonal H-representations W the relation*

$$N_H^G(\tau_{H,W}) \;=\; \tau_{G,\mathrm{Ind}_H^G W}$$

holds in $\mathbf{MGr}_0^G(S^{\mathrm{Ind}_H^G W})$.

The next proposition shows that multiplication by the inverse Thom class $\tau_{G,V}$ is realized by a certain morphism of orthogonal G-spectra $j^V_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}^V \mathbf{MGr}$. The value at an inner product space U is the map

$$\begin{aligned} j^V_{\mathbf{MGr}}(U) \;:\; \mathbf{MGr}(U) \;&\longrightarrow\; \mathbf{MGr}(U \oplus V) \;=\; (\mathrm{sh}^V \mathbf{MGr})(U) \\ (x, L) \;&\longmapsto\; ((x, 0), L \oplus 0)\;. \end{aligned} \tag{6.1.5}$$

If V has dimension m, then $j^V_{\mathbf{MGr}}$ is homogeneous of degree $-m$ in terms of the internal grading of $\mathbf{MGr}$, i.e., $j^V_{\mathbf{MGr}}$ takes the wedge summand $\mathbf{MGr}^{[k]}$ to the summand $\mathrm{sh}^V \mathbf{MGr}^{[k-m]}$. The $\underline{\pi}_*$-isomorphism $\lambda^V_X : X \wedge S^V \longrightarrow \mathrm{sh}^V X$ was defined in (3.1.23).

Proposition 6.1.6. *Let V be a representation of a compact Lie group G. The composite*

$$\pi_k^G(\mathbf{MGr} \wedge A) \xrightarrow{-\cdot \tau_{G,V}} \pi_k^G(\mathbf{MGr} \wedge A \wedge S^V) \xrightarrow{(\lambda^V_{\mathbf{MGr}\wedge A})_*} \pi_k^G(\mathrm{sh}^V \mathbf{MGr} \wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{MGr}} \wedge A : \mathbf{MGr} \wedge A \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$.

Proof In (3.1.24) we defined an isomorphism $\psi^V_X : \pi_k^G(\mathrm{sh}^V X) \longrightarrow \pi_k^G(X \wedge S^V)$, natural for morphisms of orthogonal G-spectra X, essentially given by smashing with the identity of S^V. We also defined $\varepsilon_V : \pi_k^G(X \wedge S^V) \longrightarrow \pi_k^G(X \wedge S^V)$ as the effect of the 'negative' map of S^V. Now we observe that multiplication by the class $\tau_{G,V}$ factors as the composite

$$\begin{aligned} \pi_k^G(\mathbf{MGr} \wedge A) &\xrightarrow{(j^V_{\mathbf{MGr}} \wedge A)_*} \pi_k^G(\mathrm{sh}^V \mathbf{MGr} \wedge A) \\ &\xrightarrow{\psi^V_{\mathbf{MGr}\wedge A}} \pi_k^G(\mathbf{MGr} \wedge A \wedge S^V) \xrightarrow{\varepsilon_V} \pi_k^G(\mathbf{MGr} \wedge A \wedge S^V)\;. \end{aligned}$$

Besides the definitions, this uses that the map $j^V_{\mathbf{MGr}}(U) \wedge S^V$ equals the composite

$$\begin{aligned} \mathbf{MGr}(U) \wedge S^V &\xrightarrow{\mathbf{MGr}(U) \wedge t_{G,V}} \mathbf{MGr}(U) \wedge \mathbf{MGr}(V) \wedge S^V \\ &\xrightarrow{\mu_{U,V} \wedge S^{-\mathrm{Id}}} \mathbf{MGr}(U \oplus V) \wedge S^V\;, \end{aligned}$$

where $t_{G,V}$ is the defining representative of the class $\tau_{G,V}$. The claim then follows from the fact, established in Proposition 3.1.25 (i), that $(\lambda^V_{\mathbf{MGr}\wedge A})_*$ is inverse to $\varepsilon_V \circ \psi^V_{\mathbf{MGr}\wedge A}$. □

Example 6.1.7. We define two ultra-commutative ring spectra **MO** and **MOP**. The latter is a periodic version of the former, and the former is the homogeneous degree 0 summand with respect to a natural $\mathbb{Z}$-grading of the latter. Non-equivariantly, **MO** is a version of the unoriented Thom spectrum MO, and it is a global refinement of equivariant homotopical bordism, due to tom Dieck [175]; tom Dieck first considered the unitary version in [175], and the paper [27] by Bröcker and Hook studies the orthogonal version.

The spectrum **MOP** is a Thom spectrum over the orthogonal space **BOP** discussed in Example 2.4.1. We recall that the value of **BOP** at an inner product space V is

$$\mathbf{BOP}(V) \ = \ \coprod_{n\geq 0} Gr_n(V^2) \ ,$$

the disjoint union of all Grassmannians in V^2. Over the space $\mathbf{BOP}(V)$ sits a tautological euclidean vector bundle (again of non-constant rank) with total space consisting of pairs $(x, U) \in V^2 \times \mathbf{BOP}(V)$ such that $x \in U$. We define $\mathbf{MOP}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. The structure maps are given by

$$\begin{aligned} \mathbf{O}(V, W) \wedge \mathbf{MOP}(V) \ &\longrightarrow \qquad \mathbf{MOP}(W) \\ (w, \varphi) \wedge (x, U) \quad &\longmapsto \ ((w, 0) + \mathbf{BOP}(\varphi)(x),\ \mathbf{BOP}(\varphi)(U)) \ . \end{aligned}$$

Multiplication maps

$$\mu_{V,W} \ : \ \mathbf{MOP}(V) \wedge \mathbf{MOP}(W) \ \longrightarrow \ \mathbf{MOP}(V \oplus W)$$

are defined by sending $(x, U) \wedge (x', U')$ to $(\kappa_{V,W}(x, x'), \kappa_{V,W}(U \oplus U'))$ where $\kappa_{V,W} : V^2 \oplus W^2 \cong (V \oplus W)^2$ is the preferred isometry defined by

$$\kappa_{V,W}((v, v'), (w, w')) \ = \ ((v, w), (v', w')) \ .$$

Unit maps are defined by

$$S^V \ \longrightarrow \ \mathbf{MOP}(V) \ , \quad v \ \longmapsto \ ((v, 0),\ V \oplus 0) \ .$$

The multiplication maps are binatural, associative, commutative and unital, and all this structure makes **MOP** an ultra-commutative ring spectrum.

The orthogonal space **BOP** is $\mathbb{Z}$-graded, with kth homogeneous summand

$$\mathbf{BOP}^{[k]}(V) \ = \ Gr_{|V|+k}(V^2) \ .$$

The spectrum **MOP** inherits a $\mathbb{Z}$-grading, where the summand $\mathbf{MOP}^{[k]}(V)$ of degree k is the Thom space of the tautological $(|V| + k)$-plane bundle over

$\mathbf{BOP}^{[k]}(V)$; then $\mathbf{MOP}(V)$ is the one-point union of the Thom spaces $\mathbf{MOP}^{[k]}(V)$ for $-|V| \leq k \leq |V|$, and thus

$$\mathbf{MOP} \;=\; \bigvee\nolimits_{k\in\mathbb{Z}} \mathbf{MOP}^{[k]} \tag{6.1.8}$$

as orthogonal spectra.

We define $\mathbf{MO} = \mathbf{MOP}^{[0]}$ as the homogeneous degree zero wedge summand of $\mathbf{MOP}$; this is then an ultra-commutative ring spectrum in its own right. Explicitly, $\mathbf{MO}(V)$ is the Thom space of the tautological $|V|$-plane bundle over $Gr_{|V|}(V^2)$.

Remark 6.1.9. Certain variations of the construction of $\mathbf{MO}$ and $\mathbf{MOP}$ are possible, and have been used at other places in the literature. Indeed, if U is any euclidean vector space, finite or infinite-dimensional, and $u \in S(U)$ a unit vector, we obtain an ultra-commutative ring spectrum $\mathbf{MO}_{U,u}$ in exactly the same way as above, with value at V given by the Thom space of the tautological vector bundle over $Gr_{|V|}(V \otimes U)$. The chosen vector u enters in the definition of the unit and structure maps. For $U = \mathbb{R}^2$ and $u = (1, 0)$, the construction specializes to $\mathbf{MO}$ as above.

If the dimension of U is at least 2, then we always get the same global homotopy type. Indeed: any linear isometric embedding $\psi : U \longrightarrow U'$ such that $\psi(u) = u'$ induces a morphism of ultra-commutative ring spectra $\psi_* : \mathbf{MO}_{U,u} \longrightarrow \mathbf{MO}_{U',u'}$. If the dimension of U is at least 2, this morphism is a global equivalence.

Since $\mathbf{MOP}$ is an ultra-commutative ring spectrum, the equivariant homotopy groups $\underline{\pi}_0(\mathbf{MOP})$ form a global power functor. The global power functor $\underline{\pi}_0(\mathbf{MOP})$ is an interesting algebraic structure, but a complete algebraic description does not seem to be known. Since $2 = 0$ in $\pi_0^e(\mathbf{MOP})$, the global power functor $\underline{\pi}_0(\mathbf{MOP})$ takes values in $\mathbb{F}_2$-vector spaces. In $\underline{\pi}_0(\mathbf{MOP})$, the stronger relation $\mathrm{tr}_e^{C_2}(1) = 0$ holds, compare Theorem 6.1.44 below.

The orthogonal spectrum underlying $\mathbf{MOP}$ comes with a $\mathbb{Z}$-grading, i.e., a wedge decomposition (6.1.8) into summands $\mathbf{MOP}^{[k]}$. The geometric splitting induces a direct sum decomposition of $\pi_0^G(\mathbf{MOP})$ for every compact Lie group G and makes it a commutative $\mathbb{Z}$-graded ring. The mth power operation takes the summand $\mathbf{MOP}^{[k]}$ to the summand $\mathbf{MOP}^{[mk]}$.

We move on to explain the periodicity of the ultra-commutative ring spectrum $\mathbf{MOP}$. We let $t \in \pi_{-1}(\mathbf{MOP}^{[-1]})$ be the class represented by the point

$$(0, \{0\}) \;\in\; Th(Gr_0(\mathbb{R}^2)) \;=\; \mathbf{MOP}^{[-1]}(\mathbb{R})\,. \tag{6.1.10}$$

We let $\sigma \in \pi_1(\mathbf{MOP}^{[1]})$ be the class represented by the map

$$S^2 \;\longrightarrow\; Th(Gr_2(\mathbb{R}^2)) \;=\; \mathbf{MOP}^{[1]}(\mathbb{R})\,, \quad x \;\longmapsto\; (x, \mathbb{R}^2)\,. \tag{6.1.11}$$

As the next proposition shows, **MOP** is periodic in the sense that t is a unit in the graded ring $\pi_*^e(\mathbf{MOP})$, with inverse σ.

The orthogonal spectrum **MOP** has an even stronger kind of '$RO(G)$-graded' periodicity. We define the *inverse Thom class*

$$\tau_{G,V} \in \mathbf{MOP}_0^G(S^V) \tag{6.1.12}$$

as the class represented by the G-map

$$\begin{aligned} t_{G,V} \ : \ S^V \ &\longrightarrow \ Th(Gr(V^2)) \wedge S^V \ = \ \mathbf{MOP}(V) \wedge S^V \\ v \ &\longmapsto \ ((0,0), 0 \oplus 0) \wedge (-v) \ . \end{aligned}$$

Here we abuse notation by denoting the inverse Thom class in **MOP**-theory by the same symbol as the inverse Thom class in **MGr**-theory defined in (6.1.2). The justification for this abuse is that the homomorphism $c : \mathbf{MGr} \longrightarrow \mathbf{MOP}$ introduced in (6.1.22) below takes one inverse Thom class to the other. If V has dimension m, then the class $\tau_{G,V}$ has internal degree $-m$, i.e., it lies in the homogeneous summand $\mathbf{MOP}^{[-m]}$. The justification for the name 'inverse Thom class' is that it is inverse to the Thom class $\sigma_{G,V}$ in $\mathbf{MOP}_G^0(S^V)$, defined in (6.1.16) below.

The periodicity class t of (6.1.10) is essentially the inverse Thom class of the 1-dimensional representation of the trivial group. More precisely, $t \wedge S^1 = \tau_{e,\mathbb{R}}$, i.e., the suspension isomorphism

$$- \wedge S^1 \ : \ \pi_{-1}^e(\mathbf{MOP}) \xrightarrow{\cong} \pi_0^e(\mathbf{MOP} \wedge S^1) \ = \ \mathbf{MOP}_0^e(S^1)$$

takes the periodicity class t to the inverse Thom class $\tau_{e,\mathbb{R}}$. Indeed, the suspension of the defining representative (6.1.10) for t differs from the defining representative for $\tau_{e,\mathbb{R}}$ by the inversion map $-\mathrm{Id} : S^1 \longrightarrow S^1$. So $t \wedge S^1 = -\tau_{e,\mathbb{R}}$; however, since $2 = 0$ in $\pi_0^e(\mathbf{MOP})$, this yields the claim.

The next proposition shows that multiplication by any inverse Thom class is invertible in **MOP**-theory. Moreover, multiplication by $\tau_{G,V}$ is realized, in a certain precise way, by a periodicity morphism of orthogonal G-spectra $j^V_{\mathbf{MOP}}$: $\mathbf{MOP} \longrightarrow \mathrm{sh}^V \mathbf{MOP}$: the value at an inner product space U is the map

$$\begin{aligned} j^V_{\mathbf{MOP}}(U) \ : \ \mathbf{MOP}(U) \ &\longrightarrow \ \mathbf{MOP}(U \oplus V) \ = \ (\mathrm{sh}^V \mathbf{MOP})(U) \\ (x, L) \ &\longmapsto \ (i(x), i(L)) \end{aligned} \tag{6.1.13}$$

induced by the linear isometric embedding $i : U \oplus U \longrightarrow U \oplus V \oplus U \oplus V$ with $i(u, u') = (u, 0, u', 0)$. The morphism $j^V_{\mathbf{MOP}}$ is even a homomorphism of left **MOP**-module spectra. If V has dimension m, then $j^V_{\mathbf{MOP}}$ is homogeneous of degree $-m$ in terms of the $\mathbb{Z}$-grading of **MOP**, i.e., it restricts to a morphism of orthogonal G-spectra

$$j^V_{\mathbf{MOP}} \ : \ \mathbf{MOP}^{[k+m]} \ \longrightarrow \ \mathrm{sh}^V \mathbf{MOP}^{[k]}$$

where k is any integer. In the special case $V = \mathbb{R}$ with trivial G-action, the map

$$j = j^{\mathbb{R}}_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \operatorname{sh}\mathbf{MOP}$$

is a morphism of orthogonal spectra (with trivial G-action). As usual we denote by $p_G : G \longrightarrow e$ the unique group homomorphism to the trivial group, and p_G^* is the associated inflation homomorphism.

Theorem 6.1.14. (i) *The relation $t \cdot \sigma = 1$ holds in $\pi_0^e(\mathbf{MOP})$. For every compact Lie group G, every based G-space A and all $k \in \mathbb{Z}$, the maps*

$$\mathbf{MOP}^G_{k+1}(A) \xrightarrow{\cdot p_G^*(t)} \mathbf{MOP}^G_k(A) \quad \text{and} \quad \mathbf{MOP}^G_k(A) \xrightarrow{\cdot p_G^*(\sigma)} \mathbf{MOP}^G_{k+1}(A)$$

are mutually inverse isomorphisms.

(ii) *For every representation V of a compact Lie group G, the morphism*

$$j^V_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \operatorname{sh}^V \mathbf{MOP}$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. In particular, the morphism $j = j^{\mathbb{R}}_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \operatorname{sh}\mathbf{MOP}$ is a global equivalence.

(iii) *For every based G-space A, the composite*

$$\pi_0^G(\mathbf{MOP} \wedge A) \xrightarrow{-\cdot \tau_{G,V}} \pi_0^G(\mathbf{MOP} \wedge A \wedge S^V) \xrightarrow{(\lambda^V_{\mathbf{MOP}\wedge A})_*} \pi_0^G(\operatorname{sh}^V \mathbf{MOP} \wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{MOP}} \wedge A : \mathbf{MOP} \wedge A \longrightarrow \operatorname{sh}^V \mathbf{MOP} \wedge A$. In particular, exterior multiplication by the inverse Thom class $\tau_{G,V}$ is invertible in equivariant $\mathbf{MOP}$-homology.

Proof (i) The class $t \cdot \sigma$ is represented by the composite

$$S^2 \xrightarrow{x \mapsto (0,\{0\}) \wedge (x, \mathbb{R}^2)} \mathbf{MOP}^{[-1]}(\mathbb{R}) \wedge \mathbf{MOP}^{[1]}(\mathbb{R}) \xrightarrow{\mu_{\mathbb{R},\mathbb{R}}} \mathbf{MOP}^{[0]}(\mathbb{R} \oplus \mathbb{R})$$

where the first map is the smash product of the defining representatives for t and σ. Expanding the definition of $\mu_{\mathbb{R},\mathbb{R}}$ identifies this composite as the map

$$S^2 \longrightarrow \mathbf{MOP}^{[0]}(\mathbb{R} \oplus \mathbb{R}) , \quad x \longmapsto ((\mathbb{R} \oplus \tau_{\mathbb{R},\mathbb{R}} \oplus \mathbb{R})(0,0,x), 0 \oplus \mathbb{R} \oplus 0 \oplus \mathbb{R}) .$$

This differs from the representative of the unit $1 \in \pi_0^G(\mathbf{MOP})$ by the action of the linear isometry

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^4 , \quad (a, b, c, d) \longmapsto (b, d, c, a) .$$

This isometry has determinant 1, so we conclude that $t \cdot \sigma = 1$ in $\pi_0^e(\mathbf{MOP})$.

(ii) We show first that $j^V_{\mathbf{MOP}}$ induces an isomorphism on 0th G-equivariant homotopy groups. To this end we define a map

$$\Phi : \pi_0^G(\operatorname{sh}^V \mathbf{MOP}) \longrightarrow \pi_0^G(\mathbf{MOP})$$

in the opposite direction as follows. We define a G-map

$$s_{G,V} \ : \ S^{V\oplus V} \ \longrightarrow \ Th(Gr(V\oplus V)) \ = \ \mathbf{MOP}(V) \ \text{ by } \ s_{G,V}(v,w) \ = \ ((v,w),V\oplus V)\ .$$

If $f : S^U \longrightarrow \mathbf{MOP}(U\oplus V) = (\mathrm{sh}^V \mathbf{MOP})(U)$ represents a class in $\pi_0^G(\mathrm{sh}^V \mathbf{MOP})$, then we define $\Phi[f]$ as the class of the composite

$$S^{U\oplus V\oplus V} \xrightarrow{f\wedge s_{G,V}} \mathbf{MOP}(U\oplus V)\wedge \mathbf{MOP}(V) \xrightarrow{\mu_{U\oplus V,V}} \mathbf{MOP}(U\oplus V\oplus V)\ .$$

This recipe is compatible with stabilization, so Φ is indeed well-defined.

The composite

$$S^{V\oplus V} \xrightarrow{s_{G,V}} \mathbf{MOP}(V) \xrightarrow{j^V_{\mathbf{MOP}}(V)} \mathbf{MOP}(V\oplus V)$$

is G-equivariantly homotopic to the unit map $\eta(V\oplus V) : S^{V\oplus V} \longrightarrow \mathbf{MOP}(V\oplus V)$. So

$$\begin{aligned}
(j^V_{\mathbf{MOP}})_*(\Phi[f]) \ &= \ [j^V_{\mathbf{MOP}}(U\oplus V\oplus V)\circ \mu_{U\oplus V,V}\circ (f\wedge s_{G,V})] \\
&= \ [\mu_{U\oplus V,V\oplus V}\circ (\mathbf{MOP}(U\oplus V)\wedge j^V_{\mathbf{MOP}}(V))\circ (f\wedge s_{G,V})] \\
&= \ [\mu_{U\oplus V,V\oplus V}\circ (f\wedge (j^V_{\mathbf{MOP}}(V)\circ s_{G,V}))] \\
&= \ [\mu_{U\oplus V,V\oplus V}\circ (f\wedge \eta(V\oplus V))] \ = \ [f]\ .
\end{aligned}$$

The second equation is the fact that $j^V_{\mathbf{MOP}}$ is a homomorphism of left **MOP**-modules. This proves that $(j^V_{\mathbf{MOP}})_*\circ\Phi$ is the identity.

On the other hand, the composite

$$\mathbf{MOP}(U)\wedge S^{V\oplus V} \xrightarrow{j^V_{\mathbf{MOP}}(U)\wedge s_{G,V}} \mathbf{MOP}(U\oplus V)\wedge \mathbf{MOP}(V) \xrightarrow{\mu_{U\oplus V,V}} \mathbf{MOP}(U\oplus V\oplus V)$$

agrees with the opposite structure map $\sigma^{\mathrm{op}}_{U,V\oplus V}$ of the spectrum **MOP**. So if $\varphi : S^U \longrightarrow \mathbf{MOP}(U)$ represents a class in $\pi_0^G(\mathbf{MOP})$, then

$$\begin{aligned}
\Phi((j^V_{\mathbf{MOP}})_*[\varphi]) \ &= \ [\mu_{U\oplus V,V}\circ (j^V_{\mathbf{MOP}}(U)\circ\varphi)\wedge s_{G,V}] \\
&= \ [\mu_{U\oplus V,V}\circ (j^V_{\mathbf{MOP}}(U)\wedge s_{G,V})\circ (\varphi\wedge S^{V\oplus V})] \\
&= \ [\sigma^{\mathrm{op}}_{U,V\oplus V}\circ (\varphi\wedge S^{V\oplus V})] \ = \ [\varphi]\ .
\end{aligned}$$

This proves that $\Phi\circ (j^V_{\mathbf{MOP}})_*$ is the identity, and thus completes the proof that $\pi_0^G(j^V_{\mathbf{MOP}})$ is an isomorphism. Since $j^V_{\mathbf{MOP}}$ is a homomorphism of left **MOP**-modules, its effect on homotopy groups is $\pi_*^G(\mathbf{MOP})$-linear. In particular, it commutes with the action of the element $p_G^*(t)$, where $t\in\pi^e_{-1}(\mathbf{MOP})$ is the periodicity element defined in (6.1.10). Since $p_G^*(t)$ is invertible by part (i) and $\pi_0^G(j^V_{\mathbf{MOP}})$ is an isomorphism, the map $\pi_k^G(j^V_{\mathbf{MOP}})$ is then an isomorphism for every integer k. Applying the above to a closed subgroup H of G and the underlying H-representation of V shows that $j^V_{\mathbf{MOP}}$ induces isomorphisms of equivariant stable homotopy groups for all closed subgroups of G. So $j^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism.

(iii) The proof of the first claim proceeds in the same way as its analog for **MGr** in Proposition 6.1.6. Since the morphisms $j^V_{\mathbf{MOP}}$ and $\lambda^V_{\mathbf{MOP}\wedge A}$ are both $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, they induce isomorphisms on π_0^G. So exterior multiplication by the inverse Thom class $\tau_{G,V}$ is invertible. □

The orthogonal spectra **MGr** and **MOP** admit 'shift morphisms' $j^V_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}^V \mathbf{MGr}$ and $j^V_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \mathrm{sh}^V \mathbf{MOP}$ defined in (6.1.5) and (6.1.13). However, the morphism $j^V_{\mathbf{MGr}}$ is *not* a $\underline{\pi}_*$-isomorphism, whereas the morphism $j^V_{\mathbf{MOP}}$ *is* a $\underline{\pi}_*$-isomorphism, by Theorem 6.1.14 (ii). This is a reflection of the fact that the inverse Thom classes $\tau_{G,V}$ are *not* invertible in equivariant **MGr**-homology, whereas their **MOP**-counterparts are.

Construction 6.1.15 (Thom classes for representations)**.** The Thom spectrum **MOP** comes with distinguished Thom classes for representations. We let V be a representation of a compact Lie group G. We consider the G-map

$$s_{G,V} \ : \ S^{V\oplus V} \ \longrightarrow \ Th(Gr(V\oplus V)) \ = \ \mathbf{MOP}(V)\ , \quad (v,w) \ \longmapsto \ ((v,w), V\oplus V)\ .$$

If V has dimension m, then $s_{G,V}$ is a homeomorphism onto the homogeneous summand $\mathbf{MOP}^{[m]}(V)$. The adjoint $S^V \longrightarrow \mathrm{map}_*(S^V, \mathbf{MOP}(V))$ of $s_{G,V}$ represents the *Thom class*

$$\sigma_{G,V} \ \in \ \mathbf{MOP}^0_G(S^V) \ = \ \pi_0^G(\mathrm{map}_*(S^V, \mathbf{MOP})) \tag{6.1.16}$$

in the G-equivariant **MOP**-cohomology of S^V.

The following theorem is a special case of a Thom isomorphism in the equivariant cohomology theory represented by **MOP**. It also makes precise in which way the inverse Thom class $\tau_{G,V}$ is inverse to the Thom class $\sigma_{G,V}$. This relation between $\tau_{G,V}$ and $\sigma_{G,V}$ is the ultimate justification for naming $\tau_{G,V}$ the 'inverse Thom class'.

Theorem 6.1.17. *Let V be a representation of a compact Lie group G. Then the composite*

$$\mathbf{MOP}^0_G(S^V) \ = \ \pi_0^G(\Omega^V\mathbf{MOP}) \ \xrightarrow{\cdot\tau_{G,V}} \ \pi_0^G((\Omega^V\mathbf{MOP})\wedge S^V) \ \xrightarrow{(\epsilon^V_{\mathbf{MOP}})_*} \ \pi_0^G(\mathbf{MOP})$$

is inverse to multiplication by $\sigma_{G,V}$, where $\epsilon^V_{\mathbf{MOP}} : (\Omega^V\mathbf{MOP})\wedge S^V \longrightarrow \mathbf{MOP}$ is the evaluation morphism. In particular, $\mathbf{MOP}^0_G(S^V)$ is a free module of rank 1 over the ring $\pi_0^G(\mathbf{MOP})$, and the Thom class $\sigma_{G,V}$ is a generator.

Proof We consider a G-map $f : S^U \longrightarrow \Omega^V\mathbf{MOP}(U)$ that represents a class in $\mathbf{MOP}^0_G(S^V)$. Then $\epsilon^V_{\mathbf{MOP}}([f]\cdot\tau_{G,V})$ is represented by the following compos-

ite:

$$S^{U\oplus V} \xrightarrow{f\wedge t_{G,V}} (\Omega^V \mathbf{MOP}(U)) \wedge \mathbf{MOP}(V) \wedge S^V$$
$$\xrightarrow{\text{evaluate}} \mathbf{MOP}(U) \wedge \mathbf{MOP}(V) \xrightarrow{\mu^{\mathbf{MOP}}_{U,V}} \mathbf{MOP}(U\oplus V)\ ,$$

where $t_{G,V} : S^V \longrightarrow \mathbf{MOP}(V) \wedge S^V$ is the defining representative for the class $\tau_{G,V}$ from (6.1.12). If we let $f = s^\sharp_{G,V} : S^V \longrightarrow \Omega^V \mathbf{MOP}(V)$ be adjoint to the defining representative for $\sigma_{G,V}$, then the composite comes out as the map

$$S^{V\oplus V} \longrightarrow \mathbf{MOP}(V\oplus V)\ , \quad (v,w) \longmapsto ((v,0,-w,0), V\oplus 0\oplus V\oplus 0)\ .$$

This composite is equivariantly homotopic to the map

$$(v,w) \longmapsto ((v,w,0,0),\ V\oplus V\oplus 0\oplus 0)$$

which represents the multiplicative unit. So we have shown that

$$\epsilon^V_{\mathbf{MOP}}(\sigma_{G,V}\cdot\tau_{G,V}) = 1\ .$$

All maps in sight are left $\pi_0^G(\mathbf{MOP})$-linear, so we deduce that

$$\epsilon^V_{\mathbf{MOP}}(x\cdot\sigma_{G,V}\cdot\tau_{G,V}) = x\cdot\epsilon^V_{\mathbf{MOP}}(\sigma_{G,V}\cdot\tau_{G,V}) = x$$

for every class $x\in\pi_0^G(\mathbf{MOP})$. On the other hand, the composite

$$\pi_0^G(\Omega^V\mathbf{MOP}) \xrightarrow{\cdot\tau_{G,V}} \pi_0^G((\Omega^V\mathbf{MOP})\wedge S^V) \xrightarrow{(\lambda^V_{\Omega^V\mathbf{MOP}})_*} \pi_0^G(\mathrm{sh}^V\,\Omega^V\mathbf{MOP})$$

is the effect of the morphism $\Omega^V j^V_{\mathbf{MOP}} : \Omega^V\mathbf{MOP} \longrightarrow \mathrm{sh}^V\,\Omega^V\mathbf{MOP}$, by the same reasoning as in Theorem 6.1.14 (iii). Since $j^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism, so is $\Omega^V j^V_{\mathbf{MOP}}$ by Proposition 3.1.40 (ii). On the other hand, $\lambda^V_{\Omega^V\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism by Proposition 3.1.25 (ii). So multiplication by the class $\tau_{G,V}$ is an isomorphism. The morphism $\epsilon^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism, also by Proposition 3.1.25 (ii). So the composite $(\epsilon^V_{\mathbf{MOP}})_*\circ(-\cdot\tau_{G,V})$ is bijective. Since it is also left inverse to multiplication by $\sigma_{G,V}$, this proves the first claim. □

Construction 6.1.18 (Thom classes for equivariant vector bundles)**.** The Thom spectrum **MOP** comes with a distinguished orientation, given by Thom classes for equivariant vector bundles. These Thom classes generalize the classes $\sigma_{G,V}$ defined in (6.1.16), when we view a G-representation as a G-vector bundle over a one-point G-space.

We recall the definition of the Thom classes. Given a compact Lie group G, an orthogonal G-spectrum X and a compact based G-space B, we define the *G-equivariant X-cohomology group* of B as

$$X^0_G(B) = \pi_0^G(\mathrm{map}_*(B,X))\ .$$

We let $\xi : E \longrightarrow B$ be a G-equivariant vector bundle. The bundle has a classifying G-map $\psi : B \longrightarrow Gr(V)$ for some G-representation V, i.e., such that ξ is isomorphic to the pullback of the tautological G-vector bundle over the Grassmannian. We let $\bar{\psi} : E \longrightarrow V$ be a map that covers ψ, i.e., $\bar{\psi}$ is fiberwise linear and satisfies $\bar{\psi}(e) \in \psi(\xi(e))$. We define a based G-map

$$S^V \wedge Th(\xi) \longrightarrow \mathbf{MOP}(V) = Th(Gr(V^2)) \text{ by } v \wedge e \longmapsto ((v, \bar{\psi}(e)), V \oplus \psi(\xi(e))) .$$

We denote the equivariant cohomology class represented by the adjoint $S^V \longrightarrow \mathrm{map}_*(Th(\xi), \mathbf{MOP}(V))$ of this map by

$$\sigma_G(\xi) \in \mathbf{MOP}^0_G(Th(\xi))$$

and refer to it as the *Thom class* of the G-vector bundle ξ. If the bundle ξ has constant rank m, then the image of the map lies in the wedge summand $\mathbf{MOP}^{[m]}$. It is straightforward to see that the Thom classes just defined are natural for pullback of bundles, compatible with restriction along continuous homomorphisms, and the Thom class of an exterior product of bundles is the exterior product of the Thom classes.

The Thom diagonal of the G-vector bundle $\xi : E \longrightarrow B$ is the map

$$\Delta : Th(\xi) \longrightarrow B_+ \wedge Th(\xi) , \quad e \longmapsto \xi(e) \wedge e .$$

This diagonal induces an action map of equivariant cohomology groups

$$\mathbf{MOP}^0_G(B_+) \times \mathbf{MOP}^0_G(Th(\xi)) \xrightarrow{\times} \mathbf{MOP}^0_G(B_+ \wedge Th(\xi)) \xrightarrow{\Delta^*} \mathbf{MOP}^0_G(Th(\xi)) .$$

Since the diagonal is coassociative and counital, the action map makes the group $\mathbf{MOP}^0_G(Th(\xi))$ a left module over the commutative ring $\mathbf{MOP}^0_G(B_+)$. The Thom isomorphism then says that whenever B admits the structure of a finite G-CW-complex, then $\mathbf{MOP}^0_G(Th(\xi))$ is a free module of rank 1 over $\mathbf{MOP}^0_G(B_+)$, with the Thom class $\sigma_G(\xi)$ as a generator. Theorem 6.1.17 is the special case when the base consists of a single point. We refrain from giving the proof of the general Thom isomorphism in equivariant $\mathbf{MOP}$-theory.

Construction 6.1.19 (Euler classes)**.** We let G be a compact Lie group and V an m-dimensional G-representation. As usual, the Thom classes $\sigma_{G,V}$ give rise to Euler classes by 'restriction to the zero section', i.e.,

$$e(V) = i^*(\sigma_{G,V}) \in \mathbf{MOP}^0_G(S^0) = \pi^G_0(\mathbf{MOP}) .$$

Here $i : S^0 \longrightarrow S^V$ is the inclusion of the fixed-points 0 and ∞, and $i^* : \mathbf{MOP}^0_G(S^V) \longrightarrow \mathbf{MOP}^0_G(S^0)$ the induced map on equivariant cohomology groups. The Euler class is thus represented by the based G-map

$$S^V \longrightarrow \mathbf{MOP}(V) , \quad v \longmapsto ((v, 0), V \oplus V) .$$

Since the Thom class lives in the homogeneous summand $\mathbf{MOP}^{[m]}$, so does the Euler class. If V has nonzero G-fixed-points, then the inclusion $i : S^0 \longrightarrow S^V$ is G-equivariantly null-homotopic, so $e(V) = 0$ whenever $V^G \neq 0$.

Remark 6.1.20 (Shifted Thom and Euler classes in **MO**)**.** The author thinks that the periodic theory **MOP** is the most natural home for the Thom classes, the Euler classes and the inverse Thom classes, but the more traditional place to host them is the degree 0 wedge summand $\mathbf{MO} = \mathbf{MOP}^{[0]}$. Indeed, for an m-dimensional G-representation V, the Thom class $\sigma_{G,V}$ and the Euler class $e(V)$ lie in the homogeneous summand $\mathbf{MOP}^{[m]}$, and we can use the periodicity of **MOP** to move the classes into **MO**, at the expense of shifting their degrees by m. In other words, by multiplying by a suitable power of the periodicity class $t \in \pi^e_{-1}(\mathbf{MOP}^{[-1]})$, we define

$$\bar{\sigma}_{G,V} \ = \ p_G^*(t^m)\cdot\sigma_{G,V} \in \ \mathbf{MO}_G^m(S^V) \quad \text{and} \quad \bar{e}(V) \ = \ p_G^*(t^m)\cdot e(V) \in \ \pi^G_{-m}(\mathbf{MO})\ ,$$

where $p_G : G \longrightarrow e$ is the unique group homomorphism. The Thom isomorphism theorem for **MO** then says that $\mathbf{MO}_G^*(S^V)$ is a free graded module of rank 1 over the graded ring $\pi_*^G(\mathbf{MO})$, and the shifted Thom class $\bar{\sigma}_{G,V}$ is a generator. This version of the Thom isomorphism follows directly from Theorem 6.1.17 because **MOP** is globally equivalent to the wedge of all suspensions and desuspensions of **MO**. More precisely, the maps

$$\bigoplus_{n\in\mathbb{Z}} \pi_n^G(\mathbf{MO}) \ \longrightarrow \ \pi_0^G(\mathbf{MOP}) \quad \text{and} \quad \bigoplus_{n\in\mathbb{Z}} \mathbf{MO}_G^{-n}(S^V) \ \longrightarrow \ \mathbf{MOP}_G^0(S^V)\ ,$$

given on the nth summand by multiplication by $p_G^*(t^n)$, are isomorphisms; moreover the latter isomorphism takes the shifted Thom class $\bar{\sigma}_{G,V}$ to the original Thom class $\sigma_{G,V}$.

Similarly, we can use the periodicity of **MOP** to move the inverse Thom class $\tau_{G,V} \in \mathbf{MOP}_0^G(S^V)$ into **MO**, at the expense of shifting it from degree 0 to homological degree m. The class $\tau_{G,V}$ lies in the homogeneous summand $\mathbf{MOP}^{[-m]}$, so by multiplying by a suitable power of the periodicity class $\sigma \in \pi_1^e(\mathbf{MOP}^{[1]})$ we define

$$\bar{\tau}_{G,V} \ = \ p_G^*(\sigma^m)\cdot\tau_{G,V} \in \ \mathbf{MO}_m^G(S^V)\ . \tag{6.1.21}$$

Theorem 6.1.14 (iii) then implies that for every based G-space A, the map

$$-\cdot\bar{\tau}_{G,V} \ : \ \pi_*^G(\mathbf{MO}\wedge A) \ \longrightarrow \ \pi^G_{*+m}(\mathbf{MO}\wedge A\wedge S^V)$$

is an isomorphism.

Our next task is to show that the Thom spectrum **MOP** is a localization of the Thom spectrum **MGr**, obtained by formally inverting all inverse Thom classes. This result can be viewed as a 'thomification' of the fact that the morphism

of ultra-commutative monoids $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ induces a group completion of abelian monoids $[A, i]^G : [A, \mathbf{Gr}]^G \longrightarrow [A, \mathbf{BOP}]^G$ for every compact Lie group G and every G-space A, compare Proposition 2.4.5.

The ultra-commutative ring spectra $\mathbf{MGr}$ and $\mathbf{MOP}$ are connected by a homomorphism

$$a \ : \ \mathbf{MGr} \longrightarrow \mathbf{MOP} \tag{6.1.22}$$

whose value at an inner product space V is

$$a(V) \ : \ \mathbf{MGr}(V) \longrightarrow \mathbf{MOP}(V) , \quad (x, L) \longmapsto ((x, 0), L \oplus 0) .$$

For varying V, these maps form a morphism of $\mathbb{Z}$-graded ultra-commutative ring spectra. The morphism (6.1.22) induces natural transformations of equivariant homology theories

$$(a \wedge A)_* \ : \ \mathbf{MGr}^G_*(A) \longrightarrow \mathbf{MOP}^G_*(A)$$

for all compact Lie groups G and all based G-spaces A. We observe that

$$(a \wedge S^V)_*(\tau_{G,V}) \ = \ \tau_{G,V} ,$$

i.e., the morphism a takes the $\mathbf{MGr}$-inverse Thom class to the $\mathbf{MOP}$-inverse Thom class with the same name. This relation is immediate from the explicit representatives of the inverse Thom classes in (6.1.2) and (6.1.12).

We define a localized version of equivariant $\mathbf{MGr}$-homology by

$$\mathbf{MGr}^G_k(A)[1/\tau] \ = \ \operatorname{colim}_{V \in s(\mathcal{U}_G)} \mathbf{MGr}^G_k(A \wedge S^V) ;$$

for $V \subset W$, the structure map in the colimit system is the multiplication

$$\mathbf{MGr}^G_k(A \wedge S^V) \xrightarrow{- \cdot \tau_{G,W-V}} \mathbf{MGr}^G_k(A \wedge S^V \wedge S^{W-V}) \cong \mathbf{MGr}^G_k(A \wedge S^W) .$$

In equivariant $\mathbf{MOP}$-theory, the inverse Thom classes become invertible by Theorem 6.1.14 (iii). So for a G-representation V we can consider the map

$$\mathbf{MGr}^G_k(A \wedge S^V) \xrightarrow{(a \wedge A \wedge S^V)_*} \mathbf{MOP}^G_k(A \wedge S^V) \xrightarrow{(- \cdot \tau_{G,V})^{-1}} \mathbf{MOP}^G_k(A) .$$

By the multiplicativity of a, these maps are compatible as V varies over the poset $s(\mathcal{U}_G)$, so they assemble into a natural transformation

$$a^\sharp \ : \ \mathbf{MGr}^G_k(A)[1/\tau] \ = \ \operatorname{colim}_{V \in s(\mathcal{U}_G)} \mathbf{MGr}^G_k(A \wedge S^V) \longrightarrow \mathbf{MOP}^G_k(A) .$$

Theorem 6.1.23. *For every compact Lie group G, every based G-space A and every integer k the map*

$$a^\sharp \ : \ \mathbf{MGr}^G_k(A)[1/\tau] \longrightarrow \mathbf{MOP}^G_k(A)$$

is an isomorphism.

Proof To simplify the exposition we prove the claim for $k = 0$ only, the argument for a general integer being essentially the same. Alternatively, we can observe that source and target of $a^\sharp$ are periodic, so it suffices to establish bijectivity in a single dimension. We show two separate statements that amount to the injectivity and surjectivity, respectively, of the map $a^\sharp$.

(a) We show that for every class x in the kernel of the map $(a \wedge A)_* : \mathbf{MGr}_0^G(A) \longrightarrow \mathbf{MOP}_0^G(A)$, there is a G-representation V such that $x \cdot \tau_{G,V} = 0$. Indeed, we can represent any such class x by a based G-map $f : S^V \longrightarrow \mathbf{MGr}(V) \wedge A$, for some G-representation V, such that the composite

$$S^V \xrightarrow{f} \mathbf{MGr}(V) \wedge A \xrightarrow{a(V)\wedge A} \mathbf{MOP}(V) \wedge A$$

is equivariantly null-homotopic. In (6.1.28) we defined a morphism of orthogonal G-spectra $j^V_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}^V \mathbf{MGr}$. We observe that $\mathbf{MOP}(V) = \mathbf{MGr}(V \oplus V)$ and $a(V) = j^V_{\mathbf{MGr}}(V)$. So we can apply Proposition 6.1.6 and conclude that

$$\begin{aligned}(\lambda^V_{\mathbf{MGr}\wedge A})_*(x \cdot \tau_{G,V}) &= (j^V_{\mathbf{MGr}} \wedge A)_*(x) \\ &= [(j^V_{\mathbf{MGr}}(V) \wedge A) \circ f] = [(a(V) \wedge A) \circ f] = 0 .\end{aligned}$$

Since $\lambda^V_{\mathbf{MGr}\wedge A} : \mathbf{MGr} \wedge A \wedge S^V \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$ is a $\underline{\pi}_*$-isomorphism (by Proposition 3.1.25 (ii)), this implies the desired relation $x \cdot \tau_{G,V} = 0$.

(b) We show that for every class y in $\mathbf{MOP}_0^G(A)$, there is a G-representation V and a class x in $\mathbf{MGr}_0^G(A \wedge S^V)$ such that $y \cdot \tau_{G,V} = (a \wedge A \wedge S^V)_*(x)$. To this end we represent y by a based G-map $f : S^V \longrightarrow \mathbf{MOP}(V) \wedge A$. Because $\mathbf{MOP}(V) = \mathbf{MGr}(V^2) = (\mathrm{sh}^V \mathbf{MGr})(V)$, the map f also defines a class in $\pi_0^G(\mathrm{sh}^V \mathbf{MGr} \wedge A)$. Since $\lambda^V_{\mathbf{MGr}\wedge A} : \mathbf{MGr} \wedge A \wedge S^V \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$ is a $\underline{\pi}_*$-isomorphism by Proposition 3.1.25 (ii), there is a unique class $x \in \mathbf{MGr}_0^G(A \wedge S^V)$ such that

$$(\lambda^V_{\mathbf{MGr}\wedge A})_*(x) = [f] .$$

On the other hand, the map $a(V^2) : \mathbf{MGr}(V^2) \longrightarrow \mathbf{MOP}(V^2)$ is *equal* to the map $j^V_{\mathbf{MOP}}(V) : \mathbf{MOP}(V) \longrightarrow \mathbf{MOP}(V^2)$. So

$$\begin{aligned}(\lambda^V_{\mathbf{MOP}\wedge A})_*((a \wedge A \wedge S^V)_*(x)) &= (\mathrm{sh}^V a \wedge A)_*((\lambda^V_{\mathbf{MGr}\wedge A})_*(x)) \\ &= (\mathrm{sh}^V a \wedge A)_*[f] = [(a(V^2) \wedge A) \circ f] \\ &= [(j^V_{\mathbf{MOP}}(V) \wedge A) \circ f] = (j^V_{\mathbf{MOP}} \wedge A)_*(y) \\ &= (\lambda^V_{\mathbf{MOP}\wedge A})_*(y \cdot \tau_{G,V}) .\end{aligned}$$

The sixth equation is Theorem 6.1.14 (iii), the others are either definitions or naturality properties. Since $\lambda^V_{\mathbf{MOP}\wedge A}$ is a $\underline{\pi}_*$-isomorphism, we can conclude that

$$(a \wedge A \wedge S^V)_*(x) = y \cdot \tau_{G,V} . \qquad \square$$

Example 6.1.24 (The global Thom spectra **mO** and **mOP**). We define two E_∞-orthogonal ring spectra **mO** and **mOP**, the Thom spectra over the orthogonal spaces **bO** and **bOP** defined in Examples 2.4.18 and 2.4.31. The spectrum **mOP** is a periodic version of **mO**, and conversely **mO** is the homogeneous degree 0 summand with respect to a certain $\mathbb{Z}$-grading of **mOP**. Non-equivariantly, **mO** is another version of the unoriented Thom spectrum MO. The equivariant homology theory represented by **mO** is the natural target of the equivariant Thom-Pontryagin map from equivariant bordism, and that map is trying hard to be an isomorphism, see Theorem 6.2.33 below.

We recall that the value of **bOP** at an inner product space V is

$$\mathbf{bOP}(V) \ = \ \coprod\nolimits_{n\geq 0} Gr_n(V \oplus \mathbb{R}^\infty)\ ,$$

the disjoint union of all the Grassmannians in $V \oplus \mathbb{R}^\infty$. The map $\mathbf{bOP}(\varphi)$: $\mathbf{bOP}(V) \longrightarrow \mathbf{bOP}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is defined as

$$\mathbf{bOP}(\varphi)(L) \ = \ (\varphi \oplus \mathbb{R}^\infty)(L) \ + \ ((W - \varphi(V)) \oplus 0)\ .$$

In other words: we apply the linear isometric embedding $\varphi \oplus \mathbb{R}^\infty : V \oplus \mathbb{R}^\infty \longrightarrow W \oplus \mathbb{R}^\infty$ to the subspace L and add the orthogonal complement of the image of φ (sitting in the first summand of $W \oplus \mathbb{R}^\infty$).

Over the space $\mathbf{bOP}(V)$ sits a tautological euclidean vector bundle (again of non-constant rank); the total space of this bundle consist of pairs $(x, U) \in (V \oplus \mathbb{R}^\infty) \times \mathbf{bO}(V)$ such that $x \in U$. We define $\mathbf{mOP}(V)$ as the Thom space of this tautological vector bundle. The structure maps are given by

$$\begin{array}{rcl} \mathbf{O}(V, W) \wedge \mathbf{mOP}(V) & \longrightarrow & \mathbf{mOP}(W) \\ (w, \varphi) \wedge (x, U) & \longmapsto & ((w, 0) + \mathbf{bOP}(\varphi)(x), \mathbf{bOP}(\varphi)(U))\ . \end{array}$$

As we explained in Remark 2.4.25, the orthogonal spaces **bO** and **bOP** have natural E_∞-structures. Correspondingly, the orthogonal spectra **mO** and **mOP** have natural E_∞-structures, by which we mean an action of the linear isometries operad. This multiplication is, however, *not* ultra-commutative. Multiplication maps

$$\mu_{V,W} \ : \ \mathbf{L}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty)_+ \wedge \mathbf{mOP}(V) \wedge \mathbf{mOP}(W) \ \longrightarrow \ \mathbf{mOP}(V \oplus W)$$

are defined by sending $\psi \wedge (x, U) \wedge (x', U')$ to $(\psi_\sharp(x, x'), \psi_\sharp(U \oplus U'))$, where $\psi_\sharp$ is the linear isometric embedding

$$V \oplus \mathbb{R}^\infty \oplus W \oplus \mathbb{R}^\infty \ \longrightarrow \ V \oplus W \oplus \mathbb{R}^\infty\ , \quad \psi_\sharp(v, y, w, z) \ = \ (v, w, \psi(y, z))\ . \tag{6.1.25}$$

Unit maps are defined by

$$S^V \ \longrightarrow \ \mathbf{mOP}(V)\ , \quad v \ \longmapsto \ ((v, 0),\ V \oplus 0)\ .$$

All this structure makes **mOP** an E_∞-orthogonal ring spectrum.

The orthogonal spectrum **mOP** is $\mathbb{Z}$-graded, where the summand $\mathbf{mOP}^{[k]}(V)$ of degree k is defined as the Thom space of the tautological $(|V|+k)$-plane bundle over $\mathbf{bOP}^{[k]}(V) = Gr_{|V|+k}(V \oplus \mathbb{R}^\infty)$; then $\mathbf{mOP}(V)$ is the one-point union of the Thom spaces $\mathbf{mOP}^{[k]}(V)$ for $|V|+k \geq 0$. So we have a wedge decomposition

$$\mathbf{mOP} = \bigvee_{k\in\mathbb{Z}} \mathbf{mOP}^{[k]}$$

as orthogonal spectra. We define $\mathbf{mO} = \mathbf{mOP}^{[0]}$ as the homogeneous wedge summand of degree 0.

In the rest of this section we will also use products on the equivariant homology theories represented by the Thom spectra **mO** and **mOP**. As we just explained, the spectrum **mOP** comes with an E_∞-multiplication which is, however, neither strictly associative, nor strictly commutative. So we briefly explain how to define these products. We choose a linear isometric embedding $\psi : \mathbb{R}^\infty \oplus \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. We define continuous maps

$$\begin{aligned} \psi_{V,W} \ : \ \mathbf{mOP}(V) \wedge \mathbf{mOP}(W) \ &\longrightarrow \ \mathbf{mOP}(V \oplus W) \qquad \text{by} \qquad (6.1.26)\\ \psi_{V,W}((x,U),(x',U')) \ &= \ (\psi_\sharp(x,x'), \psi_\sharp(U \oplus U')) \ , \end{aligned}$$

where $\psi_\sharp$ was defined in (6.1.25). These maps form a bimorphism, which corresponds to a morphism of orthogonal spectra

$$\psi_\star \ : \ \mathbf{mOP} \wedge \mathbf{mOP} \ \longrightarrow \ \mathbf{mOP}$$

by the universal property of the smash product. Since the space $\mathbf{L}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty)$ of linear isometric embeddings is contractible, the morphism $\psi_\star$ is independent up to homotopy of the choice of ψ. Even though the multiplication map $\psi_\star$ is neither associative nor commutative, the contractibility of the space $\mathbf{L}((\mathbb{R}^\infty)^3, \mathbb{R}^\infty)$ implies that the square

$$\begin{tikzcd} \mathbf{mOP} \wedge \mathbf{mOP} \wedge \mathbf{mOP} \arrow[r, "\mathbf{mOP}\wedge\psi_\star"] \arrow[d, "\psi_\star\wedge\mathbf{mOP}"'] & \mathbf{mOP} \wedge \mathbf{mOP} \arrow[d, "\psi_\star"] \\ \mathbf{mOP} \wedge \mathbf{mOP} \arrow[r, "\psi_\star"'] & \mathbf{mOP} \end{tikzcd}$$

commutes up to homotopy, and the contractibility of the space $\mathbf{L}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty)$ implies that the composite

$$\mathbf{mOP} \wedge \mathbf{mOP} \xrightarrow{\tau_{\mathbf{mOP},\mathbf{mOP}}} \mathbf{mOP} \wedge \mathbf{mOP} \xrightarrow{\psi_\star} \mathbf{mOP}$$

is homotopic to $\psi_\star$. So whenever we pass to induced maps on equivariant homotopy groups, an E_∞-multiplication is as good as a strictly associative and

commutative multiplication. However, an E_∞-multiplication does *not* entitle us to power operations.

Given a compact Lie group G and based G-spaces A and B, we define a multiplication

$$\cdot \ : \ \mathbf{mOP}^G_k(A) \times \mathbf{mOP}^G_l(B) \ \longrightarrow \ \mathbf{mOP}^G_{k+l}(A \wedge B)$$

as the composite

$$\begin{aligned} \pi^G_k(\mathbf{mOP} \wedge A) \times \pi^G_l(\mathbf{mOP} \wedge B) \ &\xrightarrow[(3.5.13)]{\times} \ \pi^G_{k+l}(\mathbf{mOP} \wedge A \wedge \mathbf{mOP} \wedge B) \\ &\xrightarrow{(\mathrm{twist})_*} \ \pi^G_{k+l}(\mathbf{mOP} \wedge \mathbf{mOP} \wedge A \wedge B) \\ &\xrightarrow{(\psi_\star \wedge A \wedge B)_*} \ \pi^G_{k+l}(\mathbf{mOP} \wedge A \wedge B) \ . \end{aligned}$$

We move on to explain the periodicity property of $\mathbf{mOP}$. As the theories $\mathbf{MGr}$ and $\mathbf{MOP}$, the theory $\mathbf{mOP}$ also has its own inverse Thom classes and shift morphisms. We define the *inverse Thom class*

$$\tau_{G,V} \ \in \ \mathbf{mOP}^G_0(S^V) \tag{6.1.27}$$

as the class represented by the G-map

$$S^V \ \longrightarrow \ Th(Gr(V \oplus \mathbb{R}^\infty)) \wedge S^V \ = \ \mathbf{mOP}(V) \wedge S^V \ , \quad v \ \longmapsto \ ((0,0), 0 \oplus 0) \wedge (-v) \ .$$

Here we abuse notation one more time and also denote the inverse Thom class in $\mathbf{mOP}$-theory by the same symbol as its counterparts in $\mathbf{MGr}$-theory and $\mathbf{MOP}$-theory defined in (6.1.2) and (6.1.12). The justification for this abuse is that the inverse Thom classes match up under certain homomorphisms relating $\mathbf{MGr}$, $\mathbf{MOP}$ and $\mathbf{mOP}$. As usual, if V has dimension m, then the class $\tau_{G,V}$ lies in the homogeneous summand $\mathbf{mOP}^{[-m]}$. A shift morphism of orthogonal G-spectra $j^V_{\mathbf{mOP}} : \mathbf{mOP} \longrightarrow \mathrm{sh}^V \mathbf{mOP}$ is defined as for $\mathbf{MGr}$ and $\mathbf{MOP}$: the value at an inner product space U is the map

$$\begin{aligned} j^V_{\mathbf{mOP}}(U) \ : \ \mathbf{mOP}(U) \ &\longrightarrow \ \mathbf{mOP}(U \oplus V) \ = \ (\mathrm{sh}^V \mathbf{mOP})(U) \\ (x, L) \quad &\longmapsto \quad (i(x), i(L)) \end{aligned} \tag{6.1.28}$$

induced by the linear isometric embedding $i : U \oplus \mathbb{R}^\infty \longrightarrow U \oplus V \oplus \mathbb{R}^\infty$ with $i(u, x) = (u, 0, x)$. If V has dimension m, then $j^V_{\mathbf{mOP}}$ is homogeneous of degree $-m$. In the special case $V = \mathbb{R}$ with trivial G-action, the map

$$j \ = \ j^{\mathbb{R}}_{\mathbf{mOP}} \ : \ \mathbf{mOP} \ \longrightarrow \ \mathrm{sh}\, \mathbf{mOP}$$

is a morphism of orthogonal spectra (with trivial G-action).

On the level of homotopy groups, the periodicity is realized by multiplication with a periodicity element $t \in \pi^e_{-1}(\mathbf{mOP}^{[-1]})$ represented by the point

$$(0, \{0\}) \ \in \ Th(Gr_0(\mathbb{R} \oplus \mathbb{R}^\infty)) \ = \ \mathbf{mOP}^{[-1]}(\mathbb{R}) \ . \tag{6.1.29}$$

Proposition 6.1.30. (i) *The morphism* $j = j^{\mathbb{R}}_{\mathbf{mOP}} : \mathbf{mOP} \longrightarrow \mathrm{sh}\,\mathbf{mOP}$ *is a homotopy equivalence of orthogonal spectra, hence a global equivalence. For every compact Lie group G and every based G-space A, the induced map*

$$(j \wedge A)_* \ :\ \pi^G_*(\mathbf{mOP} \wedge A) \longrightarrow \pi^G_*(\mathrm{sh}\,\mathbf{mOP} \wedge A)$$

is an isomorphism.

(ii) *For every compact Lie group G, every G-representation V and every based G-space A, the composite*

$$\pi^G_0(\mathbf{mOP} \wedge A) \xrightarrow{-\cdot \tau_{G,V}} \pi^G_0(\mathbf{mOP} \wedge A \wedge S^V) \xrightarrow{(\lambda^V_{\mathbf{mOP}\wedge A})_*} \pi^G_0(\mathrm{sh}^V \mathbf{mOP} \wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{mOP}} \wedge A : \mathbf{mOP} \wedge A \longrightarrow \mathrm{sh}^V \mathbf{mOP} \wedge A$. *In particular, exterior multiplication by the inverse Thom class* $\tau_{G,\mathbb{R}}$ *of the trivial 1-dimensional G-representation is invertible in equivariant* **mOP***-homology.*

(iii) *For every compact Lie group G, every based G-space A and every integer k the multiplication map*

$$\mathbf{mOP}^G_{k+1}(A) \xrightarrow{\cdot p^*_G(t)} \mathbf{mOP}^G_k(A)$$

is an isomorphism. In particular, the class $t \in \pi^e_{-1}(\mathbf{mOP})$ *is a unit in the graded homotopy ring* $\pi^e_*(\mathbf{mOP})$.

Proof (i) The morphism j is based on the linear isometric embedding $i : \mathbb{R}^\infty \longrightarrow \mathbb{R}\oplus\mathbb{R}^\infty$ defined by $i(x) = (0, x)$. This linear isometric embedding is homotopic, through linear isometric embeddings, to the linear isometry $\mathbb{R}^\infty \cong \mathbb{R}\oplus \mathbb{R}^\infty$ sending $(x_1, x_2, x_3, \dots)$ to $(x_1, (x_2, x_3, \dots))$. Such a homotopy induces a homotopy from the morphism j to an isomorphism between **mOP** and sh **mOP**. So j is homotopic to an isomorphism, hence a homotopy equivalence.

The first statement in (ii) is proved by the same argument as the analogous statement for **MGr** in Proposition 6.1.6. For $V = \mathbb{R}$, the trivial 1-dimensional G-representation, the morphisms $j^{\mathbb{R}}_{\mathbf{mOP}} \wedge A$ and $\lambda_{\mathbf{mOP}\wedge A}$ are both global equivalences by part (i) and Proposition 4.1.4 (i). So they induce isomorphisms on π^G_0. Hence exterior multiplication by the class $\tau_{G,\mathbb{R}}$ is invertible.

(iii) The composite

$$\mathbf{mOP}^G_{k+1}(A) \xrightarrow{-\cdot p^*_G(t)} \mathbf{mOP}^G_k(A) \underset{\cong}{\xrightarrow{-\wedge S^1}} \mathbf{mOP}^G_{k+1}(A \wedge S^1)$$

differs from multiplication by the inverse Thom class $\tau_{G,\mathbb{R}}$ by the effect of the involution $\mathbf{mOP} \wedge A \wedge S^{-\mathrm{Id}}$ of $\mathbf{mOP} \wedge A \wedge S^1$. Multiplication by $\tau_{G,\mathbb{R}}$ is an isomorphism by part (ii), and the suspension isomorphism is bijective by Proposition 3.1.30. So multiplication by $p^*_G(t)$ is bijective as well. □

In the earlier Theorem 6.1.23 we showed that equivariant **MOP**-homology is obtained from equivariant **MGr**-homology by inverting all inverse Thom classes. Now we add **mOP** to this picture, which turns out to be an intermediate localization. As we will now explain, **mOP**-theory is obtained from **MGr**-theory by inverting the inverse Thom classes of all *trivial* representation. Then **MOP**-theory is obtained from **mOP**-theory by inverting the remaining inverse Thom classes, i.e., the ones of non-trivial representations. Informally speaking, the first localization turns **MGr**-theory into a theory that is periodic in the $\mathbb{Z}$-graded sense; the second localization then turns **mOP**-theory into a theory that is periodic in the $RO(G)$-graded sense. Schematically:

$$\mathbf{MGr}_*^G(A) \xRightarrow{\text{invert } \tau_{G,\mathbb{R}}} \mathbf{mOP}_*^G(A) \xRightarrow{\text{invert all } \tau_{G,V}} \mathbf{MOP}_*^G(A)$$

The ring spectra **MGr** and **mOP** are connected by a homomorphism

$$b \ : \ \mathbf{MGr} \longrightarrow \mathbf{mOP} \tag{6.1.31}$$

whose value at an inner product space V is

$$b(V) \ : \ \mathbf{MGr}(V) \longrightarrow \mathbf{mOP}(V)\ , \quad (x, L) \longmapsto ((x, 0), L \oplus 0)\ .$$

For varying V, these maps form a morphism of $\mathbb{Z}$-graded orthogonal E_∞-ring spectra. The morphism $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$ induces natural transformations of equivariant homology theories

$$(b \wedge A)_* \ : \ \mathbf{MGr}_*^G(A) \longrightarrow \mathbf{mOP}_*^G(A)$$

for all compact Lie groups G and all based G-spaces A. We observe that

$$(b \wedge S^V)_*(\tau_{G,V}) \ = \ \tau_{G,V}\ ,$$

i.e., the morphism b takes the **MGr**-inverse Thom class to the **mOP**-inverse Thom class with the same name. This relation is again immediate from the explicit representatives of the inverse Thom classes in (6.1.2) and (6.1.27).

We define a localized version of equivariant **MGr**-homology by

$$\mathbf{MGr}_k^G(A)[\tau_{G,\mathbb{R}}^{-1}] \ = \ \operatorname{colim}_{n\geq 0} \mathbf{MGr}_k^G(A \wedge S^n)\ ,$$

the colimit of the sequence

$$\mathbf{MGr}_k^G(A) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \mathbf{MGr}_k^G(A \wedge S^1) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \mathbf{MGr}_k^G(A \wedge S^2) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \dots$$

along multiplication by the inverse Thom class $\tau_{G,\mathbb{R}} \in \mathbf{MGr}_0^G(S^1)$. In equivariant **mOP**-theory, the class $\tau_{G,\mathbb{R}}$ becomes invertible by Theorem 6.1.30 (ii). So we can consider the maps

$$\mathbf{MGr}_k^G(A \wedge S^n) \xrightarrow{(b\wedge A\wedge S^n)_*} \mathbf{mOP}_k^G(A \wedge S^n) \xrightarrow{(-\cdot\tau_{G,\mathbb{R}}^n)^{-1}} \mathbf{mOP}_k^G(A)\ .$$

By the multiplicativity of b, these maps are compatible, so they assemble into a natural transformation $b^\sharp : \mathbf{MGr}_k^G(A)[\tau_{G,\mathbb{R}}^{-1}] \longrightarrow \mathbf{mOP}_k^G(A)$.

Theorem 6.1.32. *For every compact Lie group G, every based G-space A and every integer k the map*

$$b^\sharp \;:\; \mathbf{MGr}_k^G(A)[\tau_{G,\mathbb{R}}^{-1}] \;\longrightarrow\; \mathbf{mOP}_k^G(A)$$

is an isomorphism.

Proof The 'standard' linear isometric embedding

$$\mathbb{R}^n \;\longrightarrow\; \mathbb{R}^\infty\,, \quad (x_1,\dots,x_n) \;\longmapsto\; (x_1,\dots,x_n,0,0,\dots)$$

induces a continuous map

$$\psi^n(V) \;:\; (\mathrm{sh}^n\,\mathbf{MGr})(V) \;=\; Th(Gr(V\oplus\mathbb{R}^n)) \;\longrightarrow\; Th(Gr(V\oplus\mathbb{R}^\infty)) \;=\; \mathbf{mOP}(V)\,.$$

As V varies, these maps form a morphism of orthogonal spectra $\psi^n : \mathrm{sh}^n\,\mathbf{MGr} \longrightarrow \mathbf{mOP}$. Since $\psi^n = \psi^{n+1}\circ(\mathrm{sh}^n\, j_{\mathbf{MGr}})$, the morphisms ψ^n are compatible with the sequence of morphisms

$$\mathbf{MGr} \xrightarrow{\;j_{\mathbf{MGr}}\;} \mathrm{sh}\,\mathbf{MGr} \xrightarrow{\;\mathrm{sh}\, j_{\mathbf{MGr}}\;} \mathrm{sh}^2\,\mathbf{MGr} \xrightarrow{\;\mathrm{sh}^2\, j_{\mathbf{MGr}}\;} \cdots \;.$$

Moreover, the morphisms ψ^n express $\mathbf{mOP}$ as the colimit of this sequence. So $\mathbf{mOP}\wedge A$ is the colimit of the sequence of orthogonal G-spectra $\mathrm{sh}^n\,\mathbf{MGr}\wedge A$. The map $j_{\mathbf{MGr}}(V)$ is an h-cofibration of based $O(V)$-spaces. So if G acts on V by linear isometries, then $j_{\mathbf{MGr}}(V)\wedge A$ is an h-cofibration of based G-spaces, hence a closed embedding by Proposition A.31. The morphism $j_{\mathbf{MGr}}\wedge A$ and all its shifts are thus level-wise closed embeddings.

Proposition 3.1.41 (i) shows that equivariant homotopy groups commute with sequential colimits over closed embeddings; so the canonical map

$$\mathrm{colim}_{n\geq 0}\, \pi_k^G(\mathrm{sh}^n\,\mathbf{MGr}\wedge A) \;\longrightarrow\; \pi_k^G(\mathbf{mOP}\wedge A)$$

is an isomorphism. The diagram

$$\begin{tikzcd}
\pi_k^G(\mathbf{MGr}\wedge A\wedge S^{n-1}) \arrow[r, "\cdot\tau_{G,\mathbb{R}}"] \arrow[d, "\lambda^{n-1}_{\mathbf{MGr}\wedge A}"', "\cong"] & \pi_k^G(\mathbf{MGr}\wedge A\wedge S^n) \arrow[r, "\pi_k^G(b\wedge A\wedge S^n)"] \arrow[d, "\cong"', "\lambda^n_{\mathbf{MGr}\wedge A}"] & \pi_k^G(\mathbf{mOP}\wedge A\wedge S^n) \arrow[d, "\cong"', "(-\cdot\tau^n_{G,\mathbb{R}})^{-1}"] \\
\pi_k^G(\mathrm{sh}^{n-1}\,\mathbf{MGr}\wedge A) \arrow[r, "\pi_k^G(\mathrm{sh}^{n-1}\, j_{\mathbf{MGr}}\wedge A)"'] & \pi_k^G(\mathrm{sh}^n\,\mathbf{MGr}\wedge A) \arrow[r, "\pi_k^G(\psi^n\wedge A)"'] & \pi_k^G(\mathbf{mOP}\wedge A)
\end{tikzcd}$$

commutes by Proposition 6.1.6 and because $(\mathrm{sh}^{n-1}\, j_{\mathbf{MGr}})\circ\lambda^{n-1}_{\mathbf{MGr}} = \lambda^{n-1}_{\mathrm{sh}\,\mathbf{MGr}}\circ(j_{\mathbf{MGr}}\wedge S^{n-1})$. The three vertical maps are isomorphisms. So $\pi_k^G(\mathbf{MOP})$ is also a colimit of the sequence of multiplication maps by $\tau_{G,\mathbb{R}}$, with respect to the maps that define $b^\sharp$. □

There is one localization result left: it remains to exhibit **MOP**-theory as the localization of **mOP**-theory, by inverting the inverse Thom classes of arbitrary representations. This is in fact a formal consequence of Theorem 6.1.23 and Theorem 6.1.32 which exhibit both $\mathbf{MOP}^G_*(A)$ and $\mathbf{mOP}^G_*(A)$ as localizations of $\mathbf{MGr}^G_*(A)$, the former being a more drastic localization than the latter.

We spell this out in more detail. We define a localized version of equivariant **mOP**-homology by

$$\mathbf{mOP}^G_0(A)[1/\tau] \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\ \mathbf{mOP}^G_0(A \wedge S^V) \ ;$$

for $V \subset W$, the structure map in the colimit system is the multiplication

$$\mathbf{mOP}^G_0(A \wedge S^V) \xrightarrow{-\cdot\tau_{G,W-V}} \mathbf{mOP}^G_0(A \wedge S^V \wedge S^{W-V}) \ \cong\ \mathbf{mOP}^G_0(A \wedge S^W)\ .$$

In (6.1.22) we introduced a morphism of ultra-commutative ring spectra a : **MGr** $\longrightarrow$ **MOP**. In (6.1.31) we introduced the morphism of E_∞-ring spectra b : **MGr** $\longrightarrow$ **mOP**. These morphisms induce multiplicative natural transformations

$$(a \wedge A)_* \ :\ \mathbf{MGr}^G_*(A) \ \longrightarrow\ \mathbf{MOP}^G_*(A) \quad \text{and}$$
$$(b \wedge A)_* \ :\ \mathbf{MGr}^G_*(A) \ \longrightarrow\ \mathbf{mOP}^G_*(A)$$

for all compact Lie groups G and all based G-spaces A. Moreover, the morphisms match up the inverse Thom classesnd in the sense that

$$(a \wedge S^V)_*(\tau_{G,V}) \ = \ \tau_{G,V} \qquad \text{and} \qquad (b \wedge S^V)_*(\tau_{G,V}) \ = \ \tau_{G,V}\ ;$$

indeed, this is the justification for our abuse of notation of using the same name for the inverse Thom classes in **MGr**, **MOP** and **mOP**.

Theorem 6.1.32 says that the map $(b \wedge A)_*$ becomes an isomorphism after inverting the inverse Thom class $\tau_{G,\mathbb{R}}$ of the trivial 1-dimensional G-representation. So it also becomes an isomorphism after inverting the inverse Thom classes of all representations, i.e., the induced transformation

$$(b \wedge A)_*[1/\tau] \ :\ \mathbf{MGr}^G_*(A)[1/\tau] \ \longrightarrow\ \mathbf{mOP}^G_*(A)[1/\tau]$$

is an isomorphism. On the other hand, the transformation $(a \wedge A)_*$ induces an isomorphism $a^\sharp$ from $\mathbf{MGr}^G_*(A)[1/\tau]$ to $\mathbf{MOP}^G_*(A)$, by Theorem 6.1.23. So combining these two theorems yields:

Corollary 6.1.33. *For every compact Lie group G, every based G-space A and every integer k the map*

$$a^\sharp \circ ((b \wedge A)_*[1/\tau])^{-1} \ :\ \mathbf{mOP}^G_k(A)[1/\tau] \ \longrightarrow\ \mathbf{MOP}^G_k(A)$$

is an isomorphism.

While the authors thinks that the periodic theories **mOP** and **MOP** give the most convenient formulation of the localization result, the more traditional formulation is in terms of the degree 0 summands $\mathbf{mO} = \mathbf{mOP}^{[0]}$ and $\mathbf{MO} = \mathbf{MOP}^{[0]}$. So we also take the time to reformulate Corollary 6.1.33 in terms of **mO** and **MO**. Since the inverse Thom classes do *not* lie in the degree zero summands, we instead use the shifted inverse Thom classes

$$\bar{\tau}_{G,V} \ = \ p_G^*(\sigma^m) \cdot \tau_{G,V} \in \ \mathbf{MO}_m^G(S^V)$$

introduced in (6.1.21), and its **mO**-analog

$$\bar{\tau}_{G,V} \ = \ p_G^*(\sigma^m) \cdot \tau_{G,V} \in \ \mathbf{mO}_m^G(S^V)\ . \tag{6.1.34}$$

In the **MOP**-case, the periodicity class $\sigma \in \pi_1^e(\mathbf{MOP}^{[1]})$ was defined in (6.1.11), and it is inverse to the class t. In the **mOP**-case, the periodicity class $\sigma \in \pi_1^e(\mathbf{mOP}^{[1]})$ was not yet defined, and we take it to be the inverse of the **mOP** periodicity class t from (6.1.29). Both theories **mOP** and **MOP** are periodic (in the $\mathbb{Z}$-graded sense), i.e., the maps

$$\bigoplus_{m\in\mathbb{Z}} \mathbf{MO}_m^G(A) \ \longrightarrow \ \mathbf{MOP}_0^G(A) \quad \text{and} \quad \bigoplus_{m\in\mathbb{Z}} \mathbf{mO}_m^G(A) \ \longrightarrow \ \mathbf{mOP}_0^G(A)$$

that multiply by the appropriate powers of the periodicity classes are isomorphisms.

We define a localized version of equivariant **mO**-homology by

$$\mathbf{mO}_k^G(A)[1/\bar{\tau}] \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)} \mathbf{mO}_{k+|V|}^G(A \wedge S^V)\ ;$$

for $V \subset W$, the structure map in the colimit system is the multiplication

$$\mathbf{mO}_{k+|V|}^G(A \wedge S^V) \xrightarrow{-\cdot\bar{\tau}_{G,W-V}} \mathbf{mO}_{k+|W|}^G(A \wedge S^V \wedge S^{W-V}) \ \cong \ \mathbf{mO}_{k+|W|}^G(A \wedge S^W)\ .$$

The periodicity of **mOP** is inherited by the localized theory, i.e., the upper horizontal map in the following commutative square

$$\begin{tikzcd}
\bigoplus_{m\in\mathbb{Z}} \mathbf{mO}_m^G(A)[1/\bar{\tau}] \arrow[r] \arrow[d, "a^\sharp\circ((b\wedge A)_*[1/\bar{\tau}])^{-1}"'] & \mathbf{mOP}_0^G(A)[1/\tau] \arrow[d, "a^\sharp\circ((b\wedge A)_*[1/\tau])^{-1}"] \\
\bigoplus_{m\in\mathbb{Z}} \mathbf{MO}_m^G(A) \arrow[r] & \mathbf{MOP}_0^G(A)
\end{tikzcd}$$

is an isomorphism. The lower horizontal map is an isomorphism by the periodicity of **MOP**, and the right vertical map is an isomorphism by Corollary 6.1.33. So the left vertical map is an isomorphism. Since the left map is homogeneous with respect to the $\mathbb{Z}$-grading, it is an isomorphism in every degree. So we conclude:

Corollary 6.1.35. *For every compact Lie group G, every based G-space A and every integer m the map*

$$a^\sharp \circ ((b \wedge A)_*[1/\bar{\tau}])^{-1} \ : \ \mathbf{mO}^G_m(A)[1/\bar{\tau}] \ \longrightarrow \ \mathbf{MO}^G_m(A)$$

is an isomorphism.

Now we investigate the global homotopy type of the Thom spectrum **mO** in more detail; one reason for wanting to understand **mO** better is the close connection to equivariant bordism, compare Theorem 6.2.33 below. Our main tool is the following 'rank filtration'; there is an analog of the rank filtration for **mOP**, but we refrain from making it explicit.

Construction 6.1.36 (Rank filtration of **mO**)**.** The orthogonal spectrum **mO** is a global Thom spectrum over an orthogonal space **bO**; in Proposition 2.4.24 we identified **bO** as a certain global homotopy colimit of the global classifying spaces $B_{\mathrm{gl}}O(m)$. More precisely, the filtration of $\mathbb{R}^\infty$ by the subspaces $\mathbb{R}^m$ induces a filtration of **bO** by orthogonal subspaces $\mathbf{bO}_{(m)}$, and $\mathbf{bO}_{(m)}$ receives a global equivalence from $B_{\mathrm{gl}}O(m)$. We now define and study the corresponding orthogonal Thom spectrum $\mathbf{mO}_{(m)}$ over $\mathbf{bO}_{(m)}$, which turns out to be an m-fold suspension of the orthogonal spectrum $M_{\mathrm{gl}}T(m)$, the global refinement of the spectrum traditionally denoted $MT(m)$.

Construction 4.1.23 defines the semifree orthogonal spectrum $F_{G,V}$ generated by a G-representation V. We are interested in the tautological $O(m)$-representation ν_m with underlying inner product space $\mathbb{R}^m$, and to simplify the notation we abbreviate the corresponding semifree spectrum to

$$F_m \ = \ F_{O(m),\nu_m} \ .$$

The shift functor $\mathrm{sh}^m = \mathrm{sh}^{\mathbb{R}^m}$ by the inner product space $\mathbb{R}^m$ was defined in Construction 3.1.21. We set

$$\mathbf{mO}_{(m)} \ = \ \mathrm{sh}^m F_m \ ,$$

the mth shift of F_m. Unpacking this definition reveals the value of $\mathbf{mO}_{(m)}$ at an inner product space V as the space

$$\mathbf{mO}_{(m)}(V) \ = \ \mathbf{O}(\nu_m, V \oplus \mathbb{R}^m)/O(m) \ .$$

To justify the notation $\mathbf{mO}_{(m)}$ we clarify the connection to the orthogonal space $\mathbf{bO}_{(m)}$ defined in (2.4.23). The value of $\mathbf{bO}_{(m)}$ at V is

$$\mathbf{bO}_{(m)}(V) \ = \ Gr_{|V|}(V \oplus \mathbb{R}^m) \ ,$$

the Grassmannian of $|V|$-planes in $V \oplus \mathbb{R}^m$. Over the space $\mathbf{bO}_{(m)}(V)$ sits a tautological euclidean $|V|$-plane bundle, with total space consisting of pairs

$(x, U) \in (V \oplus \mathbb{R}^m) \times \mathbf{bO}_{(m)}(V)$ such that $x \in U$. Passage to orthogonal complements provides a homeomorphism:

$$\begin{aligned} \mathbf{mO}_{(m)}(V) \ = \ \mathbf{O}(\nu_m, V \oplus \mathbb{R}^m)/O(m) \ &\cong \ Th(Gr_{|V|}(V \oplus \mathbb{R}^m)) \\ [x, \varphi] \quad &\longmapsto \ (x, \varphi(\nu_m)^\perp) \end{aligned}$$

In this sense, $\mathbf{mO}_{(m)}(V)$ 'is' the Thom space over $\mathbf{bO}_{(m)}(V)$. Just as the orthogonal spaces $\mathbf{bO}_{(m)}$ form an exhaustive filtration of $\mathbf{bO}$, the orthogonal spectra $\mathbf{mO}_{(m)}$ form an exhaustive filtration of the Thom spectrum $\mathbf{mO}$, compare Proposition 6.1.38 below.

As we explained in Remark 4.1.25, the semifree orthogonal spectrum $F_m = F_{O(m),\nu_m}$ is a global refinement of the Thom spectrum $MT(m)$ of the negative of the tautological m-plane bundle over $Gr_m(\mathbb{R}^\infty)$. Since shift and suspension are globally equivalent (by Proposition 4.1.4 (i)), $\mathbf{mO}_{(m)}$ is globally equivalent to the m-fold suspension of the orthogonal spectrum $M_{\mathrm{gl}}T(m) = F_{O(m),\nu_m}$

$$\mathbf{mO}_{(m)} \ = \ \mathrm{sh}^m F_m \ \simeq_{\mathrm{gl}} \ F_m \wedge S^m \ = \ M_{\mathrm{gl}}T(m) \wedge S^m \ .$$

We define a morphism of orthogonal spectra

$$i \ : \ F_m \ \longrightarrow \ \mathrm{sh}\, F_{m+1} \ ;$$

the value at an inner product space V is the closed embedding

$$\begin{aligned} i(V) \ : \ F_m(V) = \mathbf{O}(\nu_m, V)/O(m) \ &\xrightarrow{-\oplus\mathbb{R}} \ \mathbf{O}(\nu_{m+1}, V \oplus \mathbb{R})/O(m+1) = (\mathrm{sh}\, F_{m+1})(V) \\ [x, \varphi] \qquad &\longmapsto \qquad [(x, 0), \varphi \oplus \mathbb{R}] \end{aligned}$$

where we identify $\nu_m \oplus \mathbb{R}$ with ν_{m+1} by sending $((x_1, \dots, x_m), y)$ to $(x_1, \dots, x_m, y)$. In fact, there are very few morphisms of orthogonal spectra from F_m to $\mathrm{sh}\, F_{m+1}$: by the representing property of the semifree spectrum F_m, such morphisms biject with $O(m)$-fixed-points of $(\mathrm{sh}\, F_{m+1})(\nu_m) = \mathbf{O}(\nu_{m+1}, \nu_m \oplus \mathbb{R})/O(m+1)$; this space only has two elements, and the morphism i corresponds to the non-basepoint element. We define

$$j^m \ = \ \mathrm{sh}^m i \ : \ \mathbf{mO}_{(m)} = \mathrm{sh}^m F_m \ \longrightarrow \ \mathrm{sh}^m(\mathrm{sh}\, F_{m+1}) \ = \ \mathrm{sh}^{m+1} F_{m+1} \ = \ \mathbf{mO}_{(m+1)} \ .$$

We define a morphism

$$\psi^m \ : \ \mathbf{mO}_{(m)} \ = \ \mathrm{sh}^m F_m \ \longrightarrow \ \mathbf{mO}$$

at an inner product space V as the map

$$\begin{aligned} \psi^m(V) \ : \ \mathbf{O}(\nu_m, V \oplus \mathbb{R}^m)/O(m) \ &\longrightarrow \ Th(Gr_{|V|}(V \oplus \mathbb{R}^\infty)) \\ [x, \varphi] \quad &\longmapsto \quad (i(x), i(\varphi^\perp)) \ , \end{aligned}$$

where $\varphi^\perp = (V \oplus \mathbb{R}^m) - \varphi(v_m)$ is the orthogonal complement of the image of φ and $i : V \oplus \mathbb{R}^m \longrightarrow V \oplus \mathbb{R}^\infty$ is the 'standard' embedding given by

$$i(v, x_1, \dots, x_m) \;=\; (v, x_1, \dots, x_m, 0, 0, \dots)\ .$$

The rank filtration expresses the orthogonal spectrum **mO** as the colimit of the sequence of closed embeddings

$$\mathbb{S} \;\cong\; \mathbf{mO}_{(0)} \xrightarrow{j^0} \mathbf{mO}_{(1)} \xrightarrow{j^1} \dots \longrightarrow \mathbf{mO}_{(m)} \xrightarrow{j^m} \dots\ . \tag{6.1.37}$$

The following proposition is straightforward from the definitions. The periodic analog was already used in the proof of Theorem 6.1.32, since it expresses **mOP** as a sequential colimit of the spectra $\mathrm{sh}^m\,\mathbf{MGr}$. We omit the proof.

Proposition 6.1.38. *For every $m \geq 0$ the morphism $j^m :\ \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ is level-wise a closed embedding. These morphisms satisfy $\psi^{m+1} \circ j^m = \psi^m$. With respect to the morphisms $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$, the orthogonal spectrum* **mO** *is a colimit of the sequence* (6.1.37).

Since colimits along sequences of closed embeddings are invariant under global equivalences (Proposition 4.1.4 (v)), Proposition 6.1.38 says that **mO** is a homotopy colimit of the sequence (6.1.37). The underlying non-equivariant statement, i.e., that MO is a homotopy colimit of the spectra $\Sigma^m MT(m)$, can for example be found in [59, Sec. 3]. The identification of **mO** as a homotopy colimit of semifree orthogonal spectra now allows an algebraic description of $[\![\mathbf{mO}, E]\!]$, the group of morphisms in the global stable homotopy category $\mathcal{GH}$, into any orthogonal spectrum E.

We define a distinguished class

$$\tau_m \;\in\; \pi_m^{O(m)}(\mathbf{mO}_{(m)} \wedge S^{\nu_m})$$

in the mth $O(m)$-equivariant $\mathbf{mO}_{(m)}$-homology group of S^{ν_m} as the class represented by the based $O(m)$-map

$$\begin{aligned} S^{\nu_m \oplus \mathbb{R}^m} \;&\longrightarrow\; \mathbf{O}(\nu_m, \nu_m \oplus \mathbb{R}^m)/O(m) \wedge S^{\nu_m} \;=\; \mathbf{mO}_{(m)}(\nu_m) \wedge S^{\nu_m} \\ (v, x) \;&\longmapsto\; \qquad [(0, x), i] \wedge (-v)\ , \end{aligned}$$

where $i : \nu_m \longrightarrow \nu_m \oplus \mathbb{R}^m$ is the embedding as the first summand.

Proposition 6.1.39. *Let $m \geq 0$ be a natural number.*

(i) *The pair $(\mathbf{mO}_{(m)}, \tau_m)$ represents the functor*

$$\mathcal{GH} \longrightarrow (\text{sets})\ , \quad E \;\longmapsto\; E_m^{O(m)}(S^{\nu_m}) \;=\; \pi_m^{O(m)}(E \wedge S^{\nu_m})\ .$$

(ii) *The morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ satisfies the relation*

$$(j^m \wedge S^{\nu_m})_*(\tau_m) \wedge S^1 \;=\; \mathrm{res}^{O(m+1)}_{O(m)}(\tau_{m+1})$$

in the group $\pi_{m+1}^{O(m)}(\mathbf{mO}_{(m+1)} \wedge S^{\nu_m} \wedge S^1)$.

(iii) *The morphism* $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$ *sends* τ_m *to the shifted inverse Thom class* $\bar{\tau}_{O(m),\nu_m}$ *defined in* (6.1.34).

Proof (i) In (4.4.16) we defined a distinguished equivariant homotopy class

$$a_m \;=\; a_{O(m),\nu_m} \;\in\; \pi_0^{O(m)}(F_m \wedge S^{\nu_m}) \,.$$

Expanding the definitions of a_m and of the morphism

$$\lambda^m_{F_m \wedge S^{\nu_m}} \;:\; F_m \wedge S^{\nu_m} \wedge S^m \;\longrightarrow\; \mathrm{sh}^m F_m \wedge S^{\nu_m} \;=\; \mathbf{mO}_{(m)} \wedge S^{\nu_m}$$

shows that

$$\tau_m \;=\; (\lambda^m_{F_m \wedge S^{\nu_m}})_*(a_m \wedge S^m) \,.$$

By Theorem 4.4.18, the pair (F_m, a_m) represents the functor

$$\mathcal{GH} \longrightarrow (\text{sets}) \,, \quad E \;\longmapsto\; E_0^{O(m)}(S^{\nu_m}) \;=\; \pi_0^{O(m)}(E \wedge S^{\nu_m}) \,.$$

We claim that the following composite

$$\begin{aligned}
[\![\mathbf{mO}_{(m)}, E]\!] \;=\; [\![\mathrm{sh}^m F_m, E]\!] \;&\xrightarrow{[\![\lambda^m_{F_m}, E]\!]}\; [\![F_m \wedge S^m, E]\!] \\
&\xrightarrow{\text{adjunction}}\; [\![F_m, \Omega^m E]\!] \;\xrightarrow{\text{eval at } a_m}\; \pi_0^{O(m)}((\Omega^m E) \wedge S^{\nu_m}) \\
&\xrightarrow{\text{assembly}_*}\; \pi_0^{O(m)}(\Omega^m(E \wedge S^{\nu_m})) \;\xrightarrow{\alpha^m}\; \pi_m^{O(m)}(E \wedge S^{\nu_m})
\end{aligned}$$

coincides with evaluation at the class τ_m. Here the fourth map is induced by the assembly morphism

$$(\Omega^m E) \wedge S^{\nu_m} \;\longrightarrow\; \Omega^m(E \wedge S^{\nu_m}) \,,$$

and

$$\alpha^m \;:\; \pi_0^{O(m)}(\Omega^m Y) \;\cong\; \pi_m^{O(m)}(Y)$$

is the analog of the loop isomorphism (3.1.28). Since all the maps are natural in E, it suffices to check this claim for the identity of the universal example $E = \mathbf{mO}_{(m)} = \mathrm{sh}^m F_m$. Since the square

$$\begin{tikzcd}
E \wedge S^{\nu_m} \arrow[r, "\eta"] \arrow[d, "\tilde{\lambda}^m_E \wedge S^{\nu_m}"'] & \Omega^m(E \wedge S^{\nu_m} \wedge S^m) \arrow[d, "\Omega^m(\lambda^m_{E \wedge S^{\nu_m}})"] \\
(\Omega^m \,\mathrm{sh}^m E) \wedge S^{\nu_m} \arrow[r, "\text{assembly}"'] & \Omega^m(\mathrm{sh}^m E \wedge S^{\nu_m})
\end{tikzcd}$$

commutes, we obtain

$$\begin{aligned}\alpha^m(\mathrm{assembly}_*((\tilde{\lambda}^m_{F_m} \wedge S^{\nu_m})_*(a_m))) &= \alpha^m(\Omega^m(\lambda^m_{F_m\wedge S^{\nu_m}})_*(\eta_*(a_m)))\\ &= (\lambda^m_{F_m\wedge S^{\nu_m}})_*(\alpha^m(\eta_*(a_m)))\\ &= (\lambda^m_{F_m\wedge S^{\nu_m}})_*(a_m \wedge S^m) \;=\; \tau_m\ .\end{aligned}$$

This verifies the relation in the universal example. Since all the individual maps in the above composite are bijective, so is the composite, which proves the representability property of the pair $(\mathbf{mO}_{(m)}, \tau_m)$.

(ii) The class $(j^m \wedge S^{\nu_m})_*(\tau_m) \wedge S^1$ is represented by the composite:

$$\begin{aligned}S^{\nu_m\oplus\mathbb{R}^{m+1}} &\xrightarrow{t_m\wedge S^1} \mathbf{O}(\nu_m, \nu_m \oplus \mathbb{R}^m)/O(m) \wedge S^{\nu_m} \wedge S^1\\ &\xrightarrow{j^m(\nu_m\oplus\mathbb{R}^m)\wedge S^{\nu_m}\wedge S^1} \mathbf{O}(\nu_{m+1}, \nu_m \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_m} \wedge S^1\\ (v, x, s) &\longmapsto [(0, x), i] \wedge (-v) \wedge s \longmapsto [(0, x, 0), i \oplus \mathbb{R}] \wedge (-v) \wedge s\end{aligned}$$

If we stabilize this representative along the linear isometric embedding

$$j \;:\; \nu_m \oplus \mathbb{R}^{m+1} \longrightarrow \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1}\ , \quad (v, x, s) \longmapsto (v, 0, x, s)$$

we obtain another representative, namely

$$\begin{aligned}S^{\nu_m\oplus\mathbb{R}\oplus\mathbb{R}^{m+1}} &\longrightarrow \mathbf{O}(\nu_{m+1}, \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_m} \wedge S^1\\ (v, u, x, s) &\longmapsto \quad [(0, u, x, 0), i \oplus \mathbb{R}] \wedge (-v) \wedge s\ .\end{aligned}$$

Here $i \oplus \mathbb{R} : \nu_{m+1} \longrightarrow \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1}$ sends (v, u) to $(v, 0, 0, u)$.

On the other hand, $\mathrm{res}^{O(m+1)}_{O(m)}(\tau_{m+1})$ is represented by the underlying $O(m)$-map of the $O(m+1)$-map

$$\begin{aligned}S^{\nu_{m+1}\oplus\mathbb{R}^{m+1}} &\xrightarrow{t_{m+1}} \mathbf{O}(\nu_{m+1}, \nu_{m+1} \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_{m+1}}\\ (v, u, x, s) &\longmapsto \quad [(0, 0, x, s), i'] \wedge (-v, -u)\ ,\end{aligned}$$

where $i' : \nu_{m+1} \longrightarrow \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1}$ sends (v, u) to $(v, u, 0, 0)$. The two representatives differ by conjugation with the $O(m)$-equivariant linear isometry

$$\nu_m \oplus \mathbb{R} \oplus \mathbb{R}^m \oplus \mathbb{R} \longrightarrow \nu_m \oplus \mathbb{R} \oplus \mathbb{R}^m \oplus \mathbb{R}\ , \quad (v, u, x, s) \longmapsto (v, -s, x, u)\ ,$$

so they represent the same class in the group $\pi^{O(m)}_{m+1}(\mathbf{mO}_{(m+1)} \wedge S^{\nu_m} \wedge S^1)$.

(iii) Substituting the definitions of τ_m and of the morphism ψ^m shows that $(\psi^m \wedge S^{\nu_m})_*(\tau_m)$ is represented by the map

$$S^{\nu_m\oplus\mathbb{R}^m} \longrightarrow \mathbf{mO}(\nu_m) \wedge S^{\nu_m}\ , \quad (v, x) \longmapsto \quad [(0, i(x)), 0 \oplus i(\mathbb{R}^m)] \wedge (-v)\ ,$$

where $i : \mathbb{R}^m \longrightarrow \mathbb{R}^\infty$ is the 'standard' embedding as the leading m coordinates. The periodicity class σ is represented by the map

$$s \;:\; S^1 \longrightarrow Th(Gr(\mathbb{R}^\infty)) \;=\; \mathbf{mOP}(0)\ , \quad x \longmapsto ((x, 0, 0, \dots), \mathbb{R} \oplus 0)\ .$$

The multiplication of **mOP** is an E_∞-multiplication, so as explained in (6.1.26), multiplication of classes in **mOP**-theory involves choices of linear isometric embeddings. We choose a linear isometric embedding

$$\psi \;:\; (\mathbb{R}^\infty)^m \longrightarrow \mathbb{R}^\infty$$

that satisfies

$$\psi((x_1, 0, 0, \dots), \dots, (x_m, 0, 0, \dots)) \;=\; (x_1, \dots, x_m, 0, 0, \dots)\,.$$

If we base the multiplication of **mOP** on such a choice ψ, then the product of the defining representative for $\tau_{O(m),\nu_m}$ and m factors of the map s above is precisely the previous representative for the class $(\psi^m \wedge S^{\nu_m})_*(\tau_m)$. This proves the relation $(\psi^m \wedge S^{\nu_m})_*(\tau_m) = \tau_{O(m),\nu_m} \cdot p^*_{O(m)}(\sigma^m)$ in $\mathbf{mO}_m^{O(m)}(S^{\nu_m})$. ◻

The fact that **mO** is the global homotopy colimit of the sequence of orthogonal spectra $\mathbf{mO}_{(m)}$ (see Proposition 6.1.38) has the following consequence.

Corollary 6.1.40. *For every orthogonal spectrum E the following sequence is short exact:*

$$0 \longrightarrow \lim_m{}^1 E_{m+1}^{O(m)}(S^{\nu_m}) \longrightarrow [\![\mathbf{mO}, E]\!] \longrightarrow \lim_m E_m^{O(m)}(S^{\nu_m}) \longrightarrow 0$$

Here the right map is evaluation at the shifted inverse Thom classes $\bar{\tau}_{O(m),\nu_m}$ and the inverse limit and derived limit are formed along the maps

$$E_{m+1}^{O(m+1)}(S^{\nu_{m+1}}) \xrightarrow{\mathrm{res}^{O(m+1)}_{O(m)}} E_{m+1}^{O(m)}(S^{\nu_m} \wedge S^1) \xrightarrow{(-\wedge S^1)^{-1}} E_m^{O(m)}(S^{\nu_m})\,.$$

Proof Since **mO** is the sequential homotopy colimit, in the triangulated global stable homotopy category, of the sequence of orthogonal spectra (6.1.37), the Milnor exact sequence takes the form:

$$0 \longrightarrow \lim_m{}^1 [\![\mathbf{mO}_{(m)} \wedge S^1, E]\!] \longrightarrow [\![\mathbf{mO}, E]\!] \longrightarrow \lim_m [\![\mathbf{mO}_{(m)}, E]\!] \longrightarrow 0$$

By Proposition 6.1.39 the pair $(\mathbf{mO}_{(m)}, \tau_m)$ then represents the functor

$$\mathcal{GH} \longrightarrow (\text{sets})\,, \quad E \longmapsto E_m^{O(m)}(S^{\nu_m})\,,$$

and the morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ has the correct behavior on the universal classes. ◻

Now that we recognized **mO** as the global homotopy colimit of the rank filtration (6.1.37), we study how one filtration term is obtained from the previous one. The answer given by Theorem 6.1.42 below takes the form of a distinguished triangle in the global stable homotopy category, witnessing that the mapping cone of $j^m : \mathbf{mO}_{(m-1)} \longrightarrow \mathbf{mO}_{(m)}$ 'is' the m-fold suspension of the suspension spectrum of the global classifying space $B_{\mathrm{gl}}O(m)$.

We define a morphism of orthogonal spectra

$$T_m \;=\; T^{O(m+1)}_{O(m)} \;:\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \;\longrightarrow\; F_m$$

as the adjoint of the $O(m+1)$-equivariant map

$$\begin{aligned} r \;:\; S^{\nu_{m+1}} \;&\longrightarrow\; \mathbf{O}(\nu_m, \nu_{m+1})/O(m) \;=\; F_m(\nu_{m+1}) \\ r(A \cdot (0, \dots, 0, t)) \;&=\; A \cdot ((0, \dots, 0, (t^2-1)/t), \mathrm{incl}) \cdot O(m) \;, \end{aligned}$$

where $A \in O(m+1)$ and $t \in [0, \infty]$. Theorem 4.4.21 (ii) shows that the morphism T_m represents the dimension shifting transfer from $O(m)$ to $O(m+1)$, in the sense of the relation

$$(T_m)_*(e_{O(m+1)}) \;=\; \mathrm{Tr}^{O(m+1)}_{O(m)}(a_m) \tag{6.1.41}$$

between the tautological classes. From T_m we define another morphism of orthogonal spectra

$$\begin{aligned} \partial \;=\; \lambda^m_{F_m} \circ (T_m \wedge S^m) \;&=\; (\mathrm{sh}^m T_m) \circ \lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \;: \\ &\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \;\longrightarrow\; \mathrm{sh}^m F_m \;=\; \mathbf{mO}_{(m)} \;. \end{aligned}$$

The space

$$((\Sigma^\infty_+ \mathbf{L}_{O(m+1),\nu_{m+1}})(\nu_{m+1}))^{O(m+1)} \;=\; (S^{\nu_{m+1}} \wedge \mathbf{L}(\nu_{m+1}, \nu_{m+1})/O(m+1))^{O(m+1)}$$

has two points, the basepoint and $0 \wedge \mathrm{Id}_{\nu_{m+1}} \cdot O(m+1)$. So there is a unique non-trivial morphism of orthogonal spectra

$$a \;:\; F_{m+1} \;\longrightarrow\; \Sigma^\infty_+ \mathbf{L}_{O(m+1),\nu_{m+1}} = \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \;.$$

For every orthogonal spectrum E the morphism $\lambda^{m+1}_E : E \wedge S^{m+1} \longrightarrow \mathrm{sh}^{m+1} E$ is a global equivalence (by an iteration of Proposition 4.1.4 (i)), so it becomes invertible in the global stable homotopy category. We can thus define a morphism in $\mathcal{GH}$ as

$$q \;=\; (\lambda^{m+1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \;:\; \mathbf{mO}_{(m+1)} \;\longrightarrow\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1} \;.$$

Theorem 6.1.42. *The sequence*

$$\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m \;\xrightarrow{\;\partial\;}\; \mathbf{mO}_{(m)} \;\xrightarrow{\;j^m\;}\; \mathbf{mO}_{(m+1)} \;\xrightarrow{\;q\;}\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1}$$

is a distinguished triangle in the global stable homotopy category. The behavior of the first morphism on the stable tautological class is given by

$$\partial_*(e_{O(m+1)} \wedge S^m) \;=\; \mathrm{Tr}^{O(m+1)}_{O(m)}(\tau_m) \;.$$

Proof The tautological $O(m+1)$-representation ν_{m+1} is faithful, and the action of $O(m+1)$ on the unit sphere $S(\nu_{m+1})$ is transitive. The stabilizer group of the unit vector $(0, \dots, 0, 1)$ identifies with the group $O(m)$, and the orthogonal

complement of this vector becomes the tautological representation ν_m. We can thus apply Theorem 4.4.21 and obtain a distinguished triangle:

$$F_{m+1} \xrightarrow{\ a\ } \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \xrightarrow{\ T_m\ } F_m \xrightarrow{-\lambda^{-1}_{F_{m+1}} \circ i} F_{m+1} \wedge S^1$$

By Example 4.4.1 shifting preserves distinguished triangles; so the following sequence is also distinguished:

$$\mathrm{sh}^m F_{m+1} \xrightarrow{\mathrm{sh}^m a} \mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \xrightarrow{\mathrm{sh}^m T_m} \mathrm{sh}^m F_m \xrightarrow{-\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}} \circ i)} \mathrm{sh}^m F_{m+1} \wedge S^1$$

The rotation of this triangle is the lower sequence in the following diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m \arrow[r, "\partial"] \arrow[d, "\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)}"'] & \mathbf{mO}_{(m)} \arrow[r, "j^m"] \arrow[d, equal] & \mathbf{mO}_{(m+1)} \arrow[r, "q"] \arrow[d, "\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}})"] & \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1} \arrow[d, "\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1"] \\
\mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \arrow[r, "\mathrm{sh}^m T_m"'] & \mathrm{sh}^m F_m \arrow[r, "\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}} \circ i)"'] & \mathrm{sh}^m F_{m+1} \wedge S^1 \arrow[r, "\mathrm{sh}^m a \wedge S^1"'] & \mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^1
\end{tikzcd}$$

The left and middle squares commute by definition of the morphisms ∂ and j^m, respectively; the right square commutes by the relation

$$\begin{aligned}
(\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1) \circ q \ &= \ (\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1) \circ (\lambda^{m+1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \\
&= \ (\mathrm{sh}^m \lambda_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \\
&= \ \mathrm{sh}^m(\lambda^{-1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \circ (\mathrm{sh}\, a)) \ = \ \mathrm{sh}^m((a \wedge S^1) \circ \lambda^{-1}_{F_{m+1}}) \ .
\end{aligned}$$

So the upper sequence is a distinguished triangle.

The final relation is a consequence of various other previously established relations:

$$\begin{aligned}
\partial_*(e_{O(m+1)} \wedge S^m) \ &= \ (\lambda^m_{F_m})_*((T_m \wedge S^m)_*(e_{O(m+1)} \wedge S^m)) \\
&= \ (\lambda^m_{F_m})_*((T_m)_*(e_{O(m+1)}) \wedge S^m) \\
\overset{(6.1.41)}{=} \ &\ (\lambda^m_{F_m})_*(\mathrm{Tr}^{O(m+1)}_{O(m)}(a_m) \wedge S^m) \\
&= \ (\lambda^m_{F_m})_*(\mathrm{Tr}^{O(m+1)}_{O(m)}((F_m \wedge \tau_{\nu_m, \mathbb{R}^m})_*(a_m \wedge S^m))) \\
&= \ \mathrm{Tr}^{O(m+1)}_{O(m)}((\lambda^m_{F_m} \wedge S^{\nu_m})_*((F_m \wedge \tau_{\nu_m, \mathbb{R}^m})_*(a_m \wedge S^m))) \\
&= \ \mathrm{Tr}^{O(m+1)}_{O(m)}((\lambda^m_{F_m \wedge S^{\nu_m}})_*(a_m \wedge S^m)) \ = \ \mathrm{Tr}^{O(m+1)}_{O(m)}(\tau_m) \ .
\end{aligned}$$

The second and fifth equations are naturality. The fourth equation is the compatibility of transfer and suspension isomorphism, see Proposition 3.2.27. □

For calculations of equivariant homotopy groups of **mO** we also need to understand the composite:

$$\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m \xrightarrow{\ \partial\ } \mathbf{mO}_{(m)} \xrightarrow{\ q\ } \Sigma^\infty_+ B_{\mathrm{gl}} O(m) \wedge S^m$$

We start from the calculation

$$
\begin{aligned}
(a \circ T_m)_* \left(e_{O(m+1)}\right) &= a_*\left(\mathrm{Tr}_{O(m)}^{O(m+1)}(a_m)\right) = \mathrm{Tr}_{O(m)}^{O(m+1)}\left(a_*(a_m)\right) \\
&= \mathrm{Tr}_{O(m)}^{O(m+1)}\left(a \cdot e_{O(m)}\right) = \mathrm{tr}_{O(m)}^{O(m+1)}(e_{O(m)}) \,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
q \circ \partial &= \left(\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m)}\right)^{-1} \circ (\mathrm{sh}^m a) \circ \lambda^m_{F_m} \circ (T_m \wedge S^m) \\
&= \left(\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m)}\right)^{-1} \circ \lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m)} \circ (a \wedge S^m) \circ (T_m \wedge S^m) = (a \circ T_m) \wedge S^m \,,
\end{aligned}
$$

by definition. Combining these two facts gives

$$
\begin{aligned}
(q \circ \partial)_*(e_{O(m+1)} \wedge S^m) &= ((a \circ T_m) \wedge S^m)_*(e_{O(m+1)} \wedge S^m) \\
&= (a \circ T_m)_*(e_{O(m+1)}) \wedge S^m = \mathrm{tr}_{O(m)}^{O(m+1)}(e_{O(m)}) \wedge S^m \,.
\end{aligned}
\tag{6.1.43}
$$

In other words, the composite $q \circ \partial$ represents the degree zero transfer $\mathrm{tr}_{O(m)}^{O(m+1)}$.

Now we can easily show that $\mathbf{mO}$ is globally connective and describe the global functor $\underline{\pi}_0(\mathbf{mO})$. We denote by $\langle \mathrm{tr}_e^{O(1)} \rangle$ the global subfunctor of the Burnside ring functor $\mathbb{A}$ generated by $\mathrm{tr}_e^{O(1)} \in \mathbb{A}(O(1))$.

Theorem 6.1.44. *The orthogonal spectrum* $\mathbf{mO}$ *is globally connective and the action of the Burnside ring global functor on the unit element* $1 \in \pi_0^e(\mathbf{mO})$ *induces an isomorphism of global functors*

$$
\mathbb{A}/\langle \mathrm{tr}_e^{O(1)} \rangle \cong \underline{\pi}_0(\mathbf{mO}) \,.
$$

Proof The suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)$ is globally connective (Proposition 4.1.11); so the distinguished triangle of Theorem 6.1.42 implies that the morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ induces an isomorphism of global functors

$$
\underline{\pi}_k(\mathbf{mO}_{(m)}) \cong \underline{\pi}_k(\mathbf{mO}_{(m+1)})
$$

for $k \leq m - 1$ and an exact sequence of global functors

$$
\mathbf{A}(O(m+1), -) \longrightarrow \underline{\pi}_m(\mathbf{mO}_{(m)}) \longrightarrow \underline{\pi}_m(\mathbf{mO}_{(m+1)}) \longrightarrow 0 \,.
$$

Here we used the isomorphisms

$$
\begin{aligned}
\mathbf{A}(O(m+1), -) &\xrightarrow[\cong]{\tau \mapsto \tau(e_{O(m+1)})} \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)) \\
&\xrightarrow[\cong]{- \wedge S^m} \underline{\pi}_m(\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m) \,.
\end{aligned}
$$

Since $\mathbf{mO}_{(0)}$ is isomorphic to the global sphere spectrum, which is globally connective, we conclude inductively that $\mathbf{mO}_{(m)}$ is globally connective for all $m \geq 0$, and that the inclusion $\mathbf{mO}_{(1)} \longrightarrow \mathbf{mO}_{(m)}$ induces an isomorphism on

$\underline{\pi}_0$ for all $m \geq 1$. Since **mO** is a colimit of the sequence of closed embeddings $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$, the map

$$\mathrm{colim}_m\ \underline{\pi}_k(\mathbf{mO}_{(m)}) \longrightarrow \underline{\pi}_k(\mathbf{mO})$$

induced by the morphisms $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$ is an isomorphism of global functors for every integer k. So **mO** is globally connective and the morphism $\psi^1 : \mathbf{mO}_{(1)} \longrightarrow \mathbf{mO}$ induces an isomorphism

$$\underline{\pi}_0(\mathbf{mO}_{(1)}) \cong \underline{\pi}_0(\mathbf{mO}) .$$

The unit morphism identifies $\mathbf{mO}_{(0)}$ with the global sphere spectrum, so the action on the class $1 \in \mathbf{mO}_{(0)}$ is an isomorphism of global functors $\mathbb{A} \cong \underline{\pi}_0(\mathbf{mO}_{(0)})$. For $m = 1$, the exact sequence thus becomes an exact sequence of global functors

$$\mathbf{A}(O(1), -) \xrightarrow{\mathbf{A}(\mathrm{tr}_e^{O(1)}, -)} \mathbf{A}(e, -) \longrightarrow \underline{\pi}_0(\mathbf{mO}_{(1)}) \longrightarrow 0 ;$$

here we used (6.1.43) to identify the first morphism as the one induced by the transfer $\mathrm{tr}_e^{O(1)}$. This proves the claim about $\underline{\pi}_0(\mathbf{mO})$. □

Theorem 6.1.44 gives a nice compact description of the 0th equivariant homotopy groups of the Thom spectrum **mO**, but we may still ask for a more explicit calculation of the group $\pi_0^G(\mathbf{mO})$ for an individual compact Lie group G. Given the presentation of $\underline{\pi}_0(\mathbf{mO})$ as the quotient of $\mathbb{A}$ by the global subfunctor generated by $\mathrm{tr}_e^{O(1)}$, this is a purely algebraic exercise. Since the element $\mathrm{tr}_e^{O(1)}(1)$ is trivial in the group $\pi_0^{O(1)}(\mathbf{mO})$, also

$$2 = \mathrm{res}_e^{O(1)}(\mathrm{tr}_e^{O(1)}(1)) = 0$$

in $\pi_0^e(\mathbf{mO})$; thus all equivariant homotopy groups $\pi_*^G(\mathbf{mO})$ are $\mathbb{F}_2$-vector spaces. The next proposition pins down an $\mathbb{F}_2$-basis of $\pi_0^G(\mathbf{mO})$ in terms of the subgroup structure of G. For a closed subgroup H of G we use the familiar notation

$$t_H^G = \mathrm{tr}_H^G(p_H^*(1)) \in \pi_0^G(\mathbf{mO}) ,$$

where $p_H : H \longrightarrow e$ is the unique homomorphism.

Proposition 6.1.45. *For every compact Lie group G, an $\mathbb{F}_2$-basis of $\pi_0^G(\mathbf{mO})$ is given by the classes t_H^G, indexed by conjugacy classes of those closed subgroups H of G whose Weyl group is finite and of odd order.*

Proof We abbreviate $C = O(1)$. The group $\mathbb{A}(G)$ is free abelian with basis the classes t_H^G for all conjugacy classes of subgroups H with finite Weyl group. So the claim follows if we can show that the value $\langle \mathrm{tr}_e^C \rangle(G)$ of the global functor

$\langle \mathrm{tr}_e^C \rangle$ at G is the subgroup of $\mathbb{A}(G)$ generated by $2 \cdot \mathbb{A}(G)$ and the classes t_H^G for those closed subgroups H whose Weyl group is finite of even order.

By Theorem 4.2.6, the group $\mathbf{A}(C,G)$ is freely generated by the elements $\mathrm{tr}_L^G \circ \alpha^*$ indexed by $(G \times C)$-conjugacy classes of pairs (L, α) where L is a closed subgroup of G with finite Weyl group and $\alpha : L \longrightarrow C$ is a continuous group homomorphism. So $\langle \mathrm{tr}_e^C \rangle(G)$ is generated as an abelian group by the elements $\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C$.

A homomorphism to C is either trivial or surjective, and the generating elements come in two flavors. If α is the trivial homomorphism, then

$$\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C \;=\; \mathrm{tr}_L^G \circ p_L^* \circ \mathrm{res}_e^C \circ \mathrm{tr}_e^C \;=\; 2 \cdot \mathrm{tr}_L^G \circ p_L^* \;=\; 2 \cdot t_L^G \ .$$

These elements generate the subgroup $2 \cdot \mathbb{A}(G)$. If α is surjective with kernel H, then

$$\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C \;=\; \mathrm{tr}_L^G \circ \mathrm{tr}_H^L \circ p_H^* \;=\; \mathrm{tr}_H^G \circ p_H^* \;=\; t_H^G \ .$$

The group H that arises in this way is normal of index 2 in L. So L is contained in the normalizer $N_G H$, and 2 divides $[N_G H : H]$. So the order of the Weyl group $W_G H$ is even. Conversely, if H is a subgroup of G with finite Weyl group of even order, then we can choose a subgroup $C \leq W_G H$ of order 2. The preimage L of C under the projection $N_G H \longrightarrow W_G H$ then contains H as an index 2 subgroup. By the above, the class t_H^G is then one of the generating elements of $\langle \mathrm{tr}_e^C \rangle(G)$. $\square$

Example 6.1.46 (Geometric fixed-points of **mO** and **mOP**)**.** We give a description of the geometric fixed-points of **mO** and **mOP**. The geometric fixed-points of other global Thom spectra such as $\mathbf{mO}_{(m)}$, **MO** or **MOP** can be worked out in a similar fashion, but we leave that to the interested reader. We let V be a representation of a compact Lie group G. We write $V^\perp = V - V^G$ for the orthogonal complement of the G-fixed subspace. A point $((v, x), U) \in \mathbf{mOP}(V) = Th(Gr(V \oplus \mathbb{R}^\infty))$ is G-fixed if and only if the subspace U of $V \oplus \mathbb{R}^\infty$ is G-invariant and the vector $v \in V$ is G-fixed. The first condition guarantees that $U = U^G \oplus (U \cap V^\perp)$, and U^G is a subspace of $V^G \oplus \mathbb{R}^\infty$. So we obtain a homeomorphism

$$\begin{aligned} \mathbf{mOP}(V^G) \wedge \mathbf{Gr}(V^\perp)^G &\xrightarrow{\cong} \mathbf{mOP}(V)^G \\ (x, U) \wedge W &\longmapsto (x,\ U \oplus (V^\perp - W)) \ . \end{aligned} \tag{6.1.47}$$

Under this identification, the structure map

$$(\sigma_{V,W})^G \ : \ S^{V^G} \wedge \mathbf{mOP}(W)^G \longrightarrow \mathbf{mOP}(V \oplus W)^G$$

becomes the smash product of the structure map

$$\sigma_{V^G,W^G} \ : \ S^{V^G} \wedge \mathbf{mOP}(W^G) \longrightarrow \mathbf{mOP}(V^G \oplus W^G)$$

with the map

$$\mathbf{Gr}(i)^G \; : \; \mathbf{Gr}(W^\perp)^G \longrightarrow \mathbf{Gr}(V^\perp \oplus W^\perp)^G$$

induced by the embedding $W^\perp \longrightarrow V^\perp \oplus W^\perp$ as the second summand. So in the colimit over $V \in s(\mathcal{U}_G)$ this gives an isomorphism

$$\Phi_*^G(\mathbf{mOP}) \cong \mathbf{mOP}_*\left(\mathbf{Gr}(\mathcal{U}_G^\perp)_+^G\right) \tag{6.1.48}$$

to the non-equivariant **mOP**-homology groups of the G-fixed-point space of $\mathbf{Gr}(\mathcal{U}_G^\perp)$, the disjoint union of all Grassmannians in $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$.

The formula (6.1.48) is a compact way to express the geometric fixed-points of **mOP**, but it can be decomposed and rewritten further, thereby making it more explicit. The orthogonal spectrum **mOP** is a $\mathbb{Z}$-indexed wedge of homogeneous summands $\mathbf{mOP}^{[k]}$. The space $\mathbf{Gr}(\mathcal{U}_G^\perp)$, and hence also its G-fixed-points, is the disjoint union indexed by the dimension of the subspaces, i.e.,

$$\mathbf{Gr}(\mathcal{U}_G^\perp)^G = \coprod_{j\geq 0} \left(Gr_j(\mathcal{U}_G^\perp)\right)^G = \coprod_{j\geq 0} Gr_j^{G,\perp} .$$

The two decompositions induce a direct sum decomposition of the right-hand side of the isomorphism (6.1.48) as

$$\begin{aligned} \mathbf{mOP}_*\left(\mathbf{Gr}(\mathcal{U}_G^\perp)_+^G\right) &= \bigoplus_{k\in\mathbb{Z}} \bigoplus_{j\geq 0} \mathbf{mOP}_*^{[k]}\left((Gr_j^{G,\perp})_+\right) \qquad (6.1.49)\\ &\cong \bigoplus_{k\in\mathbb{Z},\ j\geq 0} \mathbf{mO}_{*-k}\left((Gr_j^{G,\perp})_+\right) . \end{aligned}$$

The second isomorphism uses the periodicity of **mOP** to identify $\mathbf{mOP}^{[k]}$ with $\mathbf{mO} \wedge S^k$.

The condition $\dim(U \oplus (V^\perp - W)) = \dim(V)$ is equivalent to $\dim(U) = \dim(V^G) + \dim(W)$. So the homeomorphism (6.1.47) identifies $\mathbf{mO}(V)^G$ with the wedge of the spaces $\mathbf{mOP}^{[j]}(V^G) \wedge Gr_j(V^\perp)$ for $j \geq 0$. The composite of the isomorphism (6.1.48) and the isomorphism (6.1.49) thus takes the wedge summand of $\Phi_*^G(\mathbf{mOP})$ corresponding to $\mathbf{mO} = \mathbf{mOP}^{[0]}$ to the sum of the terms with $k = j$. So the isomorphisms restrict to an isomorphism

$$\Phi_*^G(\mathbf{mO}) \cong \bigoplus_{j\geq 0} \mathbf{mO}_{*-j}\left((Gr_j^{G,\perp})_+\right) . \tag{6.1.50}$$

The space $Gr_j^{G,\perp}$ can be decomposed further: every G-invariant subspace of $\mathcal{U}_G^\perp$ is the direct sum of its isotypical components, indexed by the non-trivial irreducible G-representations. The irreducibles come in three flavors (real, complex or quaternionic), and so the space $Gr_j^{G,\perp}$ is a disjoint union of products of classifying spaces of the groups $O(n)$, $U(n)$, and $Sp(n)$ for various n.

The reader may want to compare the previous description of the geometric

G-fixed-points of **mO** with Proposition 2.4.21 (i), which identifies the G-fixed-points of the orthogonal space **bO** as

$$\mathbf{bO}(\mathcal{U}_G)^G \ \simeq\ \coprod_{j\geq 0} BO \times Gr_j^{G,\perp}\ .$$

This illustrates the general phenomenon that 'geometric G-fixed-points of a global Thom spectrum are the Thom spectrum over the G-fixed-points'.

Example 6.1.51. The Thom spectrum **mO** has an oriented analog. For $m \geq 0$ we define an orthogonal spectrum $\mathbf{mSO}_{(m)}$ by

$$\mathbf{mSO}_{(m)} \ = \ \mathrm{sh}^m F_{SO(m),\nu_m}\ ,$$

the mth shift of the semifree orthogonal spectrum generated by the $SO(m)$-representation ν_m. In much the same way in Construction 6.1.36, $\mathbf{mSO}_{(m)}(V)$ 'is' (by passage to orthogonal complements) the Thom space over the tautological bundle over the oriented Grassmannian $Gr^+_{|V|}(V\oplus\mathbb{R}^m)$ of oriented $|V|$-planes in $V \oplus \mathbb{R}^m$.

We define a morphism

$$i\ :\ F_{SO(m),\nu_m} \ \longrightarrow\ \mathrm{sh}\, F_{SO(m+1),\nu_{m+1}}$$

at an inner product space V as the closed embedding

$$\begin{aligned} i(V)\ :\ \mathbf{O}(\nu_m, V)/SO(m) \ &\xrightarrow{-\oplus\mathbb{R}}\ \mathbf{O}(\nu_{m+1}, V\oplus\mathbb{R})/SO(m+1) \\ [x,\varphi] \qquad &\longmapsto \qquad [(x,0), \varphi\oplus\mathbb{R}]\ . \end{aligned}$$

Then we set

$$j^m \ = \ \mathrm{sh}^m i\ :\ \mathbf{mSO}_{(m)} = \mathrm{sh}^m F_{SO(m),\nu_m} \ \longrightarrow\ \mathrm{sh}^{m+1} F_{SO(m+1),\nu_{m+1}} \ = \ \mathbf{mSO}_{(m+1)}$$

and we define **mSO** as the colimit of the sequence of closed embeddings

$$\mathbf{mSO}_{(0)} \ \xrightarrow{j^0}\ \mathbf{mSO}_{(1)} \ \xrightarrow{j^1}\ \dots \ \longrightarrow\ \mathbf{mSO}_{(m)} \ \xrightarrow{j^m}\ \dots\ .$$

The orthogonal spectrum **mSO** supports inverse Thom classes for *oriented* representations, i.e., oriented inner product spaces V equipped with an orientation preserving isometric action of a compact Lie group G.

Now we mention the unitary analogs **mU** and **MU** of the Thom spectra **mO** and **MO**. Beside the complexification, there is an extra twist to the unitary definitions, because we need to 'loop by imaginary spheres' to really get orthogonal spectra. The unitary Thom spectra have periodic versions **mUP** and **MUP**, respectively, but we will not go into any details about those.

Example 6.1.52. We define an orthogonal spectrum **mU** in analogy with **mO**. Non-equivariantly, **mU** is the unitary Thom spectrum MU. The spectrum **mU**

is essentially a Thom spectrum over the orthogonal space **bU**, the complex analog of the E_∞-orthogonal monoid space **bO** discussed in Example 2.4.18. As before we denote by $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ the complexification of a real inner product space V. The value of **bU** at V is

$$\mathbf{bU}(V) \ = \ Gr^{\mathbb{C}}_{|V|}(V_{\mathbb{C}} \oplus \mathbb{C}^\infty) \ ,$$

the Grassmannian of $\mathbb{C}$-linear subspaces of $V_{\mathbb{C}} \oplus \mathbb{C}^\infty$ of the same dimension as V. The map $\mathbf{bU}(\varphi) : \mathbf{bU}(V) \longrightarrow \mathbf{bU}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is defined as

$$\mathbf{bU}(\varphi)(L) \ = \ (\varphi_{\mathbb{C}} \oplus \mathbb{C}^\infty)(L) \ + \ ((W_{\mathbb{C}} - \varphi_{\mathbb{C}}(V_{\mathbb{C}})) \oplus 0) \ .$$

In other words: we apply the linear isometric embedding $\varphi_{\mathbb{C}} \oplus \mathbb{C}^\infty : V_{\mathbb{C}} \oplus \mathbb{C}^\infty \longrightarrow W_{\mathbb{C}} \oplus \mathbb{C}^\infty$ to the subspace L and add the orthogonal complement of the image of $\varphi_{\mathbb{C}}$ (sitting in the first summand of $W_{\mathbb{C}} \oplus \mathbb{C}^\infty$).

We let iV denote the imaginary subspace of $V_{\mathbb{C}}$, i.e., the $\mathbb{R}$-subspace consisting of the elements $i \otimes v$ for $v \in V$. Over the space $\mathbf{bU}(V)$ sits a tautological hermitian vector bundle with total space consisting of the pairs $(x, U) \in (V_{\mathbb{C}} \oplus \mathbb{C}^\infty) \times \mathbf{bU}(V)$ such that $x \in U$. We define $\mathbf{mU}(V)$ as

$$\mathbf{mU}(V) \ = \ \mathrm{map}_*(S^{iV}, Th(\mathbf{bU}(V))) \ ,$$

the space of based maps from the imaginary sphere S^{iV} to the Thom space of this tautological vector bundle. The structure map of the spectrum **mU** starts from the $(O(V) \times O(W))$-equivariant map

$$\begin{aligned} S^{V_{\mathbb{C}}} \wedge Th(\mathbf{bU}(W)) \ &\longrightarrow \quad Th(\mathbf{bU}(V \oplus W)) \\ v \wedge (x, U) \qquad &\longmapsto \ ((v, x), \ U \oplus (0 \oplus W_{\mathbb{C}} \oplus 0)) \ . \end{aligned}$$

The structure map

$$\sigma_{V,W} \ : \ S^V \wedge \mathbf{mU}(W) \ \longrightarrow \ \mathbf{mU}(V \oplus W)$$

is then adjoint to the composite

$$\begin{aligned} &S^V \wedge \mathrm{map}_*(S^{iW}, Th(\mathbf{bU}(W))) \wedge S^{i(V\oplus W)} \\ &\qquad\qquad \cong \quad S^{V_{\mathbb{C}}} \wedge \mathrm{map}_*(S^{iW}, Th(\mathbf{bU}(W))) \wedge S^{iW} \\ &\qquad\qquad \xrightarrow{S^{V_{\mathbb{C}}} \wedge \mathrm{ev}} S^{V_{\mathbb{C}}} \wedge Th(\mathbf{bU}(W)) \ \longrightarrow \ Th(\mathbf{bU}(V \oplus W)) \ . \end{aligned}$$

Looping by S^{iV} is essential for obtaining an orthogonal spectrum; without it, we would end up with a structure one may call a 'unitary spectrum'.

Most of our results about **mO** have analogues for **mU**. The orthogonal spec-

trum **mU** comes with an E_∞-structure, which is, however, *not* ultra-commutative. There are unitary versions of the shifted inverse Thom classes

$$\bar{\tau}^U_{G,V} \in \mathbf{mU}^G_{2n}(S^V) ,$$

in the G-equivariant **mU**-homology groups of S^V, defined for *unitary* representations V of a compact Lie group G. Here n is the complex dimension of V, so that $2n$ is its real dimension. The orthogonal spectrum **mU** is the union of a sequence of orthogonal subspectra

$$\mathbf{mU}_{(0)} \subset \mathbf{mU}_{(1)} \subset \ldots \subset \mathbf{mU}_{(m)} \subset \ldots$$

and **mU** is also the global homotopy colimit of this sequence. The unit morphism is a global equivalence $\mathbb{S} \simeq \mathbf{mU}_{(0)}$ and $\mathbf{mU}_{(m)}$ is globally equivalent to the $2m$th suspension of the semifree orthogonal spectrum generated by the tautological unitary representation ν^U_m of $U(m)$ on $\mathbb{C}^m$:

$$\mathbf{mU}_{(m)} \simeq F_{U(m),\nu^U_m} \wedge S^{2m}$$

There are distinguished triangles in the global stable homotopy category:

$$\Sigma^\infty_+ B_{\mathrm{gl}} U(m+1) \wedge S^{2m+1} \longrightarrow \mathbf{mU}_{(m)} \longrightarrow \mathbf{mU}_{(m+1)} \longrightarrow \Sigma^\infty_+ B_{\mathrm{gl}} U(m+1) \wedge S^{2m+2}$$

and the first map is classified by the $U(m+1)$-equivariant homotopy class

$$\mathrm{Tr}^{U(m+1)}_{U(m)}(\tau_{U(m),\nu^U_m} \wedge S^1) \qquad \text{in} \quad \pi^{U(m+1)}_{2m+1}(\mathbf{mU}_{(m)}).$$

This uses that the tangent $U(m)$-representation of $U(m+1)/U(m)$ is isomorphic to $\nu^U_m \oplus \mathbb{R}$. So loosely speaking, $\mathbf{mU}_{(m+1)}$ is obtained from $\mathbf{mU}_{(m)}$ by coning off this transfer class. A consequence is then that all $\mathbf{mU}_{(m)}$ and **mU** are globally connective. Since the first unitary relation $\mathrm{Tr}^{U(1)}_e(1 \wedge S^1) = 0$ in $\pi^{U(1)}_1(\mathbf{mU})$ lives in a positive dimension, the description of the global functor $\underline{\pi}_0(\mathbf{mU})$ is easier than the corresponding calculation of $\underline{\pi}_0(\mathbf{mO})$ in Theorem 6.1.44. Indeed, the action of the Burnside ring global functor on the element $1 \in \pi^e_0(\mathbf{mU})$ induces an isomorphism of global functors

$$\mathbb{A} \cong \underline{\pi}_0(\mathbf{mU}) .$$

Moreover, there is an exact sequence of global functors

$$\mathbf{A}(U(1), -) \longrightarrow \underline{\pi}_1(\mathbb{S}) \longrightarrow \underline{\pi}_1(\mathbf{mU}) \longrightarrow 0 ,$$

where the first morphism is the action on the class $\mathrm{Tr}^{U(1)}_e(1 \wedge S^1)$ in $\pi^{U(1)}_1(\mathbb{S})$.

Corollary 6.1.40 has a unitary analog that describes morphisms in the global stable homotopy category from **mU**: for every orthogonal spectrum E the sequence

$$0 \longrightarrow \lim_m{}^1 E^{U(m)}_{2m+1}(S^{\nu^U_m}) \longrightarrow [\![\mathbf{mU}, E]\!] \longrightarrow \lim_m E^{U(m)}_{2m}(S^{\nu^U_m}) \longrightarrow 0$$

is short exact. The right map is evaluation at the unitary shifted inverse Thom classes, and the inverse and derived limits are formed along the maps

$$E^{U(m+1)}_{2m+2}(S^{\nu^U_{m+1}}) \xrightarrow{\mathrm{res}^{U(m+1)}_{U(m)}} E^{U(m)}_{2m+2}(S^{\nu^U_m} \wedge S^2) \xrightarrow{(-\wedge S^2)^{-1}} E^{U(m)}_{2m}(S^{\nu^U_m}) .$$

We leave it to the interested reader to formulate the analogous properties of the rank filtrations for the special unitary and symplectic global Thom spectra **mSU** and **mSp**.

Example 6.1.53. We define another unitary global Thom spectrum **MU**, an ultra-commutative ring spectrum and the unitary analog of **MO**. Non-equivariantly, **MU** is another version of the complex bordism spectrum MU; for a compact Lie group G, the orthogonal G-spectrum $\mathbf{MU}_G$ is a model for tom Dieck's homotopical equivariant bordism [175]. Closely related, strictly commutative ring spectrum models for these homotopy types have been discussed in various places, for example [112], [68, Ex. 5.8], [166, App. A] or [31, Sec. 8].

For an inner product space V we consider the complex Grassmannian

$$\mathbf{BU}(V) \;=\; Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}}) .$$

Over the space $\mathbf{BU}(V)$ sits a tautological hermitian vector bundle and we set

$$\mathbf{MU}(V) \;=\; \mathrm{map}_*(S^{iV}, Th(\mathbf{BU}(V))) ,$$

the iV-loop space of the Thom space of this tautological vector bundle. The structure maps are defined in a similar way as for **mU**, and a commutative multiplication is given by

$$\mu_{V,W} \;:\; \mathbf{MU}(V) \wedge \mathbf{MU}(W) \;\longrightarrow\; \mathbf{MU}(V \oplus W) , \quad f \wedge g \;\longmapsto\; f \cdot g .$$

Here $f : S^{iV} \longrightarrow Th(\mathbf{BU}(V))$, $g : S^{iW} \longrightarrow Th(\mathbf{BU}(W))$, and $f \cdot g$ denotes the composite

$$S^{iV} \wedge S^{iW} \xrightarrow{f\wedge g} Th(\mathbf{BU}(V)) \wedge Th(\mathbf{BU}(W))$$
$$\xrightarrow{(x,U)\wedge(x',U')\mapsto(\kappa_{V,W}(x,x'),\kappa_{V,W}(U\oplus U'))} Th(\mathbf{BU}(V \oplus W)) ,$$

where $\kappa_{V,W}$ is the preferred isometry from $V^2_{\mathbb{C}} \oplus W^2_{\mathbb{C}} \cong (V \oplus W)^2_{\mathbb{C}}$ sending (v, v', w, w') to (v, w, v', w'). Unit maps are defined by

$$S^V \;\longrightarrow\; \mathbf{MU}(V) , \quad v \;\longmapsto\; [v' \mapsto ((v + v', 0),\ V_{\mathbb{C}} \oplus 0)] .$$

These multiplication maps unital, associative and commutative, and make **MU** an ultra-commutative ring spectrum.

The global Thom spectrum **MU** comes with distinguished Thom classes

$$\sigma^U_{G,V} \;\in\; \mathbf{MU}^{2n}_G(S^V) ,$$

for *unitary* representations V of compact Lie groups G, where n is the complex dimension of V. As in the orthogonal situation in Theorem 6.1.17 the Thom class $\sigma^U_{G,V}$ is inverse to the image of the inverse Thom class $\tau^U_{G,V}$, and $\sigma^U_{G,V}$ restricts to a unitary Euler class in $\pi^G_{-2n}(\mathbf{MU})$. The proof of Corollary 6.1.35 generalizes to the unitary situation and proves that the G-equivariant homology represented by **MU** is the localization at the unitary inverse Thom classes of the theory represented by **mU**.

Some general facts about the equivariant homotopy groups of **MU** are known for *abelian* compact Lie groups A. In that case, $\pi^A_*(\mathbf{MU})$ is a free module on even dimensional generators over the non-equivariant homotopy ring $\pi^e_*(\mathbf{MU})$; this calculation was announced by Löffler in [104], and a proof by Comezaña can be found in [113, XXVIII Thm. 5.3]. Since the graded ring $\pi^e_*(\mathbf{MU})$ is concentrated in even degrees and the periodic version **MUP** is a wedge of even suspensions of **MU**, the groups $\pi^A_*(\mathbf{MUP})$ are concentrated in even degrees. Since these groups are also 2-periodic, for abelian compact Lie groups all the information is concentrated in the ring $\pi^A_0(\mathbf{MUP})$. The non-equivariant homotopy groups $\pi^e_0(\mathbf{MUP})$ are a polynomial ring in countably many generators. For cyclic groups of prime order, Kriz [93] has described $\pi^{C_p}_0(\mathbf{MUP})$ as a pullback of two explicit ring homomorphisms. For the cyclic group of order 2, Strickland [165] has turned this into an explicit presentation of $\pi^{C_2}_0(\mathbf{MUP})$ as an algebra over $\pi^e_0(\mathbf{MUP})$.

6.2 Equivariant bordism

In this section we recall equivariant bordism groups and their relationship to the equivariant homology groups defined by the global Thom spectrum **mO** introduced in Example 6.1.24. The main result is Theorem 6.2.33 which says that when G is isomorphic to a product of a finite group and a torus, then the Thom-Pontryagin map is an isomorphism from G-equivariant bordism to G-equivariant **mO**-homology. Theorem 6.2.33 is usually credited to Wasserman because it can be derived from his equivariant transversality theorem [184, Thm. 3.11]; as far as I know, the only place where the translation is spelled out in detail is the unpublished part of Costenoble's thesis, see [40, Thm. 11.1]. Wasserman's theorem is based on equivariant differential topology; for finite groups, tom Dieck [177, Satz 5] gives a different proof of Theorem 6.2.33 based on the geometric and homotopy theoretic isotropy separation sequences. We generalize tom Dieck's proof, translated into our present language, with an emphasis on global aspects.

In Theorem 6.2.37 we also present a localized version of this result: the Thom-Pontryagin map is an isomorphism from *stable* equivariant bordism to

$\mathbf{mO}[1/\tau]$-theory, without any restriction on the compact Lie group. Given that $\mathbf{mO}[1/\tau]$-theory is isomorphic to $\mathbf{MO}$-theory (by Corollary 6.1.35), this is equivalent to a result of Bröcker and Hook [27, Thm. 4.1] that identifies stable equivariant bordism with equivariant $\mathbf{MO}$-homology.

The serious study of equivariant bordism groups was initiated by the work [39] of Conner and Floyd. We recall equivariant bordism as a homology theory for G-spaces X, where G is a compact Lie group. A *singular G-manifold* over X is a pair (M, h) consisting of a closed smooth G-manifold M and a continuous G-map $h : M \longrightarrow X$. Two singular G-manifolds (M, h) and (M', h') are *bordant* if there is a triple (B, H, ψ) consisting of a compact smooth G-manifold B, a continuous G-map $H : B \longrightarrow X$ and an equivariant diffeomorphism

$$\psi \ : \ M \cup M' \ \cong \ \partial B$$

such that $(H \circ \psi)|_M = h$ and to $(H \circ \psi)|_{M'} = h'$.

Bordism of singular G-manifolds over X is an equivalence relation. Reflexivity and symmetry are straightforward; transitivity is established by gluing two bordisms along a common piece of the boundary. To get a smooth structure on the glued bordism that is compatible with the G-action one needs smooth equivariant collars; the existence of such collars is guaranteed by [39, Thm. 21.2].

We denote by $\mathcal{N}_n^G(X)$ the set of bordism classes of n-dimensional singular G-manifolds over X. This set becomes an abelian group under disjoint union. Every element x of $\mathcal{N}_n^G(X)$ satisfies $2x = 0$. The groups $\mathcal{N}_n^G(X)$ are covariantly functorial in continuous G-maps, by post-composition.

Proposition 6.2.1. *Let G be a compact Lie group.*

(i) *Let $\varphi, \varphi' : X \longrightarrow Y$ be equivariantly homotopic continuous G-maps. Then $\varphi_* = \varphi'_*$ as homomorphisms from $\mathcal{N}_n^G(X)$ to $\mathcal{N}_n^G(Y)$.*

(ii) *For every G-weak equivalence $\varphi : X \longrightarrow Y$ the induced homomorphism $\varphi_* : \mathcal{N}_n^G(X) \longrightarrow \mathcal{N}_n^G(Y)$ is an isomorphism.*

(iii) *Let $\{X_i\}_{i \in I}$ be a family of G-spaces. Then the canonical map*

$$\bigoplus\nolimits_{i\in I} \mathcal{N}_n^G(X_i) \ \longrightarrow \ \mathcal{N}_n^G\left(\coprod\nolimits_{i\in I} X_i\right)$$

is an isomorphism.

Proof (i) We let $H : X \times [0, 1] \longrightarrow Y$ be an equivariant homotopy from φ to φ' and (M, h) a singular G-manifold over X. Then $(M \times [0, 1], H \circ (h \times [0, 1]), \psi)$ is a bordism from $(M, \varphi h)$ to $(M, \varphi' h)$, where $\psi \ : \ M \cup M \ \longrightarrow \ \partial(M \times [0, 1])$ identifies one copy of M with $M \times \{0\}$ and the other copy with $M \times \{1\}$. So $\varphi_*[M, h] = [M, \varphi \circ h] = [M, \varphi' \circ h] = \varphi'_*[M, h]$.

(ii) For surjectivity of φ_* we consider any singular G-manifold (M, g) over Y. Illman's theorem [84, Cor. 7.2] shows that M admits the structure of a G-CW-complex. Since φ is a G-weak equivalence there exists a continuous G-map $h : M \longrightarrow X$ such that φh is equivariantly homotopic to g. Part (i) then shows that $\varphi_*[M, h] = (\varphi h)_*[M, \mathrm{Id}_M] = g_*[M, \mathrm{Id}_M] = [M, g]$.

The argument for injectivity is similar. We consider a singular G-manifold (M, h) over X that represents an element in the kernel of φ_*. There is then a null-bordism (B, H, ψ) of $(M, \varphi h)$. By Illman's theorem [84, Cor. 7.2] and the discussion immediately preceding it there is a G-CW-structure on B for which the boundary is an equivariant subcomplex. So the map $\psi : M \longrightarrow B$ that identifies M with the boundary of B is a G-cofibration. Since φ is a G-weak equivalence, there exists a continuous G-map $H' : B \longrightarrow X$ such that $H' \circ \psi = h$. The triple (B, H', ψ) thus witnesses that $[M, h] = 0$. Since φ_* is a group homomorphism, it is injective.

Property (iii) holds because compact manifolds only have finitely many connected components, so all continuous reference maps from singular manifolds or bordisms have image in a finite union. □

Now we state the key exactness property of equivariant bordism in the form of a Mayer-Vietoris sequence. The definition of the boundary map needs the existence of G-invariant separating functions as provided by the following lemma.

Lemma 6.2.2. *Let M be a compact smooth G-manifold, C and C' two disjoint, closed, G-invariant subsets of M, and*

$$s : \partial M \longrightarrow \mathbb{R}$$

a smooth G-invariant map such that

$$C \cap \partial M \subseteq s^{-1}(0) \qquad \textit{and} \qquad C' \cap \partial M \subseteq s^{-1}(1) .$$

Then there exists a smooth G-invariant extension $r : M \longrightarrow \mathbb{R}$ of s such that $C \subseteq r^{-1}(0)$ and $C' \subseteq r^{-1}(1)$.

Proof Since M is compact, hence normal, the Tietze extension theorem provides a continuous map $r_0 : M \longrightarrow \mathbb{R}$ that extends s and satisfies $C \subseteq r_0^{-1}(0)$ and $C' \subseteq r_0^{-1}(1)$. We average r_0 to make it G-invariant, i.e., we define $r_1 : M \longrightarrow \mathbb{R}$ by

$$r_1(x) = \int_G r_0(g \cdot x)\, dg .$$

The integral is taken with respect to the normalized invariant measure (Haar measure) on G. The new map r_1 is again continuous, compare [26, Ch. 0 Prop. 3.2]. Since r_0 is already G-invariant on $C \cup C' \cup \partial M$, the new map r_1 coincides

with r_0 on this subset. In particular, r_1 is smooth on $C \cup C' \cup \partial M$. By [26, VI Thm. 4.2] we can then find a smooth G-invariant map $r : M \longrightarrow \mathbb{R}$ that coincides with r_1 on $C \cup C' \cup \partial M$; this is the desired separating function. □

Construction 6.2.3 (Boundary map in equivariant bordism)**.** We define a boundary homomorphism for a Mayer-Vietoris sequence. We let X be a G-space and $A, B \subset X$ open G-invariant subsets with $X = A \cup B$. Then a homomorphism

$$\partial \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \mathcal{N}_{n-1}^G(A \cap B)$$

is defined as follows.

We let (M, h) be a singular G-manifold that represents a class in $\mathcal{N}_n^G(X)$. The sets $h^{-1}(X - A)$ and $h^{-1}(X - B)$ are G-invariant, disjoint closed subsets of M; we let $r : M \longrightarrow \mathbb{R}$ be a G-invariant smooth separating function as provided by Lemma 6.2.2, i.e., such that $h^{-1}(X - A) \subseteq r^{-1}(0)$ and $h^{-1}(X - B) \subseteq r^{-1}(1)$. We let $t \in (0, 1)$ be any regular value of r. Then

$$M_t \ = \ r^{-1}(t)$$

is a smooth closed G-submanifold of M of dimension $n - 1$ (possibly empty), and $h_t = h|_{M_t}$ lands in $A \cap B$; so (M_t, h_t) is a singular G-manifold over $A \cap B$.

Proposition 6.2.4. *In the situation above, the bordism class $[M_t, h_t]$ is independent of the choice of regular value t, of the choice of separating function and of the representative for the given class in $\mathcal{N}_n^G(X)$. The resulting map*

$$\partial \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \mathcal{N}_{n-1}^G(A \cap B) \ , \quad [M, h] \ \longmapsto \ [M_t, h_t]$$

is a group homomorphism.

Proof We let $t < t'$ be two regular values in $(0, 1)$ of the separating function r. Then

$$(r^{-1}[t, t'],\ h|_{r^{-1}[t,t']},\ \text{incl})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(r^{-1}(t'), h|_{r^{-1}(t')})$, so the bordism class does not depend on the regular value.

Now we let (M, h) and (N, g) be two singular G-manifolds over X in the same bordism class, and we let (B, H, ψ) be a G-bordism from (M, h) to (N, g). We let $r : M \longrightarrow \mathbb{R}$ and $\bar{r} : N \longrightarrow \mathbb{R}$ be two G-invariant separating functions. Lemma 6.2.2 lets us extend this data to a smooth G-invariant separating function

$$\Psi \ : \ B \ \longrightarrow \ \mathbb{R}$$

such that $\Psi \circ \psi|_M = r$, $\Psi \circ \psi|_N = \bar{r}$,

$$H^{-1}(X - A) \subseteq \Psi^{-1}(0) \quad \text{and} \quad H^{-1}(X - B) \subseteq \Psi^{-1}(1) \ .$$

We choose a simultaneous regular value $t \in (0, 1)$ for Ψ, r and $\bar{r}$. Then

$$(\Psi^{-1}(t),\ H|_{\Psi^{-1}(t)},\ \psi|_{r^{-1}(t)\cup\bar{r}^{-1}(t)})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(\bar{r}^{-1}(t), g|_{\bar{r}^{-1}(t)})$. This shows at the same time that the bordism class is independent of the choice of separating function and of the choice of representing singular G-manifold. Additivity of the resulting boundary map is then clear: a separating function for a disjoint union can be taken as the union of separating functions for each summand. □

Now we formulate the property that makes equivariant bordism a homology theory for G-spaces.

Proposition 6.2.5. *Let G be a compact Lie group, X a G-space and $A, B \subset X$ open G-invariant subsets with $X = A\cup B$. Let $i^A : A\cap B \longrightarrow A$, $i^B : A\cap B \longrightarrow B$, $j^A : A \longrightarrow X$ and $j^B : B \longrightarrow X$ denote the inclusions. Then the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n^G(A\cap B) \xrightarrow{(i^A_*, i^B_*)} \mathcal{N}_n^G(A)\oplus\mathcal{N}_n^G(B) \xrightarrow{\binom{j^A_*}{-j^B_*}} \mathcal{N}_n^G(X) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(A\cap B) \longrightarrow \ldots$$

While Proposition 6.2.5 has been frequently used, there does not seem to be any published account with a complete proof in the generality of compact Lie groups; the argument is somewhat involved, but it proceeds along the lines of the non-equivariant argument as given for example in [180, Prop. 21.1.7]. As input one needs that certain basic tools from differential topology generalize to the G-equivariant context, such as for example the existence of equivariant collars and bicollars, and that non-equivariant smoothing of corners is compatible with G-actions. We honor the tradition in this area and refrain from giving further details.

We define the *reduced bordism group* of a based G-space X as

$$\widetilde{\mathcal{N}}_n^G(X) \ = \ \mathrm{coker}\left(\mathcal{N}_n^G(*) \longrightarrow \mathcal{N}_n^G(X)\right) ,$$

the cokernel of the homomorphism induced by the basepoint inclusion. If (M, h) is a singular G-manifold over X, then we use the notation $[\![M, h]\!]$ for the class it represents in the reduced bordism group $\widetilde{\mathcal{N}}_n^G(X)$. The unique map $u : X \longrightarrow *$ is retraction to the basepoint inclusion, so the map

$$(\mathrm{proj}, u_*) \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \widetilde{\mathcal{N}}_n^G(X) \oplus \mathcal{N}_n^G(*)$$

is an isomorphism. On the other hand, if we add a disjoint G-fixed basepoint to an unbased G-space Y, then the composite

$$\mathcal{N}_n^G(Y) \xrightarrow{\mathrm{incl}_*} \mathcal{N}_n^G(Y_+) \xrightarrow{\mathrm{proj}} \widetilde{\mathcal{N}}_n^G(Y_+)$$

is an isomorphism.

Construction 6.2.6. We consider a continuous G-map $f : X \longrightarrow Y$ and let

$$Cf \;=\; CX \cup_f Y \;=\; (X \times [0,1] \cup_f Y)/X \times \{0\}$$

denote its unreduced mapping cone. The two open sets

$$A \;=\; X \times [0,1)/X \times \{0\} \qquad \text{and} \qquad B \;=\; X \times (0,1] \cup_f Y$$

are G-invariant and together cover the mapping cone. The intersection $A \cap B$ is homeomorphic to $X \times (0,1)$, so the open covering has an associated boundary homomorphism

$$\partial \;:\; \mathcal{N}_n^G(Cf) \;\longrightarrow\; \mathcal{N}_{n-1}^G(X \times (0,1))$$

as in Construction 6.2.3. We take the cone point as the basepoint of Cf; this is contained in the subset A, so the map $\iota : \mathcal{N}_*(*) \longrightarrow \mathcal{N}_*(Cf)$ induced by the basepoint inclusion factors through $j_*^A : \mathcal{N}_n^G(A) \longrightarrow \mathcal{N}_n^G(Cf)$, and the composite $\partial \circ \iota$ is trivial by exactness of the excision sequence. The boundary map thus factors over the reduced bordism group. We define a 'reduced boundary map' $\bar{\partial}$ as the composite

$$\widetilde{\mathcal{N}}_n^G(Cf) \xrightarrow{\;\partial\;} \mathcal{N}_{n-1}^G(X \times (0,1)) \xrightarrow{\;\text{proj}_*\;} \mathcal{N}_{n-1}^G(X)\;.$$

Proposition 6.2.7. *Let G be a compact Lie group and $f : X \longrightarrow Y$ a continuous G-map. Then the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n^G(X) \xrightarrow{\;f_*\;} \mathcal{N}_n^G(Y) \xrightarrow{\;i_*\;} \widetilde{\mathcal{N}}_n^G(Cf) \xrightarrow{\;\bar{\partial}\;} \mathcal{N}_{n-1}^G(X) \longrightarrow \ldots$$

Proof We use the open covering of the mapping cone Cf as in the definition of the boundary map. In the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x \mapsto (x,1/2)"'] & Y \arrow[r, "i"] \arrow[d, "i"'] & Cf \arrow[d, equal] \\
A \cap B \arrow[r, "\text{incl}"'] & B \arrow[r, "\text{incl}"'] & Cf
\end{tikzcd}$$

the right square commutes and the left square commutes up to equivariant homotopy. Moreover, all vertical maps are equivariant homotopy equivalences, so they induce isomorphisms in equivariant bordism, by Proposition 6.2.1. So the resulting diagram of bordism groups commutes:

$$\begin{tikzcd}
\mathcal{N}_n^G(X) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathcal{N}_n^G(Y) \arrow[r, "i_*"] \arrow[d, "\cong"', "i_*"] & \mathcal{N}_n^G(Cf) \arrow[d, equal] \\
\mathcal{N}_n^G(A \cap B) \arrow[r] & \mathcal{N}_n^G(B) \arrow[r] & \mathcal{N}_n^G(Cf)
\end{tikzcd}$$

Moreover, all vertical maps in this diagram are isomorphisms, so we can substitute $\mathcal{N}_n^G(X)$ and $\mathcal{N}_n^G(Y)$ into the long exact excision sequence of Proposition 6.2.5. Since A is equivariantly contractible to the cone point, we can also replace the corresponding summand by the coefficient group, and the result is an exact sequence

$$\ldots \longrightarrow \mathcal{N}_n^G(X) \xrightarrow{(f_*,u_*)} \mathcal{N}_n^G(Y) \oplus \mathcal{N}_n^G(*) \longrightarrow \mathcal{N}_n^G(Cf) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(X) \longrightarrow \ldots$$

The sequence then remains exact if we divide out the summand $\mathcal{N}_n^G(*)$ and replace the absolute bordism group of Cf by the reduced group $\widetilde{\mathcal{N}}_n^G(Cf)$. □

If $f : A \longrightarrow B$ is an h-cofibration of G-spaces, then the projection $q : Cf \longrightarrow B/A$ from the mapping cone to the quotient is a based equivariant homotopy equivalence. So we can substitute $\widetilde{\mathcal{N}}_*^G(B/A)$ into the long exact mapping cone sequence of Proposition 6.2.7 and obtain a long exact sequence of abelian groups:

$$\ldots \longrightarrow \mathcal{N}_n^G(A) \xrightarrow{f_*} \mathcal{N}_n^G(B) \xrightarrow{q_*} \widetilde{\mathcal{N}}_n^G(B/A) \longrightarrow \mathcal{N}_{n-1}^G(A) \longrightarrow \ldots$$

The next proposition says, loosely speaking, that in a reduced bordism group the part of a singular G-manifold that sits over the basepoint can be ignored.

Proposition 6.2.8. *Let G be a compact Lie group and $h : N \longrightarrow X$ a singular G-manifold over a based G-space X. Let V be a G-representation and M a closed smooth G-manifold such that* $\dim(M) + \dim(V) = \dim(N)$. *Let $j : M \times D(V) \longrightarrow N$ be a smooth G-equivariant embedding. Suppose that h sends $N - j(M \times \mathring{D}(V))$ to the basepoint. Then*

$$[\![N, h]\!] = [\![M \times S(\mathbb{R} \oplus V), f]\!] \quad \text{in} \quad \widetilde{\mathcal{N}}_n^G(X),$$

where $f : M \times S(\mathbb{R} \oplus V) \longrightarrow X$ is defined by

$$f(m, (x, v)) = \begin{cases} h(j(m, v)) & \text{if } x \leq 0, \text{ and} \\ * & \text{if } x \geq 0. \end{cases}$$

Proof We define a G-space by

$$B = (N \times [-1, 0] \cup N \times [0, 1])/\sim,$$

where the equivalence relation identifies the two copies of $(n, 0)$ for every $n \in N - j(M \times \mathring{D}(V))$. The group G acts by $g \cdot [n, t] = [gn, t]$. The space B is a topological $(n + 1)$-manifold whose boundary consists of three disjoint parts that we now parametrize. There are two obvious embeddings

$$\psi, \psi' : N \longrightarrow B \quad \text{by} \quad \psi(n) = [n, -1] \quad \text{and} \quad \psi'(n) = [n, 1]$$

as the two endpoints in the direction of the internal $[-1, 1]$. We define another embedding

$$i \ : \ M \times S(\mathbb{R} \oplus V) \ \longrightarrow \ B \qquad \text{by}$$

$$i(m,(x,v)) \ = \ \begin{cases} [(j(m,v),0]^{\text{left}} & \text{for } x \le 0, \text{ and} \\ [(j(m,v),0]^{\text{right}} & \text{for } x \ge 0. \end{cases}$$

Here the superscripts 'left' and 'right' indicate whether we refer to the point $[n, 0]$ as the image of $(n, 0)$ in $N\times[-1, 0]$ or in $N\times[0, 1]$. The manifold boundary of B is then the disjoint union of the images of ψ, ψ' and i.

The topological manifold B admits a smooth structure for which the given G-action is smooth and such that the embeddings ψ, ψ' and i are smooth; the construction involves 'smoothing of corners' (also called 'straightening of angles') near the image of $j(M \times S(V)) \times 0$ and is explained for example in Construction 15.10.3 of tom Dieck's textbook [180]. Tom Dieck has no group actions around, but we also have to ensure that the given G-action on B is smooth. This can be arranged by insisting that the collars used in [180, 15.10.3] are G-equivariant collars, which is possible for example by [39, Thm. 21.2].

Now we define a continuous G-map $H : B \longrightarrow X$ by $H(n,t) = h(n)$ on $N \times [-1, 0]$ and as the constant map to the base point of X on $N \times [0, 1]$. Then

$$H \circ \psi \ = \ h \qquad \text{and} \qquad H \circ i \ = \ f \ ,$$

so the bordism $(B, H, \psi + \psi' + i)$ witnesses the relation

$$[N,h] \ = \ [N, H \circ \psi'] \ + \ [M \times S(\mathbb{R} \oplus V), f]$$

in the unreduced bordism group $\mathcal{N}_n^G(X)$. Since $H \circ \psi'$ is constant to the basepoint of X, the class $[N, H \circ \psi']$ vanishes in the reduced bordism group $\widetilde{\mathcal{N}}_n^G(X)$; this proves the claim. □

The equivariant bordism groups come with natural products, given by the biadditive maps

$$\times \ : \ \mathcal{N}_m^G(X) \times \mathcal{N}_n^G(Y) \ \longrightarrow \ \mathcal{N}_{m+n}^G(X \times Y) \, , \quad [M,h] \times [N,g] \ = \ [M \times N, h \times g] \, .$$

These products are suitably associative, commutative and unital. The product pairing descends to a pairing on reduced bordism groups if the G-spaces X and Y are based. Indeed, the composite

$$\mathcal{N}_m^G(X) \otimes \mathcal{N}_n^G(Y) \ \xrightarrow{\ \times\ } \ \mathcal{N}_{m+n}^G(X \times Y) \ \xrightarrow{\ q_*\ } \ \mathcal{N}_{m+n}^G(X \wedge Y) \ \xrightarrow{\ \text{proj}\ } \ \widetilde{\mathcal{N}}_{m+n}^G(X \wedge Y)$$

annihilates the image of $\mathcal{N}_m^G(*) \otimes \mathcal{N}_n^G(Y)$ and the image of $\mathcal{N}_m^G(X) \otimes \mathcal{N}_n^G(*)$, where $q : X \times Y \longrightarrow X \wedge Y$ is the quotient map; so the composite factors

uniquely over a homomorphism

$$\wedge \; : \; \widetilde{\mathcal{N}}^G_m(X) \otimes \widetilde{\mathcal{N}}^G_n(Y) \longrightarrow \widetilde{\mathcal{N}}^G_{m+n}(X \wedge Y) .$$

We will frequently use certain distinguished bordism classes associated with representations. We let V be an m-dimensional representation of a compact Lie group G. Stereographic projection is a G-equivariant homeomorphism

$$\Pi_V \; : \; S(\mathbb{R} \oplus V) \xrightarrow{\cong} S^V , \quad (x, v) \longmapsto \frac{v}{1 - x}$$

from the unit sphere of $\mathbb{R} \oplus V$ to the one-point compactification S^V. We define a reduced G-bordism class over S^V by

$$d_{G,V} \; = \; [\![S(\mathbb{R} \oplus V), \Pi_V]\!] \quad \in \widetilde{\mathcal{N}}^G_m(S^V) . \tag{6.2.9}$$

The following multiplicativity property is expected, but not entirely obvious.

Proposition 6.2.10. *Let V and W be two representations of a compact Lie group G. Then the relation*

$$d_{G,V} \wedge d_{G,W} \; = \; d_{G,V\oplus W}$$

holds in $\widetilde{\mathcal{N}}^G_{m+n}(S^{V\oplus W})$.

Proof We define a 'distorted' version $\tau_V : S(\mathbb{R} \oplus V) \longrightarrow S^V$ of the stereographic projection as the composite

$$S(\mathbb{R} \oplus V) \xrightarrow{\Pi_V} S^V \xrightarrow{J} S^V ,$$

where the second map J is given by

$$J(v) \; = \; \begin{cases} \frac{v}{1-|v|} & \text{for } |v| < 1, \text{ and} \\ \infty & \text{for } |v| \geq 1. \end{cases}$$

The map J is equivariantly based homotopic to the identity of S^V, so the pair $(S(\mathbb{R} \oplus V), \tau_V)$ is another representative for the bordism class $d_{G,V}$. The map

$$\begin{aligned} j \; : \; V \oplus W \; &\longrightarrow \; S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus W) \\ j(v, w) \; &= \; \left(\frac{|v|^2 - 1}{|v|^2 + 1}, \frac{2v}{|v|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1}, \frac{2w}{|w|^2 + 1} \right) \end{aligned}$$

is a smooth G-equivariant embedding. The map j is the product of the inverse stereographic projections $\Pi_V^{-1} \times \Pi_W^{-1} : S^V \times S^W \longrightarrow S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus W)$, restricted to $V \oplus W$.

If a quadruple (x, v, y, w) is in the image $j(D(V \oplus W))$ of the unit disc, then in particular $x \leq 0$ and $y \leq 0$. Equivalently, the points (x, v, y, w) of $S(\mathbb{R} \oplus V) \times$

$S(\mathbb{R} \oplus W)$ that are in the complement of $j(D(V \oplus W))$ have $x > 0$ or $y > 0$. So the map

$$q \circ (\tau_V \times \tau_W) \; : \; S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus W) \; \longrightarrow \; S^{V \oplus W}$$

sends the complement of $j(\mathring{D}(V \oplus W))$ to the basepoint at infinity, where $q : S^V \times S^W \longrightarrow S^{V \oplus W}$ is the projection. Proposition 6.2.8 thus shows that

$$\begin{aligned} q_*(d_{G,V} \wedge d_{G,W}) \; &= \; q_*([\![S(\mathbb{R} \oplus V), \tau_V]\!] \wedge [\![S(\mathbb{R} \oplus W), \tau_W]\!]) \\ &= \; [\![S(\mathbb{R} \oplus V) \times S(\mathbb{R} \oplus W), \; q \circ (\tau_V \times \tau_W)]\!] \; = \; [\![S(\mathbb{R} \oplus V \oplus W), \; f]\!] \; , \end{aligned}$$

where $f : S(\mathbb{R} \oplus V \oplus W) \longrightarrow S^{V \oplus W}$ is defined by

$$f(x, v, w) \; = \; \begin{cases} q((\tau_V \times \tau_W)(j(v, w))) & \text{if } x \leq 0, \text{ and} \\ \quad\quad \infty & \text{if } x \geq 0. \end{cases}$$

We claim that for all $(x, v, w) \in S(\mathbb{R} \oplus V \oplus W)$ the relation

$$f(x, v, w) \; \neq \; \Pi_{V \oplus W}(-x, -v, -w)$$

holds in $S^{V \oplus W}$. Assuming this claim, we can finish the proof as follows: since the G-map $\Pi^{-1}_{V \oplus W} \circ f$ never takes a point to its antipode, the linear homotopy between $\Pi^{-1}_{V \oplus W} \circ f$ and the identity in the ambient vector space $\mathbb{R} \oplus V \oplus W$ can be normalized to land in the unit sphere. This yields an equivariant based homotopy between $\Pi^{-1}_{V \oplus W} \circ f$ and the identity of $S(\mathbb{R} \oplus V \oplus W)$. Hence f is equivariantly based homotopic to the stereographic projection $\Pi_{V \oplus W}$, and so $(S(\mathbb{R} \oplus V \oplus W), f)$ represents the bordism class $d_{G, V \oplus W}$.

It remains to prove the claim. The only point of $S(\mathbb{R} \oplus V \oplus W)$ that $\Pi_{V \oplus W}$ sends to the point at infinity is $(1, 0, 0)$. Since $f(-1, 0, 0) = (0, 0)$, the claim is true for all (x, v, w) such that $f(x, v, w) = \infty$. It remains to consider those tuples for which $f(x, v, w) \neq \infty$, which means that $x < 0$ and $|v| < 1$ and $|w| < 1$. On such points the map f is given by

$$\begin{aligned} f(x, v, w) \; &= \; q\left(\tau_V\left(\frac{|v|^2 - 1}{|v|^2 + 1}, \frac{2v}{|v|^2 + 1}\right), \; \tau_W\left(\frac{|w|^2 - 1}{|w|^2 + 1}, \frac{2w}{|w|^2 + 1}\right)\right) \\ &= \; q(J(v), J(w)) \; , \end{aligned}$$

whereas

$$\Pi_{V \oplus W}(-x, -v, -w) \; = \; \left(\frac{-v}{1 + x}, \quad \frac{-w}{1 + x}\right) .$$

If these two expressions were the same, then

$$\frac{v}{1 - |v|} \; = \; \frac{-v}{1 + x} \qquad \text{and} \qquad \frac{w}{1 - |w|} \; = \; \frac{-w}{1 + x} \; .$$

For $v \neq 0$ this implies

$$|v| - 1 \; = \; 1 + x \; ,$$

which is impossible since $|v| < 1$ and $x \geq -1$. If $w \neq 0$ we obtain the same kind of contradiction. The final case is $(x, v, w) = (-1, 0, 0)$, in which case

$$f(1-, 0, 0) \;=\; (0, 0) \;\neq\; \infty \;=\; \Pi_{V \oplus W}(1, 0, 0)\,. \qquad \square$$

We shall now recall that if G acts trivially on V, then exterior multiplication by $d_{G,V}$ is an isomorphism, the suspension isomorphism in reduced equivariant bordism. In general, however, the class $d_{G,V}$ is not invertible and exterior multiplication by $d_{G,V}$ need not be an isomorphism. The theory obtained by formally inverting all the classes $d_{G,V}$ is *stable equivariant bordism*, to which we return in Remark 6.2.36 below. The projection $p : Cf \longrightarrow X \wedge S^1$ from the reduced mapping to the suspension was defined in (3.1.32).

Proposition 6.2.11. *If G acts trivially on V and X is a cofibrant based G-space, then the exterior product map*

$$- \wedge d_{G,V} \;:\; \widetilde{\mathcal{N}}^G_n(X) \;\longrightarrow\; \widetilde{\mathcal{N}}^G_{n+|V|}(X \wedge S^V)$$

is an isomorphism. For every continuous G-map $f : X \longrightarrow Y$ between based G-spaces the connecting homomorphism in the mapping cone sequence equals the composite

$$\widetilde{\mathcal{N}}^G_n(Cf) \xrightarrow{\;p_*\;} \widetilde{\mathcal{N}}^G_n(X \wedge S^1) \xrightarrow{(-\wedge d_{G,\mathbb{R}})^{-1}} \widetilde{\mathcal{N}}^G_{n-1}(X)\,.$$

Proof We start with the special case $V = \mathbb{R}$. We apply Proposition 6.2.7 to the map $f : X \longrightarrow *$ to a one-point G-space. The cone of this map is

$$X^\diamond \;=\; X \times [0, 1]/\sim\,,$$

the unreduced suspension of X, where $X \times \{0\}$ and $X \times \{1\}$ are collapsed to one point each. Since X has a G-fixed-point, the map $f_* : \mathcal{N}^G_*(X) \longrightarrow \mathcal{N}^G_*(*)$ is a split epimorphism. So the long exact sequence provided by Proposition 6.2.7 reduces to a short exact sequence:

$$0 \longrightarrow \widetilde{\mathcal{N}}^G_{n+1}(X^\diamond) \xrightarrow{\;\bar\partial\;} \mathcal{N}^G_n(X) \xrightarrow{\;f_*\;} \mathcal{N}^G_n(*) \longrightarrow 0$$

Since X is cofibrant in the based sense, the projection

$$\psi \;:\; X^\diamond \;\longrightarrow\; X \wedge S^1\,, \quad q[x, s] \;=\; x \wedge \frac{2s-1}{s(1-s)}$$

that collapses $\{x_0\} \times [0, 1]$ is an equivariant homotopy equivalence.

Then the composite

$$\widetilde{\mathcal{N}}^G_{n+1}(X \wedge S^1) \xrightarrow[\cong]{\;\psi_*^{-1}\;} \widetilde{\mathcal{N}}^G_{n+1}(X^\diamond) \xrightarrow{\;\bar\partial\;} \mathcal{N}^G_n(X) \xrightarrow{\;\text{proj}\;} \widetilde{\mathcal{N}}^G_n(X)$$

is an isomorphism. We claim that the relation

$$\text{proj}(\bar\partial(\psi_*^{-1}(x \wedge d_{G,\mathbb{R}}))) \;=\; x$$

holds for all classes $x \in \widetilde{\mathcal{N}}_n^G(X)$. Since $\mathrm{proj} \circ \bar{\partial} \circ \psi_*^{-1}$ is an isomorphism, so is smash product with the class $d_{G,\mathbb{R}}$.

This relation, in turn, is a consequence of the geometric origin of the class $d_{G,\mathbb{R}}$, the product in bordism and the boundary map. In more detail, we suppose that $x = [\![M, h]\!]$ for a singular G-manifold (M, h) over X. We define a continuous map $H : M \times S(\mathbb{R} \oplus \mathbb{R}) \longrightarrow X^\diamond$ by

$$H(m, (x, y)) \ = \ \begin{cases} [h(m), (y+1)/2] & \text{for } x \leq 0, \text{ and} \\ [x_0, (y+1)/2] & \text{for } x \geq 0. \end{cases}$$

Then the following square commutes up to G-equivariant homotopy:

$$\begin{tikzcd} M \times S(\mathbb{R} \oplus \mathbb{R}) \arrow[r, "H"] \arrow[d, "h \times \Pi_{\mathbb{R}}"'] & X^\diamond \arrow[d, "\psi"] \\ X \times S^1 \arrow[r, "q"'] & X \wedge S^1 \end{tikzcd}$$

Hence

$$\psi_* [\![M \times S(\mathbb{R} \oplus \mathbb{R}), H]\!] \ = \ [\![M \times S(\mathbb{R} \oplus \mathbb{R}), q \circ (h \times \Pi_{\mathbb{R}})]\!] \ = \ [\![M, h]\!] \wedge d_{G,\mathbb{R}} \ ,$$

and thus

$$\bar{\partial}(\psi_*^{-1}([\![M, h]\!] \wedge d_{G,\mathbb{R}})) \ = \ \bar{\partial} [\![M \times S(\mathbb{R} \oplus \mathbb{R}), H]\!] \ .$$

To calculate this geometric boundary we use the smooth separating function

$$r \ : \ M \times S(\mathbb{R} \oplus \mathbb{R}) \ \longrightarrow \ [0, 1] \ , \quad r(m, (x, y)) \ = (y+1)/2 \ .$$

Then $1/2$ is a regular value of this separating function, and the preimage over this regular value is $r^{-1}(1/2) = M \times \{(1, 0), (-1, 0)\}$, two disjoint copies of M. The function H takes the copy $M \times \{(1, 0)\}$ to the basepoint of X, so this copy does not contribute to the reduced bordism group. The restriction of H to the other copy $M \times \{(-1, 0)\}$ is the original map h, so we obtain

$$\bar{\partial}[M \times S(\mathbb{R} \oplus \mathbb{R}), H] \ = \ [\![M \times \{(1, 0), (-1, 0)\}, H|_{M \times \{(1,0),(-1,0)\}}]\!] \ \equiv \ [\![M, h]\!]$$

in the reduced bordism group of X.

The general case now follows easily. Since the claim is true for $V = \mathbb{R}$, it also holds for $V = \mathbb{R}^n$ by the associativity of the smash product pairing and the classes $d_{G,\mathbb{R}}$, compare Proposition 6.2.10. If G acts trivially on V, then it is equivariantly isomorphic to $\mathbb{R}^n$ for some n. □

The bordism theories $\mathcal{N}_*^G$ for different compact Lie groups are related by geometrically defined restriction and induction maps. Every continuous group

homomorphism $\alpha : K \longrightarrow G$ is automatically smooth (see for example [28, Prop. I.3.12]), and thus induces a restriction homomorphism

$$\alpha^* \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \mathcal{N}_n^K(\alpha^*(X)) \ , \quad [M, h] \ \longmapsto \ [\alpha^*(M), \alpha^*(h)]$$

by restricting all actions along α. Restriction maps preserve the distinguished classes (6.2.9) in the sense that $\alpha^*(d_{G,V}) = d_{K,\alpha^*(V)}$. The product in equivariant bordism is compatible with restriction maps in the sense that

$$\alpha^*(x \times y) \ = \ \alpha^*(x) \times \alpha^*(y) \ .$$

For every closed subgroup H of G and every closed smooth H-manifold M of dimension n, the induced space $G \times_H M$ has a preferred smooth structure making it a smooth closed G-manifold of dimension $d + n$ with $d = \dim(G/H) = \dim(G) - \dim(H)$. Indeed, since the diagonal H-action on $G \times M$ is smooth and free, there is a unique smooth structure on $G \times_H M$ such that the projection $G \times M \longrightarrow G \times_H M$ is a submersion, see for example [180, Thm. 15.3.4], and the G-action is smooth with respect to this smooth structure. We can also apply $G \times_H -$ to bordisms, so this gives a well-defined induction homomorphism

$$G \times_H - \ : \ \mathcal{N}_n^H(Y) \ \longrightarrow \ \mathcal{N}_{d+n}^G(G \times_H Y) \ , \quad [M, h] \ \longmapsto \ [G \times_H M, G \times_H h] \ .$$

For $Y = *$ we can compose the induction map with the effect of the projection $G \times_H * \longrightarrow *$ and arrive at an induction homomorphism on coefficient groups

$$\mathrm{ind}_H^G \ : \ \mathcal{N}_n^H(*) \ \longrightarrow \ \mathcal{N}_{d+n}^G(*) \ , \quad [M] \ \longmapsto \ [G \times_H M] \ .$$

The induction map ind_H^G is compatible with inflations, in the sense of the formula

$$\alpha^* \circ \mathrm{ind}_H^G \ = \ \mathrm{ind}_L^K \circ (\alpha|_L)^* \ : \ \mathcal{N}_n^H(*) \ \longrightarrow \ \mathcal{N}_{d+n}^K(*)$$

for a continuous epimorphism $\alpha : K \longrightarrow G$, where $L = \alpha^{-1}(H)$. If H has finite index in G, then the induction ind_H^G preserves the dimension, and then it satisfies the double coset formula. So for fixed $n \geq 0$ the coefficient groups $\mathcal{N}_n^G(*)$ *almost* form a global functor; the only missing structure are the transfer maps for closed inclusions that are not of finite index.

Multiplication, restriction and induction satisfy a reciprocity relation. We let H be a closed subgroup of G, X an H-space and Y a G-space. Then the

following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}_m^H(X) \times \mathcal{N}_n^G(Y) \arrow[r, "(G\times_H -)\times \mathrm{Id}"] \arrow[d, "\mathrm{Id}\times \mathrm{res}^G_H"'] & \mathcal{N}^G_{d+m}(G\times_H X) \times \mathcal{N}_n^G(Y) \arrow[d, "\times"] \\
\mathcal{N}_m^H(X) \times \mathcal{N}_n^H(\mathrm{res}^G_H(Y)) \arrow[d, "\times"'] & \mathcal{N}^G_{d+m+n}((G\times_H X)\times Y) \arrow[d, "\chi_*"] \\
\mathcal{N}^H_{m+n}(X \times \mathrm{res}^G_H(Y)) \arrow[r, "G\times_H -"'] & \mathcal{N}^G_{d+m+n}(G \ltimes_H (X\times \mathrm{res}^G_H(Y)))
\end{tikzcd}$$

Here

$$\chi \ : \ (G\times_H X)\times Y \ \longrightarrow \ G \ltimes_H (X\times \mathrm{res}^G_H(Y)) \ , \quad ([g,x],y) \ \longmapsto \ [g,(x,g^{-1}y)]$$

is the G-equivariant shearing isomorphism. The proof of the commutativity is straightforward from the definitions, using the shearing diffeomorphism for equivariant manifolds. If we specialize to $X = Y = *$ and post-compose with the projection to the one-point G-space, we obtain the reciprocity formula

$$\mathrm{ind}^G_H(x\times \mathrm{res}^G_H(y)) \ = \ \mathrm{ind}^G_H(x)\times y$$

for classes $x \in \mathcal{N}^H_m(*)$ and $y \in \mathcal{N}^G_n(*)$ in the coefficient groups.

The next proposition shows that the distinguished bordism classes $d_{G,V}$ measure the failure of the Wirthmüller isomorphism in equivariant bordism, see Remark 6.2.13 below. We consider a closed subgroup H of a compact Lie group G and continue to write

$$L \ = \ T_{eH}(G/H)$$

for the tangent H-representation. For an H-space Y the H-equivariant continuous map

$$l_Y \ : \ G\times_H Y \ \longrightarrow \ Y_+ \wedge S^L$$

was defined in Construction 3.2.1.

Proposition 6.2.12. *For every closed subgroup H of a compact Lie group G and every H-space Y, the composite*

$$\mathcal{N}^H_n(Y) \ \xrightarrow{G\times_H -} \ \mathcal{N}^G_{n+d}(G\times_H Y) \ \xrightarrow{\mathrm{res}^G_H} \ \mathcal{N}^H_{n+d}(G\times_H Y) \ \xrightarrow{(l_Y)_*} \ \widetilde{\mathcal{N}}^H_{n+d}(Y_+\wedge S^L)$$

is exterior multiplication by the class $d_{H,L} \in \widetilde{\mathcal{N}}^H_d(S^L)$, where $d = \dim(G/H)$.

Proof We let (M,h) be a singular H-manifold that represents a class in $\mathcal{N}^H_n(Y)$. The $G\times_H h : G\times_H M \longrightarrow G\times_H Y$ represents the class $G\times_H [M,h]$. As in the

construction of the collapse map l_Y we choose a slice around $1 \in G$ orthogonal to H, i.e., a smooth embedding $s : D(L) \longrightarrow G$ satisfying $s(0) = 1$ and

$$s(h \cdot l) \; = \; h \cdot s(l) \cdot h^{-1}$$

for all $(h, l) \in H \times D(L)$ and such that the differential at 0 of the composite

$$D(L) \xrightarrow{s} G \xrightarrow{\text{proj}} G/H$$

is the identity of L. The slice property implies that the map

$$\bar{s} \; : \; D(L) \times H \; \longrightarrow \; G \; , \quad (l, h) \; \longmapsto \; s(l) \cdot h$$

is a tubular neighborhood of H inside G. Moreover, this embedding is equivariant for the action of H^2, acting on source and target by

$$(h_1, h_2) \cdot (l, h) \; = \; (h_1 l, \; h_1 h h_2^{-1}) \quad \text{and} \quad (h_1, h_2) \cdot g \; = \; h_1 g h_2^{-1} \; .$$

The map

$$l_H^G \; : \; G_+ \; \longrightarrow \; S^L \wedge H_+$$

was defined as the H^2-equivariant collapse map with respect to the tubular neighborhood $\bar{s}$. So explicitly,

$$l_H^G(g) \; = \; \begin{cases} (l/(1 - |l|)) \wedge h & \text{if } g = s(l) \cdot h \text{ with } (l, h) \in D(L) \times H, \text{ and} \\ \quad \infty & \text{if } g \text{ is not in the image of } \bar{s}. \end{cases}$$

We obtain a smooth H-equivariant embedding

$$j \; : \; M \times D(L) \; \longrightarrow \; G \times_H M \qquad \text{by} \qquad j(m, l) \; = \; [s(l), m] \; ,$$

where H acts diagonally on the left. The composite

$$G \times_H M \xrightarrow{G \times_H h} G \times_H Y \xrightarrow{l_H^G \wedge_H Y_+} (S^L \wedge H_+) \wedge_H Y_+ \; \cong \; Y_+ \wedge S^L$$

sends the complement of $j(M \times D(L))$ to the basepoint at infinity. Proposition 6.2.8 (for (H, L) instead of (G, V)) shows that then

$$\begin{aligned} ((l_Y)_* \circ \operatorname{res}_H^G \circ (G \times_H -))[M, h] \; &= \; [\![G \times_H M, \; l_Y \circ (G \times_H h)]\!] \\ &= \; [\![M \times S(\mathbb{R} \oplus L), f]\!] \end{aligned}$$

in the reduced bordism group of $Y_+ \wedge S^L$, where $f : M \times S(\mathbb{R} \oplus L) \longrightarrow Y_+ \wedge S^L$ is defined by

$$f(m, (x, l)) \; = \; \begin{cases} (l_Y \circ (G \times_H h))(j(m, l)) & \text{if } x \leq 0, \text{ and} \\ \qquad \infty & \text{if } x \geq 0. \end{cases}$$

So f equals the composite

$$M \times S(\mathbb{R} \oplus L) \xrightarrow{h \times \Psi} Y \times S^L \xrightarrow{q} Y_+ \wedge S^L$$

where

$$\Psi(x, l) = l/(1 - |l|)$$

for $x \leq 0$, and $\Psi(x, l) = \infty$ for $x \geq 0$. The map $\Psi : S(\mathbb{R} \oplus L) \longrightarrow S^L$ is homotopic, in the equivariant based sense, to the stereographic projection Π_L. We can thus conclude that

$$\begin{aligned} ((l_Y)_* \circ \operatorname{res}^G_H \circ (G \times_H -))[M, h] &= [\![M \times S(\mathbb{R} \oplus L), f]\!] \\ &= [M, h] \wedge [\![S(\mathbb{R} \oplus L), \Psi]\!] = [M, h] \wedge d_{H,L}. \quad \square \end{aligned}$$

Remark 6.2.13 (Failure of the Wirthmüller isomorphism in equivariant bordism)**.** As we now explain, there is no general Wirthmüller isomorphism in equivariant bordism, i.e., the Wirthmüller map

$$\mathrm{Wirth}^G_H = (l_Y)_* \circ \operatorname{res}^G_H : \mathcal{N}^G_*(G \times_H Y) \longrightarrow \widetilde{\mathcal{N}}^H_*(Y_+ \wedge S^L)$$

is *not* in general bijective for closed subgroups H of G. Here Y is an H-space and l_Y is the H-equivariant collapse map defined in Construction 3.2.1. This immediately shows that there *cannot* be a natural isomorphism, compatible with restriction to subgroups, between equivariant bordism and the equivariant homology theory represented by any orthogonal G-spectrum. Indeed, Proposition 6.2.12 says that the composite

$$\mathrm{Wirth}^G_H \circ (G \times_H -) : \mathcal{N}^H_n(Y) \longrightarrow \widetilde{\mathcal{N}}^H_{n+d}(Y_+ \wedge S^L)$$

is exterior multiplication by the class $d_{H,L} \in \widetilde{\mathcal{N}}^H_d(S^L)$. Since the induction map $G \times_H -$ is an isomorphism, the Wirthmüller map is bijective if any only if the class $d_{H,L}$ acts invertibly on $\mathcal{N}^H_*(Y)$. In this sense, the distinguished bordism class $d_{H,L}$ precisely measures the failure of the Wirthmüller isomorphism in equivariant bordism. One can show that the class $d_{H,L}$ is *not* invertible as soon as H acts non-trivially on L, i.e., in this situation the class $d_{H,L}$ does *not* generate $\widetilde{\mathcal{N}}^H_*(S^L)$ as an $\mathcal{N}^H_*$-module.

If G is a product of a finite group and a torus, then H acts trivially on L for every closed subgroup H of G, and hence the Wirthmüller isomorphism *does hold* for such compact Lie groups G. And in fact, for this class of groups, the Thom-Pontryagin construction provides an natural isomorphism between equivariant bordism and the equivariant homology theory represented by the Thom spectrum **mO**, compare Theorem 6.2.33 below.

Our argument to compare the geometric bordism theory with the equivariant

homotopy groups of the Thom spectrum **mO** is based on the isotropy separation sequence. We will now identify the 'geometric fixed-point term' in the isotropy separation sequence for equivariant bordism.

Construction 6.2.14. As before we let $E\mathcal{P}$ be a universal space for the family of proper closed subgroups of G. So $E\mathcal{P}$ is a cofibrant G-space without G-fixed-points, and $(E\mathcal{P})^H$ is contractible for every proper closed subgroup of G. We let $\tilde{E}\mathcal{P}$ denote the unreduced suspension of $E\mathcal{P}$. Then $(\tilde{E}\mathcal{P})^G = S^0$ consists of the two cone points, and $(\tilde{E}\mathcal{P})^H$ is contractible for every proper subgroup of G.

We recall the 'geometric fixed-point' homomorphism, see (6.2.15) below, that identifies the reduced bordism group $\widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P})$ in terms of non-equivariant bordism groups. Loosely speaking, the class $\Phi_{\text{geom}}[\![M, h : M \longrightarrow \tilde{E}\mathcal{P}]\!]$ remembers the bordism class of the part of the fixed-point manifold that lies over the fixed-point 0, together with the normal data of the embedding into M. As before, we use the abbreviation

$$Gr_j^{G,\perp} \ = \ \left(Gr_j(\mathcal{U}_G^{\perp})\right)^G$$

for the space of j-dimensional G-invariant subspaces of $\mathcal{U}_G^{\perp} = \mathcal{U}_G - (\mathcal{U}_G)^G$. We first define a fixed-point map

$$\bar{\Phi} \ : \ \mathcal{N}_n^G(\tilde{E}\mathcal{P}) \ \longrightarrow \ \bigoplus\nolimits_{j\geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp})$$

into the unreduced, non-equivariant bordism group as follows. We let $(M, h : M \longrightarrow \tilde{E}\mathcal{P})$ be an n-dimensional singular G-manifold. Since G acts smoothly, M^G is a disjoint union of smooth submanifolds of varying dimensions, compare [26, VI Cor. 2.5]. Since $\tilde{E}\mathcal{P}$ has exactly two fixed-points, and both are isolated, h must map every path component of the fixed-point manifold M^G to either 0 or ∞. We denote by

$$M_0^G \ = \ M^G \cap h^{-1}(0)$$

the union of those components of M^G that lie over the point 0. We let $M^{(j)}$ be the union of all $(n-j)$-dimensional components of M_0^G.

The Mostow-Palais embedding theorem [124, 130] provides a smooth G-equivariant embedding $i : M \longrightarrow V$, for some G-representation V; we can assume that V is in fact a subrepresentation of the complete universe $\mathcal{U}_G$. Then for every fixed-point $x \in M^{(j)}$,

$$(di)(T_x(M^{(j)})) \ = \ ((di)(T_xM))^G \ ,$$

i.e., the tangent space inside $M^{(j)}$ 'is' the G-fixed part of the tangent space in M. So we can define a continuous map

$$\nu_j \ : \ M^{(j)} \ \longrightarrow \ (Gr_j(V^{\perp}))^G \ \xrightarrow{\text{incl}} \ Gr_j^{G,\perp}$$

by sending a fixed-point $x \in M^{(j)}$ to the orthogonal complement of $(di)(T_x(M^{(j)}))$ inside $(di)(T_x M)$. By its very construction, the map ν_j classifies the normal bundle of the inclusion $M^{(j)} \longrightarrow M$. The geometric fixed-point map is then given by

$$\bar{\Phi}[M, h] \;=\; \sum\nolimits_{j=0}^{n} [M^{(j)}, \nu_j]\ .$$

Since the image of the map $i_* : \mathcal{N}_n^G(*) \longrightarrow \mathcal{N}_n^G(\tilde{E}\mathcal{P})$ is concentrated over the basepoint ∞, it is annihilated by the map $\bar{\Phi}$. So $\bar{\Phi}$ factors over the reduced bordism group of $\tilde{E}\mathcal{P}$ as a homomorphism

$$\Phi_{\text{geom}} \;:\; \widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P}) \;\longrightarrow\; \bigoplus\nolimits_{j\geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp})\ . \tag{6.2.15}$$

The fact that the geometric fixed-point map (6.2.15) is an isomorphism is ubiquitous in calculations of equivariant bordism groups, and it goes back to Conner and Floyd [39]. In the classical literature, the reduced equivariant bordism groups of $\tilde{E}\mathcal{P}$ usually appear in a different guise, namely as the groups $\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}]$ of bordism classes of smooth compact G-manifolds with boundary, but where there are no fixed-points on the boundary, compare Remark 6.2.17 below.

Proposition 6.2.16. *For every compact Lie group G and every $n \geq 0$ the geometric fixed-point map* (6.2.15) *is an isomorphism.*

Proof The proof is by explicit geometric constructions. We start with surjectivity. We consider a non-equivariant closed smooth $(n-j)$-manifold N and a continuous map $f : N \longrightarrow Gr_j^{G,\perp}$. The space $Gr_j^{G,\perp}$ is the filtered colimit of its closed subspaces $(Gr_j(V))^G$ for $V \in s(\mathcal{U}_G^\perp)$. Since N is compact, the image of f lands in $(Gr_j(V))^G$ for some finite-dimensional G-representation V with $V^G = 0$. The space $(Gr_j(V))^G$ is a smooth manifold, and by smooth approximation we can assume without loss of generality that f is a smooth map. We define a closed smooth n-dimensional manifold by

$$M \;=\; \{(n, x, v) \in N \times S(\mathbb{R} \oplus V) \mid v \in f(n)\}\ .$$

Another way to say this is that M is a double of the unit disc bundle of the pullback of the tautological j-plane bundle along $f : N \longrightarrow (Gr_j(V))^G$. The group G acts smoothly on M by $g \cdot (n, x, v) = (n, x, gv)$.

At this point it will be convenient to use a specific model for the space $\tilde{E}\mathcal{P}$, namely the unit sphere in $\mathbb{R} \oplus \mathcal{U}_G^\perp$, compare Example 3.3.7. We can then define a continuous G-map

$$h \;:\; M \;\longrightarrow\; \tilde{E}\mathcal{P} \;=\; S(\mathbb{R} \oplus \mathcal{U}_G^\perp) \qquad \text{by} \qquad h(n, x, v) \;=\; (x, v)\ .$$

So the pair (M, h) represents a bordism class

$$[M, h] \quad \in \; \mathcal{N}_n^G(\tilde{E}\mathcal{P})\ .$$

Since $V^G = 0$, the G-fixed-points of M are a disjoint union of two copies of N embedded as $N \times (-1, 0)$ and $N \times (1, 0)$, and h maps one copy to each of the two fixed-points of $\tilde{E}\mathcal{P}$. The normal bundle of the copy of N over the non-base point is the bundle classified by the original map f, so we obtain

$$\Phi_{\mathrm{geom}}[M, h] \;=\; [N, f]\,.$$

This shows that every class in $\mathcal{N}_{n-j}(Gr_j^{G,\perp})$ is in the image of the geometric fixed-point map, and so the map (6.2.15) is surjective.

Now we consider a reduced equivariant bordism class $[\![M, h]\!]$ over $\tilde{E}\mathcal{P}$ in the kernel of the fixed-point map (6.2.15). Being in the kernel of Φ_{geom} means that for every $0 \leq j \leq n$ there is a non-equivariant null-bordism B_j with $\partial B_j = M^{(j)}$ and a continuous map $F_j : B \longrightarrow Gr_j^{G,\perp}$ whose restriction to $M^{(j)}$ classifies the normal bundle of $M^{(j)}$ inside M. As in the first part of the proof we can compress F_j to a G-map $B \longrightarrow (Gr_j(V))^G$ for some finite-dimensional G-subrepresentation V of $\mathcal{U}_G^\perp$ and replace this factorization by a homotopic smooth map $F_j : B \longrightarrow (Gr_j(V))^G$. We use this data to 'cut out' the fixed-points M_0^G from M by replacing a tubular neighborhood by the sphere bundles of the maps F_j; this produces a new singular G-manifold over $\tilde{E}\mathcal{P}$, bordant to (M, h), that has no more fixed-points over 0.

The construction is done separately and disjointly over each of the components $M^{(j)}$ of the fixed-points M_0^G. To simplify the exposition we restrict to the special case where the fixed-points M_0^G are of constant dimension $n - j$ (i.e., all other components $M^{(i)}$ for $i \neq j$ are empty). Then we proceed as follows. We let ν denote the normal bundle of $M^{(j)}$ inside M. The equivariant tubular neighborhood theorem provides a smooth G-equivariant embedding

$$\psi \;:\; D(\nu) \;\longrightarrow\; M$$

of the unit disc bundle of ν whose composite with the zero section $s : M^{(j)} \longrightarrow D(\nu)$ is the inclusion, see for example [26, VI Thm. 2.2]. In particular, the image of ψ is a closed G-invariant tubular neighborhood of $M^{(j)}$. By shrinking the neighborhood, if necessary, we can make its image disjoint from all other components of M^G except $M^{(j)}$. The bundle arising by pulling back the tautological bundle along $F : B \longrightarrow (Gr_j(V))^G$ has its own disc bundle with total space

$$D(F) \;=\; \{(b, v) \in B \times V \mid v \in F(b),\ |v| \leq 1\}\,.$$

The sphere bundle $S(F)$ is then a smooth compact G-manifold with boundary

$$\partial(S(F)) \;=\; S(F|_{\partial B}) \;=\; S(\nu)\,.$$

Now we form the G-manifold

$$\bar{M} \;=\; (M - \psi(\mathring{D}(\nu)) \cup_{S(\nu)} S(F)\,,$$

where the gluing uses the restriction $\psi : S(\nu) \longrightarrow M$ of the tubular neighborhood to the sphere bundle. A smooth structure on $\bar{M}$ is provided by choices of G-equivariant collars of $\psi(S(\nu))$ inside $M - \psi(\mathring{D}(\nu))$ and of $S(\nu)$ inside $S(F)$.

The boundary of the disc bundle of F decomposes as

$$\partial(D(F)) \ = \ S(F) \cup_{S(F|_{\partial B})} D(F|_{\partial B}) \ = \ S(F) \cup_{S(\nu)} D(\nu) \ .$$

By equivariant smoothing of corners the disc bundle $D(F)$ can be given a smooth structure such that the given G-action is smooth and that the embeddings of $D(\nu)$ and $S(F)$ into $D(F)$ are smooth. We define a bordism as the G-space

$$W \ = \ (M \times [0,1]) \cup_{D(\nu)} D(F)$$

where the gluing is along $(\psi, 1) : D(\nu) \longrightarrow M \times [0,1]$. The space W is a topological $(n+1)$-manifold whose boundary is the union of two disjoint parts that we now parametrize. An obvious embedding is given by

$$\psi \ : \ M \ \longrightarrow \ W \ , \quad \psi(m) = [m, 0] \ .$$

A second embedding

$$i \ : \ \bar{M} \ = \ (M - \psi(\mathring{D}(\nu))) \cup_{S(\nu)} S(F) \ \longrightarrow \ W$$

identifies $M - \psi(\mathring{D}(\nu))$ with $M - \psi(\mathring{D}(\nu)) \times 1$ and includes the sphere bundle $S(F)$ into the disc bundle $D(F)$. The boundary ∂W is then the disjoint union of the images of ψ and i.

The topological manifold W admits a smooth structure for which the given G-action is smooth and such that the embeddings ψ and i are smooth; in the non-equivariant version, this is explained in Construction 15.10.3 of [180]. To ensure that the given G-action on W is smooth, we must insist that the collars involved in the construction are G-equivariant collars, which is possible for example by [39, Thm. 21.2].

Now we have to arrange the equivariant reference maps to $\tilde{E}\mathcal{P}$. In fact, this extra data goes along for the ride, as the only homotopical information it encodes is a decomposition of the G-fixed-points into two disjoint subspaces, the preimages of the two fixed-points of $\tilde{E}\mathcal{P}$. In more detail, we let W be a compact smooth G-manifold, possibly with boundary. Then W admits the structure of a finite G-CW-complex by Illman's triangulation theorem [84, Thm. 7.1]. By the proof of Proposition 3.3.8, the fixed-point map

$$(-)^G \ : \ \mathrm{map}^G(W, \tilde{E}\mathcal{P}) \ \longrightarrow \ \mathrm{map}(W^G, \{0, \infty\})$$

is a weak equivalence and Serre fibration. A continuous map to the discrete space $\{0, \infty\}$ is equivalent to a decomposition into two disjoint open subsets. So altogether we conclude that for every disjoint union decomposition $W^G = A \cup B$

there is a G-map $h : W \longrightarrow \tilde{E}\mathcal{P}$ with $h(A) = \{0\}$ and $h(B) = \{\infty\}$, and any two such maps are equivariantly homotopic. We use this property three times, namely for the G-manifolds W, M and $\bar{M}$.

The G-fixed-points of W are a disjoint union of $M_\infty^G \times [0,1]$ and $(M^{(j)} \times [0,1]) \cup_{M^{(j)}\times 1} B$. So there is a G-map $H : W \longrightarrow \tilde{E}\mathcal{P}$ such that $H(M_\infty^G \times [0,1]) = \{\infty\}$ and $H((M^{(j)} \times [0,1]) \cup_{M^{(j)}\times 1} B) = \{0\}$. The triple $(W, H, \psi + i)$ is then a bordism that witnesses the relation $[\![M, H\psi]\!] = [\![\bar{M}, Hi]\!]$ in the group $\widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P})$. The map $Hi : \bar{M} \longrightarrow \tilde{E}\mathcal{P}$ sends all of $\bar{M}^G$ to the fixed-point ∞, so Hi is equivariantly homotopic to the constant map with value ∞. By homotopy invariance, the class $[\![\bar{M}, Hi]\!]$ thus vanishes in the reduced bordism group. On the other hand, the G-map $H\psi$ agrees with the original map h on the fixed-points M^G. So $H\psi$ is equivariantly homotopic to h, and homotopy invariance yields

$$[\![M, h]\!] \ = \ [\![M, H\psi]\!] \ = \ [\![\bar{M}, H|_{\bar{M}}]\!] \ = \ 0$$

in the group $\widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P})$. This shows that the map Φ_{geom} is injective. □

Remark 6.2.17. In the classical literature, the reduced equivariant bordism groups of $\tilde{E}\mathcal{P}$ usually appear in an isomorphic form, namely as the groups $\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}]$ of bordism classes of smooth compact G-manifolds with boundary, but where there are no fixed-points on the boundary. A homomorphism

$$\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}] \ \longrightarrow \ \widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P}) \tag{6.2.18}$$

is defined as follows. We let M be a smooth compact G-manifold without G-fixed-points on the boundary. The double of M is the smooth closed G-manifold

$$DM \ = \ M \cup_{\partial M} M$$

obtained by gluing two copies of M along their boundary. A smooth structure in the neighborhood of the gluing locus is provided by a choice of G-equivariant collar (see [39, Thm. 21.2]). By Illman's theorem [84, Cor. 7.2], DM admits the structure of a finite G-CW-complex. Since the original manifold M had no G-fixed-points on the boundary, $(DM)^G$ is the disjoint union of two copies of M^G, one from each of the two copies of M in the double. There is thus a continuous G-map $h : DM \longrightarrow \tilde{E}\mathcal{P}$, unique up to equivariant homotopy, that takes the 'left' copy of M^G to the fixed-point 0 and the 'right' copy of M^G to the fixed-point ∞. The pair (DM, h) is then a singular G-manifold over $\tilde{E}\mathcal{P}$, and it represents a bordism class

$$[DM, h] \quad \in \ \mathcal{N}_n^G(\tilde{E}\mathcal{P}) \ .$$

The fact that the map (6.2.18) is an isomorphism is Satz 3 in [176].

For finite groups, Stong shows in [164, Cor. 5.1] that the geometric fixed-point map

$$\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}] \longrightarrow \bigoplus_{j \geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp})$$

is an isomorphism. Combined with tom Dieck's isomorphism (6.2.18) this gives a different proof of Proposition 6.2.16 in the special case of finite groups. I was unable to find a convenient reference for Proposition 6.2.16 in the generality of compact Lie groups, which is the main reason for including a proof.

Example 6.2.19 (Bordism of manifolds with involution)**.** We look at the geometric isotropy separation sequence in the simplest non-trivial case of the two-element group $G = C_2 = C$, the case originally considered by Conner and Floyd [39, Thm. 28.1]. In this case $\mathcal{P}$ consists only of the trivial subgroup, and so $\mathcal{N}_*^C(E\mathcal{P}) = \mathcal{N}_*^C(EC)$ is the bordism ring of manifolds with free C-action. If C acts freely and smoothly on M, then we can form the smooth manifold

$$[-1, 1] \times_C M \ = \ ([-1, 1] \times M)/(x, m) \sim (-x, \tau m)$$

with C-action by $\tau \cdot [x, m] = [-x, \tau m]$; the boundary of this manifold is equivariantly diffeomorphic to the original manifold M. So every C-manifold with free action is null-bordant, and thus the forgetful map $\mathcal{N}_*^C(E\mathcal{P}) \longrightarrow \mathcal{N}_*^C$ is zero. The long exact mapping cone sequence of the C-map $EC \longrightarrow *$ (see Proposition 6.2.7) thus decomposes into short exact sequences. For every compact Lie group G, the map

$$\mathcal{N}_n^G(EG) \longrightarrow \mathcal{N}_n(BG)\,, \quad [M, h] \longmapsto [G\backslash M, G\backslash h]$$

is an isomorphism from the bordism group of G-manifolds with free action to the non-equivariant bordism group of the classifying space $BG = G\backslash EG$. So in the case at hand, $\mathcal{N}_*^C(E\mathcal{P}) = \mathcal{N}_*^C(EC)$ is isomorphic to $\mathcal{N}_*(\mathbb{R}P^\infty)$.

We use Proposition 6.2.16 to replace the group $\widetilde{\mathcal{N}}_*^C(\tilde{E}C)$ by the direct sum of non-equivariant bordism groups. Since there is only one non-trivial irreducible C-representation, the 1-dimensional sign representation, every linear subspace of $\mathcal{U}_C^\perp$ is C-invariant. Hence $Gr_j^{C,\perp}$ is just a Grassmannian of j-planes in an infinite-dimensional $\mathbb{R}$-vector space, hence a classifying space of the orthogonal group $O(j)$. Altogether, this yields a short exact sequence

$$0 \longrightarrow \mathcal{N}_*^C \xrightarrow{\ \Phi_{\text{geom}}\ } \bigoplus_{j \geq 0} \mathcal{N}_{*-j}(Gr_j(\mathbb{R}^\infty)) \xrightarrow{\ J\ } \mathcal{N}_{*-1}(\mathbb{R}P^\infty) \longrightarrow 0\,.$$

The map J is defined as follows. Given an $(n-j)$-dimensional singular manifold $(F, \eta : F \longrightarrow Gr_j(\mathbb{R}^\infty))$, we factor the continuous map η through a finite-dimensional Grassmannian $F \longrightarrow Gr_j(\mathbb{R}^m)$, choose a homotopic smooth approximation $\eta' : F \longrightarrow Gr_j(\mathbb{R}^m)$ and let $P(\eta')$ denote the projectivized bundle of the pullback of the tautological j-plane bundle along η'. The total space of

$P(\eta')$ is then a smooth closed $(n-1)$-manifold equipped with a map to $\mathbb{R}P^\infty$ that classifies the tautological line bundle over $P(\eta')$; this data represents the class $J[F, \eta]$.

One can deduce from the above short exact sequence that $\mathcal{N}_*^C$ is free as a module over the non-equivariant bordism ring $\mathcal{N}_*$. This is in fact true much more generally: Stong shows in [164, Prop. 9.4] that $\mathcal{N}_*^G(X)$ is free as an $\mathcal{N}_*$-module for every finite group G and every G-space X. The paper [2] by Alexander exhibits an explicit $\mathcal{N}_*$-basis of $\mathcal{N}_*^C$. Some basic C-bordism classes are represented by the projective spaces $\mathbb{R}P^n$ equipped with the involution

$$\tau \cdot [x_0 : x_1 : \ldots : x_n] \ = \ [-x_0 : x_1 : \ldots : x_n] \ .$$

We denote by $y_n = [\mathbb{R}P^n, \tau]$ the bordism class of this C-manifold in $\mathcal{N}_n^C$. An $\mathcal{N}_*$-linear map

$$\Gamma \ : \ \mathcal{N}_*^C \ \longrightarrow \ \mathcal{N}_{*+1}^C$$

of degree 1 is given by sending the class of a manifold M with involution $\tau : M \longrightarrow M$ to the manifold

$$S(\mathbb{C}) \times_C M \ = \ (S(\mathbb{C}) \times M)/(z, m) \sim (-z, \tau m) \ .$$

So $S(\mathbb{C}) \times_C M$ is diffeomorphic to the mapping torus of the involution τ. The involution on this manifold is by

$$\tau \ : \ S(\mathbb{C}) \times_C M \ \longrightarrow \ S(\mathbb{C}) \times_C M \ , \quad \tau \cdot [z, m] \ = \ [\bar{z}, \tau m] \ .$$

In our present notation, the operator can also be expressed as

$$\Gamma \ = \ \mathrm{res}_C^{O(2)} \circ \mathrm{ind}_{O(1)\times O(1)}^{O(2)} \circ p^* \ ,$$

where we embed C into $O(2)$ as a reflection, and where $p : O(1) \times O(1) \longrightarrow C$ is the epimorphism with kernel $e \times O(1)$.

Alexander shows in [2, Thm. 1.1] that the multiplicative unit 1 together with the classes

$$\Gamma^n(y_{i_1} \cdot \ldots \cdot y_{i_r})$$

for all $n \geq 0$, $r \geq 1$ and $i_j \geq 2$ form a basis of $\mathcal{N}_*^C$ as a module over $\mathcal{N}_*$. Besides the trivial group and C, the equivariant bordism groups have been calculated for various finite abelian groups, see [7, 8, 9, 54].

Now we work our way towards the equivariant Thom-Pontryagin construction that assigns to every equivariant bordism class over a G-space X an equivariant homology class in $\mathbf{mO}_*^G(X_+)$. We break the construction up into two steps, and we first discuss the *normal class*, a basic invariant associated with a closed smooth G-manifold.

Construction 6.2.20 (Normal class of a G-manifold)**.** In Example 6.1.1 we defined the ultra-commutative ring spectrum **MGr** that consists of the Thom spaces of the tautological vector bundles over Grassmannians. To every smooth closed G-manifold M we associate a *normal class*

$$\langle M \rangle \ \in \ \mathbf{MGr}_0^G(M_+) \ .$$

This class records the equivariant homotopical information in the stable normal bundle of M, and it is the geometric input for the Thom-Pontryagin map to equivariant **mO**-homology. If M has dimension m, then the class lives in the homogeneous summand $\mathbf{MGr}^{[-m]}$ of **MGr**.

The construction starts from the Mostow-Palais embedding theorem [124, 130] that provides a smooth G-equivariant embedding $i : M \longrightarrow V$, for some G-representation V. We can assume without loss of generality that V is a subrepresentation of the chosen complete G-universe $\mathcal{U}_G$. We use the inner product on V to define the normal bundle ν of the embedding at $x \in M$ by

$$\nu_x \ = \ V - (di)(T_x M) \ ,$$

the orthogonal complement of the image of the tangent space $T_x M$ in V. By multiplying with a suitably large scalar, if necessary, we can assume that the embedding is *wide* in the sense that the exponential map

$$D(\nu) \ \longrightarrow \ V \ , \quad (x, v) \ \longmapsto \ i(x) \ + \ v$$

is injective on the unit disc bundle of the normal bundle, and hence a closed G-equivariant embedding. The image of this map is a tubular neighborhood of radius 1 around $i(M)$, and it determines a G-equivariant Thom-Pontryagin collapse map

$$c_M \ : \ S^V \ \longrightarrow \ Th(Gr(V)) \wedge M_+ \ = \ \mathbf{MGr}(V) \wedge M_+$$

as follows: every point outside of the tubular neighborhood is sent to the basepoint, and a point $i(x) + v$, for $(x, v) \in D(\nu)$, is sent to

$$c_M(i(x) + v) \ = \ \left(\frac{v}{1 - |v|}, \ \nu_x \right) \wedge x \ .$$

The normal class $\langle M \rangle$ is the homotopy class of the collapse map c_M.

Proposition 6.2.21. *The normal class of a smooth closed G-manifold is independent of the choice of wide embedding into a G-representation.*

Proof If we enlarge the embedding $i : M \longrightarrow V$ by post-composition with a direct summand embedding $(0, -) : V \longrightarrow U \oplus V$, then the collapse map

associated with the composite embedding $(0, -) \circ i$ is equivariantly homotopic to the composite

$$S^{U\oplus V} \xrightarrow{S^U \wedge c_M} S^U \wedge \mathbf{MGr}(V) \wedge M_+ \xrightarrow{\sigma_{U,V}} \mathbf{MGr}(U \oplus V) \wedge M_+ \ .$$

So the resulting class in $\mathbf{MGr}_0^G(M_+)$ does not change. Two classes based on two different wide embeddings $i : M \longrightarrow V$ and $j : M \longrightarrow W$ can be compared by passing to $V \oplus W$; in this larger representation, the map

$$M \times [0, 1] \longrightarrow V \oplus W \ , \quad (m, t) \longmapsto (t \cdot i(m),\ (1 - t) \cdot j(m))$$

is a smooth isotopy through wide embeddings. This isotopy induces a homotopy between the two collapse maps and shows that altogether the normal class $\langle M\rangle$ is independent of the wide embedding. □

Part (ii) of the following proposition refers to an external multiplication morphism

$$\begin{aligned}\mu_{A,B} \ :\ (\mathbf{MGr} \wedge A_+) \wedge (\mathbf{MGr} \wedge B_+) \ &\cong (\mathbf{MGr} \wedge \mathbf{MGr}) \wedge (A \times B)_+ \\ &\xrightarrow{\mu\wedge(A\times B)_+} \mathbf{MGr} \wedge (A \times B)_+ \ ,\end{aligned}$$

where A and B are two G-spaces and $\mu : \mathbf{MGr} \wedge \mathbf{MGr} \longrightarrow \mathbf{MGr}$ is the multiplication morphism. In part (iv) we consider the kth power M^k of a G-manifold M as a $(\Sigma_k \wr G)$-manifold via the action

$$(\sigma;\ g_1, \ldots, g_k) \cdot (x_1, \ldots, x_k) \ = \ (g_{\sigma^{-1}(1)} x_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(k)} x_{\sigma^{-1}(k)}) \ .$$

The morphism $j = j^{\mathbb{R}}_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}\,\mathbf{MGr}$ that shows up in part (v) was defined in (6.1.5).

Proposition 6.2.22. *Let G be a compact Lie group, and let M and N be smooth closed G-manifolds.*

(i) *Let $i^1 : M \longrightarrow M \cup N$ and $i^2 : N \longrightarrow M \cup N$ denote the inclusions into a disjoint union. Then the relation*

$$\langle M \cup N\rangle \ = \ i^1_*\langle M\rangle \ + \ i^2_*\langle N\rangle$$

holds in the group $\mathbf{MGr}_0^G((M \cup N)_+)$.

(ii) *The relation*

$$\langle M \times N\rangle \ = \ (\mu_{M,N})_*(\langle M\rangle \times \langle N\rangle)$$

holds in the group $\mathbf{MGr}_0^G((M \times N)_+)$.

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups and every smooth closed G-manifold M the relation*

$$\langle \alpha^* M\rangle \ = \ \alpha^*\langle M\rangle$$

holds in $\mathbf{MGr}_0^K(\alpha^*(M)_+)$.

(iv) *For every* $k \geq 0$*, the relation*

$$\langle M^k \rangle \ = \ P^k \langle M \rangle$$

holds in the group $\mathbf{MGr}_0^{\Sigma_k \wr G}(M^k)$.

(v) *Let* B *be a compact smooth* G*-manifold with boundary* ∂B*. Then the class* $\langle \partial B \rangle$ *is in the kernel of the homomorphism*

$$(j \wedge \iota_+)_* \ : \ \mathbf{MGr}_0^G(\partial B_+) \ \longrightarrow \ (\mathrm{sh}\,\mathbf{MGr})_0^G(B_+)\ ,$$

where $\iota : \partial B \longrightarrow B$ *is the inclusion.*

Proof (i) We let $p^1 : (M \cup N)_+ \longrightarrow M_+$ and $p^2 : (M \cup N)_+ \longrightarrow N_+$ denote the two projections. We choose any wide smooth equivariant embedding $i : M \cup N \longrightarrow V$ and observe that the composite

$$S^V \xrightarrow{c_{M\cup N}} \mathbf{MGr}(V) \wedge (M \cup N)_+ \xrightarrow{\mathbf{MGr}(V)\wedge p^1} \mathbf{MGr}(V) \wedge M$$

is on the nose the collapse map for M based on the restriction of the embedding i to M. We then obtain

$$p^1_*\langle M \cup N \rangle \ = \ \langle M \rangle \ = \ p^1_*(i^1_*\langle M \rangle + i^2_*\langle N \rangle)\ ;$$

the second relation uses that $p^1 \circ i^1$ is the identity, $p^2 \circ i^1$ is the trivial map and p^1_* is additive. The analogous argument shows that $p^2_*\langle M \cup N \rangle \ = \ p^2_*(i^1_*\langle M \rangle + i^2_*\langle N \rangle)$. Since equivariant homotopy groups are additive on wedges, the map

$$(p^1_*, p^2_*) \ : \ \mathbf{MGr}_0^G((M \cup N)_+) \ \longrightarrow \ \mathbf{MGr}_0^G(M_+) \times \mathbf{MGr}_0^G(N_+)$$

is bijective, and this proves the claim.

(ii) We choose wide smooth equivariant embeddings

$$i \ : \ M \ \longrightarrow \ V \qquad \text{and} \qquad j \ : \ N \ \longrightarrow \ W$$

into G-representations. The product map

$$i \times j \ : \ M \times N \ \longrightarrow \ V \oplus W$$

is then another wide smooth equivariant embedding that we use for the Thom-Pontryagin construction of $M \times N$. The normal bundle of $i \times j$ is the exterior direct sum of the normal bundles of i and j. The unit disc $D(V \oplus W)$ of the direct sum is contained in the product $D(V) \times D(W)$ of the unit discs, so the exponential tubular neighborhood for $i \times j$ is contained in the product of the exponential tubular neighborhoods for i and j. The collapse map

$$S^{V\oplus W} \xrightarrow{c_{M\times N}} \mathbf{MGr}(V \oplus W) \wedge (M \times N)_+$$

is equivariantly homotopic to the composite

$$S^V \wedge S^W \xrightarrow{c_M \wedge c_N} (\mathbf{MGr}(V) \wedge M_+) \wedge (\mathbf{MGr}(W) \wedge N_+) \xrightarrow{\mu^{V,W}_{M,N}} \mathbf{MGr}(V \oplus W) \wedge (M \times N)_+ \ .$$

This shows the desired relation. Part (iii) is straightforward from the definitions.

(iv) We choose a wide smooth equivariant embedding $i : M \longrightarrow V$ into a G-representation. Then

$$i^k \ : \ M^k \ \longrightarrow \ V^k$$

is a $(\Sigma_k \wr G)$-equivariant wide smooth embedding that we use to calculate the class $\langle M^k \rangle$. The collapse map

$$c_{M^k} \ : \ S^{V^k} \ \longrightarrow \ Th(Gr(V^k)) \wedge M^k_+$$

based on i^k is $(\Sigma_k \wr G)$-equivariantly homotopic to the composite

$$S^{V^k} \xrightarrow{(c_M)^{\wedge k}} (\mathbf{MGr}(V) \wedge M_+)^{\wedge k} \xrightarrow{\text{shuffle}} \mathbf{MGr}(V)^{\wedge k} \wedge M^k_+ \xrightarrow{\mu^{\mathbf{MGr}}_{V,\dots,V} \wedge M^k_+} \mathbf{MGr}(V^k) \wedge M^k_+ \ .$$

This latter composite represents the power operation $P^k\langle M \rangle$, so altogether this shows the desired relation.

(v) We let $C(j \wedge B_+)$ denote the mapping cone of the morphism $j \wedge B_+ : \mathbf{MGr} \wedge B_+ \longrightarrow \mathrm{sh}\,\mathbf{MGr} \wedge B_+$. We define a relative normal class

$$\langle B \rangle^{\mathrm{rel}} \ \in \ \pi^G_1(C(j \wedge B_+))$$

such that the relation

$$\partial(\langle B \rangle^{\mathrm{rel}}) \ = \ (\mathbf{MGr} \wedge \iota_+)_*\langle \partial B \rangle \tag{6.2.23}$$

holds in the group $\pi^G_0(\mathbf{MGr} \wedge B_+)$, where ∂ is the connecting homomorphism (3.1.34) in the long exact homotopy group sequence of the mapping cone. Because two consecutive maps in the long exact homotopy group sequence compose to zero, we can then conclude that

$$(j \wedge \iota_+)_*\langle \partial B \rangle \ = \ (j \wedge B_+)_*(\partial(\langle B \rangle^{\mathrm{rel}})) \ = \ 0 \ .$$

It remains to construct the class $\langle B \rangle^{\mathrm{rel}}$ and establish the relation (6.2.23). We choose an equivariant collar, i.e., a smooth G-equivariant embedding

$$c \ : \ \partial B \times [0, 1) \ \longrightarrow \ B$$

such that $c(-,0) : \partial B \longrightarrow B$ is the inclusion and the image of c is an open neighborhood of the boundary inside B. Then we choose a smooth function

$$\kappa \; : \; [0,1] \longrightarrow [0,1]$$

that is the identity on $[0,1/3]$, identically 1 on $[2/3,1]$ and whose restriction to $[0,2/3)$ is injective. We define the smooth function

$$\psi \; : \; [0,1) \longrightarrow [0,1) \qquad \text{by} \qquad \psi(t) \; = \; \frac{\kappa(t)-1}{t-1} \; ;$$

then $\psi(t) = 1$ for $t \in [0,1/3]$ and $\kappa(t) = 0$ for $t \in [2/3,1)$.

The Mostow-Palais embedding theorem provides a wide smooth G-equivariant embedding $j : B \longrightarrow V$, for some G-representation V. Then the smooth G-map

$$i \; : \; B \longrightarrow V \oplus \mathbb{R}$$

defined by

$$i(b) = \begin{cases} (\psi(t) \cdot j(x) + (1-\psi(t)) \cdot j(b),\ \kappa(t)) & \text{for } b = c(x,t) \text{ with } (x,t) \in \partial B \times [0,1), \\ (j(b),1) & \text{for } b \notin c(B \times [0,1)), \end{cases}$$

is a new wide smooth equivariant embedding which satisfies

$$i(\partial B) \; \subset \; V \times \{0\}$$

and which is 'orthogonal to V near the boundary', i.e., the set $U = c(\partial B \times [0,1/3))$ is an open neighborhood of ∂B in B, and

$$i(U) \; = \; i(\partial B) \times [0,1) \; .$$

Since the embedding i is wide, the exponential map

$$D(\nu) \longrightarrow V \oplus \mathbb{R} \; , \quad (x,v,t) \longmapsto i(x) + (v,t)$$

is injective on the unit disc bundle of the normal bundle ν of j, and hence a closed G-equivariant embedding. We define a continuous map

$$\kappa \; : \; D(\nu) \longrightarrow C\,(j(V) \colon \mathbf{MGr}(V) \longrightarrow \mathbf{MGr}(V \oplus \mathbb{R})) \wedge B_+$$

to the reduced mapping cone of the embedding $j(V)$ of $\mathbf{MGr}(V) \wedge B_+$ into $\mathbf{MGr}(V \oplus \mathbb{R}) \wedge B_+$ as follows. We consider $(b,v,t) \in D(\nu)$ where $b \in B$ and $(v,t) \in V \oplus \mathbb{R}$ is normal to $i(B)$ at $i(b)$. If $b \in U$, then the normal vector must lie in $V \oplus 0$, i.e., $t = 0$. The map κ then takes $(b,v,0)$ to

$$\left[\left(\frac{v}{1-|v|}, \nu_b\right), i_2(b)\right] \wedge b$$

in the cone of $\mathbf{MGr}(V) \wedge B_+$, where $i_2(b) \in [0,1)$ is the second component of $i(b)$. For $b \notin U$, the map κ sends (b,v,t) to $\left(\frac{(v,t)}{1-|v,t|}, \nu_b\right) \wedge b$ in $\mathbf{MGr}(V \oplus \mathbb{R}) \wedge B_+$.

The total space of the disc bundle $D(\nu)$ is a topological manifold with boundary, and its boundary is the union of the sphere bundle $S(\nu)$ and the subspace $D(\nu)|_{\partial B}$, the part sitting over the boundary of B. The map κ sends the subspace $D(\nu)|_{\partial B}$ to the cone point in the mapping cone, and it sends the sphere bundle $S(\nu)$ to the basepoint. So κ sends the entire boundary of $D(\nu)$ to the basepoint of the reduced mapping cone. So we can extend κ continuously to $S^{V\oplus\mathbb{R}}$ by sending the complement of $D(\nu)$ to the basepoint. The result is a continuous based G-map

$$\tilde{c}_B \ : \ S^{V\oplus\mathbb{R}} \ \longrightarrow \ C(j(V)\wedge B_+ : \mathbf{MGr}(V)\wedge B_+ \longrightarrow (\mathrm{sh}\,\mathbf{MGr})(V)\wedge B_+)\ .$$

The map $\tilde{c}_B$ represents the relative normal class $\langle B\rangle^{\mathrm{rel}}$ in $\pi_1^G(C(j\wedge B_+))$.

It remains to establish the relation (6.2.23). The composite

$$S^{V\oplus\mathbb{R}} \xrightarrow{\tilde{c}_B} C(j(V)\wedge B_+) \xrightarrow{p} \mathbf{MGr}(V)\wedge B_+\wedge S^1$$

is equivariantly homotopic to the map $((\mathbf{MGr}(V)\wedge\iota_+)\circ c_{\partial B})\wedge S^1$, where

$$c_{\partial B} \ : \ S^V \ \longrightarrow \ \mathbf{MGr}(V)\wedge(\partial B)_+$$

is the collapse map for ∂B based on the restriction of i to an embedding $\partial B\longrightarrow V$. Thus

$$p_*(\langle B\rangle^{\mathrm{rel}}) \ = \ \iota_*\langle\partial B\rangle\wedge S^1$$

in the group $\pi_1^G(\mathbf{MGr}\wedge B_+\wedge S^1)$. The relation (6.2.23) thus follows from the definition of the connecting homomorphism (3.1.34) as the composite of p_* and the inverse suspension isomorphism. □

The inverse Thom class

$$\tau_{H,W} \ \in \ \mathbf{MGr}_0^H(S^W)$$

of an H-representation W was defined in (6.1.2). The next theorem shows how the normal class of an induced equivariant manifold $G\times_H M$ is determined by the normal class of the H-manifold M and the inverse Thom class of the tangent H-representation $L = T_{eH}(G/H)$. The Wirthmüller isomorphism

$$\mathrm{Wirth}_H^G \ : \ \mathbf{MGr}_0^G((G\times_H M)_+) \xrightarrow{\cong} \mathbf{MGr}_0^H(M_+\wedge S^L)$$

was established in Theorem 3.2.15.

Theorem 6.2.24. *Let H be a closed subgroup of a compact Lie group G and M a closed smooth H-manifold. Then the relation*

$$\mathrm{Wirth}_H^G\langle G\times_H M\rangle \ = \ \langle M\rangle\wedge\tau_{H,L}$$

holds in the group $\mathbf{MGr}_0^H(M_+\wedge S^L)$, where $L = T_{eH}(G/H)$ is the tangent H-representation.

Proof The Wirthmüller map is the composite

$$\mathbf{MGr}_0^G((G\times_H M)_+) \xrightarrow{\mathrm{res}^G_H} \mathbf{MGr}_0^H((G\times_H M)_+) \xrightarrow{(\mathbf{MGr}\wedge l_M)_*} \mathbf{MGr}_0^H(M_+\wedge S^L)\ .$$

Here $l^G_H : G \longrightarrow S^L \wedge H_+$ is the H^2-equivariant collapse maps for the embedding of H into G, and $l_M : (G\times_H M)_+ \longrightarrow M_+\wedge S^L$ is the composite

$$(G\times_H M)_+ \xrightarrow{l^G_H\wedge_H M} S^L\wedge M_+ \ \cong\ M_+\wedge S^L\ ,$$

compare Construction 3.2.1. So the claim is equivalent to the relation

$$(l^G_H\wedge_H M)_*(\mathrm{res}^G_H\langle G\times_H M\rangle) \ =\ \tau_{H,L}\wedge\langle M\rangle$$

in the group $\mathbf{MGr}_0^H(S^L\wedge M_+)$.

We choose a G-equivariant wide smooth embedding $i : G/H \longrightarrow V$ into a G-representation; we also choose an H-equivariant wide smooth embedding $j : M \longrightarrow W$ into an H-representation underlying some other G-representation. Then the map

$$\psi\ :\ G\times_H M \ \longrightarrow\ V\oplus W\ , \quad [g,m]\ \longmapsto\ (i(gH),\ g\cdot j(m))$$

is a G-equivariant wide smooth embedding. We base the collapse map for the G-manifold $G\times_H M$ on the embedding ψ.

The differential at the coset H of the embedding i is a linear embedding

$$L\ =\ T_{eH}(G/H)\ \xrightarrow{di}\ V\ ;$$

we define a scalar product on L so that this embedding becomes isometric. Now we choose a slice as in the construction of the map l^G_H, i.e., a smooth embedding

$$s\ :\ D(L)\ \longrightarrow\ G$$

of the unit disc of L with $s(0) = 1$, such that $s(h\cdot l) = h\cdot s(l)\cdot h^{-1}$ for all $(h,l)\in H\times D(L)$, and such that the differential at 0 of the composite

$$D(L)\ \xrightarrow{s}\ G\ \xrightarrow{\mathrm{proj}}\ G/H$$

is the identity of L. After scaling the slice, if necessary, the map

$$\bar s\ :\ D(L)\times H\ \longrightarrow\ G\ , \quad (l,h)\ \longmapsto\ s(l)\cdot h$$

is a smooth embedding whose image is an H^2-equivariant tubular neighborhood of H inside G. Here the group H^2 acts on the source by

$$(h_1,h_2)\cdot(l,h)\ =\ (h_1 l,\ h_1 h h_2^{-1})\ .$$

The map $l^G_H : G \longrightarrow S^L\wedge H_+$ is then the collapse map for this tubular neighborhood.

The following H-equivariant composite is thus a representative for the class $(l^G_H \wedge_H M)_*(\mathrm{res}^G_H\langle G \times_H M\rangle)$

$$S^{V\oplus W} \xrightarrow{c_{G\times_H M}} \mathbf{MGr}(V\oplus W)\wedge(G\times_H M_+) \xrightarrow{\mathrm{Id}\wedge(l^G_H\wedge_H M)} \mathbf{MGr}(V\oplus W)\wedge S^L\wedge M_+ \ .$$

The map l^G_H takes the complement of the tubular neighborhood $\bar{s}(D(L)\times H)$ to the basepoint. So the map

$$l^G_H \wedge_H M_+ \ : \ (G\times_H M)_+ \ \longrightarrow \ S^L \wedge M_+$$

takes the complement of the subset $\bar{s}(D(L)\times H)\times_H M$ to the basepoint. The composite

$$D(L)\times M \ \xrightarrow{\bar{s}} \ G \ \xrightarrow{\psi} \ V\oplus W$$

is given by the formula

$$(l,m) \ \longmapsto \ (i(s(l)\cdot H), \ s(l)\cdot j(m)) \ .$$

We define a homotopy of smooth wide H-equivariant embeddings

$$[0,1]\times D(L)\times M \ \longrightarrow \ V\oplus W \quad \text{by} \quad (t,l,m) \ \longmapsto \ (i(s(l)\cdot H), \ s(tl)\cdot j(m)) \ .$$

For every time $t\in[0,1]$, the corresponding embedding has an associated collapse map

$$S^{V\oplus W} \ \longrightarrow \ \mathbf{MGr}(V\oplus W)\wedge S^L\wedge M_+ \ .$$

So the homotopy induces a homotopy of based H-equivariant maps between the above representative for $(l^G_H \wedge_H M)_*(\mathrm{res}^G_H\langle G\times_H M\rangle)$ and the collapse map for the product embedding

$$D(L)\times M \ \longrightarrow \ V\oplus W \quad \text{by} \quad (l,m) \ \longmapsto \ (i(s(l)\cdot H), \ j(m)) \ . \qquad \square$$

Example 6.2.25. We identify the normal classes of some equivariant manifolds in terms of classes that were previously defined. In the case where $M=*$ is a single point, Theorem 6.2.24 specializes to the relation

$$\mathrm{Wirth}^G_H\langle G/H\rangle \ = \ \tau_{H,L}$$

in the group $\mathbf{MGr}^H_0(S^L)$, where $L=T_{eH}(G/H)$ is the tangent H-representation. In other words, the normal class of the homogeneous G-manifold G/H is the inverse, under the Wirthmüller isomorphism, of the inverse Thom class.

We let V be a representation of a compact Lie group G. Then the unit sphere $S(\mathbb{R}\oplus V)$ in $\mathbb{R}\oplus V$ is a smooth closed G-manifold. We recall that

$$\Pi_V \ : \ S(\mathbb{R}\oplus V) \ \longrightarrow \ S^V \ , \quad (x,v) \ \longmapsto \ \frac{v}{1-x}$$

is the G-equivariant stereographic projection. We claim that

$$\langle S(\mathbb{R}\oplus V)\rangle \;=\; (\mathbf{MGr}\wedge\Pi_V^{-1})_*(\tau_{G,V})\ . \tag{6.2.26}$$

Since both sides of equation (6.2.26) commute with restriction along continuous homomorphisms, it suffices to show the relation for the tautological m-dimensional $O(m)$-representation ν_m. The composite

$$O(m+1)/O(m) \xrightarrow{\ \psi\ } S(\mathbb{R}\oplus\nu_m) \xrightarrow{\ \Pi_{\nu_m}\ } S^{\nu_m}$$

is $O(m)$-equivariantly homotopic to the map $l_{O(m)}^{O(m+1)} : O(m+1)/O(m) \longrightarrow S^{\nu_m}$ that appears in the Wirthmüller isomorphism, where $\psi(A\cdot O(m)) = A\cdot(0,\dots,0,1)$. So we can argue:

$$\begin{aligned}
&(\mathbf{MGr}\wedge\Pi_{\nu_m})_*\langle S(\mathbb{R}\oplus\nu_m)\rangle\\
&\qquad = (\mathbf{MGr}\wedge\Pi_{\nu_m})_*((\mathbf{MGr}\wedge\psi_*)\langle \mathrm{res}^{O(m+1)}_{O(m)}\,O(m+1)/O(m)\rangle)\\
&\qquad = (\mathbf{MGr}\wedge l^{O(m+1)}_{O(m)})_*(\mathrm{res}^{O(m+1)}_{O(m)}\langle O(m+1)/O(m)\rangle)\\
&\qquad = \mathrm{Wirth}^{O(m+1)}_{O(m)}\langle O(m+1)/O(m)\rangle \;=\; \tau_{O(m),\nu_m}\ .
\end{aligned}$$

Inverting $(\mathbf{MGr}\wedge\Pi_{\nu_m})_*$ gives the desired relation (6.2.26) in the universal example.

Construction 6.2.27 (Equivariant Thom-Pontryagin construction)**.** The equivariant Thom-Pontryagin construction defines a natural transformation of G-homology theories

$$\Theta^G \;=\; \Theta^G(X)\;:\; \widetilde{\mathcal{N}}^G_*(X) \longrightarrow \mathbf{mO}^G_*(X)\ ,$$

as we now recall. We let (M,h) be an m-dimensional singular G-manifold over a based G-space X. The way we have set things up, all the geometry is already encoded in the normal class $\langle M\rangle\in\mathbf{MGr}^G_0(M_+)$; the rest is a formal procedure: we push the normal class forward along the morphism $b:\mathbf{MGr}\longrightarrow\mathbf{mOP}$ defined in (6.1.31), and use the periodicity of $\mathbf{mOP}$ to move into the homogeneous summand $\mathbf{mO}$ of degree 0. While the normal class is *not* yet a bordism invariant, pushing it forward to $\mathbf{mOP}$ makes it one, see Proposition 6.2.28 below. The periodicity class $t\in\pi^e_{-1}(\mathbf{mOP}^{[-1]})$ was defined in (6.1.29), and we let $\sigma\in\pi^e_1(\mathbf{mOP}^{[1]})$ be its inverse. We define

$$\Theta^G[M,h] \;=\; (b\wedge h)_*\langle M\rangle\cdot p_G^*(\sigma^m)\;\in\;\mathbf{mO}^G_m(X)\ ,$$

i.e., we take the image of the normal class of M under the homomorphism

$$(b\wedge h)_*\;:\;\mathbf{MGr}^G_0(M_+)\longrightarrow\mathbf{mOP}^G_0(X)$$

and multiply by the unit $p_G^*(\sigma^m)$ in $\pi^G_m(\mathbf{mOP}^{[m]})$. Since M has dimension m, the

normal class lies in the homogeneous summand $\mathbf{MGr}^{[-m]}$, whereas σ^m lies in the summand $\mathbf{mOP}^{[m]}$; so the product indeed lies in the homogeneous degree 0 summand $\mathbf{mO} = \mathbf{mOP}^{[0]}$.

Proposition 6.2.28. *The class $\Theta^G[M,h]$ in $\mathbf{mO}_m^G(X)$ only depends on the bordism class of the singular G-manifold (M,h).*

Proof The morphisms $j_{\mathbf{MGr}} = j^{\mathbb{R}}_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}\,\mathbf{MGr}$ and $j_{\mathbf{mOP}} = j^{\mathbb{R}}_{\mathbf{mOP}} : \mathbf{mOP} \longrightarrow \mathrm{sh}\,\mathbf{mOP}$ were defined in (6.1.5) and (6.1.28), respectively. The square

$$\begin{tikzcd}
\mathbf{MGr} \arrow[r, "j_{\mathbf{MGr}}"] \arrow[d, "b"'] & \mathrm{sh}\,\mathbf{MGr} \arrow[d, "\mathrm{sh}\,b"] \\
\mathbf{mOP} \arrow[r, "\simeq", "j_{\mathbf{mOP}}"'] & \mathrm{sh}\,\mathbf{mOP}
\end{tikzcd}$$

commutes and the lower horizontal morphism is a homotopy equivalence of orthogonal spectra by Proposition 6.1.30 (i). Now we consider a smooth compact G-manifold B with boundary ∂B. Proposition 6.2.22 (v) shows that

$$(j_{\mathbf{mOP}} \wedge B_+)_*((b \wedge \iota_+)_*\langle \partial B\rangle) \ = \ (\mathrm{sh}\,b \wedge B_+)_*((j_{\mathbf{MGr}} \wedge \iota_+)_*\langle \partial B\rangle) \ = \ 0\ ,$$

where $\iota : \partial B \longrightarrow B$ is the inclusion. Since $j_{\mathbf{mOP}}$ is a homotopy equivalence, this implies $(b \wedge \iota_+)_*\langle \partial B\rangle = 0$. We let $(M, h : M \longrightarrow X)$ be a singular G-manifold that is null-bordant. We choose a null-bordism $(B, H : B \longrightarrow X, \psi : M \cong \partial B)$, so that $H \circ \iota \circ \psi = h$. Then

$$(b \wedge h)_*\langle M\rangle \ = \ (b \wedge (H \circ \iota \circ \psi))_*\langle M\rangle \ = \ (\mathbf{mOP} \wedge H)_*((b \wedge \iota_+)_*\langle \partial B\rangle) \ = \ 0\ .$$

Multiplying by $p_G^*(\sigma^m)$ gives that $\Theta^G[M,h] \ = \ 0$. Since the normal class is additive on disjoint unions (Proposition 6.2.22 (i)), naturality then implies that $\Theta^G[M,h]$ only depends on the bordism class of (M,h). □

Example 6.2.29. We let G be a compact Lie group and V a G-representation of dimension m. We claim that then

$$\Theta^G(d_{G,V}) \ = \ \bar{\tau}_{G,V} \qquad (6.2.30)$$

in the group $\mathbf{mO}_m^G(S^V)$. In other words, the Thom-Pontryagin construction matches the distinguished geometric bordism class $d_{G,V}$ in $\widetilde{\mathcal{N}}_m^G(S^V)$ with the shifted inverse Thom class in the Thom spectrum $\mathbf{mO}$.

Indeed, the class $d_{G,V}$ is represented by the singular G-manifold $(S(\mathbb{R} \oplus$

$V), \Pi_V)$, where $\Pi_V : S(\mathbb{R} \oplus V) \longrightarrow S^V$ is the stereographic projection. So

$$\begin{aligned} \Theta^G(d_{G,V}) &= (b \wedge \Pi_V)_* \langle S(\mathbb{R} \oplus V) \rangle \cdot p_G^*(\sigma^m) \\ \overset{(6.2.26)}{=} & (b \wedge \Pi_V)_*((\mathbf{MGr} \wedge \Pi_V^{-1})_*(\tau_{G,V})) \cdot p_G^*(\sigma^m) \\ &= (b \wedge S^V)_*(\tau_{G,V}) \cdot p_G^*(\sigma^m) \ = \ \bar{\tau}_{G,V} \ . \end{aligned}$$

The next theorem says, roughly speaking, that the Thom-Pontryagin construction is compatible 'with all global structure'. There is one caveat, though, namely in how the geometric induction in equivariant bordism compares with the homotopy theoretic transfer. Indeed, the geometric induction increases the dimension by the dimension of G/H, whereas the Wirthmüller isomorphism increases the dimension in a twisted way, namely by the sphere of the tangent representation $L = T_{eH}(G/H)$ of H in G. Multiplication by the inverse Thom class is needed to compensate this 'twist' on the homotopy theory side. In the special case where H has finite index in G, then the tangent representation is zero, so in this special case $\bar{\tau}_{H,L} = 1$ and part (v) of the following theorem specializes to the simpler relation

$$\Theta^G(G \times_H y) \ = \ G \ltimes_H \Theta^H(y) \ .$$

Theorem 6.2.31. (i) *The Thom-Pontryagin map Θ^G is additive.*

(ii) *The Thom-Pontryagin map is multiplicative, i.e., for all classes $x \in \widetilde{\mathcal{N}}_m^G(X)$ and $y \in \widetilde{\mathcal{N}}_n^G(Y)$, the relation*

$$\Theta^G(x \wedge y) \ = \ \Theta^G(x) \wedge \Theta^G(y)$$

holds in $\mathbf{mO}_{m+n}^G(X \wedge Y)$.

(iii) *The Thom-Pontryagin map is compatible with the boundary maps in the mapping cone sequences in equivariant bordism and equivariant* **mO**-*homology, i.e., Θ^G is a transformation of equivariant homology theories.*

(iv) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups, every based G-space X and all $x \in \widetilde{\mathcal{N}}_m^G(X)$ the relation*

$$\Theta^K(\alpha^*(x)) \ = \ \alpha^*(\Theta^G(x))$$

holds in $\mathbf{mO}_m^K(\alpha^*(X))$.

(v) *If H is a closed subgroup of G, then for every H-space Y and all $y \in \mathcal{N}_m^G(Y)$, the relation*

$$\Theta^G(G \times_H y) \ = \ G \ltimes_H (\Theta^H(y) \wedge \bar{\tau}_{H,L})$$

holds in $\mathbf{mO}_{m+d}^G((G \times_H Y)_+)$, where $L = T_{eH}(G/H)$ is the tangent H-representation and $d = \dim(G/H)$.

Proof (i) The two functors

$$X \longmapsto \widetilde{\mathcal{N}}_n^G(X) \qquad \text{and} \qquad X \longmapsto \mathbf{mO}_n^G(X)$$

from the category of based G-spaces to the category of abelian groups are reduced and additive. Proposition 2.2.12 thus shows that the Thom-Pontryagin map is additive.

Part (ii) is a formal consequence of the multiplicativity of the normal classes. We consider singular G-manifolds $(M, h : M \longrightarrow X)$ and $(N, g : N \longrightarrow Y)$. The class $[\![M, h]\!] \wedge [\![N, g]\!]$ is then represented by the singular G-manifold $(M \times N, q \circ (h \times g))$, where $q : X \times Y \longrightarrow X \wedge Y$ is the quotient map. Then

$$\begin{aligned}(b \wedge (q \circ (h \times g)))_* \langle M \times N \rangle &= (b \wedge (q \circ (h \times g)))_*((\mu_{M,N})_*(\langle M \rangle \times \langle N \rangle)) \\ &= (\mu_{M,N})_*((b \wedge h)_* \langle M \rangle \times (b \wedge g)_* \langle N \rangle) \\ &= (b \wedge h)_* \langle M \rangle \wedge (b \wedge g)_* \langle N \rangle\end{aligned}$$

in the group $\mathbf{mOP}_0^G(X \wedge Y)$. The first equation is Proposition 6.2.22 (ii); the second equation exploits that $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$ is a homomorphism of E_∞-orthogonal ring spectra. Now we multiply with the class $p_G^*(\sigma^{m+n})$ and obtain the desired relation

$$\begin{aligned}\Theta^G [\![M \times N, q \circ (h \times g)]\!] &= (b \wedge (q \circ (h \times g)))_* \langle M \times N \rangle \cdot p_G^*(\sigma^{m+n}) \\ &= ((b \wedge h)_* \langle M \rangle \wedge (b \wedge g)_* \langle N \rangle) \cdot p_G^*(\sigma^{m+n}) \\ &= ((b \wedge h)_* \langle M \rangle \cdot p_G^*(\sigma^m)) \wedge ((b \wedge g)_* \langle N \rangle \cdot p_G^*(\sigma^n)) \\ &= \Theta^G [\![M, h]\!] \wedge \Theta^G [\![N, g]\!] \ .\end{aligned}$$

(iii) We let $f : X \longrightarrow Y$ be a continuous G-map. Compatibility of the Thom-Pontryagin construction with the boundary homomorphism amounts to the commutativity of the following square:

$$\begin{tikzcd} \widetilde{\mathcal{N}}_{m+1}^G(Cf) \arrow[r, "p_*"] \arrow[d, "\Theta^G"'] & \widetilde{\mathcal{N}}_{m+1}^G(X_+ \wedge S^1) \arrow[r, "(-\wedge d_{G,\mathbb{R}})^{-1}", "\cong"'] \arrow[d, "\Theta^G"'] & \mathcal{N}_m^G(X) \arrow[d, "\Theta^G"] \\ \mathbf{mO}_{m+1}^G(Cf) \arrow[r, "p_*"'] & \mathbf{mO}_{m+1}^G(X_+ \wedge S^1) \arrow[r, "\cong", "(-\wedge \bar{\tau}_{G,\mathbb{R}})^{-1}"'] & \mathbf{mO}_m^G(X_+) \end{tikzcd}$$

Here $p : Cf \longrightarrow X_+ \wedge S^1$ is the projection defined in (3.1.32). Indeed, the upper composite agrees with the boundary map in equivariant bordism by Proposition 6.2.11; the lower composite is the homotopy theoretic boundary map by the definition in Construction 3.1.31 and the fact that the suspension isomorphism in G-equivariant $\mathbf{mO}$-homology is exterior multiplication with the class $\bar{\tau}_{G,\mathbb{R}} \in \mathbf{mO}_1^G(S^1)$.

The Thom-Pontryagin construction is natural for continuous G-maps, so it

remains to show the commutativity of the right square above. However, multiplicativity and (6.2.30) give

$$\Theta^G(x \wedge d_{G,\mathbb{R}}) \ = \ \Theta^G(x) \wedge \Theta^G(d_{G,\mathbb{R}}) \ = \ \Theta^G(x) \wedge \bar{\tau}_{G,\mathbb{R}}$$

for all $x \in \widetilde{\mathcal{N}}_m^G(X)$.

Part (iv) is straightforward from the definitions.

(v) This is a formal consequence of the formula for the normal class of an induced manifold in Theorem 6.2.24. We consider a singular H-manifold $(M, h : M \longrightarrow Y)$. The class $G \times_H [M, h]$ is then represented by the singular G-manifold $(G \times_H M, G \times_H h)$. Then

$$\begin{aligned} \mathrm{Wirth}_H^G((b \wedge (G \ltimes_H h))_* \langle G \times_H M \rangle) \ &= \ (b \wedge h \wedge S^L)_* (\mathrm{Wirth}_H^G \langle G \times_H M \rangle) \\ &= \ (b \wedge h \wedge S^L)_* (\langle M \rangle \wedge \tau_{H,L}) \\ &= \ (b \wedge h)_* \langle M \rangle \wedge (b \wedge S^L)_* (\tau_{H,L}) \\ &= \ (b \wedge h)_* \langle M \rangle \wedge \tau_{H,L} \ . \end{aligned}$$

Now we multiply with the class $p_G^*(\sigma^{m+n})$ and obtain

$$\begin{aligned} \mathrm{Wirth}_H^G(\Theta^G(G \times_H y)) \ &= \ \mathrm{Wirth}_H^G \left((b \wedge (G \ltimes_H h))_* \langle G \times_H M \rangle \cdot p_G^*(\sigma^{m+d}) \right) \\ &= \ \mathrm{Wirth}_H^G((b \wedge (G \ltimes_H h))_* \langle G \times_H M \rangle) \cdot p_H^*(\sigma^{m+d}) \\ &= \ ((b \wedge h)_* \langle M \rangle \wedge \tau_{H,L}) \cdot p_H^*(\sigma^{m+d}) \\ &= \ ((b \wedge h)_* \langle M \rangle \cdot p_H^*(\sigma^m)) \wedge (\tau_{H,L} \cdot p_H^*(\sigma^d)) \\ &= \ \Theta^H(y) \wedge \bar{\tau}_{H,L} \ . \end{aligned}$$

By Theorem 3.2.15 the Wirthmüller isomorphism is inverse to the composite of $\varepsilon_L : \mathbf{mO}_{m+d}^H(Y_+ \wedge S^L) \longrightarrow \mathbf{mO}_{m+d}^H(Y_+ \wedge S^L)$ (the effect of $-\mathrm{Id}_L$ on S^L) and the exterior transfer. Proposition 6.1.4 (i) implies that the inverse Thom class $\tau_{H,L}$ is fixed by the involution ε_L. So this last relation is equivalent to the desired relation $\Theta^G(G \times_H y) = G \ltimes_H (\Theta^H(y) \wedge \bar{\tau}_{H,L})$. □

As before we let $E\mathcal{P}$ be a universal G-space for the family of proper closed subgroups of G, and $\tilde{E}\mathcal{P}$ denotes its unreduced suspension. Then $(\tilde{E}\mathcal{P})^G = S^0$, consisting of the two cone points, and $(\tilde{E}\mathcal{P})^H$ is contractible for every proper subgroup of G.

Proposition 6.2.32. *The Thom-Pontryagin map*

$$\Theta^G \ : \ \widetilde{\mathcal{N}}_*^G(\tilde{E}\mathcal{P}) \ \longrightarrow \ \mathbf{mO}_*^G(\tilde{E}\mathcal{P})$$

is an isomorphism for every compact Lie group G.

Proof We claim that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathcal{N}}^G_n(\tilde{E}\mathcal{P}) \arrow[rr, "\Phi_{\mathrm{geom}}", "\cong"'] \arrow[d, "\Theta^G"'] & & \bigoplus_{j\geq 0} \mathcal{N}_{n-j}(Gr^{G,\perp}_j) \arrow[d, "\bigoplus \Theta^e"] \\
\mathbf{mO}^G_n(\tilde{E}\mathcal{P}) \arrow[r, "\cong", "\Phi"'] & \Phi^G_n(\mathbf{mO}) \arrow[r, "\cong", "(6.1.50)"'] & \bigoplus_{j\geq 0} \mathbf{mO}_{n-j}((Gr^{G,\perp}_j)_+)
\end{tikzcd}$$

Here $Gr^{G,\perp}_j = (Gr_j(\mathcal{U}^\perp_G))^G$, and the geometric fixed-point homomorphism Φ_{geom} was defined in (6.2.15). To see this we let (M, h) be an n-dimensional singular G-manifold over $\tilde{E}\mathcal{P}$. We choose a smooth G-equivariant wide embedding $i : M \longrightarrow V$ into a G-representation and let

$$c_M \ : \ S^V \ \longrightarrow \ Th(Gr_{|V|-n}(V)) \wedge \tilde{E}\mathcal{P}_+$$

be the associated collapse map. As before we let $M^{(j)}$ denote the $(n-j)$-dimensional component of the fixed-point manifold M^G_0 over the point 0, and

$$\nu_j \ : \ M^{(j)} \ \longrightarrow \ (Gr_j(V^\perp))^G$$

the classifying map for the normal bundle of $M^{(j)}$ inside M, sending a fixed-point $x \in M^{(j)}$ to the orthogonal complement of $(di)(T_x(M^{(j)}))$ inside $(di)(T_xM)$.

The trick is now to base the Thom-Pontryagin construction for $M^{(j)}$ on the non-equivariant embedding

$$M^{(j)} \ \xrightarrow{\text{incl}} \ M^G_0 \ \xrightarrow{i^G} \ V^G$$

into the G-fixed-points of V. As explained in Example 6.1.46, the G-fixed-points of the Thom space that occurs in the target of c_M decompose as a wedge. The composite

$$S^{V^G} \ \xrightarrow{(c_M)^G} (Th(Gr_{|V|-n}(V)))^G$$

with the projection to the jth summand is then on the nose the map

$$S^{V^G} \ \xrightarrow{c_{M^{(j)}} \wedge (\nu_j)_+} \ Th(Gr_{\dim(V^G)+j}(V^G)) \wedge (Gr_j(V^\perp))^G_+ \ ,$$

the smash product of the collapse map for the non-equivariant manifold $M^{(j)}$, based on the embedding i^G, and ν_j. This shows that $\Phi(\Theta^G[\![M, h]\!])$ is the non-equivariant Thom-Pontryagin construction applied to $\Phi_{\mathrm{geom}}[\![M, h]\!]$.

The upper map in the original diagram is an isomorphism by Proposition 6.2.16. The lower left homotopical geometric fixed-point map Φ identifies the equivariant homotopy groups of $\mathbf{mO} \wedge \tilde{E}\mathcal{P}$ with the geometric fixed-point groups $\Phi^G_*(\mathbf{mO})$, as shown in Proposition 3.3.8. The lower right isomorphism is the calculation of these geometric fixed-point groups in (6.1.50). The right vertical map is a direct sum of non-equivariant Thom-Pontryagin maps, hence

an isomorphism by Thom's theorem [173, Thm. IV.8]. Since the diagram commutes, the left vertical map is also an isomorphism. □

Now we come to the main result of this section, showing that the equivariant homology theory represented by the global Thom spectrum **mO** is equivariant bordism, at least for products of finite groups and tori. The result is usually credited to Wasserman, because it can be derived from his equivariant transversality theorem [184, Thm. 3.11].

Theorem 6.2.33 (Wasserman)**.** *Let G be a compact Lie group that is isomorphic to a product of a finite group and a torus. Then for every cofibrant G-space X, the Thom-Pontryagin map*

$$\Theta^G(X) \ : \ \mathcal{N}_*^G(X) \ \longrightarrow \ \mathbf{mO}_*^G(X_+)$$

is an isomorphism.

Proof We adapt tom Dieck's proof given in [177, Satz 5] to our setting. Tom Dieck only considers finite groups, where homotopy theoretic transfer and geometric induction match up under the Thom-Pontryagin construction. The general case has a new ingredient, namely the observation that for compact Lie groups of positive dimensions the difference between homotopy theoretic transfer and geometric induction is controlled by the inverse Thom class of the tangent representation, compare Theorem 6.2.24 and Theorem 6.2.31 (v).

We prove the statement by double induction over the dimension and the number of path components of G. The induction starts with the trivial group, i.e., the non-equivariant statement, which is Thom's celebrated theorem [173, Thm. IV.8]. Now we let G be a non-trivial compact Lie group that is a product of a finite group and a torus, and we assume that the theorem has been established for all such groups of smaller dimension, and for all groups of the same dimension but with fewer path components.

Every cofibrant G-space is equivariantly homotopy equivalent to a G-CW-complex. We can thus assume without loss of generality that X is a G-CW-complex. To show that Θ^G is an isomorphism, we exploit the fact that $\mathcal{N}_*^G$ and $\mathbf{mO}_*^G$ are both equivariant homology theories and Θ^G is a morphism of homology theories. This reduces the claim to the special case $X = G/H$ of an orbit for a closed subgroup H of G. The argument for an orbit falls into two cases, depending on whether H is a proper subgroup or $H = G$.

When H is a proper closed subgroup of G, the following diagram commutes

by Proposition 6.2.31 (v) for the one-point H-space:

$$\begin{tikzcd}
\mathcal{N}_m^H(*) \arrow[rr, "G\times_H -"] \arrow[d, "\Theta^H(*)"'] & & \mathcal{N}_{m+d}^G(G/H) \arrow[d, "\Theta^G(G/H)"] \\
\mathbf{mO}_m^H(S^0) \arrow[r, "-\cdot\bar\tau_{H,L}"'] & \mathbf{mO}_{m+d}^H(S^L) \arrow[r, "G\ltimes_H -"'] & \mathbf{mO}_{m+d}^G(G/H_+)
\end{tikzcd}$$

In that diagram, the left vertical map is an isomorphism by the inductive hypothesis. The induction map $G\times_H - : \mathcal{N}_m^H \longrightarrow \mathcal{N}_{m+d}^G(G/H)$ is an isomorphism, with inverse given by sending a singular G-manifold $f : M \longrightarrow G/H$ to the fiber over the coset H of an equivariant smooth approximation of f. The lower right horizontal map is an isomorphism by Theorem 3.2.15. Now we use the hypothesis that the group G is a product of a finite group and a torus. In this situation the group H acts trivially on the tangent representation L, so multiplication by the class $\bar\tau_{H,L}$ in $\mathbf{mO}_d^H(S^L)$ is the suspension isomorphism, hence bijective, by Proposition 6.2.11. Hence the right vertical Thom-Pontryagin map $\Theta^G(G/H)$ is an isomorphism.

Now we treat the case $H = G$. We let $E\mathcal{P}$ be a universal G-space for the family of proper subgroups of G, and $\tilde{E}\mathcal{P}$ its unreduced suspension. We then get compatible long exact isotropy separation sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{N}_*^G(E\mathcal{P}) \arrow[r, "p_*"] \arrow[d, "\Theta^G"] & \mathcal{N}_*^G(*) \arrow[r, "i_*"] \arrow[d, "\Theta^G"] & \widetilde{\mathcal{N}}_*^G(\tilde{E}\mathcal{P}) \arrow[r, "\partial"] \arrow[d, "\Theta^G"] & \mathcal{N}_{*-1}^G(E\mathcal{P}) \arrow[r] \arrow[d, "\Theta^G"] & \cdots \\
\cdots \arrow[r] & \mathbf{mO}_*^G(E\mathcal{P}_+) \arrow[r, "p_*"'] & \mathbf{mO}_*^G(S^0) \arrow[r, "i_*"'] & \mathbf{mO}_*^G(\tilde{E}\mathcal{P}) \arrow[r, "\partial"'] & \mathbf{mO}_{*-1}^G(E\mathcal{P}_+) \arrow[r] & \cdots
\end{tikzcd}$$

The map Θ^G is an isomorphism for the cofibrant and fixed-point free G-space $E\mathcal{P}$ by the previous paragraph, and Θ^G is an isomorphism for $\tilde{E}\mathcal{P}$ by Proposition 6.2.32. So the five lemma shows that the Thom-Pontryagin map for the one-point G-space is an isomorphism. This completes the inductive step, and hence the proof of the theorem. □

Remark 6.2.34. In dimension 0, Theorem 6.1.44 gives an explicit description of the global functor $\underline{\pi}_0(\mathbf{mO})$ as the quotient of the Burnside ring global functor $\mathbb{A}$ by the global subfunctor generated by $\mathrm{tr}_e^{O(1)} \in \mathbb{A}(O(1))$. Equivariant manifolds of dimension 0 are easy to understand, and this allows us to present the groups $\mathcal{N}_0^G$ in a global fashion very similar to (but different from) this presentation of $\underline{\pi}_0(\mathbf{mO})$. We use this description to give a direct verification that the map

$$\Theta^G \ : \ \mathcal{N}_0^G \ \longrightarrow \ \pi_0^G(\mathbf{mO})$$

is an isomorphism when G is a product of a finite group and a torus. We will

also use the calculation of $\underline{\pi}_0(\mathbf{mO})$ to show that the map Θ^G is *not* an isomorphism in general.

As we mentioned above, the groups $\mathcal{N}_0^G$ enjoy the structure of a restricted global functor, i.e., a group-like global power monoid. Moreover, the interval $[-1,1]$ with C_2-action by reflection at the origin is a C_2-equivariant null-bordism of the free transitive C_2-set. This shows that

$$\mathrm{ind}_e^{C_2}(1) \;=\; 0 \quad \text{in} \quad \mathcal{N}_0^{C_2} \;,$$

where $1 \in \mathcal{N}_0^e$ is the bordism class of a point. The action on the class 1 thus factors over a morphism of restricted global functors

$$\mathbb{A}^{\mathrm{res}}/\langle \mathrm{ind}_e^{C_2}\rangle^{\mathrm{res}} \;\longrightarrow\; \underline{\mathcal{N}}_0 \;, \tag{6.2.35}$$

from the quotient of the represented restricted global functor $\mathbb{A}^{\mathrm{res}} = \mathbf{A}^{\mathrm{res}}(e,-)$ by the restricted global subfunctor generated by $\mathrm{ind}_e^{C_2} \in \mathbb{A}^{\mathrm{res}}(C_2)$.

We claim that the morphism (6.2.35) is an isomorphism. Indeed, a smooth G-manifold of dimension 0 is just a finite G-set, so (6.2.35) is surjective. By the same algebraic argument as for unrestricted global functors in Proposition 6.1.45, the value of the restricted global functor $\langle \mathrm{ind}_e^{C_2}\rangle^{\mathrm{res}}$ at G is the subgroup of $\mathbb{A}^{\mathrm{res}}(G)$ generated by $2\cdot\mathbb{A}^{\mathrm{res}}(G)$ and the classes $\mathrm{ind}_H^G \circ p_H^*$ for those finite index subgroups H with Weyl group of even order. So the source of the map (6.2.35) at a group G is an $\mathbb{F}_2$-vector space with basis the classes $\mathrm{ind}_H^G \circ p_H^*$ for those conjugacy classes of finite index subgroups H with Weyl group of odd order. The same classes form a basis for the bordism group $\mathcal{N}_0^G$; this is shown for finite groups in [164, Prop. 13.1], and the general case follows because restriction along the projection $p : G \longrightarrow \pi_0(G)$ induces an isomorphism $p^* : \mathcal{N}_0^{\pi_0(G)} \longrightarrow \mathcal{N}_0^G$ for bordism of 0-manifolds.

Summing up, we have calculated both sides of the Thom-Pontryagin map Θ in dimension 0; under these isomorphisms Θ becomes the map

$$\mathbb{A}^{\mathrm{res}}/\langle \mathrm{ind}_e^{C_2}\rangle^{\mathrm{res}} \;\longrightarrow\; \mathbb{A}/\langle \mathrm{tr}_e^{C_2}\rangle \;,$$

i.e., the same kind of quotient in the category of restricted versus unrestricted global functors. So the only difference between $\underline{\mathcal{N}}_0$ and $\underline{\pi}_0(\mathbf{mO})$ is that the left side only has finite index transfers, whereas the right-hand side also has transfers for infinite index inclusions with finite Weyl group. In finite and abelian compact Lie groups, every subgroup inclusion with finite Weyl group is necessarily of finite index, so for finite and abelian compact Lie groups, there is no difference in the two kinds of quotients. This is an independent verification of Theorem 6.2.33 in dimension 0. Moreover, we conclude that the Thom-Pontryagin map $\Theta^G : \mathcal{N}_0^G \longrightarrow \pi_0^G(\mathbf{mO})$ in dimension 0 is always injective.

On the other hand, the map Θ^G is *not* generally surjective in dimension 0. A specific example is the group $G = SU(2)$: the normalizer $N = N_{SU(2)}T$ of a

maximal torus T of $SU(2)$ is self-normalizing, so N has trivial Weyl group in $SU(2)$. So the classes 1 and $\mathrm{tr}_N^{SU(2)}(1)$ are linearly independent in $\pi_0^{SU(2)}(\mathbf{mO})$. On the other hand $\mathcal{N}_0^{SU(2)} = \mathbb{Z}/2$ because $SU(2)$ is connected.

Remark 6.2.36 (Stable equivariant bordism and localized **mO**)**.** We showed in Corollary 6.1.35 that the localized equivariant **mO**-theory $\mathbf{mO}_*^G[1/\tau]$ is isomorphic to the theory $\mathbf{MO}_*^G$. Our next task is to show that $\mathbf{mO}[1/\tau]$, and hence also **MO**, has a geometric interpretation as *stable equivariant bordism*. In contrast to Theorem 6.2.33, this stable interpretation works for all compact Lie groups, not only for products of finite groups and tori.

In [27], Bröcker and Hook define the *stable equivariant bordism groups* $\mathfrak{N}_*^{G:S}(X)$ of a based G-space X as the localization of the geometric bordism group $\tilde{\mathcal{N}}_*^G(X)$ by formally inverting all the classes $d_{G,V}$. More precisely, their definition comes down to

$$\mathfrak{N}_m^{G:S}(X) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\ \widetilde{\mathcal{N}}^G_{m+|V|}(X\wedge S^V)\ ;$$

for $V \subset W$, the structure map in the colimit system is the multiplication

$$\widetilde{\mathcal{N}}^G_{m+|V|}(X\wedge S^V) \xrightarrow{-\wedge d_{G,W-V}} \widetilde{\mathcal{N}}^G_{m+|W|}(X\wedge S^V\wedge S^{W-V}) \ \cong\ \widetilde{\mathcal{N}}^G_{m+|W|}(X\wedge S^W)\ .$$

As we explained in Example 6.2.29, the Thom-Pontryagin construction takes the distinguished geometric bordism class $d_{G,V}$ to the shifted inverse Thom class, i.e., $\Theta^G(d_{G,V}) = \bar{\tau}_{G,V}$ in $\mathbf{mO}^G_{|V|}(S^V)$. Since the Thom-Pontryagin maps take the geometric product in equivariant bordism to the homotopy theoretic product in **mO**, we conclude that for every compact Lie group G, every G-representation V and every based G-space X, the following square commutes:

$$\begin{tikzcd}
\widetilde{\mathcal{N}}^G_m(X) \arrow[r, "\Theta^G"] \arrow[d, "-\wedge d_{G,V}"'] & \mathbf{mO}^G_m(X) \arrow[d, "-\cdot\bar{\tau}_{G,V}"] \\
\widetilde{\mathcal{N}}^G_{m+|V|}(X\wedge S^V) \arrow[r, "\Theta^G"'] & \mathbf{mO}^G_{m+|V|}(X\wedge S^V)
\end{tikzcd}$$

The Thom-Pontryagin maps thus assemble into a natural transformation

$$\Theta^G\ :\ \mathfrak{N}_m^{G:S}(X)\ \longrightarrow\ \mathbf{mO}^G_m(X)[1/\tau]$$

between the localized theories, for which we use the same letter.

If G is a product of a finite group and a torus, then the next theorem is a direct consequence of Theorem 6.2.33. The point, however, is that the following localized version holds without any restriction on the compact Lie group G. Morally, the reason for this is that formally inverting the classes $d_{G,V}$ forces the Wirthmüller isomorphism to hold, so in stable equivariant bordism this potential obstruction to representability by a global homotopy type vanishes.

Theorem 6.2.37. *For every compact Lie group G and every cofibrant based G-space X, the map*

$$\Theta^G \ : \ \tilde{\mathfrak{N}}_*^{G:S}(X) \ \longrightarrow \ \mathbf{mO}_*^G(X)[1/\tau]$$

is an isomorphism of graded abelian groups.

Proof Since filtered colimits of abelian groups are exact, the localized theories $\tilde{\mathfrak{N}}_*^{G:S}(-)$ and $\mathbf{mO}_*^G(-)[1/\tau]$ are both equivariant homology theories for every fixed group G. We can run the same inductive argument as in the proof of Proposition 6.2.33, i.e., show the claim by double induction over the dimension and the number of path components of G. But this time the case of an orbit $X = G/H$ for a proper closed subgroup H of G works without any restriction on G and H. Indeed, in the commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{N}}_m^{H:S}(S^0) \arrow[rr, "G\times_H -"] \arrow[d, "\Theta^H"'] & & \tilde{\mathfrak{N}}_{m+d}^{G:S}(G/H_+) \arrow[d, "\Theta^G"] \\
\mathbf{mO}_m^H(S^0)[1/\tau] \arrow[r, "-\cdot\bar{\tau}_{H,L}"'] & \mathbf{mO}_{m+d}^H(S^L)[1/\tau] \arrow[r, "G\ltimes_H -"'] & \mathbf{mO}_{m+d}^G(G/H_+)[1/\tau]
\end{tikzcd}$$

multiplication by the class $\bar{\tau}_{H,L}$ is now invertible in the localized theory $\mathbf{mO}_*^G[1/\tau]$. So all horizontal maps in the diagram are isomorphisms. Since the left vertical map is an isomorphism by induction, the right vertical map is an isomorphism as well.

The case $X = G/G$ is essentially taken care of by Proposition 6.2.32. We let V be any G-representation, and observe that the fixed-point inclusion $i : V^G \longrightarrow V$ induces a G-homotopy equivalence

$$\tilde{E}\mathcal{P} \wedge i \ : \ \tilde{E}\mathcal{P} \wedge S^{V^G} \ \longrightarrow \ \tilde{E}\mathcal{P} \wedge S^V \ .$$

In the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{N}}_{m+|V^\perp|}^G(\tilde{E}\mathcal{P}) \arrow[r, "\Theta^G"] \arrow[d, "-\wedge d_{G,V^G}"'] & \mathbf{mO}_{m+|V^\perp|}^G(\tilde{E}\mathcal{P}) \arrow[d, "-\cdot\tau_{G,V^G}"] \\
\widetilde{\mathcal{N}}_{m+|V|}^G(\tilde{E}\mathcal{P} \wedge S^{V^G}) \arrow[r, "\Theta^G"] \arrow[d, "(\tilde{E}\mathcal{P}\wedge i)_*"'] & \mathbf{mO}_{m+|V|}^G(\tilde{E}\mathcal{P} \wedge S^{V^G}) \arrow[d, "(\tilde{E}\mathcal{P}\wedge i)_*"] \\
\widetilde{\mathcal{N}}_{m+|V|}^G(\tilde{E}\mathcal{P} \wedge S^V) \arrow[r, "\Theta^G"'] & \mathbf{mO}_{m+|V|}^G(\tilde{E}\mathcal{P} \wedge S^V)
\end{tikzcd}$$

the lower vertical maps are thus isomorphisms. The upper vertical maps are suspension isomorphisms, hence both vertical composites are isomorphisms. The upper horizontal map is an isomorphism by Proposition 6.2.32, hence so is

the lower horizontal map. As a colimit of isomorphisms, the localized Thom-Pontryagin map

$$\Theta^G \ : \ \mathfrak{N}_*^{G:S}(\tilde{E}\mathcal{P}) \ \longrightarrow \ \mathbf{mO}_*^G(\tilde{E}\mathcal{P})[1/\tau]$$

is also an isomorphism. Now we finish the argument as in Proposition 6.2.33: we compare the isotropy separation sequence for $\tilde{\mathfrak{N}}_*^{G:S}(-)$ to that for $\mathbf{mO}_*^G(-)[1/\tau]$, and the five lemma concludes the inductive step. □

Remark 6.2.38 (Stable equivariant bordism and **MO**)**.** Corollary 6.1.35 and Theorem 6.2.37 provide natural isomorphisms

$$\tilde{\mathfrak{N}}_*^{G:S}(X) \ \xrightarrow{\Theta^G} \mathbf{mO}_*^G(X)[1/\tau] \ \xrightarrow{\cong} \ \mathbf{MO}_*^G(X)$$

for cofibrant based G-spaces X. Hence the composite is an isomorphism, which provides an alternative proof that stable equivariant bordism agrees with equivariant **MO**-homology, which is the main result of the paper [27] by Bröcker and Hook. Strictly speaking there is a bit more work involved in the translation, because our group $\mathbf{MO}_*^G(X)$ is not literally the same as the homotopy theoretic equivariant bordism group $\tilde{N}_n^G(X)$ in [27]; we invite the reader to spell out an isomorphism, which boils down to a certain rewriting of colimits, such that our composite isomorphism becomes the map Φ^S considered by Bröcker and Hook.

6.3 Connective global K-theory

In this section we define and discuss the ultra-commutative ring spectrum **ku**, the *connective global K-theory spectrum*, see Construction 6.3.9. Our construction is an elaboration of a model of non-equivariant connective K-theory by Segal [154], constructed from certain $\mathbf{\Gamma}$-spaces of 'orthogonal subspaces in the symmetric algebra'. The degree zero equivariant cohomology theory represented by **ku** tries hard to be equivariant K-theory. Indeed, Theorem 6.3.31 exhibits a natural transformation from the G-equivariant K-group that is an isomorphism for finite G-CW-complexes with finite isotropy groups. A special case is a ring homomorphism $\mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ from the complex representation ring that is an isomorphism whenever G is finite. For varying G, these homomorphisms almost form a morphism of global power functors, in the sense that they are compatible with finite index transfers, but *not* with general degree zero transfers across an infinite index inclusion, see Remark 6.3.38.

The *Bott class* is a preferred element $\beta \in \pi_2^e(\mathbf{ku})$ that we define in Construction 6.3.42. This class becomes invertible in the periodic global K-theory

spectrum **KU**, by Theorem 6.4.29 below. More generally, every G-Spin^c-representation has an associated *equivariant Bott class*

$$\beta_{G,W} \in \mathbf{ku}_G^0(S^W) ,$$

see Construction 6.3.46, that becomes an $RO(G)$-graded unit in the periodic theory **KU**.

Construction 6.3.1. We let $\mathcal{U}$ be a complex vector space of countable dimension (finite or infinite) equipped with a hermitian inner product. We recall a certain $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ of 'orthogonal subspaces in $\mathcal{U}$', due to Segal [154, Sec. 1]. For a finite based set A we let $\mathscr{C}(\mathcal{U}, A)$ be the space of tuples (E_a), indexed by the non-basepoint elements of A, of finite-dimensional, pairwise orthogonal $\mathbb{C}$-subspaces of $\mathcal{U}$. The topology on $\mathscr{C}(\mathcal{U}, A)$ is that of a disjoint union of subspaces of a product of Grassmannians. The basepoint of $\mathscr{C}(\mathcal{U}, A)$ is the tuple where each E_a is the zero subspace. For a based map $\alpha : A \longrightarrow B$ the induced map $\mathscr{C}(\mathcal{U}, \alpha) : \mathscr{C}(\mathcal{U}, A) \longrightarrow \mathscr{C}(\mathcal{U}, B)$ sends (E_a) to (F_b) where

$$F_b = \bigoplus\nolimits_{\alpha(a)=b} E_a .$$

Then $\mathscr{C}(\mathcal{U})$ is a $\mathbf{\Gamma}$-space whose underlying space is

$$\mathscr{C}(\mathcal{U}, 1_+) = \coprod\nolimits_{n\geq 0} Gr_n^{\mathbb{C}}(\mathcal{U}) ,$$

the disjoint union of the different Grassmannians of $\mathcal{U}$.

Every $\mathbf{\Gamma}$-space can be evaluated on a based space by the coend construction (4.5.14). We write $\mathscr{C}(\mathcal{U}, K) = \mathscr{C}(\mathcal{U})(K)$ for the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ on a based space K. When A is a finite based set with discrete topology, then the prolongation $\mathscr{C}(\mathcal{U})(A)$ is canonically homeomorphic to the original space $\mathscr{C}(\mathcal{U}, A)$, compare Remark B.23. Elements of $\mathscr{C}(\mathcal{U}, K)$ can be interpreted as 'labeled configurations': a point is represented by an unordered tuple

$$[E_1, \ldots, E_n;\ k_1, \ldots, k_n]$$

where $(E_1, \ldots, E_n)$ is an n-tuple of finite-dimensional, pairwise orthogonal subspaces of $\mathcal{U}$, and $k_1, \ldots, k_n$ are points of K, for some n. The topology is such that, informally speaking, the labels sum up whenever two points collide, and a label disappears whenever a point approaches the basepoint of K.

Remark 6.3.2. When K is compact, the space $\mathscr{C}(\mathcal{U}, K)$ can be described differently, compare again [154, Sec. 1], namely as the space

$$\operatorname{colim}_{V\in s(\mathcal{U})} C^*(C_0(K), \operatorname{End}_{\mathbb{C}}(V)) ,$$

where the colimit runs over all finite-dimensional $\mathbb{C}$-subspaces of $\mathcal{U}$. Here $C_0(K)$ is the C^*-algebra of continuous $\mathbb{C}$-valued functions on K that vanish at

the basepoint, $\mathrm{End}_{\mathbb{C}}(V)$ is the endomorphism C^*-algebra of V (which is isomorphic to a matrix algebra over $\mathbb{C}$), and $C^*(-,-)$ is the space of C^*-algebra homomorphisms, with the zero homomorphism as the basepoint. For $U \subset V$ the map in the colimit system is induced by the $*$-homomorphism $\mathrm{End}_{\mathbb{C}}(U) \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ that extends an endomorphism by 0 on the orthogonal complement. A homeomorphism

$$\mathscr{C}(\mathcal{U}, K) \longrightarrow \mathrm{colim}_{V \in s(\mathcal{U})} C^*(C_0(K), \mathrm{End}_{\mathbb{C}}(V))$$

is given by sending a configuration $[E_1, \dots, E_n;\ k_1, \dots, k_n]$ in $\mathscr{C}(\mathcal{U}, K)$ to the homomorphism that takes a function $\varphi \in C_0(K)$ to

$$\sum\nolimits_{i=1}^{n} \varphi(k_i) \cdot p_{E_i} \ ;$$

here V is chosen large enough to contain all the spaces E_i, and $p_{E_i} : V \longrightarrow V$ is the orthogonal projection onto the subspace E_i.

Remark 6.3.3 (Eigenspace decomposition)**.** If $\mathcal{U}$ is infinite-dimensional, then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ is special, compare Theorem 6.3.19 (i) below. The orthogonal spectrum $\mathscr{C}(\mathcal{U})(\mathbb{S})$ is then a positive Ω-spectrum by the general theory. In particular, the space $\mathscr{C}(\mathcal{U})(\mathbb{S})(\mathbb{R}) = \mathscr{C}(\mathcal{U}, S^1)$ is an infinite loop space. In fact, $\mathscr{C}(\mathcal{U}, S^1)$ is a familiar space, namely the infinite unitary group $U(\mathcal{U})$, i.e., the group of linear self-isometries of $\mathcal{U}$ that are the identity on the orthogonal complement of some finite-dimensional subspace. This *eigenspace decomposition* works for every hermitian inner product space $\mathcal{U}$, of finite or countable dimension, as follows. As before we identify S^1 with the unit circle $U(1)$ in the complex numbers via the Cayley transform

$$c \ : \ S^1 \ \cong \ U(1) \,, \quad x \ \longmapsto \ (x+i)(x-i)^{-1} \ .$$

This homeomorphism sends the basepoint at infinity to 1. Given a tuple of pairwise orthogonal subspaces $(E_1, \dots, E_n)$ of $\mathcal{U}$ and a point $(x_1, \dots, x_n) \in (S^1)^n$, we let $\psi(E_1, \dots, E_n;\ x_1, \dots, x_n)$ be the isometry of $\mathcal{U}$ that is multiplication by $c(x_i)$ on E_i and the identity on the orthogonal complement of $\bigoplus_{i=1}^n E_i$. In other words: E_i is the eigenspace of $\psi(E_1, \dots, E_n;\ x_1, \dots, x_n)$ for the eigenvalue $c(x_i)$. As n varies, these maps are compatible with the equivalence relation and so they assemble into a continuous map

$$\begin{aligned} \mathscr{C}(\mathcal{U}, S^1) \ = \ \int^{n_+ \in \mathbf{\Gamma}} \mathscr{C}(\mathcal{U}, n_+) \times (S^1)^n \ &\longrightarrow \quad U(\mathcal{U}) \\ [E_1, \dots, E_n;\ x_1, \dots, x_n] \ &\longmapsto \ \psi(E_1, \dots, E_n;\ x_1, \dots, x_n) \ . \end{aligned} \tag{6.3.4}$$

This map is a homeomorphism because every unitary transformation is diagonalizable with eigenvalues in $U(1)$ and pairwise orthogonal eigenspaces.

Now we let $\mathcal{U}$ and $\mathcal{V}$ be two hermitian vector spaces (of countable dimension, but possibly infinite-dimensional). Here and below we write $\otimes = \otimes_{\mathbb{C}}$ for the tensor product over $\mathbb{C}$. We endow $\mathcal{U} \otimes \mathcal{V}$ with a hermitian scalar product by declaring

$$(u \otimes v,\ u' \otimes v') \ = \ (u, u') \cdot (v, v') \tag{6.3.5}$$

on elementary tensors and extending biadditively. If E, F are orthogonal subspaces of $\mathcal{U}$ and E' is a subspace of $\mathcal{V}$, then $E \otimes E'$ and $F \otimes E'$ are orthogonal subspaces of $\mathcal{U} \otimes \mathcal{V}$, and similarly in the second variable. For based spaces K and L we can thus define a continuous multiplication map

$$\begin{aligned} \mathscr{C}(\mathcal{U}, K) \wedge \mathscr{C}(\mathcal{V}, L) \ &\longrightarrow \ \mathscr{C}(\mathcal{U} \otimes \mathcal{V}, K \wedge L) \\ [E_i; k_i] \wedge [F_j; l_j] \quad &\longmapsto \quad [E_i \otimes F_j; k_i \wedge l_j] \,. \end{aligned} \tag{6.3.6}$$

These multiplication maps are associative, and commutative in the sense that the following square commutes:

$$\begin{tikzcd} \mathscr{C}(\mathcal{U}, K) \wedge \mathscr{C}(\mathcal{V}, L) \arrow[r] \arrow[d, "\tau_{\mathscr{C}(\mathcal{U},K),\mathscr{C}(\mathcal{V},L)}"'] & \mathscr{C}(\mathcal{U} \otimes \mathcal{V}, K \wedge L) \arrow[d, "\mathscr{C}(\tau_{\mathcal{U},\mathcal{V}}, \tau_{K,L})"] \\ \mathscr{C}(\mathcal{V}, L) \wedge \mathscr{C}(\mathcal{U}, K) \arrow[r] & \mathscr{C}(\mathcal{V} \otimes \mathcal{U}, L \wedge K) \end{tikzcd}$$

Our construction of connective global K-theory needs induced inner products on symmetric powers. We explain the complex version; the real version works in much the same way. For a $\mathbb{C}$-vector space V we denote by

$$\mathrm{Sym}^n(V) \ = \ V^{\otimes n}/\Sigma_n$$

the nth symmetric power of V and by $\mathrm{Sym}(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$ the symmetric algebra of V. If W is another $\mathbb{C}$-vector space, then the two direct summand embeddings of V and W into $V \oplus W$ induce homomorphisms of symmetric algebras that combine (by multiplying in the target) into a natural $\mathbb{C}$-algebra isomorphism

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(W) \ \cong \ \mathrm{Sym}(V \oplus W) \,. \tag{6.3.7}$$

If V is equipped with a hermitian inner product, then the symmetric powers inherit a preferred inner product:

Proposition 6.3.8. *For every hermitian inner product space V there is a unique inner product on* $\mathrm{Sym}^n(V)$ *that satisfies*

$$(v_1 \cdot \ldots \cdot v_n,\ \bar{v}_1 \cdot \ldots \cdot \bar{v}_n) \ = \ \sum_{\sigma \in \Sigma_n} (v_1, \bar{v}_{\sigma(1)}) \cdot \ldots \cdot (v_n, \bar{v}_{\sigma(n)})$$

for all $v_i, \bar{v}_i \in V$. This inner product on $\mathrm{Sym}^n(V)$ *is natural for $\mathbb{C}$-linear isometric embeddings and it makes the algebra isomorphism* (6.3.7) *an isometry.*

Proof Uniqueness of the scalar product follows from the fact that the symmetric products $v_1 \cdot \ldots \cdot v_n$ generate $\mathrm{Sym}^n(V)$ as a $\mathbb{C}$-vector space. The tensor product $V \otimes W$ of two hermitian inner product spaces V and W has a preferred hermitian inner product as in (6.3.5). By iteration, the n-fold tensor product $V^{\otimes n}$ inherits an inner product. There is thus a unique inner product on $\mathrm{Sym}^n(V)$ that makes the normalized $\mathbb{C}$-linear 'symmetrization' embedding

$$\begin{array}{rcl} \mathrm{Sym}^n(V) & \longrightarrow & V^{\otimes n} \\ v_1 \cdot \ldots \cdot v_n & \longmapsto & \dfrac{1}{\sqrt{n!}} \cdot \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \end{array}$$

a linear isometric embedding. We omit the straightforward verification that this inner product is indeed given by the formula in the statement of the proposition, and that it is natural for linear isometric embeddings.

The algebra isomorphism (6.3.7) is the sum of the embeddings

$$\begin{array}{rcl} \mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(W) & \longrightarrow & \mathrm{Sym}^{m+n}(V \oplus W) \\ v_1 \cdot \ldots \cdot v_m \otimes w_1 \cdot \ldots \cdot w_n & \longmapsto & (v_1, 0) \cdot \ldots \cdot (v_m, 0) \cdot (0, w_1) \cdot \ldots \cdot (0, w_n) . \end{array}$$

The relation

$$\begin{aligned} &(v_1 \cdot \ldots \cdot v_m \otimes w_1 \cdot \ldots \cdot w_n,\ \bar v_1 \cdot \ldots \cdot \bar v_m \otimes \bar w_1 \cdot \ldots \cdot \bar w_n) \\ &\qquad = (v_1 \cdot \ldots \cdot v_m,\ \bar v_1 \cdot \ldots \cdot \bar v_m) \cdot (w_1 \cdot \ldots \cdot w_n,\ \bar w_1 \cdot \ldots \cdot \bar w_n) \\ &\qquad = \sum_{(\sigma,\tau) \in \Sigma_m \times \Sigma_n} (v_1, \bar v_{\sigma(1)}) \cdot \ldots \cdot (v_m, \bar v_{\sigma(m)}) \cdot (w_1, \bar w_{\tau(1)}) \cdot \ldots \cdot (w_n, \bar w_{\tau(m)}) \\ &\qquad = ((v_1, 0) \cdot \ldots \cdot (v_m, 0) \cdot (0, w_1) \cdot \ldots \cdot (0, w_n), \\ &\qquad\qquad (\bar v_1, 0) \cdot \ldots \cdot (\bar v_m, 0) \cdot (0, \bar w_1) \cdot \ldots \cdot (0, \bar w_n)) \end{aligned}$$

then proves that (6.3.7) preserves the inner product. The last equation uses that $(v, 0)$ and $(0, w)$ are orthogonal in $V \oplus W$ for all $v \in V$ and $w \in W$. □

The case $n = 2$ gives an idea of the induced inner product on $\mathrm{Sym}^n(V)$: if $\{e_1, \ldots, e_k\}$ is an orthonormal basis of V, then the vectors

$$1/\sqrt{2} \cdot e_i^2 \quad (1 \le i \le k) \qquad \text{and} \qquad e_i \cdot e_j \quad (1 \le i < j \le k)$$

form an orthonormal basis of $\mathrm{Sym}^2(V)$.

Construction 6.3.9 (Connective global K-theory)**.** We can now define an ultra-commutative ring spectrum **ku**, the *connective global K-theory spectrum*. The value of **ku** on a euclidean inner product space V is

$$\mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) ,$$

the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ on the one-point compactification of V.

Here $V_{\mathbb{C}}$ is the complexification of V with the induced hermitian inner product, and the inner product on the symmetric algebra described in Proposition 6.3.8. The action of $O(V)$ on V then extends to a unitary action on $\mathrm{Sym}(V_{\mathbb{C}})$. We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere S^V and the action on the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$. Using the tensor product pairing (6.3.6) we define an $(O(V) \times O(W))$-equivariant multiplication map

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{ku}(V) \wedge \mathbf{ku}(W) \ &= \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \wedge \mathscr{C}(\mathrm{Sym}(W_{\mathbb{C}}), S^W) \\ &\xrightarrow[(6.3.6)]{} \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}) \otimes \mathrm{Sym}(W_{\mathbb{C}}), S^V \wedge S^W) \\ &\underset{(6.3.7)}{\cong} \ \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W}) = \mathbf{ku}(V \oplus W) \ . \end{aligned}$$

The maps $\mu_{V,W}$ are associative and commutative. An $O(V)$-equivariant unit map is given by

$$\iota_V \ : \ S^V \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V) \ , \quad v \ \longmapsto \ [\mathbb{C}\cdot 1; v] \ ,$$

where $\mathbb{C}\cdot 1$ is the homogeneous summand of degree 0 in the symmetric algebra, i.e., the line spanned by the multiplicative unit. This structure makes $\mathbf{ku}$ an ultra-commutative ring spectrum.

The space $\mathbf{ku}_0 = \mathscr{C}(\mathbb{C}\cdot 1, S^0)$ consists of all subspaces of $\mathrm{Sym}(0) = \mathbb{C}\cdot 1$, so it has two points, the basepoint 0 and the point $\mathbb{C}\cdot 1$. The unit map $\iota_0 : S^0 \longrightarrow \mathbf{ku}_0$ is thus a homeomorphism.

Construction 6.3.10 (Complex conjugation on **ku**)**.** The ultra-commutative ring spectrum **ku** comes with an involution by 'complex conjugation' that preserves all the structure available. Indeed, for every euclidean inner product space V the complex symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification is canonically isomorphic to $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}(V)$, the complexification of the real symmetric algebra of V. So $\mathrm{Sym}(V_{\mathbb{C}})$ comes with an involution $\psi_{\mathrm{Sym}(V)}$ that is $\mathbb{C}$-semilinear and preserves the orthogonality relation. Applying this involution elementwise to tuples of orthogonal subspaces gives an involution $\mathscr{C}(\psi_{\mathrm{Sym}(V)}) : \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}})) \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ of the $\mathbf{\Gamma}$-space and hence a homeomorphism

$$\psi(V) \ = \ \mathscr{C}(\psi_{\mathrm{Sym}(V)}, S^V) \ : \ \mathbf{ku}(V) \ \longrightarrow \ \mathbf{ku}(V) \tag{6.3.11}$$

of order 2. As V varies, the maps $\psi(V)$ form an automorphism $\psi : \mathbf{ku} \longrightarrow \mathbf{ku}$ of the ultra-commutative ring spectrum **ku**.

Remark 6.3.12 (Connective real global K-theory)**.** There is a straightforward real analog **ko** of the complex connective global K-theory spectrum **ku**. The value of **ko** on an inner product space V is

$$\mathbf{ko}(V) \ = \ \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ ,$$

where now $\mathrm{Sym}(V)$ is the symmetric algebra of V over the real numbers, and $\mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V))$ is the $\mathbf{\Gamma}$-space of tuples of pairwise orthogonal, finite-dimensional $\mathbb{R}$-subspaces of $\mathrm{Sym}(V)$. As V varies, the spaces $\mathbf{ko}(V)$ again come with the structure of an ultra-commutative ring spectrum.

Complexification defines a morphism of ultra-commutative ring spectra

$$c \ : \ \mathbf{ko} \ \longrightarrow \ \mathbf{ku} \ . \tag{6.3.13}$$

In more detail: if V is a real inner product space, then the map

$$c(V) \ : \ \mathbf{ko}(V) = \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V)$$

sends a configuration $[E_1, \dots, E_n;\ v_1, \dots, v_n]$ to the complexified configuration

$$[(E_1)_{\mathbb{C}}, \dots, (E_n)_{\mathbb{C}};\ v_1, \dots, v_n] \ .$$

As V varies, these maps form the morphism (6.3.13) of ultra-commutative ring spectra. The complexification of a real subspace of $\mathrm{Sym}(V)$ is invariant under the complex conjugation involution $\psi_{\mathrm{Sym}(V)}$ of $\mathrm{Sym}(V_{\mathbb{C}})$, so the image of the complexification morphism is invariant under the complex conjugation involution ψ of $\mathbf{ku}$ defined in (6.3.11). Even more is true: a complex subspace of $\mathrm{Sym}(V_{\mathbb{C}})$ is ψ-invariant if and only if it is the complexification of its real part (the $+1$ eigenspace of $\psi_{\mathrm{Sym}(V)}$). This means that $\mathbf{ko}$ 'is' the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$; more formally, the complexification morphism (6.3.13) is an isomorphism from $\mathbf{ko}$ onto the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$.

The next proposition justifies the adjective 'connective' that we attached to the names of the orthogonal spectra $\mathbf{ku}$ and $\mathbf{ko}$. The proof uses a certain cofibrancy property of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ that we now introduce.

Construction 6.3.14 (Latching map)**.** We let $\mathcal{P}(n)$ denote the power set of the set $\{1, \dots, n\}$, i.e., the set of subsets. We also write $\mathcal{P}(n)$ for the associated poset category, i.e., with object set $\mathcal{P}(n)$ and exactly one morphism $U \longrightarrow T$ whenever $U \subseteq T$. Given a $\mathbf{\Gamma}$-space F, we obtain a functor from $\mathcal{P}(n)$ to based spaces by sending a subset U to $F(U_+)$, with the maps $F(U_+) \longrightarrow F(T_+)$ induced by the inclusions. We obtain a *latching map*

$$l_n \ : \ \mathrm{colim}_{U \subsetneq \{1,\dots,n\}} \ F(U_+) \ \longrightarrow \ F(n_+) \ .$$

The map l_n is equivariant for the action of Σ_n given on the target by functoriality of F. The action of a permutation $\sigma \in \Sigma_n$ on the source is induced by sending $F(U_+)$ to $F(\sigma(U)_+)$ via the map

$$F((\sigma \cdot -)_+) \ : \ F(U_+) \ \longrightarrow \ F(\sigma(U)_+) \ .$$

The latching map l_n is always a closed embedding, by Proposition B.32 (ii).

We will now consider a $\mathbf{\Gamma}$-G-space for a compact Lie group G, i.e., a reduced functor $F : \mathbf{\Gamma} \longrightarrow G\mathbf{T}_*$ to the category of based G-spaces. Then G acts on source and target of the latching map l_n, which is thus $(\Sigma_n \times G)$-equivariant. The following cofibrancy condition on a $\mathbf{\Gamma}$-G-space ensures that the prolongation is homotopically meaningful.

Definition 6.3.15. Let G be a compact Lie group. A $\mathbf{\Gamma}$-G-space F is *G-cofibrant* if for every $n \geq 1$ the latching map l_n is a $(\Sigma_n \times G)$-cofibration.

Example 6.3.16. We let G be a compact Lie group and $\mathcal{U}$ a unitary G-representation, of finite or countably infinite dimension. We argue that the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathcal{U}, -)$ is G-cofibrant. The actions of G and Σ_n on an n-tuple of orthogonal subspaces are componentwise and by permuting the entries, i.e.,

$$(\sigma, g) \cdot (E_1, \ldots, E_n) \ = \ (g \cdot E_{\sigma^{-1}(1)}, \ldots, g \cdot E_{\sigma^{-1}(n)})$$

for $(\sigma, g) \in \Sigma_n \times G$. The topology on $\mathscr{C}(\mathcal{U}, n_+)$ is that of a disjoint union of subspaces of a product of Grassmannians:

$$\mathscr{C}(\mathcal{U}, n_+) \ = \ \coprod_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \mathscr{C}(\mathcal{U}; i_1, \ldots, i_n) \ ,$$

where $\mathscr{C}(\mathcal{U}; i_1, \ldots, i_n)$ is the subspace of those tuples such that $\dim(E_j) = i_j$ for all $1 \leq j \leq n$.

The summands are invariant under the G-action, and $\mathscr{C}(\mathcal{U}; i_1, \ldots, i_n)$ is G-equivariantly homeomorphic to

$$\mathbf{L}^{\mathbb{C}}(\mathbb{C}^{i_1+\cdots+i_n}, \mathcal{U})/U(i_1) \times \cdots \times U(i_n) \ ,$$

where $\mathbf{L}^{\mathbb{C}}(-,-)$ is the space of $\mathbb{C}$-linear isometric embeddings. This G-space is G-cofibrant by the unitary analog of Proposition 1.1.19 (ii). We let $\Gamma \subset \Sigma_n$ be the stabilizer group of the dimension vector $(i_1, \ldots, i_n)$, i.e., the group of those $\sigma \in \Sigma_n$ such that $i_j = i_{\sigma(j)}$ for all $1 \leq j \leq n$. Then Γ acts trivially on $\mathscr{C}(\mathcal{U}; i_1, \ldots, i_n)$, so this space is $(\Gamma \times G)$-cofibrant. Hence the induced $(\Sigma_n \times G)$-space

$$\Sigma_n \times_\Gamma \mathscr{C}(\mathcal{U}; i_1, \ldots, i_n)$$

is $(\Sigma_n \times G)$-cofibrant. The full configuration space $\mathscr{C}(\mathcal{U}, n_+)$ decomposes as the disjoint union of $(\Sigma_n \times G)$-invariant subspaces indexed by the Σ_n-orbits on $\mathbb{N}^n$,

$$\mathscr{C}(\mathcal{U}, n_+) \ = \ \coprod_{i\Sigma_n \in \mathbb{N}^n/\Sigma_n} \Sigma_n \times_{\Gamma_i} \mathscr{C}(\mathcal{U}; i) \ ,$$

where $\Gamma_i \subset \Sigma_n$ is the stabilizer of $i = (i_1, \ldots, i_n)$. Altogether this shows that $\mathscr{C}(\mathcal{U}, n_+)$ is a disjoint union of $(\Sigma_n \times G)$-cofibrant spaces. The latching map

$$l_n \ : \ \operatorname{colim}_{U \subsetneq \{1,\ldots,n\}} \mathscr{C}(\mathcal{U}, U_+) \ \longrightarrow \ \mathscr{C}(\mathcal{U}, n_+)$$

is a closed embedding by Proposition B.32 (ii), and its image is the disjoint union of the summands $\mathscr{C}(\mathcal{U}; i_1, \dots, i_n)$ such that $i_j = 0$ for some $j \in \{1, \dots, n\}$. Since all additional summands not in the image are $(\Sigma_n \times G)$-cofibrant, the map l_n is a $(\Sigma_n \times G)$-cofibration.

Proposition 6.3.17. *The orthogonal spectra* **ku** *and* **ko** *are globally connective.*

Proof The arguments for **ku** and **ko** are completely parallel, and we concentrate on the complex case. We let G be any compact Lie group, and we show that the group $\pi_{-k}^G(\mathbf{ku})$ is trivial for all $k \geq 1$. We let V be a G-representation and set $\mathcal{U} = \mathrm{Sym}((\mathbb{R}^k \oplus V)_{\mathbb{C}})$, a unitary G-representation of countably infinite dimension. We let H be a closed subgroup of G. The $\mathbf{\Gamma}$-H-space $\mathscr{C}(\mathcal{U}, -)$ is H-cofibrant by Example 6.3.16. So the space $\mathscr{C}(\mathcal{U}, S^{\mathbb{R}^k \oplus V})^H$ is $(k + \dim(V^H) - 1)$-connected by Proposition B.43 (i).

The cellular dimension of S^V at H, in the sense of [179, II.2, p. 106], is the topological dimension of the space $(S^{V^H})/N_G H$; this cellular dimension is at most $\dim(V^H)$. Because k is positive, the cellular dimension of S^V at H does not exceed the connectivity of $\mathbf{ku}(\mathbb{R}^k \oplus V)^H$. So every based continuous G-map $S^V \longrightarrow \mathbf{ku}(\mathbb{R}^k \oplus V)$ is equivariantly null-homotopic by [179, II Prop. 2.7], and the set $[S^V, \mathbf{ku}(\mathbb{R}^k \oplus V)]^G$ has only one element. Passage to the colimit over $V \in s(\mathcal{U}_G)$ proves the claim. □

Every $\mathbb{C}$-linear isometric embedding $u : \mathcal{U} \longrightarrow \mathcal{V}$ of complex inner product spaces induces a morphism of $\mathbf{\Gamma}$-spaces $\mathscr{C}(u) : \mathscr{C}(\mathcal{U}) \longrightarrow \mathscr{C}(\mathcal{V})$ by applying u elementwise to a tuple of orthogonal subspaces. So if a compact Lie group G acts on $\mathcal{U}$ by linear isometries (for example if $\mathcal{U}$ is a G-universe), then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ inherits a G-action, so it becomes a $\mathbf{\Gamma}$-G-space.

Proposition 6.3.18. *Let G be a compact Lie group and $\mathcal{U}$ and $\mathcal{V}$ two isomorphic complex G-universes. Then for every G-equivariant linear isometric embedding $u : \mathcal{U} \longrightarrow \mathcal{V}$ and every based G-space K the map*

$$\mathscr{C}(u, K) \ : \ \mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ \mathscr{C}(\mathcal{V}, K)$$

is a G-homotopy equivalence.

Proof We start with the special case where $\mathcal{V} = \mathcal{U}$. The space of G-equivariant linear isometric embeddings from $\mathcal{U}$ to itself is contractible. A homotopy from u to the identity then induces a G-homotopy from $\mathscr{C}(u, K)$ to the identity of the G-space $\mathscr{C}(\mathcal{U}, K)$. In the general case we choose a G-equivariant linear isometry $v : \mathcal{V} \cong \mathcal{U}$. Then the two G-maps

$$\mathscr{C}(vu, K)\colon \mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ \mathscr{C}(\mathcal{U}, K) \ \text{ and } \ \mathscr{C}(uv, K)\colon \mathscr{C}(\mathcal{V}, K) \ \longrightarrow \ \mathscr{C}(\mathcal{V}, K)$$

are G-homotopic to the respective identity maps. □

If F is any $\mathbf{\Gamma}$-space and S a finite set, then we define the map

$$P_S \ : \ F(S_+) \ \longrightarrow \ \mathrm{map}(S, F(1_+))$$

by $P_S(x)(s) = F(p_s)(x)$, where $p_s : S_+ \longrightarrow 1_+$ sends s to 1 and all other elements of S_+ to the basepoint.

Now we suppose that G is a compact Lie group and F is a $\mathbf{\Gamma}$-G-space. When we evaluate F (or rather its prolongation) at a G-space K, it comes with a $(G \times G)$-action; one action comes from the 'external' action on F, the other one from the G-action on K via the continuous functoriality of F. In such a situation, we always consider $F(K)$ as a G-space via the diagonal action of this $(G \times G)$-action. The map P_S is natural for bijections in S; so whenever the group G acts on S, then $P_S : F(S_+) \longrightarrow \mathrm{map}(S, F(1_+))$ is G-equivariant for the diagonal action on the source and the conjugation action on the target.

We recall from Definition B.49 that a $\mathbf{\Gamma}$-G-space F is *special* if for every closed subgroup H of G and every finite H-set S the map

$$(P_S)^H \ : \ F(S_+)^H \ \longrightarrow \ \mathrm{map}^H(S, F(1_+))$$

is a weak equivalence. It goes without saying that H must act continuously on S with the discrete topology; so S is a disjoint union of H-sets of the form H/K for finite index subgroups K of H. Equivalently, F is special if and only if the map

$$P_n \ : \ F(n_+) \ \longrightarrow \ \mathrm{map}(\{1, \dots, n\}, F(1_+)) \ = \ F(1_+)^n$$

is an $\mathcal{F}(G; \Sigma_n)$-equivalence for every $n \geq 1$, where $\mathcal{F}(G; \Sigma_n)$ is the family of graph subgroups, see Proposition B.52.

Theorem 6.3.19. *Let G be a compact Lie group and $\mathcal{U}$ a complete complex G-universe.*

(i) *The $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathcal{U}, -)$ is special.*
(ii) *If G is finite and V and W are two G-representations with $W^G \neq \{0\}$, then the adjoint assembly map*

$$\tilde{\alpha} \ : \ \mathscr{C}(\mathcal{U}, S^W) \ \longrightarrow \ \mathrm{map}_*(S^V, \mathscr{C}(\mathcal{U}, S^{V \oplus W}))$$

is a G-weak equivalence.

Proof (i) We show that for every $n \geq 1$ the $(G \times \Sigma_n)$-map

$$P_n \ : \ \mathscr{C}(\mathcal{U}, n_+) \ \longrightarrow \ \mathscr{C}(\mathcal{U}, 1_+)^n$$

is an $\mathcal{F}(G; \Sigma_n)$-weak equivalence, where $\mathcal{F}(G; \Sigma_n)$ is the family of graph subgroups of $G \times \Sigma_n$. We define a morphism of $(G \times \Sigma_n)$-spaces

$$\lambda_n \ : \ \mathscr{C}(\mathcal{U}, 1_+)^n \ \longrightarrow \ \mathscr{C}(\mathbb{C}^n \otimes \mathcal{U}, n_+) \ ;$$

here the Σ_n-action on the target is diagonally, from the permutation action on n_+ and on the tensor factor $\mathbb{C}^n$. The map λ_n sends an n-tuple $(E_1, \dots, E_n)$ of finite-dimensional $\mathcal{U}$-subspaces to the configuration

$$(e_1 \otimes E_1, \dots, e_n \otimes E_n)$$

of pairwise orthogonal subspaces of $\mathbb{C}^n \otimes \mathcal{U}$, where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the jth vector of the standard basis of $\mathbb{C}^n$.

Now we consider the two composites $\lambda_n \circ P_n$ and $P'_n \circ \lambda_n$, where P'_n is the map P_n for the universe $\mathbb{C}^n \otimes \mathcal{U}$ (as opposed to $\mathcal{U}$):

$$\mathscr{C}(\mathcal{U}, n_+) \xrightarrow{P_n} \mathscr{C}(\mathcal{U}, 1_+)^n \xrightarrow{\lambda_n} \mathscr{C}(\mathbb{C}^n \otimes \mathcal{U}, n_+) \xrightarrow{P'_n} \mathscr{C}(\mathbb{C}^n \otimes \mathcal{U}, 1_+)^n$$

We claim that both composites are $\mathcal{F}(G; \Sigma_n)$-weak equivalences. We start by investigating the composite $\lambda_n \circ P_n$. For $1 \leq j \leq n$, we define a 1-parameter family of unit vectors

$$u_j \;:\; [0,1] \longrightarrow \mathbb{C}^n \qquad \text{by} \qquad u_j(t) \;=\; t \cdot e_j + \sqrt{\frac{1-t^2}{n-1}} \cdot \sum_{k \neq j} e_k \;.$$

This provides a homotopy

$$H \;:\; \mathscr{C}(\mathcal{U}, n_+) \times [0,1] \longrightarrow \mathscr{C}(\mathbb{C}^n \otimes \mathcal{U}, n_+)$$

by

$$H((E_1, \dots, E_n), t) \;=\; (u_1(t) \otimes E_1, \dots, u_n(t) \otimes E_n) \;.$$

We have

$$u_1(1/\sqrt{n}) \;=\; \dots \;=\; u_n(1/\sqrt{n}) \;=\; 1/\sqrt{n} \cdot (1, \dots, 1)$$

and

$$H((E_1, \dots, E_n), 1) \;=\; (e_1 \otimes E_1, \dots, e_n \otimes E_n) \;=\; (\lambda_n \circ P_n)(E_1, \dots, E_n) \;.$$

Moreover, each of the maps $H(-, t)$ is $(G \times \Sigma_n)$-equivariant, so the composite $\lambda_n \circ P_n$ is $(G \times \Sigma_n)$-equivariantly homotopic to the map $\mathscr{C}(u, n_+)$, where $u : \mathcal{U} \longrightarrow \mathbb{C}^n \otimes \mathcal{U}$ is the $(G \times \Sigma_n)$-equivariant linear isometric embedding

$$u(v) \;=\; 1/\sqrt{n} \cdot (1, \dots, 1) \otimes v \;.$$

Now we let $\alpha : H \longrightarrow \Sigma_n$ be a continuous homomorphism defined on a closed subgroup H of G. Since $\mathcal{U}$ is a complete complex G-universe, both $\mathcal{U}$ and $\alpha^*(\mathbb{C}^n) \otimes \mathcal{U}$ are complete complex H-universes, and u is an H-equivariant linear isometric embedding

$$u \;:\; \mathrm{res}^G_H(\mathcal{U}) \longrightarrow \alpha^*(\mathbb{C}^n) \otimes \mathrm{res}^G_H(\mathcal{U}) \;.$$

Interpreted in this way, the map $\mathscr{C}(u, n_+)^H$ is a homotopy equivalence by Proposition 6.3.18. This shows that the morphism $\mathscr{C}(u, n_+)$ is an $\mathcal{F}(G; \Sigma_n)$-weak equivalence, hence so is the morphism $\lambda_n \circ P_n$.

Now we show that the composite $P'_n \circ \lambda_n$ is an $\mathcal{F}(G; \Sigma_n)$-weak equivalence. We let H be a closed subgroup of G, $\alpha : H \longrightarrow \Sigma_n$ a continuous homomorphism, and $\Gamma = \{(h, \alpha(h)) \mid h \in H\} \leq G \times \Sigma_n$ the graph of α. We let $a_1, \dots, a_k \in \{1, \dots, n\}$ be a set of representatives of the orbits of the H-action on $\{1, \dots, n\}$ through α. We let $H_i \leq H$ be the stabilizer group of a_i. Then projection to the factors indexed by $a_1, \dots, a_k$ provides homeomorphisms

$$\begin{aligned} (\mathscr{C}(\mathcal{U}, 1_+)^n)^\Gamma \quad &\cong \prod\nolimits_{i=1}^k \quad \mathscr{C}(\mathcal{U}, 1_+)^{H_i} \quad \text{and} \\ (\mathscr{C}(\mathbb{C}^n \otimes \mathcal{U}, 1_+)^n)^\Gamma \quad &\cong \prod\nolimits_{i=1}^k \mathscr{C}(\alpha^*(\mathbb{C}^n) \otimes \mathcal{U}, 1_+)^{H_i} . \end{aligned}$$

Under these identifications, the map $(P'_n \circ \lambda_n)^\Gamma$ becomes the product of the maps

$$(\mathscr{C}(\mathcal{U}, 1_+))^{H_i} \longrightarrow \mathscr{C}(\alpha^*(\mathbb{C}^n) \otimes \mathcal{U}, 1_+)^{H_i} \tag{6.3.20}$$

induced by the H_i-equivariant linear isometric embeddings

$$\mathcal{U} \longrightarrow \alpha^*(\mathbb{C}^n) \otimes \mathcal{U} , \quad v \longmapsto e_{a_i} \otimes v .$$

Since $\mathcal{U}$ is a complete complex G-universe, both $\mathcal{U}$ and $\alpha^*(\mathbb{C}^n) \otimes \mathcal{U}$ are complete complex H_i-universes. So the map (6.3.20) is a homotopy equivalence by Proposition 6.3.18. Since $\lambda_n \circ P_n$ and $P'_n \circ \lambda_n$ are $\mathcal{F}(G; \Sigma_n)$-weak equivalences, so is the map $P_n : \mathscr{C}(\mathcal{U}, n_+) \longrightarrow \mathscr{C}(\mathcal{U}, 1_+)^n$.

Part (ii) is essentially a special case of Segal and Shimakawa's equivariant delooping machine based on $\mathbf{\Gamma}$-G-spaces [155, 157]. There is a bit of extra work necessary, though, because our claim is about the prolongation $\mathscr{C}(\mathcal{U}, S^{V \oplus W})$ (which is a categorical coend), whereas Segal and Shimakawa formulate their results in terms of a bar construction (which amounts to a homotopy coend). We provide the additional arguments in Theorem B.65; to apply this, we need that the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathcal{U}, -)$ is special by part (i) and G-cofibrant by Example 6.3.16. □

The orthogonal spectrum **ku** is trying to be a $\mathcal{F}in$-global Ω-spectrum. However, the global Ω-spectrum condition on the adjoint structure maps only holds for 'sufficiently large' representations.

Definition 6.3.21. Let G be a compact Lie group. An orthogonal G-representation W is *ample* if the complex symmetric algebra $\mathrm{Sym}(W_\mathbb{C})$ is a complete complex G-universe.

Remark 6.3.22. Every ample G-representation is non-zero and the action of

G is faithful. Examples of ample representations are non-zero faithful permutation representations (which in particular forces the group to be finite). Indeed, we let $\mathbb{R}A$ denote the permutation representation of a non-empty faithful finite G-set A. Then the complexified symmetric algebra $\mathrm{Sym}(\mathbb{C}A)$ is a complex permutation representation, namely of the infinite G-set $\mathbb{N}^A = \mathrm{map}(A, \mathbb{N})$ of functions from A to $\mathbb{N}$. Since G acts faithfully on A, every injective map $A \longrightarrow \mathbb{N}$ generates a free G-orbit in the G-set $\mathbb{N}^A$. Since A is non-empty, there are infinitely many injections from A to $\mathbb{N}$ with pairwise disjoint images, and these generate infinitely many distinct free G-orbits in $\mathbb{N}^A$. So $\mathrm{Sym}(\mathbb{C}A)$ contains infinitely many copies of the complex regular G-representation, and is thus a complete complex G-universe.

A more general class of ample G-representations are non-zero representations containing a vector with trivial isotropy group. Indeed, given a non-zero G-free vector $w_0 \in W$, we let ν be the normal space at w_0 to the orbit Gw_0. A choice of G-equivariant tubular neighborhood of the orbit Gw_0 allows us to G-equivariantly embed the Hilbert space $L^2(G; \mathbb{C}) \otimes \mathrm{Sym}(\nu)$ into $L^2(W; \mathbb{C})$. Since $L^2(G; \mathbb{C}) \otimes \mathrm{Sym}(\nu)$ contains every finite-dimensional G-representation infinitely often, so does $L^2(W; \mathbb{C})$. But $L^2(W; \mathbb{C})$ is equivariantly isomorphic to the Hilbert space completion of $\mathrm{Sym}(W^*_{\mathbb{C}})$: elements of $\mathrm{Sym}(W^*_{\mathbb{C}})$ are polynomial functions on W, which we can map to L^2-functions by scaling with the function $w \mapsto \exp(-|w|^2)$. This gives a G-equivariant isometric embedding $\mathrm{Sym}(W^*_{\mathbb{C}}) \longrightarrow L^2(W; \mathbb{C})$ with dense image. So if there was any complex G-representation that did not embed equivariantly into $\mathrm{Sym}(W^*_{\mathbb{C}})$, then it would also not embed into $L^2(W; \mathbb{C})$, a contradiction.

Theorem 6.3.23. *Let G be a finite group and W an ample orthogonal G-representation. Then for every G-representation V the adjoint structure map*

$$\tilde{\sigma}_{V,W} \ : \ \mathbf{ku}(W) \ \longrightarrow \ \mathrm{map}_*(S^V, \mathbf{ku}(V \oplus W))$$

is a G-weak equivalence.

Proof The adjoint structure map $\tilde{\sigma}_{V,W}$ factors as the composite

$$\begin{aligned} \mathbf{ku}(W) \ = \ \mathscr{C}(\mathrm{Sym}(W_{\mathbb{C}}), S^W) \ &\xrightarrow{\mathscr{C}(\mathrm{Sym}(i_{\mathbb{C}}), S^W)} \ \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^W) \\ &\xrightarrow{\tilde{\alpha}} \ \mathrm{map}_*(S^V, \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W})) \ = \ \mathrm{map}_*(S^V, \mathbf{ku}(V \oplus W)); \end{aligned}$$

here $i : W \longrightarrow V \oplus W$ is the inclusion of the second summand and $\tilde{\alpha}$ is the adjoint assembly map for the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))$. Since W is ample, the map $\mathrm{Sym}(i_{\mathbb{C}}) : \mathrm{Sym}(W_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$ is an equivariant linear isometric embedding between complete complex G-universes. So the first map is a G-homotopy equivalence by Proposition 6.3.18. The second map is a G-weak equivalence by Theorem 6.3.19 (ii). □

Now we will justify that for every finite group G the underlying orthogonal G-spectrum of $\mathbf{ku}$ represents connective G-equivariant topological K-theory.

Construction 6.3.24. We define a morphism of orthogonal spaces

$$c \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega^{\bullet}\mathbf{ku} \ , \tag{6.3.25}$$

where $\mathbf{Gr}^{\mathbb{C}}$ is the complex additive Grassmannian introduced in Example 2.3.16. The value at an inner product space V is the continuous map

$$c(V) \ : \ Gr^{\mathbb{C}}(V_{\mathbb{C}}) \ \longrightarrow \ \mathrm{map}_*(S^V, \mathbf{ku}(V)) \ = \ (\Omega^{\bullet}\mathbf{ku})(V)$$

that sends a complex subspace $L \subset V_{\mathbb{C}}$ to the continuous based map

$$[L; -] \ : \ S^V \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V) \ , \quad v \ \longmapsto \ [L; v] \ .$$

Here we consider $L \subset V_{\mathbb{C}}$ as sitting in the linear summand of the symmetric algebra of $V_{\mathbb{C}}$.

It will be useful to also define a delooping of the morphism c, namely a morphism of orthogonal spaces

$$\mathrm{eig} \ : \ \mathbf{U} \ \longrightarrow \ \Omega^{\bullet}(\mathrm{sh}\,\mathbf{ku}) \ , \tag{6.3.26}$$

where $\mathbf{U}$ is the ultra-commutative monoid of unitary groups (compare Example 2.3.7), and $\mathrm{sh}\,\mathbf{ku} = \mathrm{sh}^{\mathbb{R}}\,\mathbf{ku}$ is the shift of $\mathbf{ku}$ as defined in (3.1.22). The name 'eig' refers to the fact that the morphism records the eigenspace decomposition of a unitary isomorphism, compare Remark 6.3.3. The definition uses the homeomorphism

$$h \ : \ U(1) \ \cong \ S^1 \ , \quad h(\lambda) \ = \ i \cdot \frac{\lambda + 1}{\lambda - 1} \ ,$$

the inverse of the Cayley transform. Every unitary automorphism of a finite-dimensional hermitian inner product space is diagonalizable with eigenvalues in $U(1)$ and pairwise orthogonal eigenspaces. So given an inner product space V we define

$$\mathrm{eig}(V) \ : \ \mathbf{U}(V) \ = \ U(V_{\mathbb{C}}) \ \longrightarrow \ \mathrm{map}_*(S^V, \mathbf{ku}(V \oplus \mathbb{R})) = (\Omega^{\bullet}(\mathrm{sh}\,\mathbf{ku}))\,(V)$$

by

$$\mathrm{eig}(V)(A)(v) \ = \ [E(\lambda_1), \dots, E(\lambda_n); \, (v, h(\lambda_1)), \dots, (v, h(\lambda_n))] \ .$$

Here $\lambda_1, \dots \lambda_n \in U(1)$ are the eigenvalues of A and $E(\lambda_i)$ is the eigenspace of λ_i. Strictly speaking, $E(\lambda_i)$ is a subspace of $V_{\mathbb{C}}$, which we embed into the linear summand of $\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})$.

Theorem 6.3.27. *The morphism*

$$\mathrm{eig}\ :\ \mathbf{U}\ \longrightarrow\ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku})$$

is a $\mathcal{F}in$-global equivalence of orthogonal spaces.

Proof We let $\bar{\mathbf{U}}$ denote the orthogonal space with

$$\bar{\mathbf{U}}(V)\ =\ U(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}))\ .$$

The structure map induced by $\varphi : V \longrightarrow W$ is given by extending by the identity on the orthogonal complement of $\mathrm{Sym}((\varphi\oplus\mathbb{R})_{\mathbb{C}}) : \mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((W\oplus\mathbb{R})_{\mathbb{C}})$. The eigenspace decomposition map then factors as the composite

$$\mathbf{U}\ \longrightarrow\ \bar{\mathbf{U}}\ \xrightarrow{\overline{\mathrm{eig}}}\ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku})\ ,$$

where $\overline{\mathrm{eig}}$ is defined in the same way as eig, recording the set of eigenvalues and eigenspaces. Since $\bar{\mathbf{U}}(V)$ is the colimit over $n\geq 0$, along closed embeddings, of the spaces $U(\mathrm{Sym}^{\leq n}((V\oplus\mathbb{R})_{\mathbb{C}}))$, the morphism $\mathbf{U}\longrightarrow\bar{\mathbf{U}}$ is a global equivalence of orthogonal spaces by Theorem 1.1.10 and Proposition 1.1.9 (ix). We may thus show that the morphism

$$\overline{\mathrm{eig}}\ :\ \bar{\mathbf{U}}\ \longrightarrow\ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku})$$

is a $\mathcal{F}in$-global equivalence. We show the stronger statement that for every finite group G and every ample G-representation V the map $\overline{\mathrm{eig}}(V)$ is a G-weak equivalence. Indeed, the map $\overline{\mathrm{eig}}(V)$ factors as the composite

$$\begin{aligned} U(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}))\ &\xrightarrow{\cong}\ \mathscr{C}(\mathrm{Sym}((V_{\mathbb{C}}\oplus\mathbb{R})_{\mathbb{C}}), S^1)\\ &\xrightarrow{\sim}\ \mathrm{map}_*(S^V, \mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}), S^{V\oplus\mathbb{R}}))\ . \end{aligned}$$

The first map is the eigenspace decomposition, hence a homeomorphism by Remark 6.3.3. The second map is adjoint to the assembly map

$$S^V\wedge\mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}), S^1)\ \longrightarrow\ \mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}), S^{V\oplus\mathbb{R}})$$

of the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}))$. Since V is ample, $\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}})$ is a complete complex G-universe. So the adjoint assembly map is a G-weak equivalence by Theorem 6.3.19 (ii). □

The set $[A,\mathbf{Gr}^{\mathbb{C}}]^G$ has an abelian monoid structure arising from the ultra-commutative multiplication of $\mathbf{Gr}^{\mathbb{C}}$ as explained in (2.4.4). The set $[A,\Omega^\bullet\mathbf{ku}]^G$ has an abelian group structure as an equivariant stable homotopy group, i.e., through the adjunction bijection

$$[A,\Omega^\bullet\mathbf{ku}]^G\ \cong\ \pi_0^G(\mathrm{map}(A,\mathbf{ku}))\ ;$$

so this group structure arises from concatenation of loops. The morphism of orthogonal spaces $c : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega^\bullet\mathbf{ku}$ defined in (6.3.25) is *not* a homomorphism of ultra-commutative monoids, nor is it a loop map; so it is not a priori clear whether the induced map on equivariant homotopy sets is a monoid homomorphism.

Theorem 6.3.28. *Let G be a compact Lie group and A a finite G-CW-complex. Then the map*

$$[A, c]^G \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ [A, \Omega^\bullet\mathbf{ku}]^G$$

is a monoid homomorphism. If all isotropy groups of A are finite, then $[A, c]^G$ is a group completion of abelian monoids.

Proof To show that $[A, c]^G$ is a monoid homomorphism we exhibit a 'delooping' of c, namely the eigenspace morphism (6.3.26). In (2.5.38) we defined a morphism of ultra-commutative monoids $\beta : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ that is a global group completion by Theorem 2.5.40. Now we link the monoid homomorphism

$$[A, \beta]^G \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ [A, \Omega\mathbf{U}]^G$$

to the set map $[A, c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \Omega^\bullet\mathbf{ku}]^G$. The global equivalence

$$\tilde{\lambda}_{\mathbf{ku}} \ : \ \mathbf{ku} \ \longrightarrow \ \Omega(\mathrm{sh}\,\mathbf{ku})$$

was defined in (3.1.23). The adjunction isomorphisms

$$\begin{aligned} (\Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku})))(V) \ &= \ \mathrm{map}_*(S^V, \Omega\mathbf{ku}(V \oplus \mathbb{R})) \\ &\cong \ \mathrm{map}_*(S^{V\oplus\mathbb{R}}, \mathbf{ku}(V \oplus \mathbb{R})) \ = \ (\mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku}))(V) \end{aligned}$$

provide an isomorphism of orthogonal spaces between $\Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku}))$ and $\mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku})$, where $\mathrm{sh}_\oplus = \mathrm{sh}_\oplus^{\mathbb{R}}$ is the additive shift defined in Example 1.1.11. Under this isomorphism, the morphism

$$\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}} \ : \ \Omega^\bullet\mathbf{ku} \ \longrightarrow \Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku}))$$

becomes the morphism

$$(\Omega^\bullet\mathbf{ku}) \circ i \ : \ \Omega^\bullet\mathbf{ku} \ \longrightarrow \ \mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku})$$

given by pre-composition with the embeddings $i_V : V \longrightarrow V \oplus \mathbb{R}$. This morphism is a global equivalence by Theorem 1.1.10. So the morphism $\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}}$ is a global equivalence of orthogonal spaces as well.

The rigorous statement of the delooping property of the eigenspace morphism is the following commutative square of orthogonal spaces:

$$\begin{tikzcd} \mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\beta"'] & \Omega^\bullet \mathbf{ku} \arrow[d, "\Omega^\bullet \tilde{\lambda}_{\mathbf{ku}}", "\simeq"'] \\ \Omega \mathbf{U} \arrow[r, "\Omega\,\mathrm{eig}"'] & \Omega(\Omega^\bullet(\mathrm{sh}\,\mathbf{ku})) \end{tikzcd}$$

This square induces a commutative square of set maps

$$\begin{tikzcd} {[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "{[A,c]^G}"] \arrow[d, "{[A,\beta]^G}"'] & {[A, \Omega^\bullet \mathbf{ku}]^G} \arrow[d, "{[A,\Omega^\bullet \tilde{\lambda}_{\mathbf{ku}}]^G}", "\cong"'] \\ {[A, \Omega \mathbf{U}]^G} \arrow[r, "{[A,\Omega\,\mathrm{eig}]^G}"'] & {[A, \Omega\,(\Omega^\bullet(\mathrm{sh}\,\mathbf{ku}))]^G} \end{tikzcd} \qquad (6.3.29)$$

The set $[A, \Omega\mathbf{U}]^G$ can be endowed with a monoid structure in two ways, via the ultra-commutative multiplication as in (2.4.4), and by concatenation of loops. Since the ultra-commutative monoid structure of $\Omega\mathbf{U}$ is 'pointwise' (i.e., induced by the ultra-commutative monoid structure of $\mathbf{U}$), these two monoid structures satisfy the interchange law, so they coincide, and both are abelian group structures. The morphism β is a homomorphism of ultra-commutative monoids, so it induces an additive map on $[A, -]^G$. The morphisms $\Omega\,\mathrm{eig}$ and $\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}}$ are loop maps, so they induce homomorphisms with respect to the group structure by concatenation of loops. This shows that three of the four maps in (6.3.29) are homomorphisms of abelian monoids. Since the right vertical map is an isomorphism, the map $[A, c]^G$ is a homomorphism as well.

For the rest of the proof we suppose that the isotropy groups of A are finite. The morphism $\Omega\,\mathrm{eig}$ is a $\mathcal{F}in$-global equivalence by Theorem 6.3.27, so the map $[A, \Omega\,\mathrm{eig}]^G$ is bijective, and hence an isomorphism of abelian groups, by Proposition 1.5.3 (ii). The morphism $\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}} : \Omega^\bullet\mathbf{ku} \longrightarrow \Omega(\Omega^\bullet(\mathrm{sh}\,\mathbf{ku}))$ is a global equivalence of orthogonal spaces, so the map $[A, \Omega^\bullet\tilde{\lambda}_{\mathbf{ku}}]^G$ is bijective, and hence an isomorphism of abelian groups, again by Proposition 1.5.3 (ii). Since $[A,\beta]^G$ is a group completion of abelian monoids (by Corollary 2.5.42), the commutative square (6.3.29) shows that $[A, c]^G$ is a group completion. □

We draw an important consequence of Theorem 6.3.28, namely that the equivariant cohomology theory represented by the connective global K-theory spectrum **ku** is essentially equivariant K-theory. There is a caveat, however, as this is *not* true on arbitrary finite G-CW-complexes, but only under the hypothesis of finite stabilizer groups.

We let G be a compact Lie group and A a compact G-space. We define the

0th G-equivariant $\mathbf{ku}$-cohomology group of A as

$$\mathbf{ku}^0_G(A_+) \;=\; [A, \Omega^\bullet \mathbf{ku}]^G \ ,$$

the equivariant homotopy set into the orthogonal space $\Omega^\bullet\mathbf{ku}$. This set is an abelian group by concatenation of loops, i.e., via the adjunction bijection

$$[A, \Omega^\bullet \mathbf{ku}]^G \;\cong\; \pi_0^G(\mathrm{map}(A, \mathbf{ku}))$$

to an equivariant stable homotopy group. The set also has a multiplication, i.e., another commutative binary operation arising from the ring spectrum structure of $\mathbf{ku}$, which turns $\Omega^\bullet\mathbf{ku}$ into a 'multiplicative' ultra-commutative monoid as in Example 4.1.16. A conjugation involution of the ultra-commutative ring spectrum $\mathbf{ku}$ was defined in Construction 6.3.10.

We denote by $\mathbf{K}_G(A)$ the G-equivariant K-group of A, i.e., the group completion (Grothendieck group) of the abelian monoid of isomorphism classes of complex G-vector bundles over A. The map

$$\langle - \rangle \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ \mathbf{K}_G(A) \ , \quad [f : A \longrightarrow Gr^{\mathbb{C}}(V)] \ \longmapsto \ [f^\star(\gamma_V^{\mathbb{C}})]$$

that takes the pullback of the tautological vector bundles over Grassmannians is a group completion of abelian monoids by Theorem 2.4.10 (or rather its complex analog). Since $[A, c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \Omega^\bullet\mathbf{ku}]^G$ is a monoid homomorphism (by Theorem 6.3.28) to an abelian group, the universal property of group completion provides a unique additive extension

$$[-] \ : \ \mathbf{K}_G(A) \ \longrightarrow \ \mathbf{ku}^0_G(A_+) \tag{6.3.30}$$

such that the composite

$$[A, \mathbf{Gr}^{\mathbb{C}}]^G \ \xrightarrow{\langle - \rangle} \ \mathbf{K}_G(A) \ \xrightarrow{[-]} \ \mathbf{ku}^0_G(A_+) \ = \ [A, \Omega^\bullet \mathbf{ku}]^G$$

is the map induced by the morphism of orthogonal spaces $c : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega^\bullet\mathbf{ku}$.

Theorem 6.3.31. *Let G be a compact Lie group and A a finite G-CW-complex.*

(i) *For every G-representation V and every continuous G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$, the homomorphism $[-]$ sends the class of the G-vector bundle $f^\star(\gamma^{\mathbb{C}}_{\mathrm{Sym}(V_{\mathbb{C}})})$ to the homotopy class of the G-map*

$$A \ \xrightarrow{[f(-);-]} \ \mathrm{map}_*(S^V, \mathbf{ku}(V)) \ , \quad a \ \longmapsto \ \{v \ \longmapsto \ [f(a); v]\} \ .$$

(ii) *The additive map $[-]$ is a ring homomorphism, natural for G-maps in A, natural for restriction homomorphisms in G, and compatible with complex conjugation.*

(iii) *If A has finite isotropy groups, then the homomorphism $[-] : \mathbf{K}_G(A) \longrightarrow \mathbf{ku}^0_G(A_+)$ is an isomorphism.*

Proof (i) We recall from Example 2.3.18 the multiplicative Grassmannian $\mathbf{Gr}_{\otimes}^{\mathbb{C}}$ with values

$$\mathbf{Gr}_{\otimes}^{\mathbb{C}}(V) \;=\; \coprod_{n\geq 0} Gr_n^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))\ ,$$

the disjoint union of all Grassmannians in the symmetric algebra of $V_{\mathbb{C}}$. We let $i : V_{\mathbb{C}} \longrightarrow \mathrm{Sym}(V_{\mathbb{C}})$ be the embedding as the linear summand of the symmetric algebra. Then as V varies, the maps

$$i(V) \;:\; \mathbf{Gr}^{\mathbb{C}}(V) \;\longrightarrow\; \mathbf{Gr}_{\otimes}^{\mathbb{C}}(V)\ , \quad L \;\longmapsto\; i(L)$$

form a global equivalence $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}_{\otimes}^{\mathbb{C}}$ of orthogonal spaces, see Example 2.3.18. The morphism of orthogonal spaces $c : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega^{\bullet}\mathbf{ku}$ defined in (6.3.25) has an extension

$$c_{\otimes} \;:\; \mathbf{Gr}_{\otimes}^{\mathbb{C}} \;\longrightarrow\; \Omega^{\bullet}\mathbf{ku}\ ,$$

defined by the same formula as for c, namely

$$c_{\otimes}(V) \;:\; \mathbf{Gr}_{\otimes}^{\mathbb{C}}(V) \;\longrightarrow\; \mathrm{map}_*(S^V, \mathbf{ku}(V)) \;=\; (\Omega^{\bullet}\mathbf{ku})(V)\ , \quad L \;\longmapsto\; [L; -]$$

with

$$[L; -] \;:\; S^V \;\longrightarrow\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \;=\; \mathbf{ku}(V) \quad \text{given by} \quad v \;\longmapsto\; [L; v]\ .$$

The 'pullback bundle' map $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ also has a straightforward extension

$$\langle - \rangle_{\otimes} \;:\; [A, \mathbf{Gr}_{\otimes}^{\mathbb{C}}]^G \;\longrightarrow\; \mathbf{K}_G(A)$$

again given by the same recipe: we send the class represented by a G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$ to the class of the pullback $f^{\star}(\gamma^{\mathbb{C}}_{\mathrm{Sym}(V_{\mathbb{C}})})$ of the tautological bundle over $Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$. The outer square in the diagram

$$\begin{tikzcd}
{[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,i]^G}"', "\cong"] & \mathbf{K}_G(A) \arrow[d, "{[-]}"] \\
{[A, \mathbf{Gr}_{\otimes}^{\mathbb{C}}]^G} \arrow[r, "{[A,c_{\otimes}]^G}"'] \arrow[ur, "\langle - \rangle_{\otimes}"] & {[A, \Omega^{\bullet}\mathbf{ku}]^G}
\end{tikzcd}$$

commutes because $c_{\otimes} \circ i = c$ and $[A, c]^G = [-] \circ \langle - \rangle$. The upper left triangle commutes because the tautological bundle on $\mathbf{Gr}_{\otimes}^{\mathbb{C}}(V)$ restricts to the tautological bundle on $\mathbf{Gr}^{\mathbb{C}}(V)$ along the map $i(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow \mathbf{Gr}_{\otimes}^{\mathbb{C}}(V)$. The left vertical map is bijective by Proposition 1.5.3 (ii), because $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}_{\otimes}^{\mathbb{C}}$ is a global equivalence. So the lower right triangle commutes as well. The map

$$[A, c_{\otimes}]^G \;:\; [A, \mathbf{Gr}_{\otimes}^{\mathbb{C}}]^G \;\longrightarrow\; [A, \Omega^{\bullet}\mathbf{ku}]^G \;=\; \mathbf{ku}_G^0(A_+)$$

sends the homotopy class of a continuous G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$ to

the class of the map $[f(-);-] : A \longrightarrow \mathrm{map}_*(S^V, \mathbf{ku}(V))$, by the very definition of $c_\otimes$. So this proves the claim.

(ii) Naturality in G-maps is straightforward. For a G-map $h : A' \longrightarrow A$ and every G-map $f : A \longrightarrow \mathbf{Gr}^{\mathbb{C}}(V)$, the two G-vector bundles $h^\star(f^\star(\gamma_V^{\mathbb{C}}))$ and $(fh)^\star(\gamma_V^{\mathbb{C}})$ over A' are isomorphic, so the following square commutes:

$$\begin{tikzcd}
[A, \mathbf{Gr}^{\mathbb{C}}]^G \arrow[r, "\langle - \rangle"] \arrow[d, "{[h,\mathbf{Gr}^{\mathbb{C}}]^G}"'] & \mathbf{K}_G(A) \arrow[d, "\mathbf{K}_G(h)"] \\
[A', \mathbf{Gr}^{\mathbb{C}}]^G \arrow[r, "\langle - \rangle"'] & \mathbf{K}_G(A')
\end{tikzcd}$$

Together with the defining property of the homomorphism $[-]$, this implies that the two group homomorphisms

$$[-] \circ \mathbf{K}_G(h)\ ,\ \mathbf{ku}_G^0(h_+) \circ [-]\ :\ \mathbf{K}_G(A) \longrightarrow \mathbf{ku}_G^0(A'_+)$$

coincide after pre-composition with $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$. Since this map is a group completion of abelian monoids, already the homomorphisms $[-] \circ \mathbf{K}_G(h)$ and $\mathbf{ku}_G^0(h_+) \circ [-]$ agree, by the universal property of group completions.

The compatibility with complex conjugation and restriction along group homomorphisms follow the same pattern. The conjugation morphism of orthogonal spectra $\psi : \mathbf{ku} \longrightarrow \mathbf{ku}$ deloops the conjugation morphism of ultra-commutative monoids $\psi : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}$, in the sense that the square of orthogonal spaces commutes:

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\psi"'] & \Omega^\bullet \mathbf{ku} \arrow[d, "\Omega^\bullet \psi"] \\
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"'] & \Omega^\bullet \mathbf{ku}
\end{tikzcd}$$

In particular, the homomorphism $[A, c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{ku}_G^0(A_+)$ commutes with complex conjugation. On the other hand, for every G-map $f : A \longrightarrow \mathbf{Gr}^{\mathbb{C}}(V)$, the bundle $(\psi(V) \circ f)^\star(\gamma_V^{\mathbb{C}})$ is isomorphic to the complex conjugate of the bundle $f^\star(\gamma_V^{\mathbb{C}})$. So the following square commutes:

$$\begin{tikzcd}
[A, \mathbf{Gr}^{\mathbb{C}}]^G \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,\psi]^G}"'] & \mathbf{K}_G(A) \arrow[d, "\psi"] \\
[A, \mathbf{Gr}^{\mathbb{C}}]^G \arrow[r, "\langle - \rangle"'] & \mathbf{K}_G(A)
\end{tikzcd}$$

Thus the two group homomorphisms

$$[-] \circ \psi\ ,\ \psi \circ [-]\ :\ \mathbf{K}_G(A) \longrightarrow \mathbf{ku}_G^0(A_+)$$

agree after pre-composition with a group completion, hence they coincide. The analogous argument works for restriction along a continuous homomorphism $\alpha : K \longrightarrow G$, using that the underlying K-vector bundle of $f^\star(\gamma_V^{\mathbb{C}})$ equals the bundle $\alpha^*(f)(\gamma_{\alpha^*(V)}^{\mathbb{C}})$, and the effect of c (being a morphism of orthogonal spaces) commutes with restriction.

Now we show that the additive map $[-] : \mathbf{K}_G(A) \longrightarrow \mathbf{ku}_G^0(A_+)$ is a ring homomorphism. The multiplicative unit of $\mathbf{K}_G(A)$ is the class of the trivial line bundle $A \times \mathbb{C}$. This bundle is isomorphic to $f^\star(\gamma_{\mathrm{Sym}(0)}^{\mathbb{C}})$ for the constant map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(0))$ with value $\mathbb{C}$, the constant (and only non-trivial) summand in the symmetric algebra associated to the 0-dimensional G-representation. The associated G-map $[f(-); -] : A \longrightarrow \mathrm{map}_*(S^0, \mathbf{ku}(0))$ is constant with image the unit map $\iota_0 : S^0 \longrightarrow \mathbf{ku}(0)$; so by part (i) the class $\langle A \times \mathbb{C}\rangle$ of the trivial line bundle is the multiplicative unit in the ring $\mathbf{ku}_G^0(A_+)$. So the map $[-]$ preserves multiplicative units.

It remains to show that the map $[-]$ preserves products. Because $[-]$ is additive and the group $\mathbf{K}_G(A)$ is generated by classes of actual vector bundles (as opposed to virtual bundles), it suffices to show multiplicativity for two classes in $\mathbf{K}_G(A)$ represented by G-vector bundles. We may assume that the two bundles are classified by continuous G-maps

$$f : A \longrightarrow Gr^{\mathbb{C}}(V_{\mathbb{C}}) \quad \text{and} \quad g : A \longrightarrow Gr^{\mathbb{C}}(W_{\mathbb{C}})\ ,$$

where V and W are G-representations. The tensor product bundle $f^\star(\gamma_V^{\mathbb{C}}) \otimes g^\star(\gamma_W^{\mathbb{C}})$ is then classified by the composite

$$\begin{aligned} A \xrightarrow{(f,g)} Gr^{\mathbb{C}}(V_{\mathbb{C}}) \times Gr^{\mathbb{C}}(W_{\mathbb{C}}) &\xrightarrow{-\otimes-} Gr^{\mathbb{C}}(V_{\mathbb{C}} \otimes W_{\mathbb{C}}) \\ &\xrightarrow{Gr^{\mathbb{C}}(j)} Gr^{\mathbb{C}}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))\ , \end{aligned}$$

where

$$j : V_{\mathbb{C}} \otimes W_{\mathbb{C}} \longrightarrow \mathrm{Sym}((V \oplus W)_{\mathbb{C}}) \quad \text{is defined by} \quad j(v \otimes w) = (v, 0) \cdot (0, w)\ .$$

By part (i) the associated homotopy class $[f^\star(\gamma_V^{\mathbb{C}}) \otimes g^\star(\gamma_W^{\mathbb{C}})]$ is represented by the G-map

$$[(j \circ (f \otimes g))(-); -] : A \longrightarrow \mathrm{map}_*(S^{V \oplus W}, \mathbf{ku}(V \oplus W))\ .$$

This map is the adjoint of the composite

$$A_+ \wedge S^{V \oplus W} \xrightarrow{a \wedge v \wedge w \mapsto [f(a); v] \wedge [g(a); w]} \mathbf{ku}(V) \wedge \mathbf{ku}(W) \xrightarrow{\mu_{V,W}} \mathbf{ku}(V \oplus W)\ .$$

This composite represents the product of the classes $[f^\star(\gamma_V^{\mathbb{C}})]$ and $[g^\star(\gamma_W^{\mathbb{C}})]$, so this establishes the relation

$$[f^\star(\gamma_V^{\mathbb{C}})] \cdot [g^\star(\gamma_W^{\mathbb{C}})] = [f^\star(\gamma_V^{\mathbb{C}}) \otimes g^\star(\gamma_W^{\mathbb{C}})]$$

in the group $\mathbf{ku}_G^0(A)$.

(iii) If A is a finite G-CW-complex with finite isotropy groups, then the map $[A, c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{ku}_G^0(A_+)$ is a group completion of abelian monoids by Theorem 6.3.28. The map $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is also a group completion (by Theorem 2.4.10, or rather its complex analog), so the unique extension $[-] : \mathbf{K}_G(A) \longrightarrow \mathbf{ku}_G^0(A_+)$ is an isomorphism. □

We specialize Theorem 6.3.31 to the case $A = *$, i.e., when the base is a single point. In this case the bundle projection is no information, G-vector bundles specialize to G-representations, and the ring $\mathbf{K}_G(*)$ becomes the unitary representation ring $\mathbf{RU}(G)$. On the other hand, $\mathbf{ku}_G^0(S^0)$ specializes to $\pi_0^G(\mathbf{ku})$. The ring homomorphism

$$[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$$

is then determined by its effect on the classes of actual representations. If W is a unitary G-representation, we get an explicit representative for the class $[W]$ by choosing a G-equivariant $\mathbb{C}$-linear isometric embedding $j : W \longrightarrow V_{\mathbb{C}}$ into the complexification of an orthogonal G-representations. For example, the map

$$j_W : W \longrightarrow (uW)_{\mathbb{C}} , \quad j_W(w) \longmapsto 1/\sqrt{2} \cdot (1 \otimes w - i \otimes (iw))$$

into the complexification of the underlying orthogonal G-representation of W does the job. Then $[W]$ is the homotopy class of the G-map

$$S^V \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V) , \quad v \longmapsto [j(W); v] . \tag{6.3.32}$$

Both $\mathbf{RU}(G)$ and $\pi_0^G(\mathbf{ku})$ have restriction maps, transfers and multiplicative power operations in G, i.e., they are global power functors in the sense of Definition 5.1.6. The maps $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ preserve most of this additional structure, but that is not apparent from what we discussed so far.

Theorem 6.3.33. *For every compact Lie group G, the map*

$$[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku}) \tag{6.3.34}$$

is a ring homomorphism. As G varies over all compact Lie groups, the homomorphisms (6.3.34) *are compatible with restriction maps, with complex conjugation, with finite index transfers, and with multiplicative power operations. Moreover, the map* $[-]$ *is an isomorphism whenever the group G is finite.*

Proof The fact that the map is a ring homomorphism, compatible with restrictions, compatible with complex conjugation and an isomorphism for finite groups is a special case of Theorem 6.3.31 for a one-point G-space.

Now we show that the maps $[-]$ are compatible with finite index transfers. We let H be a finite index subgroup of G and introduce an orthogonal G-spectrum $\mathbf{ku}[G/H]$ by

$$\mathbf{ku}[G/H](V) \;=\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge (G/H)_+)\,.$$

The structure maps of $\mathbf{ku}[G/H]$ are defined in much the same way as for $\mathbf{ku}$, with the extra smash factor $(G/H)_+$ acting as a dummy; the G-action comes entirely from the translation action on G/H. As V varies over all inner product spaces, the assembly maps

$$\begin{aligned}\mathbf{ku}(V) \wedge (G/H)_+ \;&= \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \wedge (G/H)_+ \\ &\longrightarrow\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge (G/H)_+) \;=\; \mathbf{ku}[G/H](V)\end{aligned}$$

form a morphism of orthogonal G-spectra $\alpha : \mathbf{ku} \wedge (G/H)_+ \longrightarrow \mathbf{ku}[G/H]$.

We let $\nabla : (G/H)_+ \longrightarrow 1_+$ be the G-equivariant fold map that sends all of G/H to 1. We let $l : G/H_+ \longrightarrow 1_+$ be the H-equivariant 'projection' onto the preferred coset, i.e., $l(eH) = 1$ and $l(gH) = 0$ for $g \notin H$. Then the following diagram commutes:

$$\begin{tikzcd}
\pi_0^H(\mathbf{ku}) & \pi_0^H(\mathbf{ku} \wedge G/H_+) \arrow[l, "\pi_0^H(\mathbf{ku}\wedge l)"'] \arrow[d, "\pi_0^H(\alpha)"] & \pi_0^G(\mathbf{ku} \wedge G/H_+) \arrow[l, "\mathrm{res}^G_H"'] \arrow[d, "\pi_0^G(\alpha)"] \arrow[r, "\pi_0^G(\mathbf{ku}\wedge\nabla)"] \arrow[ll, bend right, "\mathrm{Wirth}^G_H"'] & \pi_0^G(\mathbf{ku}) \\
 & \pi_0^H(\mathbf{ku}[G/H]) \arrow[ul, bend left, "\pi_0^H(\mathbf{ku}[l])"] & \pi_0^G(\mathbf{ku}[G/H]) \arrow[l, "\mathrm{res}^G_H"] \arrow[ur, bend right, "\pi_0^G(\mathbf{ku}[\nabla])"'] &
\end{tikzcd} \tag{6.3.35}$$

The upper left horizontal composite is the Wirthmüller map (3.2.6). Since the Wirthmüller map is an isomorphism (Theorem 3.2.15), the map $\pi_0^H(\mathbf{ku}[l]) \circ \mathrm{res}^G_H$ is surjective. Now we let $f : S^V \longrightarrow \mathbf{ku}[G/H](V)$ be a G-map that represents an element in the kernel of the map $\pi_0^H(\mathbf{ku}[l]) \circ \mathrm{res}^G_H$, where V is a G-representation. After increasing V, if necessary, we can assume that V is ample and that the composite

$$S^V \xrightarrow{\;f\;} \mathbf{ku}[G/H](V) \xrightarrow{\mathbf{ku}[l](V)} \mathbf{ku}(V)$$

is H-equivariantly null-homotopic. By adjointness, this means that the com-

posite

$$S^V \xrightarrow{f} \mathbf{ku}[G/H](V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V \wedge (G/H)_+)$$
$$\xrightarrow{l^\flat} \mathrm{map}^H(G, \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V)) = \mathrm{map}^H(G, \mathbf{ku}(V))$$

is G-equivariantly null-homotopic, where $l^\flat$ is adjoint to the H-equivariant map

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V \wedge l) : \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V \wedge (G/H)_+) \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V) .$$

The map $l^\flat$ coincides with the Wirthmüller map

$$\omega_{S^V} : \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); G \ltimes_H S^V) \longrightarrow \mathrm{map}^H(G, \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}); S^V))$$

defined in (B.51), up to the effect of the shearing isomorphism $S^V \wedge (G/H)_+ \cong G \ltimes_H S^V$. Since V is ample, the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ is special by Theorem 6.3.19 (i), and it is G-cofibrant by Example 6.3.16. So the Wirthmüller map ω_{S^V}, and hence also the map $l^\flat$, is a G-weak equivalence by Theorem B.54 (ii). Hence already f is G-equivariantly null-homotopic. Altogether this proves that the map $\pi_0^H(\mathbf{ku}[l]) \circ \mathrm{res}^G_H$ is also injective, hence bijective. Since $\pi_0^H(\mathbf{ku}[l]) \circ \mathrm{res}^G_H$ and the Wirthmüller map are both bijective, the commutativity of (6.3.35) shows that the map $\pi_0^G(\alpha) : \pi_0^G(\mathbf{ku} \wedge (G/H)_+) \longrightarrow \pi_0^G(\mathbf{ku}[G/H])$ is also bijective.

Now we let W be any unitary H-representation and $j : \mathrm{map}^H(G, W) \longrightarrow V_{\mathbb{C}}$ a G-equivariant $\mathbb{C}$-linear embedding into the complexification of some G-representation. We define a class

$$[W]^G_H \in \pi_0^G(\mathbf{ku}[G/H])$$

by specifying a representative G-map

$$S^V \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge (G/H)_+) = \mathbf{ku}[G/H](V)$$

by

$$v \longmapsto [j(\mathrm{map}^H(Hg_1^{-1}, W)), \ldots, j(\mathrm{map}^H(Hg_m^{-1}, W));\ v \wedge g_1H, \ldots, v \wedge g_mH] .$$

Here $g_1, \ldots, g_m$ is a set of coset representatives for H in G. The relation

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge l_+)[j(\mathrm{map}^H(Hg_k, W));\ v \wedge g_kH]_{k=1,\ldots,m}$$
$$= [j(\mathrm{map}^H(H, W));\ v]$$

shows that

$$\pi_0^H(\mathbf{ku}[l])(\mathrm{res}^G_H([W]^G_H)) = [W] .$$

The commutativity of the left part of diagram (6.3.35) then shows that

$$[W] = \pi_0^H(\mathbf{ku}[l])(\mathrm{res}^G_H([W]^G_H)) = \mathrm{Wirth}^G_H(\pi_0^G(\alpha)^{-1}([W]^G_H)) .$$

Since the Wirthmüller map is inverse to the external transfer (Theorem 3.2.15), this is equivalent to the relation

$$[W]^G_H \ = \ \pi_0^G(\alpha)(G \ltimes_H [W])$$

in the group $\pi_0^G(\mathbf{ku}[G/H])$.

On the other hand, if we fold $S^V \wedge (G/H)_+$ onto S^V, then

$$\begin{aligned}
&\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge \nabla_+)[\mathrm{map}^H(Hg_1, W), \dots, \mathrm{map}^H(Hg_m, W);\ v \wedge g_1 H, \dots, v \wedge g_m H] \\
&\qquad = \ [\mathrm{map}^H(Hg_1, W), \dots, \mathrm{map}^H(Hg_m, W);\ v, \dots, v] \\
&\qquad = \ [\mathrm{map}^H(Hg_1, W) \oplus \dots \oplus \mathrm{map}^H(Hg_m, W);\ v] \\
&\qquad = \ [\mathrm{map}^H(G, W);\ v] \ .
\end{aligned}$$

This shows that

$$\pi_0^G(\mathbf{ku}[\nabla])([W]^G_H) \ = \ [\mathrm{map}^H(G, W)] \ .$$

The commutativity of the right part of diagram (6.3.35) then yields the desired relation for classes of actual representations:

$$\begin{aligned}
[\mathrm{map}^H(G, W)] \ = \ \pi_0^G(\mathbf{ku}[\nabla])([W]^G_H) \ &= \ \pi_0^G(\mathbf{ku}[\nabla])(\pi_0^G(\alpha)(G \ltimes_H [W])) \\
&= \ \pi_0^G(\mathbf{ku} \wedge \nabla)(G \ltimes_H [W]) \ = \ \mathrm{tr}_H^G[W] \ .
\end{aligned}$$

Since transfer maps are additive, the relation persists to classes of virtual representations.

Now we treat the compatibility with power operations, i.e., that the following square commutes for every compact Lie group G and all $m \geq 1$:

$$\begin{tikzcd}
\mathbf{RU}(G) \arrow[r, "{[-]}"] \arrow[d, "P^m"'] & \pi_0^G(\mathbf{ku}) \arrow[d, "P^m"] \\
\mathbf{RU}(\Sigma_m \wr G) \arrow[r, "{[-]}"'] & \pi_0^{\Sigma_m \wr G}(\mathbf{ku})
\end{tikzcd}$$

We consider a unitary G-representation W and a G-equivariant $\mathbb{C}$-linear isometric embedding $j : W \longrightarrow V_{\mathbb{C}}$ into the complexification of an orthogonal G-representation. The class $[W]$ is then represented by the G-map

$$[j(W); -] \ : \ S^V \ \longrightarrow \ \mathbf{ku}(V) \ .$$

The map

$$\begin{aligned}
J \ : \ W^{\otimes m} \quad &\longrightarrow \quad \mathrm{Sym}(V_{\mathbb{C}}^m) \\
w_1 \otimes \dots \otimes w_m &\longmapsto (j(w_1), 0, \dots, 0) \cdot (0, j(w_2), 0, \dots, 0) \cdot \ \dots \ \cdot (0, \dots, 0, j(w_m))
\end{aligned}$$

is a $(\Sigma_m \wr G)$-equivariant $\mathbb{C}$-linear isometric embedding. So the class $[W^{\otimes m}] = [P^m\langle W\rangle]$ is represented by the G-map

$$[J(W^{\otimes m}); -] \ : \ S^{V^m} \ \longrightarrow \ \mathbf{ku}(V^m) \ .$$

This map coincides with the composite

$$S^{V^m} \xrightarrow{[j(W);-]^{\wedge m}} \mathbf{ku}(V)^{\wedge m} \xrightarrow{\mu_{V,\dots,V}} \mathbf{ku}(V^m)$$

which represents the power operation $P^m[W]$, so we have shown the relation

$$P^m[W] \ = \ [W^{\otimes m}] \ = \ [P^m\langle W\rangle]$$

in the group $\pi_0^{\Sigma_m\wr G}(\mathbf{ku})$. Since the map $[-]$ is additive, compatible with finite index transfers and the classes of actual representations generate $\mathbf{RU}(G)$ as an abelian group, the additivity formula for power operations

$$P^m(x+y) \ = \ \sum\nolimits_{k=0}^{m} \operatorname{tr}_{k,m-k}(P^k(x)\times P^{m-k}(y))$$

implies that the map $[-]$ is compatible with power operations of virtual representations. □

Example 6.3.36 (Dimension homomorphism)**.** The *dimension homomorphism* is a homomorphism of ultra-commutative ring spectra

$$\dim \ : \ \mathbf{ku} \ \longrightarrow \ Sp^\infty$$

from the connective global K-theory spectrum to the infinite symmetric product spectrum defined in Example 5.3.10. The value of this homomorphism at an inner product space V is the map

$$\begin{aligned} \dim(V) \ : \ \mathbf{ku}(V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ &\longrightarrow \quad Sp^\infty(S^V) \\ [E_1,\dots,E_n;\, v_1,\dots,v_n] \ &\longmapsto \ \sum\nolimits_{i=1}^{n} \dim(E_i)\cdot v_i \ , \end{aligned}$$

i.e., a configuration of vector spaces is mapped to the configurations of its dimensions (i.e., the sum in the abelian monoid structure of $Sp^\infty(S^V)$). The map is multiplicative because dimension is multiplicative on tensor products.

The infinite symmetric product spectrum includes by a global equivalence of ultra-commutative ring spectra $Sp^\infty \longrightarrow \mathcal{H}\mathbb{Z}$ into the Eilenberg-Mac Lane spectrum of the integers, see Proposition 5.3.12 (ii). So we sometimes take $\mathcal{H}\mathbb{Z}$ (instead of Sp^∞) as the target of the dimension homomorphism.

We record that the effect of the dimension homomorphism on equivariant homotopy groups 'is' the augmentation

$$\dim \ : \ \mathbf{RU}(G) \ \longrightarrow \ \mathbb{Z}$$

that sends a virtual representation to its virtual dimension. We warn the reader

that the two vertical maps in the following commutative square are isomorphisms for finite groups, but *not* for general compact Lie groups.

Proposition 6.3.37. *For every compact Lie group G, the following square of commutative rings commutes:*

$$\begin{tikzcd} \mathbf{RU}(G) \arrow[r, "\dim"] \arrow[d, "{[-]}"'] & \mathbb{Z} \arrow[d, "n \mapsto n\cdot 1"] \\ \pi_0^G(\mathbf{ku}) \arrow[r, "\pi_0^G(\dim)"'] & \pi_0^G(Sp^\infty) \end{tikzcd}$$

Proof For $n \geq 1$ we consider the 'geometric' degree n morphism

$$\delta^n \ : \ \mathbb{S} \ \longrightarrow \ Sp^\infty \ , \quad x \ \longmapsto \ n \cdot x \ .$$

In level 0, the map

$$\delta^n(0) \ : \ S^0 \ \longrightarrow \ Sp^\infty(S^0) \ \cong \ \mathbb{N}\{0\}$$

sends the non-basepoint $0 \in S^0$ to $n \cdot 0$, the n-fold multiple of the corresponding generator of the free abelian monoid $\mathbb{N}\{0\}$. Since the ring homomorphism $\mathbb{Z} \longrightarrow \pi_0^G(Sp^\infty)$ sends n to the class of $n \cdot 0$, this shows that $\delta^n_*(1) = n \cdot 1$ in the group $\pi_0^G(Sp^\infty)$.

Now we let W be a unitary G-representation of dimension n. A representative $[j_W(W); -]$ for the class $[W]$ in $\mathbf{RU}(G)$ was specified in (6.3.32). The composite

$$S^{uW} \xrightarrow{[j_W(W);-]} \mathbf{ku}(uW) \xrightarrow{\dim(uW)} Sp^\infty(S^{uW})$$

is the map

$$\delta^n(uW) \ : \ S^{uW} \longrightarrow Sp^\infty(uW) \ .$$

Hence we conclude that

$$\pi_0^G(\dim)[W] \ = \ [\delta^n(uW)] \ = \ \delta^n_*(1) \ = \ n \cdot 1 \ = \ \dim(W) \cdot 1 \ .$$

This proves the proposition for the classes of actual representations. Since all four maps in the square are homomorphisms of abelian groups, the relation also holds for the classes of virtual representations. □

Remark 6.3.38. Segal [150, §2] discusses a 'smooth transfer' $i_! : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ where $i : H \longrightarrow G$ is the inclusion of a closed subgroup of a compact Lie group G. This smooth transfer is *not* to be confused with the holomorphic transfer, which is defined when G/H is a complex algebraic variety. One could

hope that the maps $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ take the smooth transfer to the homotopy theoretic transfer

$$\mathrm{tr}_H^G : \pi_0^H(\mathbf{ku}) \longrightarrow \pi_0^G(\mathbf{ku}) .$$

This (false!) expectation is suggested by the fact that the smooth transfer has all the right formal properties, and the complex representations rings do form a global functor. As we now illustrate by a specific example, the collection of ring homomorphisms $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ comes close to being a morphism of global functors, but it fails to commute with infinite index transfers. For *periodic* global K-theory $\mathbf{KU}$, discussed below in Construction 6.4.9, the two transfers do correspond also for infinite index inclusions; in fact, $\underline{\pi}_0(\mathbf{KU})$ is isomorphic, as a global power functor, to the representation ring global functor, see Theorem 6.4.24 below.

To illustrate the issue we recall a specific smooth transfer to $G = SU(2)$ from the normalizer $N = N_{SU(2)}T$ of the diagonal maximal torus

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in U(1) \right\} .$$

The maximal torus normalizer N is generated by T and the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We use Segal's character formula [150, p. 119] to calculate the character of the smooth transfer $i_!(1)$ of the trivial 1-dimensional N-representation, where $i : N \longrightarrow SU(2)$ is the inclusion. An element $g \in SU(2)$ is regular in the sense of [150, Def. 1.9] if and only if it has infinite order, and then the closed subgroup generated by g is a conjugate ${}^{\gamma}T$ of T. The fixed set of such a regular element g on G/N then consists of a single coset, namely γN. So

$$\chi_{i_!(1)}(g) = \chi_1(\gamma^{-1} g \gamma) = 1 .$$

Since the character is continuous and regular elements are dense, the character of $i_!(1)$ is constant with value 1. Since the character determines the representation up to isomorphism, we have $i_!(1) = 1$ in $\mathbf{RU}(SU(2))$. We claim that in contrast to this relation in $\mathbf{RU}(SU(2))$, the elements $\mathrm{tr}_N^{SU(2)}(1)$ and 1 are linearly independent (so in particular different) in $\pi_0^{SU(2)}(\mathbf{ku})$. We can detect this through the dimension homomorphism $\dim : \mathbf{ku} \longrightarrow \mathcal{H}\mathbb{Z}$ introduced in Example 6.3.36. As we discussed in Example 5.3.14, the elements $\mathrm{tr}_N^{SU(2)}(1)$ and 1 are linearly independent in $\pi_0^{SU(2)}(H\mathbb{Z})$, so they must also be linearly independent in $\pi_0^{SU(2)}(\mathbf{ku})$.

The dimension homomorphism can also be used to detect some odd-dimensional classes in the coefficient ring $\pi_*^{U(1)}(\mathbf{ku})$: we showed in Theorem 5.3.16 that the group $\pi_1^{U(1)}(\mathcal{H}\mathbb{Z})$ is isomorphic to $\mathbb{Q}$, and that the dimension shifting transfer from the trivial group to $U(1)$, applied to the suspension of the multi-

plicative unit $1 \in \pi_0^e(\mathcal{H}\mathbb{Z})$, is non-zero in $\pi_1^{U(1)}(\mathcal{H}\mathbb{Z})$. So the class

$$\mathrm{Tr}_e^{U(1)}(1 \wedge S^1) \ \in \ \pi_1^{U(1)}(\mathbf{ku})$$

has infinite order, and is in particular non-zero.

Remark 6.3.39. There is a morphism of orthogonal spaces $c : \mathbf{Gr} \longrightarrow \Omega^\bullet \mathbf{ko}$ defined in much the same way as its complex analog (6.3.25). Derived from this are ring homomorphisms

$$[-] \ : \ \mathbf{KO}_G(A) \ \longrightarrow \ \mathbf{ko}_G^0(A_+)$$

for every compact G-space A, analogous to the homomorphism (6.3.30) in the complex case. For $A = *$ this specializes to a ring homomorphism

$$[-] \ : \ \mathbf{RO}(G) \ \longrightarrow \ \pi_0^G(\mathbf{ko}) \ ,$$

where $\mathbf{RO}(G)$ is the orthogonal representation ring of G. The analog of Theorem 6.3.33 then says that these homomorphisms are compatible with restriction, finite index transfer and multiplicative power operations, and an isomorphism for finite groups G. However, in the real situation there is no eigenspace decomposition, hence no direct analog of the delooping $\mathrm{eig} : \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}\,\mathbf{ku})$ of the morphism c. So to establish the real analog of Theorem 6.3.28 (which is used in the proof of Theorem 6.3.33), one has to use a different proof.

Construction 6.3.40 (Rank filtrations)**.** The connective global K-theory spectra come with exhaustive filtrations

$$\mathbf{ko}^{[1]} \longrightarrow \mathbf{ko}^{[2]} \longrightarrow \ldots \longrightarrow \mathbf{ko}^{[m]} \longrightarrow \ldots \ \longrightarrow \ \mathbf{ko}$$

and

$$\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}^{[2]} \longrightarrow \ldots \longrightarrow \mathbf{ku}^{[m]} \longrightarrow \ldots \ \longrightarrow \ \mathbf{ku}$$

by orthogonal subspectra. We define the rank filtration for $\mathbf{ku}$, the case of $\mathbf{ko}$ being similar. For $m \geq 1$ we let

$$\mathbf{ku}^{[m]}(V) \ = \ \mathscr{C}^{[m]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \subset \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V)$$

be the subspace of those configurations $[E_1, \ldots, E_n;\ v_1, \ldots, v_n]$ such that

$$\sum\nolimits_{i=1}^n \dim(E_i) \ \leq \ m \ .$$

As V varies, the spaces $\mathbf{ku}^{[m]}(V)$ form an orthogonal subspectrum $\mathbf{ku}^{[m]}$ of $\mathbf{ku}$.

The dimension function is multiplicative on tensor products, so the multiplication of $\mathbf{ku}$ restricts to associative and unital pairings

$$\mathbf{ku}^{[m]} \wedge \mathbf{ku}^{[n]} \ \longrightarrow \ \mathbf{ku}^{[mn]} \ .$$

For $m = 1$ this gives $\mathbf{ku}^{[1]}$ the structure of an ultra-commutative ring spectrum

and it gives $\mathbf{ku}^{[n]}$ a module structure over $\mathbf{ku}^{[1]}$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ is multiplicative, i.e., a morphism of ultra-commutative ring spectra.

The first pieces $\mathbf{ko}^{[1]}$ and $\mathbf{ku}^{[1]}$ are multiplicative models for the suspension spectra of global classifying spaces of the cyclic group $C_2 = O(1)$ and the circle group $U(1)$, respectively. Indeed, the complex global projective space $\mathbf{P}^{\mathbb{C}} = \mathbf{Gr}_{\otimes}^{\mathbb{C},[1]}$, introduced in (2.3.20), is a multiplicative model of the global classifying space of $U(1)$. A configuration of points labeled by vector spaces of total dimension 1 has to be concentrated on at most one point. So the map

$$\begin{array}{ccc} S^V \wedge \mathbf{P}^{\mathbb{C}}(V)_+ \ = \ S^V \wedge P(\mathrm{Sym}(V_{\mathbb{C}}))_+ & \longrightarrow & \mathscr{C}^{[1]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}^{[1]}(V) \\ v \wedge L & \longmapsto & [L; v] \end{array}$$

is a homeomorphism. As V varies through real inner product spaces, these maps form an isomorphism of ultra-commutative ring spectra

$$\Sigma^\infty_+ \mathbf{P}^{\mathbb{C}} \ \cong \ \mathbf{ku}^{[1]} \ .$$

Similarly, the ultra-commutative ring spectrum $\mathbf{ko}^{[1]}$ is globally a suspension spectrum of a global classifying space of the group C_2.

The underlying non-equivariant spectrum of $\Sigma^\infty_+ B_{\mathrm{gl}} U(1)$, hence of $\mathbf{ku}^{[1]}$, has the stable homotopy type of the suspension spectrum of $\mathbb{C}P^\infty$; moreover, on underlying non-equivariant spectra, the morphism

$$\mathbf{ku}^{[1]} \ \longrightarrow \ \mathbf{ku} \tag{6.3.41}$$

is a rational stable equivalence. However, the morphism (6.3.41) is *not* a rational global equivalence, which can be seen already on the level of $\underline{\pi}_0$ for the group C_2. Indeed, by Proposition 4.2.5 the homotopy group global functor $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} U(1))$, and hence also the global functor $\underline{\pi}_0(\mathbf{ku}^{[1]})$, is the represented global functor $\mathbf{A}(U(1), -)$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ induces a morphism of global power functors $\mathbf{A}(U(1), -) \cong \underline{\pi}_0(\mathbf{ku}^{[1]}) \longrightarrow \underline{\pi}_0(\mathbf{ku})$. An element of infinite order in the kernel of the map $\mathbf{A}(U(1), C_2) \longrightarrow \pi_0^{C_2}(\mathbf{ku})$ is

$$\mathrm{tr}_e^{C_2} \circ \mathrm{res}_e^{U(1)} - z^* - \mathrm{res}_{C_2}^{U(1)} \ \in \ \mathbf{A}(U(1), C_2) \ ,$$

where $z : U(1) \longrightarrow C_2$ is the trivial homomorphism. This element maps trivially to $\underline{\pi}_0(\mathbf{ku})$ because the regular representation of C_2 splits as the sum of the 1-dimensional trivial and sign representations.

For finite groups G, the ring homomorphism $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is an isomorphism. It will follow from Theorem 6.4.24 below that the map $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is always a split monomorphism, also when G has positive dimension. On the other hand, Remark 6.3.38 shows that the map is not always surjective, as the class $\mathrm{tr}_N^{SU(2)}(1)$ in $\pi_0^{SU(2)}(\mathbf{ku})$ is not in the image. In [74], Hausmann and Ostermayr give a complete calculation of $\underline{\pi}_0(\mathbf{ku})$ as a

global functor. The strategy of [74] is to identify the global homotopy types of the subquotients of the rank filtration and deduce from it a presentation $\underline{\pi}_0(\mathbf{ku})$ by generators and relations. The final answer is that $\underline{\pi}_0(\mathbf{ku})$ is generated as a global functor by the classes

$$x_m \;=\; [\tau_m] \;\in\; \pi_0^{U(m)}(\mathbf{ku}) \;,$$

where τ_m denotes the tautological $U(m)$-representation on $\mathbb{C}^m$. There are two kinds of relations; on the one hand, the relations

$$\mathrm{res}^{U(m+n)}_{U(m)\times U(n)}(x_{m+n}) \;=\; p^*(x_m) + q^*(x_n) \quad \in \pi_0^{U(m)\times U(n)}(\mathbf{ku})$$

for all $m, n \geq 1$, where $p : U(m) \times U(n) \longrightarrow U(m)$ and $q : U(m) \times U(n) \longrightarrow U(n)$ are the two projections. These relations follow from the fact that the maps $[-]$ are additive and compatible with restrictions. The other set of relations equates finite index transfers of representations with the corresponding finite index transfer in homotopy theory.

Construction 6.3.42 (Bott class)**.** The *Bott class*

$$\beta \;\in\; \pi_2^e(\mathbf{ku})$$

is an important non-equivariant homotopy class of the spectrum **ku** that we recall now. As we shall show in Theorem 6.4.29 below, the Bott class becomes invertible in the homotopy ring of the periodic K-theory spectrum **KU**.

We define the Bott class by specifying an explicit representative. We define a continuous map $m : S^{\mathbb{R}\oplus\mathbb{C}} \longrightarrow SU(2)$ by the formula

$$m(v) \;=\; m(x,z) \;=\; \frac{1}{|v|^2+1}\begin{pmatrix} |v|^2 - 1 - i2x & 2i\bar{z} \\ 2iz & |v|^2 - 1 + i2x \end{pmatrix} . \tag{6.3.43}$$

The eigenspace decomposition

$$\mathrm{eig} \;:\; U(2) \;\cong\; \mathscr{C}(\mathbb{C}^2; S^1)$$

is the inverse of the homeomorphism (6.3.4); it sends a unitary matrix to the configuration of its eigenvalues, with the inverse Cayley transform applied, labeled by the corresponding eigenspaces. The Bott class is then represented by the composite

$$\begin{aligned} S^3 \;=\; S^{\mathbb{R}\oplus\mathbb{C}} \xrightarrow{\;m\;} U(2) &\xrightarrow[\cong]{\;\mathrm{eig}\;} \mathscr{C}(\mathbb{C}^2; S^1) \\ &\xrightarrow{\;\mathrm{incl}\;} \mathscr{C}(\mathrm{Sym}(\mathbb{C}); S^1) \;=\; \mathbf{ku}(\mathbb{R}) \,. \end{aligned} \tag{6.3.44}$$

In the last step we identify $\mathbb{C}^2$ with the subspace of $\mathrm{Sym}(\mathbb{C})$ spanned by the constant and linear summand in the symmetric algebra, in terms of the preferred basis $1 \in \mathrm{Sym}^0(\mathbb{C})$ and $1 \in \mathrm{Sym}^1(\mathbb{C})$.

Proposition 6.3.45. *The group $\pi_2^e(\mathbf{ku})$ is infinite cyclic, and the Bott class β is a generator.*

Proof The map $m : S^{\mathbb{R}\oplus\mathbb{C}} \longrightarrow SU(2)$ is bijective; indeed, an explicit formula for the inverse is

$$SU(2) \longrightarrow S^{\mathbb{R}\oplus\mathbb{C}}, \quad \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix} \longmapsto \frac{\mathrm{Im}(a)}{\mathrm{Re}(a)-1} \wedge \frac{ib}{1-\mathrm{Re}(a)} ,$$

where $a, b \in \mathbb{C}^2$ are complex numbers satisfying $|a|^2+|b|^2 = 1$. As a continuous bijection between compact spaces, m is thus a homeomorphism from S^3 to $SU(2)$. Since the inclusion $SU(2) \longrightarrow U(2)$ induces an isomorphism on π_3, the map m represents a generator of the infinite cyclic group $\pi_3(U(2), 1)$. The standard embedding $U(2) \longrightarrow U$ into the infinite unitary group is 4-connected, so the composite (6.3.44) represents a generator of the infinite cyclic group $\pi_3(\mathbf{ku}(\mathbb{R}), *)$. Theorem 6.3.23 says that $\mathbf{ku}$ is a positive Ω-spectrum (in the non-equivariant sense), so in particular the stabilization map

$$\pi_3(\mathbf{ku}(\mathbb{R}), *) \;=\; [S^3, \mathbf{ku}(\mathbb{R})] \;\longrightarrow\; \mathrm{colim}_{n\geq 0}\, [S^{n+2}, \mathbf{ku}(\mathbb{R}^n)] \;=\; \pi_2^e(\mathbf{ku})$$

is an isomorphism. So the group $\pi_2^e(\mathbf{ku})$ is infinite cyclic, and the Bott class is a generator. □

Construction 6.3.46 (Equivariant Bott classes)**.** There are more general *equivariant Bott classes*

$$\beta_{G,W} \;\in\; \mathbf{ku}_G^0(S^W)$$

defined for G-Spin^c-representations, i.e., hermitian inner product spaces W equipped with a lift $G \longrightarrow \mathrm{Spin}^c(uW)$ of the representation homomorphism $G \longrightarrow O(uW)$ through the adjoint representation $\mathrm{ad}(uW) : \mathrm{Spin}^c(uW) \longrightarrow SO(uW)$, see (2.3.11). These equivariant Bott classes become invertible in the $RO(G)$-graded homotopy ring of $\mathbf{KU}$, compare Remark 6.4.31 below; this equivariant generalization of Bott periodicity goes back to Atiyah [3, Thm. 4.3].

We recall the definition as given, for example, in [88, III p. 44]. The construction depends heavily on a canonical isomorphism, for every hermitian inner product space W, between the complexified Clifford algebra $\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(uW)$ of the underlying euclidean vector space of W, and $\mathrm{End}_{\mathbb{C}}(\Lambda^*(W))$, the endomorphism algebra of the exterior algebra of W. The underlying $\mathbb{R}$-vector space uW of the given hermitian inner product space has a euclidean inner product defined by

$$\langle w, w' \rangle \;=\; \mathrm{Re}(w, w') ,$$

the real part of the complex inner product. In what follows, we use the hermi-

tian inner product on the exterior algebra $\Lambda^*(W)$ characterized by the formula

$$(v_1 \wedge \ldots \wedge v_n,\ w_1 \wedge \ldots \wedge w_n) \ = \ \det((v_i, w_j)_{i,j})$$

for all $v_i, w_j \in W$. Another way to say this is that if $(e_i)_{i=1,\ldots,k}$ is an orthonormal basis of W, then the vectors

$$e_{i_1} \wedge \ldots \wedge e_{i_n}$$

form an orthonormal basis of $\Lambda^*(W)$ as the indices run through all tuples with $1 \leq i_1 < \cdots < i_n \leq k$.

For $w \in W$ we let

$$d_w \ : \ \Lambda^*(W) \ \longrightarrow \ \Lambda^*(W) \ , \quad d_w(x) \ = \ w \wedge x$$

denote left exterior multiplication by w. We let $d_w^* : \Lambda^*(W) \longrightarrow \Lambda^*(W)$ denote the adjoint of d_w, i.e., the $\mathbb{C}$-linear map characterized by

$$(x, d_w^*(y)) \ = \ (d_w(x), y) \ = \ (w \wedge x, y)$$

for all $x, y \in \Lambda^*(W)$.

The exterior algebra $\Lambda^*(W)$ is $\mathbb{Z}/2$-graded by even and odd exterior powers. The endomorphism algebra $\mathrm{End}_{\mathbb{C}}(\Lambda^*(W))$ then inherits a $\mathbb{Z}/2$-grading by even and odd operators. Exterior multiplication by w takes $\Lambda^n(W)$ to $\Lambda^{n+1}(W)$, so the operator d_w is odd for every $w \in W$. Hence the adjoint d_w^* is odd, and so is the endomorphism

$$\delta_w \ = \ d_w - d_w^* \ . \tag{6.3.47}$$

Lemma 6.3.48. *Let W be a hermitian inner product space. For every $w \in W$, the endomorphisms d_w and δ_w of $\Lambda^*(W)$ satisfy the relations*

$$d_w \circ d_w^* \ + \ d_w^* \circ d_w \ = \ |w|^2 \cdot \mathrm{Id} \qquad \textit{and} \qquad \delta_w \circ \delta_w \ = \ -|w|^2 \cdot \mathrm{Id} \ .$$

Proof If we multiply w by a real scalar $\lambda \in \mathbb{R}$, then both sides of both equations scale by λ^2. So it suffices to show the claims for unit vectors, i.e., we may assume that $|w| = 1$.

If w is a unit vector, then $\Lambda^*(W)$ decomposes as an orthogonal direct sum $\Lambda^*(W^\perp) \oplus (w \wedge \Lambda^*(W^\perp))$, where $W^\perp$ is the orthogonal complement of w in W. Moreover, d_w is an isometry from the summand $\Lambda^*(W^\perp)$ onto the summand $w \wedge \Lambda^*(W^\perp)$, and it vanishes on $w \wedge \Lambda^*(W^\perp)$. So the adjoint d_w^* is inverse to d_w on the summand $w \wedge \Lambda^*(W^\perp)$, and it vanishes on $\Lambda^*(W^\perp)$. Hence $d_w^* \circ d_w$ is the orthogonal projection onto the summand $\Lambda^*(W^\perp)$, and $d_w \circ d_w^*$ is the orthogonal projection onto the other summand $w \wedge \Lambda^*(W^\perp)$. This shows that $d_w \circ d_w^* + d_w^* \circ d_w$ is the identity.

We have $d_w \circ d_w = 0$, hence also $d_w^* \circ d_w^* = 0$. This gives

$$\delta_w \circ \delta_w \ = \ (d_w - d_w^*) \circ (d_w - d_w^*) \ = \ -(d_w \circ d_w^* + d_w^* \circ d_w) \ = \ -\mathrm{Id} \ . \qquad \square$$

Because of the previous lemma, the $\mathbb{R}$-linear map

$$uW \longrightarrow \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) , \quad w \longmapsto \delta_w$$

satisfies $\delta_w^2 = -|w|^2 \cdot \mathrm{Id}$; so the universal property of the Clifford algebra provides a unique homomorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras

$$\delta_W \ : \ \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(uW) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) \tag{6.3.49}$$

sending $w \in W$ to δ_w. The homomorphism is also compatible with passage to adjoints; indeed, for $w \in W$ we have

$$\delta_w^* = (d_w - d_w^*)^* = d_w^* - d_w = d_{-w} - d_{-w}^* = \delta_{-w} = \delta_{w^*} .$$

Since $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(uW)$ is generated by the elements of uW, the compatibility with adjoints holds in general. The homomorphism δ_W thus makes the exterior algebra $\Lambda^*(W)$ a $\mathbb{Z}/2$-graded $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(uW)$-module.

Example 6.3.50 (The complexified Clifford algebra of $u\mathbb{C}$)**.** We make the homomorphism δ_W explicit in the simplest non-trivial case, namely for $W = \mathbb{C}$. In this case the elements $1, i \in \mathbb{C}$ form an orthonormal $\mathbb{R}$-basis of $u\mathbb{C}$; we let

$$e = 1 \otimes [1] \qquad \text{and} \qquad f = 1 \otimes [i]$$

be their images in the complexified Clifford algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(u\mathbb{C})$. This Clifford algebra then has a $\mathbb{C}$-basis $(1, e, f, ef)$, with multiplicative relations

$$e^2 = f^2 = -1 \qquad \text{and} \qquad fe = -ef .$$

On the other hand, the exterior algebra $\Lambda^*(\mathbb{C})$ is two dimensional, and an orthonormal $\mathbb{C}$-basis is given by the classes $1 \in \Lambda^0(\mathbb{C})$ (the multiplicative unit) and $x \in \Lambda^1(\mathbb{C})$, the class of $1 \in \mathbb{C}$. A direct calculation shows that in terms of the basis $(1, x)$ of $\Lambda^*(\mathbb{C})$, the map $\delta_{\mathbb{C}}$ is given by

$$\delta_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \delta_f = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \delta_e \circ \delta_f = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} . \tag{6.3.51}$$

In particular, the homomorphism $\delta_{\mathbb{C}}$ sends the basis $(1, e, f, ef)$ of $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(u\mathbb{C})$ to a basis of $\mathrm{End}_{\mathbb{C}}(\Lambda^*(\mathbb{C}))$, so $\delta_{\mathbb{C}}$ is an isomorphism.

We recall that the homomorphism δ_W is in fact an isomorphism in general. Indeed, given two hermitian inner product spaces V and W, the following

square commutes:

$$\begin{tikzcd}
\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(uV\oplus uW) \arrow[rr, "\delta_{V\oplus W}"] \arrow[dd, "\cong"'] & & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V\oplus W)) \arrow[d, "\cong"'] \\
& & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V)\otimes\Lambda^*(W)) \\
(\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(uV))\otimes_{\mathbb{C}} (\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(uW)) \arrow[rr, "\delta_V\otimes\delta_W"'] & & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V))\otimes \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) \arrow[u, "\cong"', "\otimes"]
\end{tikzcd}$$

The left vertical isomorphism sends $1\otimes[v,w]$ to $(1\otimes v)\otimes(1\otimes 1)+(1\otimes 1)\otimes(1\otimes w)$. The upper right vertical map is induced by the isomorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras

$$\Lambda^*(V\oplus W) \xrightarrow{\cong} \Lambda^*(V)\otimes\Lambda^*(W)$$

that extends the linear map

$$V\oplus W \longrightarrow \Lambda^*(V)\otimes\Lambda^*(W)\ , \quad (v,w) \longmapsto v\otimes 1 + 1\otimes w\ .$$

The vertical maps in the above diagram are isomorphisms, and every hermitian inner product space is isometrically isomorphic to $\mathbb{C}^n$ with standard inner product; so this reduces the claim to the special case $W=\mathbb{C}$, in which case the morphism $\delta_{\mathbb{C}}$ is an isomorphism by Example 6.3.50.

To construct the equivariant Bott class we now start with a hermitian inner product space W and a continuous homomorphism $G \longrightarrow \mathrm{Spin}^c(uW)$. The group G then acts on the underlying euclidean inner product space of W via the adjoint representation $\mathrm{ad}(uW) : \mathrm{Spin}^c(uW) \longrightarrow SO(uW)$. The equivariant Bott class most naturally lives in the relative K-group $\mathbf{K}_G(D(W), S(W))$; elements in this group are represented by triples (ξ,η,α), where ξ and η are G-vector bundles over $D(W)$, and $\alpha : \xi|_{S(W)} \cong \eta|_{S(W)}$ is an equivariant isomorphism between the restrictions of the two bundles to the unit sphere $S(W)$. In this description, the Bott class $\beta_{G,W}$ is represented by the triple

$$(D(W)\times\Lambda^{\mathrm{ev}}(W), D(W)\times\Lambda^{\mathrm{odd}}(W), \alpha)$$

consisting of the trivial vector bundles over $D(W)$ with fibers $\Lambda^{\mathrm{ev}}(W)$ and $\Lambda^{\mathrm{odd}}(W)$, and the equivariant bundle isomorphism α is given by the Clifford action (6.3.49):

$$\alpha : S(W)\times\Lambda^{\mathrm{ev}}(W) \longrightarrow S(W)\times\Lambda^{\mathrm{odd}}(W)\ , \quad (w,x) \longmapsto (w,\delta_w(x))$$

Here the group G acts on $\Lambda^{\mathrm{ev}}(W)$ and $\Lambda^{\mathrm{odd}}(W)$ via the given Spin^c-structure, i.e., through the composite

$$G \longrightarrow \mathrm{Spin}^c(uW) \subset \mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(uW) \underset{\cong}{\xrightarrow{\delta_W}} \mathrm{End}_{\mathbb{C}}(\Lambda^*(W))\ . \qquad (6.3.52)$$

Equivariant Bott periodicity [3, Thm. 4.3] says that for every compact G-space A, exterior product with the Bott class is an isomorphism

$$- \times \beta_{G,W} \ : \ \mathbf{K}_G(A) \longrightarrow \mathbf{K}_G(A \times D(W), A \times S(W)) .$$

To represent the Bott class in $\mathbf{ku}_G^0(S^W)$ we interpret the Clifford action as a clutching function for a vector bundle. We define a G-vector bundle $\xi(W)$ over S^W by gluing the trivial bundle with fiber $\Lambda^{\mathrm{ev}}(W)$ over $D(W)$ with the trivial bundle with fiber $\Lambda^{\mathrm{odd}}(W)$ over $S^W - \mathring{D}(W)$ using the map

$$S(W) \longrightarrow \mathbf{L}^{\mathbb{C}}(\Lambda^{\mathrm{ev}}(W), \Lambda^{\mathrm{odd}}(W)) , \quad w \longmapsto \delta_w .$$

The homomorphism

$$[-] \ : \ \mathbf{K}_G(S^W) \longrightarrow \mathbf{ku}_G^0(S^W_+)$$

defined in (6.3.30) turns this G-vector bundle into an unreduced equivariant **ku**-cohomology class. By construction, the fiber of $\xi(W)$ over the basepoint at infinity is the G-representation $\Lambda^{\mathrm{odd}}(W)$, with G acting via the composite (6.3.52). So by subtracting the class of the trivial vector bundle with fiber $\Lambda^{\mathrm{odd}}(W)$ we obtain the equivariant Bott class as a reduced equivariant **ku**-cohomology class,

$$\beta_{G,W} \ = \ [\xi(W)] - [S^W \times \Lambda^{\mathrm{odd}}(W)] \ \in \ \mathbf{ku}_G^0(S^W) .$$

We invite the reader to perform a reality check and establish the relation

$$\beta_{e,\mathbb{C}} \ = \ -\beta$$

in $\pi_2^e(\mathbf{ku})$. In other words, the equivariant Bott class of the trivial group acting on $\mathbb{C}$ specializes to the non-equivariant Bott class as defined in Construction 6.3.42, up to a sign.

6.4 Periodic global K-theory

Our main object of study in this section is *periodic global K-theory* **KU**, an ultra-commutative ring spectrum whose G-homotopy type realizes G-equivariant periodic K-theory, see Construction 6.4.9. The model we use is due to M. Joachim [86], and made of spaces of homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras. Corollary 6.4.23 shows that the equivariant cohomology theory represented by **KU** on finite G-CW-complexes 'is' equivariant K-theory; Theorem 6.4.24 shows that $\underline{\pi}_0(\mathbf{KU})$ is isomorphic, as a global power functor, to the complex representation ring functor **RU**. Periodic global K-theory receives a morphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ from connective global K-theory defined in the previous section. Theorem 6.4.29 shows

that **KU** is Bott periodic, i.e., the Bott class in $\pi_2^e(\mathbf{ku})$ becomes invertible in $\pi_2^e(\mathbf{KU})$.

Global connective K-theory $\mathbf{ku}^c$ is a certain homotopy pullback of the periodic theory **KU**, its associated global Borel theory, and the global Borel theory of connective K-theory, see Construction 6.4.32. The global homotopy type $\mathbf{ku}^c$ is a refinement of Greenlees 'equivariant connective K-theory' [66]. One should note the different order of the adjectives 'global' and 'connective', indicating that **ku** and $\mathbf{ku}^c$ are quite different global homotopy types (with the same underlying non-equivariant homotopy type).

The spectrum **KU** consists of spaces of homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras. Consequently, in this section we will use basic results from the theory of C^*-algebras; the textbooks [126, 186] can serve as general references. Since we are not assuming the reader to be fluent with C^*-algebras, we recall a certain amount of material in some detail.

Construction 6.4.1 (The C^*-algebra s)**.** The construction of the orthogonal spectrum **KU** is based on a certain C^*-algebra and on spaces of $*$-homomorphisms out of it. We consider *graded* C^*-algebras, i.e., C^*-algebras A equipped with a $*$-automorphism $\alpha : A \longrightarrow A$ such that $\alpha^2 = \mathrm{Id}$. We can then decompose A into the ± 1 eigenspaces of α and obtain a $\mathbb{Z}/2$-grading of the underlying $\mathbb{C}$-algebra by setting

$$A_{\mathrm{ev}} \ = \ \{a \in A \mid \alpha(a) = a\} \quad \text{and} \quad A_{\mathrm{odd}} \ = \ \{a \in A \mid \alpha(a) = -a\} \ .$$

The conjugation of A preserves the grading into even and odd parts.

We let s denote the C^*-algebra of complex valued continuous functions on $\mathbb{R}$ vanishing at infinity; this is a $\mathbb{Z}/2$-graded C^*-algebra with involution $\alpha : s \longrightarrow s$ defined by

$$\alpha(f)(t) \ = \ f(-t) \ .$$

With respect to this grading, 'even' and 'odd' have their usual meaning: a function $f \in s$ is even (or odd) if and only if $f(-t) = f(t)$ (or $f(-t) = -f(t)$) for all $t \in \mathbb{R}$.

Graded $*$-morphisms out of s correspond to special elements in the target C^*-algebra. The continuous function

$$r(x) \ = \ \frac{2i}{x-i} \ : \ \mathbb{R} \ \longrightarrow \ \mathbb{C}$$

is an element of s that satisfies

$$rr^* \ + \ r \ + \ r^* \ = \ 0 \qquad \text{and} \qquad \alpha(r) \ = \ r^* \ .$$

The element r is not homogeneous; its even and odd components are given by

$$r_+(x) \ = \ \frac{-2}{x^2+1} \quad \text{and} \quad r_-(x) \ = \ \frac{2xi}{x^2+1} \ . \tag{6.4.2}$$

Moreover, s is the universal $\mathbb{Z}/2$-graded C^*-algebra generated by such an element, i.e., evaluation at r is a bijection

$$C^*_{\text{gr}}(s, A) \ \cong \ \{x \in A \ : \ xx^* = x^*x = -x - x^* \ , \ \alpha(x) = x^*\} \tag{6.4.3}$$

for every graded C^*-algebra A. Indeed, after adjoining a unit to the algebra s and identifying S^1 with $U(1)$ via the Cayley transform $c : S^1 \cong U(1)$ given by $c(x) = (x+i)(x-i)^{-1}$, the algebra $\mathbb{C} \oplus s$ becomes isomorphic to the unital C^*-algebra $C(U(1))$ of continuous $\mathbb{C}$-valued functions on $U(1)$; this isomorphism takes $1 + r$ to the inclusion $z : U(1) \longrightarrow \mathbb{C}$, which is a unitary element in $C(U(1))$. So the universal property of s follows from the fact that $C(U(1))$ is freely generated, as an ungraded, unital C^*-algebra, by the unitary element z; indeed, unitary elements are in particular normal and have their spectrum contained in $U(1)$, so the bijectivity of (6.4.3) becomes a special case of functional calculus for normal elements in unital C^*-algebras, see for example [126, Thm. 2.1.13].

In what follows, $\hat{\otimes}$ denotes the *spatial tensor product* of graded C^*-algebras, see [126, Sec. 6.3], [186, Def. T.5.16] or [76, Def. 1.10]. The spatial tensor product is the completion of the algebraic tensor product with respect to the 'spatial norm'; it is also called the *minimal tensor product* because the spatial norm is minimal among all C^*-norms on the algebraic tensor product, compare [126, Thm. 6.4.18]. A comprehensive discussion of the spatial and other tensor products of C^*-algebras can be found in [186, App. T]. We need the *graded* tensor product of $\mathbb{Z}/2$-graded algebras, which involves a sign in the formula for multiplication. If $a, a' \in A$ and $b, b' \in B$ are homogeneous elements of two $\mathbb{Z}/2$-graded algebras, then the multiplication in $A \otimes B$ is defined by

$$(a \otimes b) \cdot (a' \otimes b') \ = \ (-1)^{|b||a'|} \cdot (aa') \otimes (bb') \ .$$

In other words, if $a \in A$ and $b \in B$ are odd, then $a \otimes 1$ and $1 \otimes b$ *anti-commute* in the graded tensor product.

The underlying algebra of the graded tensor of two $\mathbb{Z}/2$-graded algebras is *not* the tensor product of the underlying ungraded algebras, as soon as both factors have non-zero odd elements. For example, while the ungraded C^*-algebra underlying s is commutative, the underlying C^*-algebra of the graded tensor product $s\hat{\otimes}s$ is *not* commutative anymore. In particular, $s\hat{\otimes}s$ does not embed into the algebra of continuous functions on any space.

The algebra s has another important piece of extra structure, namely a comultiplication $\Delta : s \longrightarrow s\hat{\otimes}s$ in the category of $\mathbb{Z}/2$-graded C^*-algebras. The

algebra s is also generated by the self-adjoint functions

$$u_+(t) \;=\; e^{-t^2} \qquad \text{and} \qquad u_-(t) \;=\; t\cdot e^{-t^2}\;, \tag{6.4.4}$$

and the comultiplication is completely determined by the values on these; indeed, there is a unique graded $*$-homomorphism $\Delta : s \longrightarrow s\hat{\otimes}s$ that satisfies

$$\Delta(u_+) \;=\; u_+\otimes u_+ \qquad \text{and} \qquad \Delta(u_-) \;=\; u_-\otimes u_+ \;+\; u_+\otimes u_-\;,$$

see for example Lemma 1.3.1 and Remark 1.3.3 of [76]. This characterization readily implies the cocommutativity and coassociativity of Δ. The cocommutativity of Δ is with respect to the *graded* symmetry automorphism of $s\hat{\otimes}s$, which involves a sign whenever two odd elements are interchanged. An explicit definition of the diagonal morphism Δ can be found in [69, Sec. 1, (13)].

Construction 6.4.5 (Complex Clifford algebras)**.** We let V be a euclidean inner product space. We define the complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ by

$$\mathbb{C}\mathrm{l}(V) \;=\; (TV)_{\mathbb{C}}/(v\otimes v - |v|^2\cdot 1)\;,$$

the quotient of the complexified tensor algebra of V by the ideal generated by the elements $v\otimes v - |v|^2\cdot 1$ for all $v\in V$. We write $[-] : V \longrightarrow \mathbb{C}\mathrm{l}(V)$ for the $\mathbb{R}$-linear and injective composite

$$V \xrightarrow{\text{linear summand}} TV \xrightarrow{1\otimes -} (TV)_{\mathbb{C}} \longrightarrow \mathbb{C}\mathrm{l}(V)\;.$$

With this notation the relation $[v]^2 = |v|^2\cdot 1$ holds in $\mathbb{C}\mathrm{l}(V)$ for all $v\in V$. The Clifford algebra construction is functorial for $\mathbb{R}$-linear isometric embeddings, so in particular $\mathbb{C}\mathrm{l}(V)$ inherits an action of the orthogonal group $O(V)$. The Clifford algebra is $\mathbb{Z}/2$-graded, coming from the grading of the tensor algebra by even and odd tensor powers.

The complex Clifford algebra is in fact a $\mathbb{Z}/2$-graded $O(V)$-C^*-algebra. The $*$-involution on $\mathbb{C}\mathrm{l}(V)$ is defined by declaring $[v]^* = [v]$ for all $v\in V$ and extending this to a $\mathbb{C}$-semilinear anti-automorphism. This makes the elements $[v]$ for $v\in S(V)$ into unitary elements of $\mathbb{C}\mathrm{l}(V)$. The norm on $\mathbb{C}\mathrm{l}(V)$ arises from an embedding into the endomorphism algebra of the exterior algebra $\Lambda^*(V_{\mathbb{C}})$. Indeed, we consider the $\mathbb{R}$-linear map

$$V \;\longrightarrow\; \mathrm{End}_{\mathbb{C}}(\Lambda^*(V_{\mathbb{C}}))\;, \quad v \;\longmapsto\; i\cdot\delta_{1\otimes v}\;,$$

where $\delta_{1\otimes v}$ was defined in (6.3.47). Lemma 6.3.48 provides the relation

$$(i\cdot\delta_{1\otimes v})\circ(i\cdot\delta_{1\otimes v}) \;=\; -\delta^2_{1\otimes v} \;=\; |v|^2\cdot\mathrm{Id}\;.$$

The universal property of $\mathbb{C}\mathrm{l}(V)$ provides a morphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras

$$\mathbb{C}\mathrm{l}(V) \;\longrightarrow\; \mathrm{End}_{\mathbb{C}}(\Lambda^*(V_{\mathbb{C}}))$$

that sends $[v]$ to $i \cdot \delta_{1\otimes v}$. This homomorphism is an embedding and compatible with passage to adjoints. The target is a C^*-algebra via the operator norm, so we can endow the Clifford algebra with a C^*-norm via this embedding. For this norm, the embedding $[-] : V \longrightarrow \mathbb{C}\mathrm{l}(V)$ is isometric. Indeed,

$$\delta^*_{1\otimes v} \circ \delta_{1\otimes v} \;=\; (d^*_{1\otimes v} - d_{1\otimes v}) \circ (d_{1\otimes v} - d^*_{1\otimes v}) \;=\; d_{1\otimes v} \circ d^*_{1\otimes v} + d^*_{1\otimes v} \circ d_{1\otimes v} \;=\; |v|^2 \cdot \mathrm{Id}\,,$$

by Lemma 6.3.48, and so

$$\|i \cdot \delta_{1\otimes v}\|^2 \;=\; \|\delta^*_{1\otimes v} \circ \delta_{1\otimes v}\| \;=\; |v|^2\,.$$

The Clifford algebra functor sends orthogonal direct sum to graded tensor product, in the following sense. The $\mathbb{R}$-linear map

$$V \oplus W \;\longrightarrow\; \mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W)\,, \quad (v, w) \;\longmapsto\; [v] \otimes 1 + 1 \otimes [w]$$

satisfies

$$([v] \otimes 1 + 1 \otimes [w])^2 \;=\; [v]^2 \otimes 1 + 1 \otimes [w]^2 \;=\; (|v|^2 + |w|^2) \cdot 1 \otimes 1$$

because the elements $[v] \otimes 1$ and $1 \otimes [w]$ anti-commute in $\mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W)$. The universal property then provides a unique extension to a morphism of graded $\mathbb{C}$-algebras

$$\mu_{V,W} \;:\; \mathbb{C}\mathrm{l}(V \oplus W) \;\cong\; \mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W) \tag{6.4.6}$$

characterized by $\mu_{V,W}[v, w] = [v] \otimes 1 + 1 \otimes [w]$ for all $(v, w) \in V \oplus W$. Moreover, this morphism is an isomorphism, and the square

$$\begin{tikzcd}
\mathbb{C}\mathrm{l}(V \oplus W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\mathbb{C}\mathrm{l}(\tau_{V,W})"'] & \mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W) \arrow[d, "\cong"] \\
\mathbb{C}\mathrm{l}(W \oplus V) \arrow[r, "\mu_{W,V}"'] & \mathbb{C}\mathrm{l}(W) \otimes \mathbb{C}\mathrm{l}(V)
\end{tikzcd}$$

commutes, where the right vertical map is the symmetry isomorphism for $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras (which involves a sign whenever two odd degree elements interchange places).

The complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ is isomorphic to the complexification of the real Clifford algebra $\mathrm{Cl}(V)$ considered in the definition of the ultra-commutative monoids **Pin** and **Spin** in Example 2.3.10. However, the isomorphism involves a slight twist that we want to make explicit. Indeed, the real Clifford algebra $\mathrm{Cl}(V)$ was defined from the tensor algebra by imposing the relations $v \otimes v = -|v|^2 \cdot 1$, so that unit vectors of V square to -1 in $\mathrm{Cl}(V)$. Over the field $\mathbb{R}$ it makes a difference whether we make $v \otimes v$ equal to $|v|^2 \cdot 1$ or $-|v|^2 \cdot 1$, but the presence of the imaginary unit makes the difference disappear over $\mathbb{C}$. Indeed, the $\mathbb{R}$-linear map

$$\psi \;:\; V \;\longrightarrow\; \mathbb{C}\mathrm{l}(V)\,, \quad v \;\longmapsto\; i \cdot [v]$$

satisfies $\psi(v)^2 = -|v|^2 \cdot 1$, so it extends to a morphism of $\mathbb{C}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V) \longrightarrow \mathbb{C}\mathrm{l}(V)$ by the universal property of the former, and this morphism is an isomorphism.

Another key piece of structure is a continuous based map

$$\mathrm{fc} \;:\; S^V \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V))\,, \quad v \longmapsto (-)[v]\,, \tag{6.4.7}$$

often referred to as 'functional calculus'. For $v \in V$ the $*$-homomorphism $\mathrm{fc}(v)$ is given on homogeneous elements of s by

$$f[v] \;=\; \mathrm{fc}(v)(f) \;=\; \begin{cases} f(|v|)\cdot 1 & \text{when } f \text{ is even, and} \\ \frac{f(|v|)}{|v|}\cdot [v] & \text{when } f \text{ is odd.} \end{cases}$$

For $v = 0$ the formula for odd functions is to be interpreted as $f[0] = 0$; this is continuous because for $v \neq 0$, the norm of $f(|v|)/|v| \cdot [v]$ is $f(|v|)$, which tends to $f(0) = 0$ if v tends to 0. If the norm of v tends to infinity, then $f(|v|)$ tends to 0, so $\mathrm{fc}(v)$ tends to the constant $*$-homomorphism with value 0, the basepoint of $C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V))$. Hence fc extends to a continuous map on the one-point compactification S^V. The functional calculus map is $O(V)$-equivariant for the $O(V)$-action on the mapping space $C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V))$ through the action on the target. The functional calculus maps are multiplicative in the sense that the following diagram commutes:

$$\begin{tikzcd} S^V \wedge S^W \arrow[rr, "\mathrm{fc}\wedge\mathrm{fc}"] \arrow[dd, "\cong"'] & & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)) \wedge C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(W)) \arrow[d, "\hat{\otimes}"] \\ & & C^*_{\mathrm{gr}}(s\hat{\otimes}s, \mathbb{C}\mathrm{l}(V)\otimes\mathbb{C}\mathrm{l}(W)) \arrow[d, "\Delta^*"] \\ S^{V\oplus W} \arrow[r, "\mathrm{fc}"'] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V\oplus W)) \arrow[r, "(6.4.6)", "(\mu_{V,W})_*"'] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)\otimes\mathbb{C}\mathrm{l}(W)) \end{tikzcd} \tag{6.4.8}$$

To see this we consider $v \in V$ and $w \in W$, chase the element $v \wedge w$ both ways around the square and compare the results on the two functions (6.4.4) that generate the algebra s. Indeed, for the even generating function we have

$$\begin{aligned} \mu_{V,W}(\mathrm{fc}(v,w)(u_+)) \;&=\; \mu_{V,W}(e^{-|v|^2-|w|^2}\cdot 1) \;=\; (e^{-|v|^2}\cdot 1)\otimes(e^{-|w|^2}\cdot 1) \\ &=\; \mathrm{fc}(v)(u_+)\otimes \mathrm{fc}(w)(u_+) \;=\; (\mathrm{fc}(v)\hat{\otimes}\,\mathrm{fc}(w))(u_+\otimes u_+) \\ &=\; (\mathrm{fc}(v)\hat{\otimes}\,\mathrm{fc}(w))(\Delta(u_+)) \;\; ; \end{aligned}$$

and similarly for the odd generating function:

$$\begin{aligned}
\mu_{V,W}(\,\mathrm{fc}(v,w)(u_-)) \;&=\; \mu_{V,W}(e^{-|v|^2-|w|^2}\cdot[v,w]) \\
&=\; e^{-|v|^2-|w|^2}\cdot([v]\otimes 1 + 1\otimes[w]) \\
&=\; (e^{-|v|^2}\cdot[v])\otimes(e^{-|w|^2}\cdot 1) \;+\; (e^{-|v|^2}\cdot 1)\otimes(e^{-|w|^2}\cdot[w]) \\
&=\; \mathrm{fc}(v)(u_-)\otimes\mathrm{fc}(w)(u_+) \;+\; \mathrm{fc}(v)(u_+)\otimes\mathrm{fc}(w)(u_-) \\
&=\; (\mathrm{fc}(v)\hat{\otimes}\,\mathrm{fc}(w))(u_-\otimes u_+ + u_+\otimes u_-) \;=\; (\mathrm{fc}(v)\hat{\otimes}\,\mathrm{fc}(w))(\Delta(u_-))\,.
\end{aligned}$$

Now we have all ingredients for the periodic global K-theory spectrum **KU**.

Construction 6.4.9 (Periodic global K-theory)**.** We let V be a euclidean inner product space. As we explained in Proposition 6.3.8, the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification inherits a hermitian inner product and an $O(V)$-action by $\mathbb{C}$-linear isometries. The inner product space $\mathrm{Sym}(V_{\mathbb{C}})$ is usually infinite-dimensional (unless $V = 0$), but it is not complete. We denote by $\mathcal{H}_V$ the Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$. Since the action of $O(V)$ on $\mathrm{Sym}(V_{\mathbb{C}})$ is by linear isometries, it extends to an analogous action on the completion $\mathcal{H}_V$. So $\mathcal{H}_V$ becomes a complex Hilbert space representation of the orthogonal group $O(V)$. We denote by $\mathcal{K}_V$ the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}_V$, see for example [126, Sec. 2.4].

The orthogonal spectrum **KU** assigns to a euclidean inner product space V the space

$$\mathbf{KU}(V) \;=\; C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)\otimes\mathcal{K}_V)$$

of $\mathbb{Z}/2$-graded $*$-homomorphisms from s to the tensor product, over $\mathbb{C}$, of $\mathbb{C}\mathrm{l}(V)$ and $\mathcal{K}_V$. Here we consider $\mathcal{K}_V$ as evenly graded, so the grading comes entirely from the grading of the Clifford algebra. The topology is the topology of pointwise convergence in the operator norm of $\mathcal{K}_V$; the basepoint is the zero $*$-homomorphism.

The continuous action of the orthogonal group $O(V)$ by linear isometries of $\mathcal{H}_V$ induces an action on the algebra $\mathcal{K}_V$ by conjugation, and together with the action on $\mathbb{C}\mathrm{l}(V)$ it gives an $O(V)$-action on $\mathbb{C}\mathrm{l}(V)\otimes\mathcal{K}_V$ by graded $*$-automorphisms. This induces an $O(V)$-action on the mapping space $\mathbf{KU}(V)$.

The multiplication of the spectrum **KU** starts from the $(O(V)\times O(W))$-equivariant isometric isomorphism

$$\mathrm{Sym}(V_{\mathbb{C}})\otimes\mathrm{Sym}(W_{\mathbb{C}}) \;\cong\; \mathrm{Sym}(V_{\mathbb{C}}\oplus W_{\mathbb{C}}) \;\cong\; \mathrm{Sym}((V\oplus W)_{\mathbb{C}})\,,$$

compare (6.3.7). This extends to an isometry of the Hilbert space completions

$$\begin{aligned}
\mathcal{H}_V\hat{\otimes}\mathcal{H}_W \;=\; \widehat{\mathrm{Sym}}(V_{\mathbb{C}})\hat{\otimes}\widehat{\mathrm{Sym}}(W_{\mathbb{C}}) \;&\cong\; (\mathrm{Sym}(V_{\mathbb{C}})\otimes\mathrm{Sym}(W_{\mathbb{C}}))^\wedge \\
&\cong\; \widehat{\mathrm{Sym}}((V\oplus W)_{\mathbb{C}}) \;=\; \mathcal{H}_{V\oplus W}\,.
\end{aligned}$$

Tensor product of compact operators and conjugation with this isometry is an injective homomorphism of $\mathbb{C}$-algebras

$$\mathcal{K}_V \otimes \mathcal{K}_W \longrightarrow \mathcal{K}(\mathcal{H}_V \hat{\otimes} \mathcal{H}_W) \cong \mathcal{K}(\mathcal{H}_{V\oplus W}) = \mathcal{K}_{V\oplus W} . \tag{6.4.10}$$

The source of this map is the algebraic tensor product, which is not complete; the homomorphism (6.4.10) extends to an isomorphism of C^*-algebras

$$\mathcal{K}_V \hat{\otimes} \mathcal{K}_W \cong \mathcal{K}_{V\oplus W} \tag{6.4.11}$$

from the spatial (minimal) tensor product; this is in fact tautologically true, as $\mathcal{K}_V$ acts on the Hilbert space $\mathcal{H}_V$ and so the spatial norm on the algebraic tensor product can be defined via the embedding (6.4.10). Combining (6.4.11) with the isomorphism between $\mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W)$ and $\mathbb{C}\mathrm{l}(V \oplus W)$ specified in (6.4.6), we obtain an isomorphism of graded C^*-algebras

$$\begin{aligned}(\mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V)\hat{\otimes}(\mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_W) &\cong (\mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(W)) \otimes (\mathcal{K}_V \hat{\otimes} \mathcal{K}_W) \\ &\cong \mathbb{C}\mathrm{l}(V \oplus W) \otimes \mathcal{K}_{V\oplus W} . \end{aligned} \tag{6.4.12}$$

The first step is the symmetry isomorphism, which in this special case does not involve any signs because $\mathcal{K}_V$ is concentrated in even grading. The multiplication map

$$\mu_{V,W} \ : \ \mathbf{KU}(V) \wedge \mathbf{KU}(W) \longrightarrow \mathbf{KU}(V \oplus W)$$

is now defined as the composite

$$\begin{aligned} &C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \wedge C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_W) \\ &\qquad \xrightarrow{\hat{\otimes}} C^*_{\mathrm{gr}}(s \hat{\otimes} s,\ (\mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V)\hat{\otimes}(\mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_W)) \\ &\qquad \xrightarrow{C^*_{\mathrm{gr}}(\Delta,(6.4.12))} C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V \oplus W) \otimes \mathcal{K}_{V\oplus W}) . \end{aligned}$$

These multiplication maps are associative and commutative. The unit map

$$\eta_V \ : \ S^V \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \ = \ \mathbf{KU}(V)$$

is defined as the composite

$$S^V \xrightarrow{\mathrm{fc}} C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)) \xrightarrow{(-\otimes p_0)_*} C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \ = \ \mathbf{KU}(V) ,$$

where $p_0 \in \mathcal{K}_V$ is the orthogonal projection onto the constant summand in the symmetric algebra. In other words,

$$\eta_V(v)(f) \ = \ f[v] \otimes p_0 .$$

The multiplicativity of the unit maps follows from the multiplicativity (6.4.8) of the functional calculus maps and the fact that the isomorphism (6.4.11) sends $p_0 \otimes p_0$ to p_0.

Construction 6.4.13. The connective and periodic global K-theory spectra are related by a morphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$. To define the value

$$j(V) \ : \ \mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \ = \ \mathbf{KU}(V)$$

at an inner product space V we consider a configuration

$$[E_1, \dots, E_n;\ v_1, \dots, v_n] \ \in \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)$$

of pairwise orthogonal, finite-dimensional subspaces on S^V. The associated $*$-homomorphism

$$j(V)[E_1, \dots, E_n;\ v_1, \dots, v_n] \ : \ s \ \longrightarrow \ \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V$$

is then defined on a function $f \in s$ by

$$j(V)[E_1, \dots, E_n;\ v_1, \dots, v_n](f) \ = \ \sum\nolimits_{i=1}^n f[v_i] \otimes p_{E_i} \ ,$$

where $f[v_i] = \mathrm{fc}(v_i)(f)$ is the functional calculus (6.4.7) and p_E denotes the orthogonal projection onto a subspace E. The fact that the morphism $j(V)[E_k;\ v_k]$ is $\mathbb{Z}/2$-graded is a direct consequence of the same property for the functional calculus map. The verification that the map $j(V)$ is $O(V)$-equivariant is straightforward, and we omit it. The verification that the maps $j(V)$ are multiplicative amounts to the multiplicativity (6.4.8) of the functional calculus maps and the fact that the isomorphism (6.4.11) sends $p_E \otimes p_F$ to $p_{E\otimes F}$. Moreover, the composite of $j(V)$ with the unit map $S^V \longrightarrow \mathbf{ku}(V)$ is the unit map of $\mathbf{KU}$, so the maps j indeed form a morphism of ultra-commutative ring spectra.

Remark 6.4.14. We have defined $\mathbf{KU}(V)$ in a slightly different way, compared to the presentation of Joachim [86]. We use the Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$, instead of the Hilbert space $L^2(V)$ of $\mathbb{C}$-valued square integrable functions on V. These two Hilbert spaces are naturally isomorphic, as follows. We use the inner product on V to identify it with its dual space V^*, and hence the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ with $\mathrm{Sym}(V^*_{\mathbb{C}})$. Elements of $\mathrm{Sym}(V^*_{\mathbb{C}})$ are complex valued polynomial functions on V. We make them square integrable by multiplying with the rapidly decaying function $\epsilon : V \longrightarrow \mathbb{R}$, $\epsilon(v) = e^{-|v|^2}$. This provides a linear isometric embedding

$$\mathrm{Sym}(V^*_{\mathbb{C}}) \ \longrightarrow \ L^2(V) \ , \quad f \ \longmapsto \ f \cdot \epsilon$$

with dense image. Altogether this exhibits $L^2(V)$ as a Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$.

The reader should beware that for us the C^*-algebra $\mathcal{K}_V$ is evenly graded, whereas Joachim uses a non-trivial grading on $L^2(V)$ and $\mathcal{K}(L^2(V))$ by even and odd functions. Our grading convention is necessary to ensure that the map $j(V) : \mathbf{ku}(V) \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) = \mathbf{KU}(V)$ defined in Construction 6.4.13 takes values in *graded* $*$-homomorphisms.

Remark 6.4.15 (Homotopy type of $\mathbf{KU}(V)$)**.** Up to isomorphism, there are only three different graded C^*-algebras of the form $\mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V$, and only three different homeomorphism types of spaces $\mathbf{KU}(V)$. The case $V = 0$ is degenerate in that $\mathrm{Sym}(0)$ is just the copy of $\mathbb{C}$ generated by 1. This is already complete, so $\mathcal{H}_0 = \mathbb{C}$ and $\mathbb{C}\mathrm{l}(0) = \mathcal{K}_0 = \mathbb{C}$. The space $\mathbf{KU}(0) = C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(0) \otimes \mathcal{K}_0)$ is thus discrete with two points: the zero homomorphism as basepoint and the augmentation

$$s \longrightarrow \mathbb{C}\mathrm{l}(0) \otimes \mathcal{K}_0 , \quad f \longmapsto f(0) \otimes 1 .$$

This non-basepoint is a unit for the multiplication maps $\mu_{V,W}$. As we explain in Remark 6.4.31 below, the space $\mathbf{KU}(\mathbb{R}^n)$ is non-canonically homeomorphic to $\mathbf{KU}(\mathbb{R}^{n+2})$ for all $n \geq 1$. So all odd-dimensional terms of $\mathbf{KU}$ are homeomorphic, and all even-dimensional terms in positive degrees are homeomorphic.

We can also identify the space $\mathbf{KU}(\mathbb{R})$ as follows. We claim that the map

$$j(\mathbb{R}) : \mathbf{ku}(\mathbb{R}) \longrightarrow \mathbf{KU}(\mathbb{R})$$

is a homotopy equivalence. Indeed, if A is an (ungraded) C^*-algebra, then $*$-homomorphisms from s to A (in the ungraded sense) biject with graded $*$-homomorphisms from s to $\mathbb{C}\mathrm{l}(\mathbb{R}) \otimes A$, via

$$C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes A) \cong C^*(s, A) , \quad \varphi \longmapsto \pi \circ \varphi ,$$

where $\pi : \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes A \longrightarrow A$ is the $*$-homomorphism that sends both $1 \otimes a$ and $[1] \otimes a$ to a. The composite

$$S^1 \xrightarrow{\ \mathrm{fc}\ } C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R})) \underset{\cong}{\xrightarrow{\ \pi\circ -\ }} C^*(s, \mathbb{C})$$

sends $x \in S^1$ to evaluation at x. This implies that the following square commutes:

$$\begin{tikzcd}
\mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^1) \ = \ \mathbf{ku}(\mathbb{R}) \arrow[r, "j(\mathbb{R})"] \arrow[d, "\cong"'] & \mathbf{KU}(\mathbb{R}) = C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes \mathcal{K}_{\mathbb{R}}) \arrow[d, "\cong"', "\pi\circ -"] \\
\mathrm{colim}_n\, C^*(s, \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^{[n]}(\mathbb{C}))) \arrow[r] & C^*(s, \mathcal{K}_{\mathbb{R}})
\end{tikzcd}$$

The lower horizontal map is induced by the $*$-homomorphisms $\mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^{[n]}(\mathbb{C})) \longrightarrow \mathcal{K}_{\mathbb{R}}$ that extend an endomorphism by zero on the orthogonal complement, and

the left vertical map sends a configuration $[E_1, \dots, E_n; x_1, \dots, x_n]$ to the $*$-homomorphism

$$f \longmapsto \sum\nolimits_{i=1}^n f(x_i) \cdot p_{E_i} .$$

The two vertical maps are homeomorphisms (compare Remark 6.3.2) and the lower horizontal map is a homotopy equivalence by [154, Prop. 1.2] or [79, Prop. 4.6] (or rather its complex analog). So the map $j(\mathbb{R})$ is a homotopy equivalence, and $\mathbf{KU}(\mathbb{R})$ is homotopy equivalent to the infinite unitary group U.

The spectrum $\mathbf{KU}$ is positive Ω-spectrum (in the non-equivariant sense) by Theorem 6.4.20 below. So the adjoint structure map $\mathbf{KU}(\mathbb{R}^2) \longrightarrow \Omega\mathbf{KU}(\mathbb{R}^3)$ is a weak equivalence, and the target is homeomorphic to $\Omega\mathbf{KU}(\mathbb{R}) \simeq \Omega U$. So $\mathbf{KU}(\mathbb{R}^2)$ has the weak homotopy type of $\mathbb{Z} \times BU$, by Bott periodicity.

The analysis of the global homotopy type of $\mathbf{KU}$ depends on the operator theoretic formulation of equivariant Bott periodicity that we now recall. We let G be a compact Lie group and V an orthogonal G-representation. We denote by $C_0(V, \mathbb{Cl}(V))$ the G-C^*-algebra of continuous $\mathbb{Cl}(V)$-valued functions on V that vanish at infinity. Since the Clifford algebra is finite-dimensional, the map

$$C_0(V) \otimes \mathbb{Cl}(V) \longrightarrow C_0(V, \mathbb{Cl}(V)) , \quad f \otimes x \longmapsto f(-) \cdot x \qquad (6.4.16)$$

is an isomorphism of C^*-algebras. Functional calculus (6.4.7) provides a distinguished graded $*$-homomorphism

$$\beta_V \ : \ s \longrightarrow C_0(V, \mathbb{Cl}(V)) , \quad \beta_V(f)(v) \ = \ f[v] .$$

Since the functional calculus map is G-equivariant, β_V takes values in the G-fixed-points of $C_0(V, \mathbb{Cl}(V))$ for the conjugation action of G.

We let $\mathcal{H}_G$ be any complete G-Hilbert space universe, i.e., a Hilbert G-representation that is isometrically isomorphic to the completion of a complete unitary G-universe. We let $\mathcal{K}_G$ be the G-C^*-algebra of (not necessarily equivariant) compact operators on $\mathcal{H}_G$, with G acting by conjugation.

Theorem 6.4.17 (Equivariant Bott periodicity)**.** *Let G be a compact Lie group and V an orthogonal G-representation. Then for every graded G-C^*-algebra A, the map*

$$\beta_V \cdot - \ : \ C^*_{\mathrm{gr}}(s, A \otimes \mathcal{K}_G) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, C_0(V, \mathbb{C}\mathrm{l}(V)) \otimes A \otimes \mathcal{K}_G)$$

is a G-weak equivalence.

When $G = e$ is a trivial group, $\mathcal{K}_G$ reduces to the C^*-algebra of compact operators on a separable Hilbert space, and then this formulation of Bott periodicity can be found in [76, Thm. 1.14]. Unfortunately, we are lacking a reference in the generality of compact Lie groups. If one specializes [75, Sec. 3, Thm. 5] to finite-dimensional representations of finite groups, one obtains a formulation very close to (but not exactly the same as) Theorem 6.4.17.

Since the unit map $\eta_V : S^V \longrightarrow \mathbf{KU}(V)$ is essentially the functional calculus map, a direct consequence of the formulation of equivariant Bott periodicity in Theorem 6.4.17 is that $\mathbf{KU}$ is 'eventually' a global Ω-spectrum, see Theorem 6.4.20 below.

We showed in Proposition 6.3.18 that an equivariant linear isometric embedding between two complete complex G-universes induces a G-homotopy equivalence between configuration spaces with labels in the two universes. We will now establish an analogous property for the C^*-algebras of compact operators on the Hilbert space completions.

Construction 6.4.18. We recall that a $\mathbb{C}$-linear isometric embedding $\varphi : \mathcal{H} \longrightarrow \mathcal{H}'$ between complex separable Hilbert spaces gives rise to a preferred $*$-homomorphism $\mathcal{K}(\varphi)$ between the C^*-algebras of compact operators. We refer to $\mathcal{K}(\varphi)$ as the *conjugation homomorphism* induced by φ. We emphasize that there is no further hypothesis on φ besides $\mathbb{C}$-linearity and the requirement

$$(x, y)_{\mathcal{H}} \ = \ (\varphi(x), \varphi(y))_{\mathcal{H}'}$$

for all $x, y \in \mathcal{H}$; in particular, φ is *not* assumed to be bounded, complemented, or adjointable. To characterize $\mathcal{K}(\varphi)$ we need the following notation: for a finite-dimensional subspace L of $\mathcal{H}$ we write $p_L \in \mathcal{K}(\mathcal{H})$ for the orthogonal projection onto L. There is then a unique $*$-homomorphism

$$\mathcal{K}(\varphi) \ : \ \mathcal{K}(\mathcal{H}) \ \longrightarrow \ \mathcal{K}(\mathcal{H}') \qquad \text{such that} \qquad \mathcal{K}(\varphi)(p_L) \ = \ p_{\varphi(L)}$$

for all finite-dimensional L in $\mathcal{H}$. The uniqueness is a consequence of the fact that the linear span of the finite rank projections is the space of finite rank operators [126, Thm. 2.4.6] and the finite rank operators are dense in $\mathcal{K}(\mathcal{H})$ [126, Thm. 2.4.5]. The construction of $\mathcal{K}(\varphi)$ can be found in [119, Lemma 4.1].

Informally, one can think of $\mathcal{K}(\varphi)$ as 'conjugation by φ', combined with extension by 0 on the orthogonal complement, which justifies the name. Indeed, if φ happens to be complemented, i.e., $\mathcal{H}'$ is the direct sum of $\varphi(\mathcal{H})$ and the orthogonal complement $\varphi(\mathcal{H})^\perp$, then this is literally true. In particular, if φ is a linear isometric isomorphism, then $\mathcal{K}(\varphi)$ is conjugation by φ.

The construction is covariantly functorial in the linear isometric embedding. Indeed, if $\psi : \mathcal{H}' \longrightarrow \mathcal{H}''$ is another linear isometric embedding, then

$$\mathcal{K}(\psi) \circ \mathcal{K}(\varphi) \;=\; \mathcal{K}(\psi \circ \varphi)$$

by the uniqueness clause.

The construction generalizes to the G-equivariant context, where G is any compact Lie group. More precisely, we assume that $\mathcal{H}$ and $\mathcal{H}'$ are separable complex G-Hilbert space representations. The C^*-algebra $\mathcal{K}(\mathcal{H})$ then inherits a continuous G-action by conjugation with the G-action on $\mathcal{H}$. If $\varphi : \mathcal{H} \longrightarrow \mathcal{H}'$ is G-equivariant, then the functoriality of the homomorphism $\mathcal{K}(\varphi)$ as a function of φ directly implies that the $*$-homomorphism $\mathcal{K}(\varphi) : \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{K}(\mathcal{H}')$ is also G-equivariant.

Proposition 6.4.19. *Let G be a compact Lie group and $u : \mathcal{U} \longrightarrow \mathcal{V}$ a G-equivariant $\mathbb{C}$-linear isometric embedding between two complete complex G-universes. Then the morphism of G-C^*-algebras*

$$\mathcal{K}(\hat{u}) \;:\; \mathcal{K}(\widehat{\mathcal{U}}) \;\longrightarrow\; \mathcal{K}(\widehat{\mathcal{V}})$$

is a G-equivariant homotopy equivalence.

Proof We start with the special case where $\mathcal{V} = \mathcal{U}$. The space of G-equivariant linear isometric embeddings from $\mathcal{U}$ to itself is contractible. By passage to completions, a G-equivariant homotopy from u to the identity induces a G-homotopy

$$\Phi \;:\; \widehat{\mathcal{U}} \times [0,1] \;\longrightarrow\; \widehat{\mathcal{U}}\,,$$

i.e., the map $\Phi(x,-) : [0,1] \longrightarrow \widehat{\mathcal{U}}$ is continuous for every $x \in \widehat{\mathcal{U}}$, and the map $\Phi(-,t)$ is a linear isometric embedding for every $t \in [0,1]$. The map

$$[0,1] \;\longrightarrow\; C^*(\mathcal{K}(\widehat{\mathcal{U}}), \mathcal{K}(\widehat{\mathcal{U}}))\,, \quad t \;\longmapsto\; \mathcal{K}(\Phi(-,t))$$

is then continuous, see for example [119, p. 211]. So $\mathcal{K}(\hat{u}) : \mathcal{K}(\widehat{\mathcal{U}}) \longrightarrow \mathcal{K}(\widehat{\mathcal{U}})$ is homotopic, through G-equivariant $*$-homomorphisms, to the identity.

In the general case we choose an equivariant linear isometry $v : \mathcal{V} \cong \mathcal{U}$. By the previous paragraph the G-C^*-homomorphisms $\mathcal{K}(\hat{v}) \circ \mathcal{K}(\hat{u}) = \mathcal{K}(\widehat{vu})$ and $\mathcal{K}(\hat{u}) \circ \mathcal{K}(\hat{v}) = \mathcal{K}(\widehat{uv})$ are G-homotopic to the respective identity maps. □

We recall from Definition 6.3.21 that an orthogonal G-representation is *ample* if its complexified symmetric algebra is a complete complex G-universe.

Theorem 6.4.20. *For every orthogonal G-representation V and every ample G-representation W, the adjoint structure map*

$$\tilde{\sigma}_{V,W} \ : \ \mathbf{KU}(W) \ \longrightarrow \ \mathrm{map}_*(S^V, \mathbf{KU}(V \oplus W))$$

is a G-weak equivalence.

Proof We let $u : \mathrm{Sym}(W_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$ be the linear isometric embedding induced by the direct summand embedding $W \longrightarrow V \oplus W$. Since W is ample, u is an equivariant $\mathbb{C}$-linear isometric embedding between complete complex G-universes. So the induced map $\mathcal{K}(\hat{u}) : \mathcal{K}_W \longrightarrow \mathcal{K}_{V\oplus W}$ of compact operators is a G-equivariant homotopy equivalence of C^*-algebras by Proposition 6.4.19.

The adjoint structure map $\tilde{\sigma}_{V,W}$ factors as the composite:

$$\begin{aligned}
\mathbf{KU}(W) \ &= C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_W) \\
&\xrightarrow{\beta_V \cdot -} C^*_{\mathrm{gr}}(s, C_0(V, \mathbb{C}\mathrm{l}(V)) \otimes \mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_W) \\
&\xrightarrow{\mathcal{K}(\hat{u})_*} C^*_{\mathrm{gr}}(s, C_0(V, \mathbb{C}\mathrm{l}(V)) \otimes \mathbb{C}\mathrm{l}(W) \otimes \mathcal{K}_{V\oplus W}) \\
&\xrightarrow[\cong]{(6.4.16)} C^*_{\mathrm{gr}}(s, C_0(V) \otimes \mathbb{C}\mathrm{l}(V \oplus W) \otimes \mathcal{K}_{V\oplus W}) \\
&\cong \quad \mathrm{map}_*(S^V, C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V \oplus W) \otimes \mathcal{K}_{V\oplus W})) \ = \ \mathrm{map}_*(S^V, \mathbf{KU}(V \oplus W))
\end{aligned}$$

Since W is ample, $\mathcal{H}_W$ is a complete Hilbert G-universe and so $\mathcal{K}_W$ is a $\mathcal{K}_G$. Bott periodicity (Theorem 6.4.17) for the G-C^*-algebra $\mathbb{C}\mathrm{l}(W)$ shows that the first map is a G-weak equivalence. The second map is a G-homotopy equivalence by the first paragraph, so altogether $\tilde{\sigma}_{V,W}$ is a G-weak equivalence. □

The shift $\mathrm{sh}\, X = \mathrm{sh}^{\mathbb{R}} X$ of an orthogonal spectrum X was defined in (3.1.22). Shifting an orthogonal spectrum provides a delooping by Proposition 4.1.4 (i). The eigenspace morphism was defined in (6.3.26).

Theorem 6.4.21. *The composite*

$$\mathbf{U} \ \xrightarrow{\mathrm{eig}} \ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku}) \ \xrightarrow{\Omega^\bullet(\mathrm{sh}\, j)} \ \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$$

is a global equivalence of orthogonal spaces.

Proof As in the proof of Theorem 6.3.27 we let $\bar{\mathbf{U}}$ denote the orthogonal space with $\bar{\mathbf{U}}(V) = U(\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}}))$, and we factor the eigenspace decomposition morphism as the composite

$$\mathbf{U} \ \longrightarrow \ \bar{\mathbf{U}} \ \xrightarrow{\overline{\mathrm{eig}}} \ \Omega^\bullet(\mathrm{sh}\,\mathbf{ku}) \ ,$$

where the first morphism is the global equivalence of orthogonal spaces induced by the natural linear isometric embedding of $V_{\mathbb{C}}$ into the linear summand in $\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})$.

We claim that composite

$$\bar{\mathbf{U}} \xrightarrow{\overline{\mathrm{eig}}} \Omega^\bullet(\mathrm{sh}\,\mathbf{ku}) \xrightarrow{\Omega^\bullet(\mathrm{sh}\,j)} \Omega^\bullet(\mathrm{sh}\,\mathbf{KU})$$

is an 'eventually strong level equivalence'. More precisely, we show that for every ample G-representation V the map

$$\begin{aligned} U(\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})) \ = \ \bar{\mathbf{U}}(V) \ &\longrightarrow \ (\Omega^\bullet(\mathrm{sh}\,\mathbf{KU}))(V) \\ &= \ \mathrm{map}_*(S^V, C^*_{\mathrm{gr}}(s, \mathbb{Cl}(V \oplus \mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}})) \end{aligned}$$

is a G-weak equivalence. This map factors through the G-map

$$\begin{aligned} U(\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})) \cong \mathscr{C}(\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}}), S^1) \ &\longrightarrow \ \hat{\mathscr{C}}(\mathcal{H}_{V\oplus\mathbb{R}}, S^1) \\ &\cong \ C^*(s, \mathcal{K}_{V\oplus\mathbb{R}}) \ \cong \ C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}}) \ . \end{aligned}$$

Here $\hat{\mathscr{C}}(\mathcal{H}_{V\oplus\mathbb{R}}, S^1)$ is the space of configurations, not necessarily finite, of pairwise orthogonal finite-dimensional subspaces of $\mathcal{H}_{V\oplus\mathbb{R}}$, that are allowed to accumulate around ∞. This map is a G-equivariant homotopy equivalence; the non-equivariant argument (or rather its real analog) can be found in [154, Prop. 1.2] or [79, Prop. 4.6], and the explicit homotopies in [79, Prop. 4.6] work just the same way in the presence of an isometric action of a compact Lie group.

By the operator theoretic equivariant Bott periodicity theorem (Theorem 6.4.17), multiplication by the graded $*$-homomorphism $\beta_V : s \longrightarrow C_0(V, \mathbb{Cl}(V))$ is a G-weak equivalence

$$\beta_V \cdot - \ : \ C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}}) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, C_0(V, \mathbb{Cl}(V)) \otimes \mathbb{Cl}(\mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}}) \ .$$

We use here that $\mathcal{K}_{V\oplus\mathbb{R}}$ is a $\mathcal{K}_G$ because the G-representation V is ample. This proves the claim because the target is G-equivariantly homeomorphic to

$$\begin{aligned} &C^*_{\mathrm{gr}}(s, C_0(V) \otimes \mathbb{Cl}(V) \otimes \mathbb{Cl}(\mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}}) \\ &\overset{(6.4.6)}{\cong} \ C^*_{\mathrm{gr}}(s, C_0(V) \otimes \mathbb{Cl}(V \oplus \mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}}) \\ &\cong \ \mathrm{map}_*(S^V, C^*_{\mathrm{gr}}(s, \mathbb{Cl}(V \oplus \mathbb{R}) \otimes \mathcal{K}_{V\oplus\mathbb{R}})) \\ &= \ \mathrm{map}_*(S^V, \mathrm{sh}\,\mathbf{KU}(V)) \ . \qquad \square \end{aligned}$$

We emphasize that in contrast to Theorem 6.3.27, the composite in Theorem 6.4.21 is a global equivalence for all compact Lie groups (as opposed to only a $\mathcal{F}in$-global equivalence). Another key difference is that the spectrum **ku** is globally connective; in contrast, we will see in Theorem 6.4.29 below that **KU** is Bott periodic, so applying $\Omega^\bullet$ is losing information.

Remark 6.4.22 (**KU** globally deloops **BUP**). A corollary of the previous theorem is that **KU** globally deloops the orthogonal space **BUP** defined in Example 2.4.33. Indeed, the global formulation of Bott periodicity in Theorem 2.5.41 provides a global equivalence of ultra-commutative monoids $\bar{\beta} : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$. Combined with Theorem 6.4.21 this yields a chain of global equivalences of orthogonal spaces

$$\mathbf{BUP} \xrightarrow[\simeq]{\bar{\beta}} \Omega(\mathrm{sh}_\otimes \mathbf{U}) \xleftarrow[\simeq]{} \Omega\mathbf{U} \xrightarrow[\simeq]{\Omega((\Omega^\bullet(\mathrm{sh}\, j))\circ \mathrm{eig})} \Omega(\Omega^\bullet(\mathrm{sh}\,\mathbf{KU})) \xleftarrow[\simeq]{\Omega^\bullet(\tilde{\lambda}_{\mathbf{KU}})} \Omega^\bullet\mathbf{KU}\,.$$

We spell out another corollary of Theorem 6.4.21. We recall that $\mathbf{K}_G(A)$ denotes the equivariant K-group of a G-space A, i.e., the Grothendieck group of isomorphism classes of G-vector bundles over A. A ring homomorphism $[-]$ from this Grothendieck group to the equivariant cohomology group $\mathbf{ku}^0_G(A_+)$ was defined in (6.3.30).

Corollary 6.4.23. *For every compact Lie group G and every finite G-CW-complex A the composite*

$$\mathbf{K}_G(A) \xrightarrow{[-]} \mathbf{ku}^0_G(A_+) \xrightarrow{j_*} \mathbf{KU}^0_G(A_+)$$

is an isomorphism.

Proof We contemplate the commutative diagram of orthogonal spaces:

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\beta"'] & \Omega^\bullet\mathbf{ku} \arrow[r, "\Omega^\bullet j"] \arrow[d, "\simeq"', "\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}}"] & \Omega^\bullet\mathbf{KU} \arrow[d, "\simeq"', "\Omega^\bullet\tilde{\lambda}_{\mathbf{KU}}"] \\
\Omega\mathbf{U} \arrow[r, "\Omega\,\mathrm{eig}"'] & \Omega\,(\Omega^\bullet(\mathrm{sh}\,\mathbf{ku})) \arrow[r, "\Omega(\Omega^\bullet(\mathrm{sh}\,j))"'] & \Omega(\Omega^\bullet(\mathrm{sh}\,\mathbf{KU}))
\end{tikzcd}$$

The morphism $(\Omega^\bullet(\mathrm{sh}\, j))\circ \mathrm{eig}$ is a global equivalence by Theorem 6.4.21, so the lower horizontal composite is also a global equivalence. The same argument as for $\Omega^\bullet\tilde{\lambda}_{\mathbf{ku}}$ in the proof of Theorem 6.3.28 shows that the right vertical morphism $\Omega^\bullet\tilde{\lambda}_{\mathbf{KU}}$ is a global equivalence of orthogonal spaces. In the commutative diagram of abelian monoids

$$\begin{tikzcd}
{[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "{[A,c]^G}"] \arrow[d, "{[A,\beta]^G}"'] & {[A, \Omega^\bullet\mathbf{ku}]^G} \arrow[r, "{[A,\Omega^\bullet j]^G}"] & {[A, \Omega^\bullet\mathbf{KU}]^G} \arrow[r, phantom, "="] \arrow[d, "\cong"', "{[A,\Omega^\bullet\tilde{\lambda}_{\mathbf{KU}}]^G}"] & \mathbf{KU}^0_G(A_+) \\
{[A, \Omega\mathbf{U}]^G} \arrow[rr, "\cong", "{[A,\Omega((\Omega^\bullet(\mathrm{sh}\, j))\circ\mathrm{eig})]^G}"'] & & {[A, \Omega(\Omega^\bullet(\mathrm{sh}\,\mathbf{KU}))]^G} &
\end{tikzcd}$$

the lower horizontal and right vertical maps are thus isomorphisms by Proposition 1.5.3 (ii). The left vertical map $[A,\beta]^G$ is a group completion of abelian monoids by Corollary 2.5.42. So the upper horizontal composite $[A,(\Omega^\bullet j)\circ$

$c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{KU}^0_G(A_+)$ is also a group completion of abelian monoids. On the other hand, the homomorphism $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is yet another group completion of abelian monoids, by the complex analog of Theorem 2.4.10. The universal property of group completions yields an isomorphism of abelian groups

$$\Psi \ : \ \mathbf{K}_G(A) \ \longrightarrow \ \mathbf{KU}^0_G(A_+)$$

such that $\Psi \circ \langle - \rangle = [A, (\Omega^\bullet j) \circ c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{KU}^0_G(A_+)$. The defining relation $[A, c]^G = [-] \circ \langle - \rangle$ of the homomorphism (6.3.30) then yields

$$\Psi \circ \langle - \rangle \ = \ [A, (\Omega^\bullet j) \circ c]^G \ = \ [A, \Omega^\bullet j]^G \circ [-] \circ \langle - \rangle \ .$$

Since $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is a group completion, this forces the relation $\Psi = [A, \Omega^\bullet j]^G \circ [-]$. □

The special case $A = *$ of the previous corollary is worth spelling out explicitly. In this case the group $\mathbf{K}_G(*)$ becomes the unitary representation ring $\mathbf{RU}(G)$, and $\mathbf{KU}^0_G(*)$ becomes the 0th equivariant homotopy group $\pi_0^G(\mathbf{KU})$.

Theorem 6.4.24. *As G ranges of all compact Lie groups, the composite maps*

$$\mathbf{RU}(G) \xrightarrow{[-]} \pi_0^G(\mathbf{ku}) \xrightarrow{\pi_0^G(j)} \pi_0^G(\mathbf{KU})$$

form an isomorphism of global power functors between $\mathbf{RU}$ *and* $\underline{\pi}_0(\mathbf{KU})$*.*

Proof By Corollary 6.4.23 for $A = *$ the composite is a ring isomorphism for every compact Lie group G. The second map is induced by a homomorphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ of ultra-commutative ring spectra, so the maps $\pi_0^G(j)$ form a morphism of global power functors.

The maps $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are ring homomorphisms, compatible with restriction homomorphisms, with finite index transfers and multiplicative power operations by Theorem 6.3.33. There is still something to show, though, because the maps $[-]$ are definitely *not* compatible with general transfers (i.e., of infinite index). So an additional argument is needed to see that the composite *does* commute with all transfers.

We consider a closed subgroup H of a compact Lie group G, not necessarily of finite index, and we want to show that

$$\mathrm{tr}_H^G(j_*[x]) \ = \ j_*[\mathrm{tr}_H^G(x)] \tag{6.4.25}$$

in $\pi_0^G(\mathbf{KU})$ for all classes $x \in \mathbf{RU}(H)$. Since representations are detected by characters, two classes in $\mathbf{RU}(G)$ are equal already if their restrictions to all finite abelian subgroups of G coincide. Since the composite maps $\mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{KU})$ are all isomorphisms and compatible with restriction, the analogous property holds for the global functor $\underline{\pi}_0(\mathbf{KU})$. So it suffices to show that the

relation (6.4.25) holds after restriction to every finite abelian subgroup A of G. The double coset formula

$$\mathrm{res}^G_A \circ \mathrm{tr}^G_H \;=\; \sum_{[M]} \chi^\sharp(M) \cdot \mathrm{tr}^A_{A\cap{}^gH} \circ g_\star \circ \mathrm{res}^H_{A^g\cap H}$$

holds for **RU** by [160, Thm. 2.4], and it holds in the homotopy global functor of every orthogonal spectrum. Since A is finite, the right-hand side of the double coset formula only involves finite index transfers. Since the maps from **RU** to $\underline{\pi}_0(\mathbf{KU})$ under consideration do commute with finite index transfers, they commute with the right-hand side of the double coset formula, hence also with the left-hand side. This shows that (6.4.25) holds after restriction to every finite abelian subgroup, so it holds altogether. □

Remark 6.4.26. In Construction 6.3.40 we discussed the rank filtration of the connective global K-theory spectrum **ku**, and we identified the first stage $\mathbf{ku}^{[1]}$ with the suspension spectrum of the ultra-commutative monoid $\mathbf{P}^{\mathbb{C}}$. On $\underline{\pi}_0$, the morphisms of ultra-commutative ring spectra

$$\Sigma^\infty_+ \mathbf{P}^{\mathbb{C}} \;\cong\; \mathbf{ku}^{[1]} \xrightarrow{\text{incl}} \mathbf{ku} \xrightarrow{j} \mathbf{KU}$$

induce morphisms of global power functors. Since $\mathbf{P}^{\mathbb{C}}$ is a global classifying space of the circle group $U(1)$, the global functor $\underline{\pi}_0(\Sigma^\infty_+\mathbf{P}^{\mathbb{C}})$ is representable by $U(1)$, by Proposition 4.2.5. On the other hand, $\underline{\pi}_0(\mathbf{KU})$ is isomorphic to the representation ring global functor **RU**, by Theorem 6.4.24. Under these identifications, the composite morphism $\Sigma^\infty_+\mathbf{P}^{\mathbb{C}} \longrightarrow \mathbf{KU}$ of ultra-commutative ring spectra becomes the morphism

$$\mathrm{ev}_x \;:\; \mathbf{A}(U(1), -) \;\longrightarrow\; \mathbf{RU}$$

that sends the generator $1_{U(1)} \in \mathbf{A}(U(1), U(1))$ to the class of the tautological $U(1)$-representation on $\mathbb{C}$. As we recalled in Remark 5.3.19, this morphism is surjective, and 'explicit Brauer induction' provides a specific section.

Example 6.4.27. Our language allows a reformulation of the generalization, due to Adams, Haeberly, Jackowski and May [1], of the Atiyah-Segal completion theorem: the global K-theory spectrum **KU** is right induced from the global family *cyc* of finite cyclic groups.

If G is a compact Lie group, then a virtual G-representation that restricts to zero on every finite cyclic subgroup is already zero. In other words, the intersection of the kernels of all restriction maps $\mathrm{res}^G_C : R(G) \longrightarrow R(C)$ for all finite cyclic subgroups C of G, is trivial. Then by [1, Cor. 2.1], the projection $A \times E(cyc \cap G) \longrightarrow A$ induces an isomorphism

$$\mathbf{K}^*_G(A) \;\cong\; \mathbf{K}^*_G(A \times E(cyc \cap G))$$

on equivariant K-groups for every finite G-CW-complex A, where $E(cyc \cap G)$ is a universal G-space for the family of finite cyclic subgroups of G. The Milnor short exact sequence lets us extend this to infinite G-CW-complexes, so the criterion provided by Proposition 4.5.16 for being right induced from the global family cyc is satisfied.

Our next aim is to establish Bott periodicity for the global K-theory spectrum **KU**.

Construction 6.4.28 (Inverse Bott class)**.** The Bott class $\beta \in \pi_2^e(\mathbf{ku})$ was defined in Construction 6.3.42. We will now define the *inverse Bott class* $\lambda \in \pi_{-2}^e(\mathbf{KU})$ and show that it is multiplicatively inverse to the image of β under the homomorphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$.

We recall that $u\mathbb{C}$ denotes the underlying euclidean inner product space of $\mathbb{C}$. We let $e = [1]$ and $f = [i]$ be the images in $\mathrm{Cl}(u\mathbb{C})$ of the standard orthonormal $\mathbb{R}$-basis $\{1, i\}$ of $u\mathbb{C}$. We denote by $q \in \mathrm{Cl}(u\mathbb{C})$ the projection (i.e., self-adjoint idempotent)

$$q \;=\; (1 - i \cdot ef)/2 \;.$$

The isomorphism $\delta_{\mathbb{C}} : \mathrm{Cl}(u\mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Lambda^*\mathbb{C})$ defined in (6.3.49) takes the projection q to the orthogonal projection onto $\Lambda^0\mathbb{C}$, the constant summand of the exterior algebra, see Example 6.3.50.

Moreover, we denote by $p_0 \in \mathcal{K}_{u\mathbb{C}}$ the orthogonal projection onto the subspace $\mathbb{C} \cdot 1$ in $\mathcal{H}_{u\mathbb{C}} = \widehat{\mathrm{Sym}}((u\mathbb{C})_{\mathbb{C}})$. Then the element $q \otimes p_0$ of $\mathrm{Cl}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}$ is another projection. So the map

$$s \;\longrightarrow\; \mathrm{Cl}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}} \;, \quad f \;\longmapsto\; f(0) \cdot q \otimes p_0$$

is a $\mathbb{Z}/2$-graded $*$-homomorphism, i.e., an element in the space

$$C^*_{\mathrm{gr}}(s, \mathrm{Cl}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}) \;=\; \mathbf{KU}(u\mathbb{C}) \;=\; \mathbf{KU}(\mathbb{R}^2) \;.$$

We denote by

$$\lambda \;\in\; \pi_{-2}^e(\mathbf{KU})$$

the homotopy class represented by this point.

Theorem 6.4.29. *The relation $j_*(\beta) \cdot \lambda = 1$ holds in $\pi_0^e(\mathbf{KU})$.*

Proof Step 1: We start with an identification of $\mathrm{Cl}(\mathbb{R}) \otimes M_2$ with $\mathrm{Cl}(\mathbb{R} \oplus u\mathbb{C})$, where M_2 is the C^*-algebra of 2×2 complex matrices, concentrated in even grading. We let $d = [1] \in \mathrm{Cl}(\mathbb{R})$ denote the odd unit corresponding to $1 \in \mathbb{R}$. The map

$$\Psi\left(1 \otimes \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)\right) \;=\; (w+z)/2 \cdot 1 \;+\; (z-w)/2 \cdot i \cdot ef \;+\; (y-x)/2 \cdot de \;-\; (x+y)/2 \cdot i \cdot df$$

identifies the even summand $1 \otimes M_2$ of $\mathbb{Cl}(\mathbb{R}) \otimes M_2$ with the even summand of $\mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C})$. The odd element $i \cdot def$ of $\mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C})$ squares to 1 and commutes with all even elements of $\mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C})$. So we can extend the isomorphism to the odd summands by setting

$$\Psi\left(d \otimes \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)\right) \;=\; i \cdot def \cdot \Psi\left(1 \otimes \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)\right) \;.$$

The result is an isomorphism of $\mathbb{Z}/2$-graded C^*-algebras

$$\Psi \;:\; \mathbb{Cl}(\mathbb{R}) \otimes M_2 \xrightarrow{\cong} \mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C}) \;.$$

While C^*-algebras are not required to have multiplicative units, and $*$-homomorphisms are ignorant of units, these two C^*-algebras are unital and Ψ happens to preserve multiplicative units.

Step 2: We let $j' : \mathscr{C}(\mathbb{C}^2, S^1) \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R}) \otimes M_2)$ be the restriction of the map

$$j(\mathbb{R}) \;:\; \mathbf{ku}(\mathbb{R}) = \mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^1) \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R}) \otimes \mathcal{K}_{\mathbb{R}}) = \mathbf{KU}(\mathbb{R})$$

to those configurations contained in $\mathbb{C}^2 \subset \mathrm{Sym}(\mathbb{C})$, defined by the same formula

$$j'[E_1, E_2;\, v_1, v_2](u) \;=\; u[v_1] \otimes p_{E_1} \;+\; u[v_2] \otimes p_{E_2} \;.$$

Our next claim is that the composite

$$\begin{aligned} S^{\mathbb{R}\oplus\mathbb{C}} \;&\xrightarrow[\cong]{m}\; SU(2) \;\xrightarrow{\mathrm{eig}}\; \mathscr{C}(\mathbb{C}^2, S^1) \\ &\xrightarrow{j'}\; C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R}) \otimes M_2) \;\xrightarrow{\Psi_*}\; C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C})) \end{aligned}$$

is the functional calculus map $\mathrm{fc} : S^{\mathbb{R}\oplus\mathbb{C}} \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{Cl}(\mathbb{R} \oplus u\mathbb{C}))$, where m is the homeomorphism (6.3.43) defined by the formula

$$m(v) \;=\; m(x, z) \;=\; \frac{1}{|v|^2 + 1} \begin{pmatrix} |v|^2 - 1 - i2x & 2i\bar{z} \\ 2iz & |v|^2 - 1 + i2x \end{pmatrix} \;.$$

The map $\mathrm{eig} \circ m$ sends (x, z) to the configuration of Cayley transforms of the eigenvalues of $m(x, z)$, labeled by their eigenspaces. So to identify the composite $j' \circ \mathrm{eig} \circ m$ we must calculate these eigenvalues and eigenspaces, and their orthogonal projections. This is a straightforward exercise in linear algebra: a direct calculation shows that the matrices

$$p_+ \;=\; \frac{1}{2|v|} \begin{pmatrix} |v| - x & \bar{z} \\ z & |v| + x \end{pmatrix} \quad \text{and} \quad p_- \;=\; \frac{1}{2|v|} \begin{pmatrix} |v| + x & -\bar{z} \\ -z & |v| - x \end{pmatrix}$$

are projections and orthogonal to each other, i.e., they satisfy the relations

$$p_+^2 \;=\; p_+^* \;=\; p_+ \;, \quad p_-^2 \;=\; p_-^* \;=\; p_- \quad \text{and} \quad p_+ \cdot p_- \;=\; p_- \cdot p_+ \;=\; 0 \;.$$

Moreover,

$$c(|v|) \cdot p_+ \ + \ c(-|v|) \cdot p_- \ = \ m(x, z) \ ,$$

so p_+ and p_- are the orthogonal projections onto the eigenspaces of $m(x, z)$, with eigenvalues

$$c(|v|) \ = \ (|v| + i)(|v| - i)^{-1} \quad \text{and} \quad c(-|v|) \ = \ c(|v|)^{-1} \ = \ (|v| - i)(|v| + i)^{-1} \ .$$

The map $j' \circ \mathrm{eig} \circ m$ thus sends $v = (x, z)$ to the $*$-homomorphism

$$(j' \circ \mathrm{eig} \circ m)(x, z)(u) \ = u[|v|] \otimes p_+ \ + \ u[-|v|] \otimes p_- \ .$$

The C^*-algebra s is generated by the function $r(x) = 2i(x - i)^{-1}$. So to verify that two graded $*$-homomorphisms with source s agree, it suffices to show that they coincide on the even and odd components of the generator r, which we spelled out explicitly in (6.4.2). For the even component r_+ the calculation is

$$\begin{aligned}(\Psi_* \circ j' \circ \mathrm{eig} \circ m)(x, z)(r_+) \ &= \ \Psi(r_+[|v|] \otimes p_+ + r_+[-|v|] \otimes p_-) \\ &= \ \Psi(r_+[|v|] \otimes (p_+ + p_-)) \ = \ r_+[|v|] \cdot 1 \ = \ r_+[x, z] \ .\end{aligned}$$

For the odd component r_- we first observe that

$$\begin{aligned}\Psi\left(d \otimes \left(\begin{smallmatrix} -x & \bar{z} \\ z & x \end{smallmatrix}\right)\right) \ &= \ i \cdot def \cdot (x \cdot ief + (z - \bar{z})/2 \cdot de - (z + \bar{z})/2 \cdot i \cdot df) \\ &= \ -x \cdot defef \ - \ \mathrm{Im}(z) \cdot defde \ + \ \mathrm{Re}(z) \cdot defdf \\ &= \ x \cdot d + \mathrm{Im}(z) \cdot f \ + \mathrm{Re}(z) \cdot e \ = \ [x, z]\end{aligned}$$

in $\mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C})$. So

$$\begin{aligned}(\Psi_* \circ j' \circ \mathrm{eig} \circ m)(x, z)(r_-) \ &= \ \Psi(r_-[|v|] \otimes p_+ + r_-[-|v|] \otimes p_-) \\ &= \ \Psi(r_-[|v|] \otimes (p_+ - p_-)) \\ &= \ \Psi\left(\frac{2i[|v|]}{|v|^2 + 1} \otimes \frac{1}{|v|}\begin{pmatrix} -x & \bar{z} \\ z & x \end{pmatrix}\right) \\ &= \ \frac{2i}{|v|^2 + 1} \cdot \Psi\left(d \otimes \begin{pmatrix} -x & \bar{z} \\ z & x \end{pmatrix}\right) \\ &= \ \frac{2i \cdot [x, z]}{|v|^2 + 1} \ = \ r_-[x, z] \ .\end{aligned}$$

This completes the verification that $\Psi_* \circ j' \circ \mathrm{eig} \circ m$ is the functional calculus map $\mathrm{fc} : S^{\mathbb{R} \oplus \mathbb{C}} \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}))$.

Step 3: We recall that $q = (1 - ief)/2$ in the Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C})$, which satisfies the relations

$$dq \ = \ qd \ , \quad eq = (1 - q)e \qquad \text{and} \qquad qeq = 0 \ . \tag{6.4.30}$$

We consider the even element u of $\mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2$ defined by

$$u \ = \ q \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ + \ qed \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ + \ edq \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ + \ (1-q) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

A direct calculation, making repeated use of the identities (6.4.30), shows that this is a unitary element, i.e.,

$$u^* \cdot u \ = \ u \cdot u^* \ = \ 1 \ .$$

So conjugation by u is a graded $*$-automorphism

$$\zeta \ : \ \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2 \ \longrightarrow \ \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2 \ , \quad x \ \longmapsto \ u^* \cdot x \cdot u \ .$$

Now we observe that the following square of graded C^*-algebras commutes:

$$\begin{tikzcd}
\mathbb{C}\mathrm{l}(\mathbb{R}) \otimes M_2 \arrow[r, "-\otimes q \otimes -"] \arrow[d, "\Psi"'] & \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes \mathbb{C}\mathrm{l}(u\mathbb{C}) \otimes M_2 \arrow[r, "(6.4.6)\otimes M_2"] & \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2 \arrow[d, "\zeta"] \\
\mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \arrow[rr, "-\otimes\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & & \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2
\end{tikzcd}$$

This, again, is a direct calculation, using the identities (6.4.30). We claim that the conjugation map ζ is homotopic, through graded $*$-homomorphisms, to the identity of $\mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2$. To show this we define a continuous path

$$u \ : \ [0, \pi/2] \ \longrightarrow \ \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2$$

by the formula

$$\begin{aligned} u(t) \ = \ &(q + \sin(t)(1-q)) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ + \ \cos(t) \cdot qed \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \ \cos(t) \cdot edq \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ + \ (\sin(t) \cdot q + (1-q)) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ . \end{aligned}$$

One more direct calculation shows that all elements in this path are even unitaries. So conjugation by the elements $u(t)$ is a path of graded $*$-automorphism. Since $u(0) = u$, this path starts with the conjugation map ζ; since $u(\pi/2) = 1$, the path ends with the identity. Since ζ is homotopic to the identity, the induced self-map of $C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \oplus u\mathbb{C}) \otimes M_2)$ is based homotopic to the identity.

Step 4: The class λ is represented by the point of $\mathbf{KU}(u\mathbb{C})$ given by the graded $*$-homomorphism

$$s \ \xrightarrow{\ \epsilon\ } \ \mathbb{C} \ \xrightarrow{\ \cdot q \otimes p_0\ } \ \mathbb{C}\mathrm{l}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}$$

associated with the projection $q \otimes p_0$; here $\epsilon(\varphi) = \varphi(0)$ is the augmentation. So the multiplication map $- \cdot \lambda : \pi^e_{k+2}(\mathbf{KU}) \longrightarrow \pi^e_k(\mathbf{KU})$ is the effect on homotopy

groups of the based continuous map

$$\begin{aligned}
\mathbf{KU}(V) \ &= C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \\
&\xrightarrow{-\wedge(-\cdot q\otimes p_0)} \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \wedge C^*_{\mathrm{gr}}(\mathbb{C}, \mathbb{C}\mathrm{l}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}) \\
&\xrightarrow{\mathrm{Id}\wedge\epsilon^*} \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V) \wedge C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}) \\
&\xrightarrow{\mu^{\mathbf{KU}}_{V,u\mathbb{C}}} \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V \oplus u\mathbb{C}) \otimes \mathcal{K}_{V\oplus u\mathbb{C}}) \ = \ \mathbf{KU}(V \oplus u\mathbb{C}) \ .
\end{aligned}$$

The composite $s\hat{\otimes}\epsilon \circ \Delta$ is the identity of s, so multiplication by λ is the effect of post-composition with the graded $*$-homomorphism

$$\begin{aligned}
\mathbb{C}\mathrm{l}(V) \otimes \mathcal{K}_V \ &\xrightarrow{-\otimes q\otimes -\otimes p_0} \ \mathbb{C}\mathrm{l}(V) \otimes \mathbb{C}\mathrm{l}(u\mathbb{C}) \otimes \mathcal{K}_V \hat{\otimes} \mathcal{K}_{u\mathbb{C}} \\
&\xrightarrow[\cong]{(6.4.6)\otimes(6.4.11)} \ \mathbb{C}\mathrm{l}(V \oplus u\mathbb{C}) \otimes \mathcal{K}_{V\oplus u\mathbb{C}} \ .
\end{aligned}$$

Now we put the pieces together and prove the theorem. We contemplate the following diagram of continuous based maps:

$$\begin{tikzcd}
S^{\mathbb{R}\oplus\mathbb{C}} \arrow[rr, "\mathrm{fc}"] \arrow[d, "\mathrm{eig}\circ m"'] & & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \otimes u\mathbb{C})) \arrow[d, "\left(-\otimes\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)_*"] \arrow[dd, bend left=60, "(-\otimes p_0)_*"] \\
\mathscr{C}(\mathbb{C}^2, S^1) \arrow[r, "j'"'] \arrow[d, "\mathscr{C}(i,S^1)"'] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes M_2) \arrow[r, "(-\otimes q\otimes -)_*"'] \arrow[d, "(-\otimes i_!)_*"] \arrow[ur, "\Psi_*"] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \otimes u\mathbb{C}) \otimes M_2) \arrow[d, "(-\otimes i_!)_*"] \\
\mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^1) \arrow[dd, equal] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R}) \otimes \mathcal{K}_{\mathbb{R}}) \arrow[r, "(-\otimes q\otimes -)_*"'] \arrow[dd, equal] & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \otimes u\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}}) \arrow[d, "(-\otimes p_0)_*"] \\
 & & C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(\mathbb{R} \otimes u\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}\oplus u\mathbb{C}}) \arrow[d, equal] \\
\mathbf{ku}(\mathbb{R}) \arrow[r, "j(\mathbb{R})"'] & \mathbf{KU}(\mathbb{R}) \arrow[r, "(-\otimes q\otimes -\otimes p_0)_*"'] & \mathbf{KU}(\mathbb{R} \oplus u\mathbb{C})
\end{tikzcd}$$

Here $i : \mathbb{C}^2 \longrightarrow \mathrm{Sym}(\mathbb{C})$ is the identification with the constant and linear summands of the symmetric algebra used throughout, and $i_! : M_2 \longrightarrow \mathcal{K}_{\mathbb{R}}$ is the $*$-homomorphism given by extension by 0 on the higher symmetric powers. By Step 2, the upper left part commutes on the nose, and by Step 3 the upper right triangle commutes up to based homotopy. The lower left part commutes because j' is the restriction of $j(\mathbb{R})$. The lower right part commutes as well. So the whole diagram commutes up to based homotopy. By Step 4, the composite through the lower left corner represents the class $j_*(\beta)\cdot\lambda$, whereas the composite along the right side of the diagram is the unit map

$\eta_{\mathbb{R}\oplus\mathbb{C}} : S^{\mathbb{R}\oplus\mathbb{C}} \longrightarrow \mathbf{KU}(\mathbb{R} \oplus u\mathbb{C})$ of the ring spectrum structure. So the diagram witnesses the desired relation

$$j_*(\beta)\cdot\lambda \;=\; [(-\otimes q\otimes-\otimes p_0)_*\circ j(\mathbb{R})\circ\mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^1)\circ\mathrm{eig}\circ m] \;=\; [\eta_{\mathbb{R}\oplus u\mathbb{C}}] \;=\; 1\,.$$

□

Remark 6.4.31 (Thom isomorphism in equivariant K-theory)**.** The spectrum **KU** enjoys a much stronger form of periodicity that generalizes the $\mathbb{Z}$-graded periodicity manifested by Theorem 6.4.29. The equivariant Bott class $\beta_{G,W}$ associated to a G-Spin^c-representation W was introduced in Construction 6.3.46, and $\beta_{G,W}$ is in fact an '$RO(G)$-graded unit' in **KU**. This kind of equivariant periodicity in K-theory goes back to Atiyah [3, Thm. 4.3]; it is formally similar to the periodicity of **MO** and **MOP** for orthogonal representations (compare Theorem 6.1.14 (iii)), or the periodicity of **MU** and **MUP** for unitary representations.

Even more is true. The equivariant Bott class is a special case of the Thom class of an equivariant Spin^c-vector bundle, and equivariant Bott periodicity is a special case of a Thom isomorphism for such bundles. This Thom isomorphism for equivariant **KU**-theory goes under the name 'Atiyah-Bott-Shapiro orientation' and was established in [4]. In [86, Thm. 6.9] Joachim shows that the Atiyah-Bott-Shapiro orientation is in fact extremely highly structured, namely 'globally and ultra-commutative'. Joachim defines a morphism of ultra-commutative ring spectra $\alpha : \mathbb{M}\,\mathrm{Spin}^c \longrightarrow \mathbf{KU}$ from a global equivariant version of the Spin^c-Thom spectrum; his morphism refines the Atiyah-Bott-Shapiro orientation, in the sense that it takes certain tautological Thom classes for Spin^c-vector bundles in $\mathbb{M}\,\mathrm{Spin}^c$ to the **KU**-theoretic Thom classes of Atiyah-Bott-Shapiro.

Construction 6.4.32 (Global connective K-theory)**.** Now we define *global connective K-theory* $\mathbf{ku}^c$, an ultra-commutative ring spectrum whose associated G-homotopy type, for compact Lie groups G, is that of *G-equivariant connective K-theory* in the sense of Greenlees [66]. This is *not* a connective equivariant theory, i.e., the equivariant homotopy groups $\pi_*^G(\mathbf{ku}^c)$ do *not* vanish in negative dimensions, as soon as the group G is non-trivial. Hence the order of the adjectives 'global' and 'connective' matters, i.e., 'global connective' K-theory is different from 'connective global' K-theory. Two of the advantages of $\mathbf{ku}^c$ are that it is equivariantly (and in fact globally) orientable, and that $\mathbf{ku}^c$ satisfies a completion theorem: for every compact Lie group G, the completion of the graded ring $\pi_*^G(\mathbf{ku}^c)$ at the augmentation ideal of the unitary representation ring is the connective **ku**-cohomology of the classifying space BG, see [66, Prop. 2.2 (i)].

Our construction of $\mathbf{ku}^c$ is a direct 'globalization' of Greenlees' definition

in [66, Def. 3.1]. We define $\mathbf{ku}^c$ as the homotopy pullback in the square of ultra-commutative ring spectra

$$\begin{tikzcd} \mathbf{ku}^c \arrow[r] \arrow[d] & b(\mathbf{ku}) \arrow[d, "bj"] \\ \mathbf{KU} \arrow[r, "i_{\mathbf{KU}}"'] & b(\mathbf{KU}) \end{tikzcd} \tag{6.4.33}$$

The morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ from connective to periodic global K-theory was defined in Construction 6.4.9; the Borel theory functor b and the natural transformation $i : \mathrm{Id} \longrightarrow b$ were defined in Construction 4.5.21. So more explicitly, we set

$$\mathbf{ku}^c \;=\; \mathbf{KU} \times_{b(\mathbf{KU})} b(\mathbf{KU})^{[0,1]} \times_{b(\mathbf{KU})} b(\mathbf{ku})\,.$$

Since the spectra $\mathbf{KU}$, $b(\mathbf{KU})$ and $b(\mathbf{ku})$ are ultra-commutative ring spectra and the two morphisms $i_{\mathbf{KU}} : \mathbf{KU} \longrightarrow b(\mathbf{KU})$ and $bj : b(\mathbf{ku}) \longrightarrow b(\mathbf{KU})$ are homomorphisms, the homotopy pullback is canonically an ultra-commutative ring spectrum and the two morphisms from $\mathbf{ku}^c$ to $\mathbf{KU}$ and $b(\mathbf{ku})$ are morphisms of ultra-commutative ring spectra. As a homotopy pullback, the square (6.4.33) does *not* commute, but the construction comes with a preferred homotopy between the two composites around the square.

The morphisms $j : \mathbf{ku} \longrightarrow \mathbf{KU}$, $i_{\mathbf{ku}} : \mathbf{ku} \longrightarrow b(\mathbf{ku})$ and the constant homotopy provide a morphism of ultra-commutative ring spectra

$$\iota \;:\; \mathbf{ku} \;\longrightarrow\; \mathbf{ku}^c$$

from connective global K-theory to global connective K-theory. For finite groups, this morphism induces an isomorphism on homotopy global functors in non-negative dimensions. The construction of $\mathbf{ku}^c$ endows it with a morphism of ultra-commutative ring spectra $\mathbf{ku}^c \longrightarrow \mathbf{KU}$. As explained by Greenlees in [66, Thm. 2.1 (iv)], this morphism becomes a global equivalence after inverting the Bott class $v \in \pi_2^e(\mathbf{ku}^c)$, the image of the Bott class $\beta \in \pi_2^e(\mathbf{ku})$ defined in Construction 6.3.42. Indeed, for every compact Lie group G, the morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ induces a morphism of graded rings

$$j^*(BG) \;:\; \mathbf{ku}^*(BG) \;\longrightarrow\; \mathbf{KU}^*(BG)$$

that becomes an isomorphism after inverting the Bott class by [33, Lemma 1.1.1]. The natural isomorphisms

$$\pi^G_{-*}(b(\mathbf{ku})) \xrightarrow{\cong} \mathbf{ku}^*(BG) \quad \text{and} \quad \pi^G_{-*}(b(\mathbf{KU})) \xrightarrow{\cong} \mathbf{KU}^*(BG)$$

of Proposition 4.5.22 then show that the map

$$\underline{\pi}_*(bj) \;:\; \underline{\pi}_*(b(\mathbf{ku})) \;\longrightarrow\; \underline{\pi}_*(b(\mathbf{KU}))$$

becomes an isomorphism of graded global functors after inverting the Bott class. Since the right vertical morphism in the defining global homotopy pullback (6.4.33) becomes a global equivalence after inverting the Bott class, the same is true for the left vertical morphism.

The composite

$$\dim^c \ : \ \mathbf{ku}^c \ \longrightarrow \ b(\mathbf{ku}) \ \xrightarrow{b(\dim)} \ b(\mathcal{H}\mathbb{Z})$$

is a morphism of ultra-commutative ring spectra, where $\mathcal{H}\mathbb{Z}$ is the Eilenberg-Mac Lane spectrum of the integers (see Construction 5.3.8) and the dimension homomorphism $\dim : \mathbf{ku} \longrightarrow \mathcal{H}\mathbb{Z}$ was defined in Example 6.3.36. The dimension morphism annihilates the Bott class $\beta \in \pi_2^e(\mathbf{ku})$. We claim that the sequence

$$\mathbf{ku}^c \wedge S^2 \ \xrightarrow{\tilde{v}} \ \mathbf{ku}^c \ \xrightarrow{\dim^c} \ b(\mathcal{H}\mathbb{Z}) \tag{6.4.34}$$

is a 'global homotopy cofiber sequence', i.e., part of a distinguished triangle in the global stable homotopy category, where $\tilde{v}$ is the extension of the Bott class $v \in \pi_2^e(\mathbf{ku}^c)$ to a morphism of $\mathbf{ku}^c$-module spectra. Indeed, the sequence

$$\mathbf{ku} \wedge S^2 \ \xrightarrow{\tilde{\beta}} \ \mathbf{ku} \ \xrightarrow{\dim} \ \mathcal{H}\mathbb{Z}$$

is a non-equivariant homotopy fiber sequence; the Borel theory functor b takes this to the global homotopy fiber sequence

$$b(\mathbf{ku}) \wedge S^2 \ \xrightarrow{b(\tilde{\beta})} \ b(\mathbf{ku}) \ \xrightarrow{b(\dim)} \ b(\mathcal{H}\mathbb{Z}) \ .$$

The spectrum $\mathbf{KU}$ is Bott periodic, i.e., the image of the Bott class in $\pi_2^e(\mathbf{KU})$ is invertible by Theorem 6.4.29. Since the morphism $\mathbf{KU} \longrightarrow b(\mathbf{KU})$ is multiplicative, the same goes for the Borel theory $b(\mathbf{KU})$. So the morphisms

$$\tilde{\beta} \ : \ \mathbf{KU} \wedge S^2 \ \longrightarrow \ \mathbf{KU} \quad \text{and} \quad b(\tilde{\beta}) \ : \ b(\mathbf{KU}) \wedge S^2 \ \longrightarrow \ b(\mathbf{KU})$$

are global equivalences, and their respective homotopy fibers are globally stably contractible. Passing to homotopy pullbacks gives the desired global homotopy fiber sequence (6.4.34).

The global homotopy fiber sequence (6.4.34) and the isomorphism

$$\pi_k^G(b(\mathcal{H}\mathbb{Z})) \ \cong \ H^{-k}(BG, \mathbb{Z})$$

of Proposition 4.5.22 (ii) give rise to a long exact sequence of global functors

$$\cdots \longrightarrow \underline{\pi}_{k+1}(b(\mathcal{H}\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{k-2}(\mathbf{ku}^c) \xrightarrow{\cdot v} \underline{\pi}_k(\mathbf{ku}^c) \xrightarrow{\dim^c_*} \underline{\pi}_k(b(\mathcal{H}\mathbb{Z})) \longrightarrow \cdots \ .$$

This way Greenlees calculates some of the homotopy group global functors of equivariant connective K-theory in [66, Prop. 2.6]. We review Greenlees'

calculations in our language. The group $H^{-k}(BG, \mathbb{Z})$ vanishes for $k > 0$. So multiplication by the Bott class

$$- \cdot v \ : \ \underline{\pi}_{k-2}(\mathbf{ku}^c) \ \longrightarrow \ \underline{\pi}_k(\mathbf{ku}^c)$$

is an isomorphism for $k > 0$ and a monomorphism for $k = 0$. In particular, we conclude that

$$\underline{\pi}_k(\mathbf{ku}^c) \ \cong \ \begin{cases} \mathbf{RU} & \text{for } k \geq 0 \text{ and } k \text{ even, and} \\ 0 & \text{for } k \geq -1 \text{ and } k \text{ odd.} \end{cases}$$

More precisely, for every $m \geq 0$, the composite

$$\mathbf{RU} \xrightarrow{[-]} \underline{\pi}_0(\mathbf{ku}) \xrightarrow{\underline{\pi}_0(\iota)} \underline{\pi}_0(\mathbf{ku}^c) \xrightarrow{\cdot v^m} \underline{\pi}_{2m}(\mathbf{ku}^c)$$

is an isomorphism of global functors.

The global functor $\underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ sends G to $\pi_0^G(b(\mathcal{H}\mathbb{Z})) = H^0(BG, \mathbb{Z})$; so $\underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ is constant with value $\mathbb{Z}$ and the morphism $\dim_*^c : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ is isomorphic to the augmentation morphism $\dim : \mathbf{RU} \longrightarrow \underline{\mathbb{Z}}$ of global functors, by Proposition 6.3.37. This is an epimorphism, so the sequence of global functors

$$0 \longrightarrow \underline{\pi}_{-2}(\mathbf{ku}^c) \xrightarrow{\cdot v} \underline{\pi}_0(\mathbf{ku}^c) \xrightarrow{\dim_*^c} \underline{\pi}_0(b(\mathcal{H}\mathbb{Z})) \longrightarrow 0$$

is short exact and

$$\underline{\pi}_{-2}(\mathbf{ku}^c) \ \cong \ \mathbf{IU} \ = \ \ker(\dim : \mathbf{RU} \longrightarrow \underline{\mathbb{Z}})$$

is the augmentation ideal global functor. Again since the map $\dim_*^c : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ is surjective, the global functor $\underline{\pi}_{-3}(\mathbf{ku}^c)$ injects into $\underline{\pi}_{-1}(\mathbf{ku}^c)$, which is trivial by the above. So we conclude that $\underline{\pi}_{-3}(\mathbf{ku}^c) = 0$.

This method can be pushed a little further to also determine the global functors $\pi_{-4}(\mathbf{ku}^c)$ and $\pi_{-5}(\mathbf{ku}^c)$; we refer to [66, Prop. 2.6] for the argument. The result is that

$$\underline{\pi}_{-4}(\mathbf{ku}^c) \ \cong \ \mathbf{ISU}(G) \ = \ \{x \in \mathbf{IU}(G) \mid \det(x) = 0\}$$

and that $\underline{\pi}_{-5}(\mathbf{ku}^c) = 0$. After this point things become less explicit.

Appendix A
Compactly generated spaces

In this appendix we recall some background material about compactly generated spaces, our basic category to work in. Compactly generated spaces are in particular 'k-spaces', a notion that seems to go back to Kelley's book [89, p. 230]. Compactly generated spaces were popularized by Steenrod in his paper [161] as a 'convenient category of spaces'; however, in contrast to our usage of the term, Steenrod includes the Hausdorff property in his definition of 'compactly generated'. But Steenrod already writes that '(...) *The Hausdorff property is imposed to ensure that compact subsets are closed.* (...)'; the *weak* Hausdorff condition it thus the next logical step, as it isolates this relevant property. The weak Hausdorff condition first appears in print in McCord's paper [118], who credits the idea to J. C. Moore. McCord also was the first to use the terminology 'compactly generated spaces' for weak Hausdorff k-spaces. The fact that much of the recent literature in equivariant and stable homotopy theory uses compactly generated spaces can be taken as evidence that the weak Hausdorff condition is even more convenient than the actual Hausdorff separation property. Section 7.9 of tom Dieck's textbook [180] is a nice summary of compactly generated spaces (where these are called *whk*-spaces), and contains most of the material that we discuss here. Two influential – but unpublished – sources about compactly generated spaces are the Appendix A of Gaunce Lewis's thesis [96] and Neil Strickland's preprint [167].

I want to emphasize that this appendix does not contain new mathematics and makes no claim to originality. I have decided to include it because I found it cumbersome to collect proofs of all the relevant properties of compactly generated spaces from the scattered literature. An additional complication stems from the fact that the basic references [89, 118, 161] all work in slightly different categories; so in the interest of a self-contained treatment, I felt obliged to fill in arguments where a reference confines itself to the statement that '(...) the proof is analogous to that of (...)'.

We fix some terminology. A topological space is *compact* if it is quasi-compact (i.e., every open cover has a finite subcover) and satisfies the Hausdorff separation property (i.e., every pair of distinct points can be separated by disjoint open subsets).

Definition A.1. Let X be a topological space.

- A subset A of X is *compactly closed* if for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image $f^{-1}(A)$ is closed in K.
- X is a *k-space* if every compactly closed subset is closed.
- X is *weak Hausdorff* if for every compact space K and every continuous map $f : K \longrightarrow X$ the image $f(K)$ is closed in X.
- X is a *compactly generated* space if it is a k-space and weak Hausdorff.

Every closed subset is also compactly closed. One can similarly define *compactly open* subsets of X by demanding that for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image is open in K. A subset is then compactly open if and only if its complement is compactly closed. Thus k-spaces can equivalently be defined by the property that all compactly open subsets are open.

We denote by **Spc** the category of topological spaces and continuous maps. We denote by **K** and **T** the full subcategories of **Spc** consisting of k-spaces and compactly generated spaces, respectively. So we have full embeddings

$$\mathbf{T} \subset \mathbf{K} \subset \mathbf{Spc} .$$

We will see below that the inclusion $\mathbf{K} \subset \mathbf{Spc}$ has a right adjoint 'Kelleyfication' $k : \mathbf{K} \longrightarrow \mathbf{Spc}$ and the inclusion $\mathbf{T} \subset \mathbf{K}$ has a left adjoint $w : \mathbf{T} \longrightarrow \mathbf{K}$.

We follow the terminology introduced by McCord in [118]. We warn the reader that the usage of the terms 'k-space' and 'compactly generated' is not consistent throughout the literature. For example, some authors define k-spaces by the property that a subset is closed if and only its intersection with every compact subspace is closed. For Hausdorff spaces, and more generally weak Hausdorff spaces, that definition agrees with the one in Definition A.1 because for such spaces the image of a compact space under a continuous map is automatically compact, by Proposition A.4 (v) below. Moreover, there are sources (for example [161]) that require a compactly generated space to be Hausdorff (as opposed to only weak Hausdorff); on the other hand, several recent references do not include the weak Hausdorff condition in 'compactly generated'.

We recall various useful properties of k-spaces, weak Hausdorff spaces and compactly generated spaces. We start by discussing k-spaces. A Hausdorff

topological space X is *locally compact* if the following condition holds: for every open subset U of X and every point $x \in U$, there is an open neighborhood V of x whose closure $\bar{V}$ is compact and contained in U. For example, every compact space is in particular locally compact. A topological space is *first countable* if every point has a countable basis of neighborhoods.

Because the product of two k-spaces need not be a k-space in the usual product topology, we need notation for two different kinds of product topologies. We shall denote by $X \times_0 Y$ the cartesian product of two spaces X and Y, endowed with the product topology (which makes it a categorical product in the category **Spc** of all topological spaces). So a basis of the topology $X \times_0 Y$ is given by products of open subsets in the two factors. We reserve the symbol $X \times Y$ for $k(X \times_0 Y)$, the Kelleyfication of the product topology, discussed in more detail below. This product $X \times Y$ is a categorical product in the categories **K** and **T**. Part (vi) of the following proposition says that these two product topologies coincide if one factor is a k-space and the other factor is locally compact Hausdorff.

Proposition A.2. (i) *Every quotient space of a k-space is a k-space.*
(ii) *Every closed subset of a k-space is a k-space in the subspace topology.*
(iii) *Every locally compact Hausdorff space, and hence every compact space, is a k-space.*
(iv) *Every first countable space is a k-space.*
(v) *Every metric space is first countable, and hence a k-space.*
(vi) *If X is a k-space and Y a locally compact Hausdorff space, then the product $X \times_0 Y$ is a k-space in the product topology.*

Proof (i) Let $p : X \longrightarrow Y$ be a quotient projection and B a compactly closed subset of Y. We claim that $p^{-1}(B)$ is compactly closed in X. Indeed, if $f : K \longrightarrow X$ is a continuous map from a compact space, then $pf : K \longrightarrow Y$ is continuous, so $f^{-1}(p^{-1}(B)) = (pf)^{-1}(B)$ is closed because B is compactly closed. Since $p^{-1}(B)$ is compactly closed and X is a k-space, the set $p^{-1}(B)$ is in fact closed in X. So B is closed in Y by definition of the quotient topology.

(ii) We let Y be a closed subset of a k-space X, and A a compactly closed subset of Y with respect to the subspace topology. We claim that then A is also compactly closed as a subset of X. Indeed, if $f : K \longrightarrow X$ is a continuous map from a compact space, then $L = f^{-1}(Y)$ is closed in K, and hence compact. The restriction $f|_L : L \longrightarrow Y$ is then continuous and so $(f|_L)^{-1}(A) = f^{-1}(A)$ is closed in L because A was assumed to be compactly closed in Y. Since L is closed in K, the set $f^{-1}(A)$ is closed in K. This shows that A is compactly closed as a subset of X. Since X is a k-space, A is closed in X. But then A is also closed in Y in the subspace topology, so this concludes the proof that Y is a k-space.

(iii) The argument goes all the way back to Kelley [89, Ch. 7, Thm. 13]. We let A be a compactly closed subset of a locally compact Hausdorff space X. We let $\bar{A}$ be the closure of A in X and $x \in \bar{A}$. Since X is locally compact, the point x has a compact neighborhood K. Since A is compactly closed and the inclusion $K \longrightarrow X$ is continuous, the set $K \cap A$ is closed inside K. Since K is compact and X is Hausdorff, K is closed in X. So $K \cap A$ is closed in X.

Now we claim that $x \in K \cap A$. We argue by contraction and suppose that $x \notin K \cap A$. Then $X - (K \cap A)$ is an open neighborhood of x, and hence $K \cap (X - (K \cap A)) = K \cap (X - A)$ is another neighborhood of x. Let U be an open subset of X with $x \in U \subset K \cap (X - A)$. Then $A \subset X - U$, and hence $x \in \bar{A} \subset X - U$ because $X - U$ is closed. But his contradicts the hypothesis $x \in U$. Altogether this proves the claim that $x \in K \cap A \subset A$. Hence $A = \bar{A}$, and so A is closed.

(iv) This, too, goes back to Kelley [89, Ch. 7, Thm. 13]. We let X be a first countable space and A a compactly closed subset of X. We let $z \in \bar{A}$ be a point in the closure of A. The point z has a countable basis of open neighborhoods $\{U_n\}_{n \geq 1}$, which we can moreover take to be nested, i.e.,

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots .$$

If the intersection of U_n and A were empty, then $\bar{A} \subset X - U_n$ which contradicts the fact that $z \in \bar{A} \cap U_n$. So for every $n \geq 1$ there is a point $x_n \in U_n \cap A$. We define a map

$$g : K = \{0\} \cup \{1/n \mid n \geq 1\} \longrightarrow X$$

by $g(0) = z$ and $g(1/n) = x_n$. The hypotheses imply that the map g is continuous if we give the source K the subspace topology of the interval $[0, 1]$. In this topology the space K is compact, so $g^{-1}(A)$ is closed since A was assumed to be compactly closed. On the other hand, all the points $1/n$ are contained in $g^{-1}(A)$, and the closure of the set of these point is all of K. So $0 \in g^{-1}(A)$, which means that $z = g(0) \in A$. So A coincides with its closure, i.e., A is closed in X.

(v) In a metric space the balls of radius $1/n$ for all $n \geq 1$ form a countable neighborhood basis of a given point. So metric spaces are first countable, hence k-spaces by (iv).

(vi) This argument goes back to D. E. Cohen [37, 3.2] who thanks J. H. C. Whitehead for 'suggesting the subject (...) and for valuable help'. We let A be a compactly closed subset of $X \times_0 Y$. We let $(x_0, y_0) \in (X \times Y) - A$ be a point in the complement. We let

$$A_0 = \{y \in Y \mid (x_0, y) \in A\}$$

be the 'slice' of A through $x_0 \in X$. We let N be a compact neighborhood of y_0 in Y. Then $A_0 \cap N$ is closed in N because A is compactly closed. Since Y is

Hausdorff, N is closed in Y, and hence $A_0 \cap N$ is closed in Y. Since $y_0 \notin A_0$, the set $Y - A_0$ is an open neighborhood of y_0. Since Y is locally compact Hausdorff, there is a compact neighborhood K of y_0 with $K \subset Y - A_0$. We let

$$B \;=\; \{x \in X \mid (\{x\} \times K) \cap A \neq \emptyset\}$$

be the projection of $(X \times K) \cap A$ to X. The condition $K \subset Y - A_0$ is then equivalent to $x_0 \notin B$.

We let $f : C \longrightarrow X$ be a continuous map from a compact space. Then $C \times K$ is compact, and so $(f \times K)^{-1}((X \times K) \cap A)$ is closed in $C \times K$ since A is compactly closed. Hence $(f \times K)^{-1}((X \times K) \cap A)$ is compact in the subspace topology inherited from $C \times K$. Since

$$f^{-1}(B) \;=\; \{c \in C \mid (\{f(c)\} \times K) \cap A \neq \emptyset\}$$

is the projection of $(f \times K)^{-1}((X \times K) \cap A)$ onto C, the set $f^{-1}(B)$ is closed in C. Altogether this shows that the set B is compactly closed. Since X is a k-space, B must be closed in X. Since $x_0 \notin B$, the set $(X - B) \times K$ is a neighborhood of (x_0, y_0). Moreover, $(X - B) \times K$ is disjoint from A by definition of the set B. This shows that the complement of the original set A is open, hence A is closed. This completes the proof that $X \times_0 Y$ is a k-space in the product topology. □

The analog of Proposition A.2 (ii) is also true for *open* subsets, i.e., every open subset of a k-space is a k-space with respect to the subspace topology. We will not use this, so we refer the reader to [180, Prop. 7.9.10].

If X is any topological space we let kX be the space which has the same underlying set as X, but such that the closed subsets of kX are the compactly closed subsets of X. This indeed defines a topology which makes kX a k-space and such that the identity $\mathrm{Id} : kX \longrightarrow X$ is continuous. Moreover, any continuous map $Y \longrightarrow X$ whose source Y is a k-space is also continuous when viewed as a map to kX. In more fancy language, the assignment $X \mapsto kX$ extends to a functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ that is right adjoint to the inclusion of the full subcategory of k-spaces. Since the inclusion $\mathbf{K} \longrightarrow \mathbf{Spc}$ has a right adjoint, the category $\mathbf{K}$ of k-spaces has small limits and colimits. Colimits can be calculated in the ambient category of all topological spaces; equivalently, any colimit of k-spaces is again a k-space. To construct limits, we can first take a limit in the ambient category of all topological spaces; this ambient limit need not be a k-space, but applying the Kelleyfication functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ yields a limit in $\mathbf{K}$. Since k does not change the underlying set, the categories $\mathbf{K}$ and $\mathbf{Spc}$ share the property that the forgetful functor to sets preserves all limits and colimits. More loosely speaking, the underlying set of a limit or colimit in $\mathbf{K}$ is what one first thinks of.

The discussion about limits above applies in particular to products, and the

product of two k-spaces need *not* be a k-space in the usual product topology. As already mentioned, we denote by $X \times_0 Y$ the cartesian product of two spaces X and Y, endowed with the product topology (which makes it a categorical product in the category **Spc** of all topological spaces). We denote by $X \times Y = k(X \times_0 Y)$ the Kelleyfication of the product topology; if X and Y are k-spaces, then $X \times Y$ is a categorical product in the category **K**. Proposition A.2 (vi) shows that Kelleyfication is unnecessary if one of the factors is locally compact Hausdorff.

An important example where products in **K** and **Spc** can differ is the product of two 'sufficiently large' CW-complexes X and Y. Every CW-complex is a k-space, for example by [57, Prop. 1.2.1]. The product $X \times_0 Y$ with the usual product topology comes with a filtration $(X \times_0 Y)_{(n)} = \cup_{p+q=n} X_{(p)} \times_0 Y_{(q)}$, where $X_{(p)}$ is the p-skeleton of the CW-structure on X. If X or Y is locally compact, then the product topology is a k-space by Proposition A.2 (vi), and then the above filtration makes $X \times_0 Y$ a CW-complex, as was already noted by J. H. C. Whitehead [190, (H), p. 227]. In general, however, $X \times_0 Y$ may not be a k-space, and hence cannot have a CW-structure. The first example of this phenomenon was given by Dowker [43, III.5, p. 563], namely where X is a countable infinite wedge of circles, and Y is an uncountable infinite wedges of circles. This is also an example where the topology on $X \times Y = k(X \times_0 Y)$ is strictly finer than the product topology. The product in the category **K**, i.e., the space $X \times Y = k(X \times_0 Y)$, is always compactly generated and a CW-complex via the above filtration, see for example [71, Thm. A.6].

A surjective continuous map $p : X \longrightarrow Y$ is a *proclusion* if whenever $O \subset Y$ is such that $p^{-1}(O)$ is open in X, then O is already open in Y. So a proclusion is a continuous map that is homeomorphic, under X, to the projection onto a quotient space. The following proposition is one of the key properties of the category of k-spaces, and is stated without proof as Proposition 2.2 of [118]. The argument goes back, at least, to Steenrod, who states it in [161, Thm. 4.4] for Hausdorff k-spaces. The analog of the following proposition does *not* hold for general topological spaces in the usual product topology, and the examples in [180, Ex. 7.9.23] illustrate what can go wrong.

Proposition A.3. *Let X and Z be k-spaces and $p : X \longrightarrow Y$ a proclusion. Then the map $p \times Z : X \times Z \longrightarrow Y \times Z$ is a proclusion.*

Proof We adapt Steenrod's argument from [161, Thm. 4.4] to the more general context, i.e., for k-spaces without any additional separation hypothesis. We start with the special case when Z is compact. In that case $X \times Z = X \times_0 Z$ and $Y \times Z = Y \times_0 Z$ by Proposition A.2 (vi), i.e., the usual product topologies coincide with their Kelleyfications. In this formulation, the special case goes

back to J. H. C. Whitehead [189, Lemma 4], and the proof can also be found in many topology textbooks, for example [139, Ch. 8, Lemma 8.9].

Now we reduce the general case to the special case. We consider a subset $A \subset Y \times Z$ such that $(p \times Z)^{-1}(A)$ is closed in $X \times Z$. We let

$$f \;=\; (f_1, f_2) \;:\; K \;\longrightarrow\; Y \times Z$$

be a continuous map from a compact space. Then

$$(p \times K)^{-1}((Y \times f_2)^{-1}(A)) \;=\; (X \times f_2)^{-1}((p \times Z)^{-1}(A))$$

is closed in $X \times K$. The map $p \times K : X \times K \longrightarrow Y \times K$ is a proclusion by the special case, so the set $(Y \times f_2)^{-1}(A)$ is closed in $Y \times K$. So

$$f^{-1}(A) \;=\; (f_1, \mathrm{Id}_K)^{-1}((Y \times f_2)^{-1}(A))$$

is closed in K. This shows that A is compactly closed, and hence closed in the k-space topology of $Y \times Z$. □

Now we turn to weak Hausdorff spaces, and collect various useful properties in the following proposition.

Proposition A.4. (i) *If $i : A \longrightarrow X$ is a continuous injection and X is weak Hausdorff, then A is also weak Hausdorff. In particular, every subspace of a weak Hausdorff space is again weak Hausdorff.*

(ii) *Every Hausdorff space is also weak Hausdorff.*

(iii) *Every finite subset of a weak Hausdorff space is closed.*

(iv) *Every continuous bijection from a compact space to weak Hausdorff space is a homeomorphism.*

(v) *Let $f : K \longrightarrow X$ be a continuous map from a compact space to a weak Hausdorff space. Then the image $f(K)$ is compact in the subspace topology.*

(vi) *Every disjoint union of weak Hausdorff spaces is weak Hausdorff.*

(vii) *Every limit in* **Spc** *of a functor with values in weak Hausdorff spaces is weak Hausdorff.*

(viii) *If X is a weak Hausdorff space, then its Kelleyfication kX is again weak Hausdorff.*

Proof (i) Let $f : K \longrightarrow A$ be a continuous map from a compact space. Then $if : K \longrightarrow X$ is also continuous, hence $(if)(K)$ is closed in X. Since i is injective we have $f(K) = i^{-1}((if)(K))$, which is thus closed in A. This shows that A is a weak Hausdorff space.

(ii) If K is compact and $f : K \longrightarrow X$ continuous, then the image $f(K)$ is always quasi-compact; if X is Hausdorff, then any quasi-compact subset such as $f(K)$ is closed. So X is weak Hausdorff.

(iii) Every one point space is compact, so every point of any space is the continuous image of a compact space. So in weak Hausdorff spaces, all points and thus all finite subsets are closed.

(iv) Let $f : X \longrightarrow Y$ be a continuous bijection from a compact space to a weak Hausdorff space. Every closed subset A of X is compact in the subspace topology, so $f(A)$ is closed in Y by the weak Hausdorff property. This shows that f is also a closed map, hence a homeomorphism.

(v) The property of being quasi-compact is automatically inherited under continuous surjections, so the main issue is the Hausdorff property of $f(K)$, for which we reproduce the argument from [118, Lemma 2.1]. We let $x, y \in f(K)$ be two distinct points. The sets $\{x\}$ and $\{y\}$ are closed in X by part (iii), so $f^{-1}(x)$ and $f^{-1}(y)$ are disjoint closed subsets of K. Since compact spaces are normal, there are disjoint open subsets U and V of K with $f^{-1}(x) \subset U$ and $f^{-1}(y) \subset V$. Since X is weak Hausdorff, the sets $f(K - U)$ and $f(K - V)$ are closed in X, and hence also in $f(K)$. But then $f(K) - f(K - U)$ and $f(K) - f(K - V)$ are disjoint open subsets of $f(K)$ that separate x and y.

(vi) We let $\{Y_i\}_{i \in I}$ be a family of weak Hausdorff spaces and $f : K \longrightarrow \coprod_{i \in I} Y_i$ a continuous map from a compact space to the disjoint union. Since K is the disjoint union of the closed subspaces $K_i = f^{-1}(Y_i)$, each K_i is compact in the subspace topology. The image of the restriction $f|_{K_i} : K_i \longrightarrow Y_i$ is closed since Y_i is weak Hausdorff. So $f(K) = \coprod_{i \in I} f(K_i)$ is closed in the disjoint union.

(vii) In a first step we show that a product of any family $\{Y_i\}_{i \in I}$ of weak Hausdorff spaces is weak Hausdorff. We let $f = (f_i)_{i \in I} : K \longrightarrow \prod^0_{i \in I} Y_i$ be a continuous map from a compact space, with respect to the product topology on the target. Because Y_i is weak Hausdorff, the subset $f_i(K)$ is closed in Y_i, and a Hausdorff space in the subspace topology by part (v). So the subset $\prod_{i \in I} f_i(K)$ is closed in $\prod^0_{i \in I} Y_i$ and Hausdorff in the subspace topology. Since K is compact and the image of f is contained in the Hausdorff space $\prod_{i \in I} f_i(K)$, the image $f(K)$ is closed in the subspace topology of $\prod_{i \in I} f_i(K)$. Since $\prod_{i \in I} f_i(K)$ is closed in $\prod^0_{i \in I} Y_i$, the image $f(K)$ is also closed in $\prod^0_{i \in I} Y_i$. This proves that the product $\prod^0_{i \in I} Y_i$ is weak Hausdorff.

A limit in **Spc** of a functor $F : I \longrightarrow \mathbf{Spc}$ is a subspace of the product of the values $F(i)$ for all object $i \in I$. Since a subspace of a weak Hausdorff space is weak Hausdorff by part (i), this proves the claim.

(viii) The identity is a continuous injection $kX \longrightarrow X$, so part (i) proves the claim. □

Now we turn to compactly generated spaces. By combining Propositions A.2 and A.4, we get some immediate corollaries:

Proposition A.5. (i) *Every closed subset of a compactly generated space is compactly generated in the subspace topology.*

(ii) *Every locally compact Hausdorff space, and hence every compact space, is compactly generated.*

(iii) *Every metric space is compactly generated.*

(iv) *Every disjoint union of compactly generated spaces is compactly generated.*

(v) *If X is compactly generated and Y locally compact Hausdorff, then $X \times_0 Y$ is compactly generated in the product topology.*

There is a useful criterion, due to McCord [118, Prop. 2.3], for when a k-space is weak Hausdorff (and hence compactly generated).

Proposition A.6. *A k-space X is weak Hausdorff if and only if the diagonal is closed in $X \times X = k(X \times_0 X)$.*

Proof We suppose first that the diagonal is closed in $X \times X$. We let $f : K \longrightarrow X$ be a continuous map from a compact space. Then the map

$$f \times X \;:\; K \times X \;\longrightarrow\; X \times X$$

is continuous, and so $(f \times X)^{-1}(\Delta_X)$ is closed in $K \times X$. Since K is compact, the topology on $K \times X$ is the usual product topology by Proposition A.2 (vi). Moreover, the projection $p : K \times X \longrightarrow X$ away from K is a closed map, also by compactness. Hence

$$f(K) \;=\; p((f \times X)^{-1})(\Delta_X)$$

is closed in X. This shows that X is weak Hausdorff.

For the converse we assume that X is weak Hausdorff. We show that the diagonal Δ_X is compactly closed in $X \times_0 X$, and hence closed in $X \times X$. We let $f = (f_1, f_2) : K \longrightarrow X \times_0 X$ be a continuous map from a compact space. Then the map

$$f_1 + f_2 \;:\; K \amalg K \;\longrightarrow\; X$$

is continuous. Since X is weak Hausdorff, the subset $A = (f_1 + f_2)(K \amalg K) = f_1(K) \cup f_2(K)$ of X is closed and a Hausdorff space in the subspace topology, by Proposition A.4 (v). So the diagonal Δ_A is closed in $A \times_0 A$. Since moreover, $f(K)$ is contained in $A \times_0 A$, the set $f^{-1}(\Delta_X) = f^{-1}(\Delta_A)$ is closed in K. This completes the proof that Δ_X is compactly closed. □

Proposition A.6 now leads to the following useful criterion for when a quotient space of a k-space is weak Hausdorff (and hence compactly generated).

Proposition A.7. *Let X be a k-space and $E \subset X \times X$ an equivalence relation. Then the quotient space X/E is compactly generated if and only if E is closed in the k-topology of $X \times X$.*

Proof We let $p : X \longrightarrow X/E$ denote the projection. Any quotient space of a k-space is automatically a k-space by Proposition A.2 (i). If X/E is also weak Hausdorff, then $\Delta_{X/E}$ is closed in $X/E \times X/E$ by Proposition A.6. Hence

$$E \;=\; (p \times p)^{-1}(\Delta_{X/E}) \tag{A.8}$$

is closed in $X \times X$. Conversely, suppose that E is closed in $X \times X$. Since X and X/E are k-spaces, the map

$$p \times p \;:\; X \times X \;\longrightarrow\; (X/E) \times (X/E)$$

is a proclusion by two applications of Proposition A.3. Since E is closed by hypothesis, the relation (A.8) shows that $\Delta_{X/E}$ is closed in $(X/E) \times (X/E)$. So X/E is weak Hausdorff by Proposition A.6. □

Corollary A.9. *Let X be a compactly generated space and A a closed subset of X. Then the quotient topology on X/A is again compactly generated.*

Proof Since A is closed in X, $A \times A$ is closed in $X \times X$. Since X is weak Hausdorff, the diagonal Δ_X is closed $X \times X$. So the equivalence relation $E = (A \times A) \cup \Delta_X$ is closed in $X \times X$, and the quotient space X/A is compactly generated by Proposition A.7. □

Proposition A.7 also suggests how to construct a 'maximal weak Hausdorff quotient' of a k-space:

Proposition A.10. *Let X be a k-space. Let $E_{\min} \subset X \times X$ be the intersection of all equivalence relations on X that are closed in the k-topology of $X \times X$. Then $X/E_{\min}$ with the quotient topology is a compactly generated space and the quotient map $X \longrightarrow X/E_{\min}$ is initial among continuous maps from X to a compactly generated space.*

Proof The intersection $E_{\min}$ is again an equivalence relation, and $E_{\min}$ is closed in $X \times X$ as an intersection of closed subsets; so the quotient topology is weak Hausdorff by Proposition A.7.

Now we let $f : X \longrightarrow Y$ be a continuous map to a compactly generated space. The equivalence relation

$$E \;=\; \{(x, x') \in X \times X \mid f(x) = f(x')\}$$

is the preimage of the diagonal under the continuous map

$$f \times f \;:\; X \times X \;\longrightarrow\; Y \times Y\,.$$

Since the diagonal is closed in $Y \times Y$ by Proposition A.6, the set E is closed in $X \times X$, and so $E_{\min} \subseteq E$. So f factors uniquely over a continuous map from $X/E_{\min}$, by the universal property of the quotient topology. □

The previous proposition implies that the assignment

$$X \longmapsto X/E_{\min} = w(X)$$

extends canonically to a functor

$$w : \mathbf{K} \longrightarrow \mathbf{T}$$

that is left adjoint to the inclusion of compactly generated spaces into k-spaces. Moreover, if X is already compactly generated, then the diagonal is closed in $X \times X$ by Proposition A.6; every equivalence relation contains the diagonal, so $E_{\min} = \Delta_X$ whenever X is compactly generated. In this situation the quotient map $X \longrightarrow X/E_{\min} = w(X)$ is a homeomorphism.

It follows formally from the existence of a left adjoint to the inclusion $\mathbf{K} \subset \mathbf{T}$ that the category of compactly generated spaces has small limits and colimits; limits can be calculated in the category $\mathbf{K}$ of k-spaces. To construct a colimit of a diagram in $\mathbf{T}$, we can first take a colimit in the category $\mathbf{K}$ of k-spaces (or equivalently in **Spc**); while a k-space, this colimit need not be weak Hausdorff, but applying the left adjoint $w : \mathbf{K} \longrightarrow \mathbf{T}$ yields a colimit in $\mathbf{T}$.

If the colimit, taken in the category $\mathbf{K}$ of k-spaces, of a diagram of compactly generated spaces is not already weak Hausdorff, then the minimal closed equivalence relation on it is strictly larger than the diagonal, so the 'maximal weak Hausdorff quotient' identifies at least two distinct points and thus changes the underlying set. So one has to be especially careful with general colimits in $\mathbf{T}$: unlike for **Spc** of $\mathbf{K}$, the forgetful functor from $\mathbf{T}$ to sets need not preserve colimits. More loosely speaking, the underlying set of colimit in $\mathbf{T}$ may be smaller than one first thinks.

The fact that colimits in $\mathbf{T}$ may be hard to identify may seem like a problem at first. However, the issue is largely irrelevant for purposes of homotopy theory because we don't expect to have homotopical control over arbitrary colimits anyhow. The colimits that we do care about turn out to be 'as expected'; in particular, for pushouts along closed embeddings (see Proposition A.13), filtered colimits along closed embeddings (see Proposition A.14), wedges (see Proposition A.18), and orbits by actions of compact topological groups (see Proposition B.13), it makes no difference if we calculate the colimit in the category $\mathbf{T}$ or in $\mathbf{K}$ (or equivalently in **Spc**).

We call a continuous map $i : A \longrightarrow X$ between topological spaces a *closed embedding* if i is injective and a closed map; equivalently, the image $i(A)$ is

closed in X and i is a homeomorphism onto its image. By Proposition A.13 below, the cobase change, in **Spc**, **K** or **T**, of a closed embedding is again a closed embedding.

There is an ambiguity with the meaning of 'embedding' in general, due to the fact that a general subset of a k-space, endowed with the subspace topology, need not be a k-space, and so one may or may not want to apply 'Kelleyfication' $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ to the subspace topology. However, closed subsets of k-spaces are again k-spaces in the usual subspace topology, so there is no such ambiguity with the notion of 'closed embedding'.

Proposition A.11. *Let $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ be closed embeddings between topological spaces. Then the product maps $i \times_0 j : A \times_0 B \longrightarrow X \times_0 Y$ and $i \times j : A \times B \longrightarrow X \times Y$ are closed embeddings.*

Proof The map $i \times j$ is clearly injective, so must show that it is a closed map with respect to the product topologies, and with respect to their Kelleyfications. We may assume that i is the inclusion of a closed subset $A \subset X$ and j is the inclusion of a closed subset $B \subset Y$. The subspace topology on $A \times B$ induced from $X \times_0 Y$ agrees with the product topology $A \times_0 B$, which is the claim for $i \times_0 j$, i.e., the usual product topologies.

For the second claim we must show that every compactly closed subset C of $A \times_0 B$ is also compactly closed in $X \times_0 Y$. So we let $f : K \longrightarrow X \times_0 Y$ be a continuous map from a compact space. Since $A \times_0 B$ is closed inside $X \times_0 Y$, the subset $L = f^{-1}(A \times_0 B)$ is closed in K, and hence compact in the subspace topology. The relation

$$f^{-1}(C) \ = \ (f|_L)^{-1}(C)$$

and the hypothesis that C is compactly closed inside $A \times_0 B$ then show that $f^{-1}(C)$ is closed in K. So C is compactly closed in $X \times_0 Y$. □

This following proposition is [96, Lemma 8.1].

Proposition A.12. *Let $i : X \longrightarrow Y$ be a continuous map between compactly generated spaces that admits a continuous retraction. Then i is a closed embedding.*

Proof Let $r : Y \longrightarrow X$ be a continuous retraction, i.e., such that $ri = \mathrm{Id}_X$. Then the map

$$(\mathrm{Id}_Y, ir) \ : \ Y \ \longrightarrow \ Y \times Y$$

is continuous and for every subset $A \subset X$ we have

$$i(A) \ = \ \{y \in Y \mid y = i(r(y)) \text{ and } r(y) \in A\} \ = \ (\mathrm{Id}_Y, ri)^{-1}(\Delta_Y) \ \cap \ r^{-1}(A)\ .$$

Since Y is compactly generated, the diagonal Δ_Y is closed in $Y \times Y$ by Proposition A.6; hence the set $(\mathrm{Id}_Y, ri)^{-1}(\Delta_Y)$ is closed in Y. If A is closed, then so is $r^{-1}(A)$, and hence also $i(A)$. This proves that i is a closed embedding. □

Proposition A.13. *Given a pushout square in the category* **Spc** *of topological spaces*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Z \arrow[d, "j"] \\ Y \arrow[r, "g"'] & P \end{tikzcd}$$

such that i is a closed embedding, then j is also a closed embedding. If moreover Y and Z are compactly generated, then so is P, and hence the square is a pushout in **T**.

Proof We adapt the argument given in [118, Prop. 2.5]. The map j is injective because i is. Indeed, we can choose a set-theoretic retraction $r : Y \longrightarrow X$ to i (not necessarily continuous), and then $(fr) \cup \mathrm{Id} : P \longrightarrow Z$ is a set-theoretic retraction to j.

For the other claims we first treat the special case where the map g is a proclusion. We let $A \subset Z$ be a closed subset; then $f^{-1}(A)$ is closed in X. Since the map i is injective, the relation

$$g^{-1}(j(A)) \;=\; i(f^{-1}(A))$$

holds as subsets of Y. So $i(f^{-1}(A))$, and hence $g^{-1}(j(A))$, is closed in Y because i is a closed embedding. Since g is a proclusion, this shows that $j(A)$ is closed, and hence j is a closed map. Altogether this proves that j is a closed embedding.

Now we suppose that Y and Z are compactly generated. Then P is a k-space by Proposition A.2 (i) because $g : Y \longrightarrow P$ is a proclusion. Moreover, the diagonals of Y and Z are closed in $Y \times Y$ and $Z \times Z$, respectively, by Proposition A.6. Hence $(f \times f)^{-1}(\Delta_Z)$ is closed in $X \times X$. Since i is a closed embedding, so is $i \times i : X \times X \longrightarrow Y \times Y$, by Proposition A.11. Because

$$(g \times g)^{-1}(\Delta_P) \;=\; \Delta_Y \cup (i \times i)\Big((f \times f)^{-1}(\Delta_Z)\Big)$$

we conclude that $(g \times g)^{-1}(\Delta_P)$ is closed in $Y \times Y$. Since g is a proclusion, so is $g \times g$, by two applications of Proposition A.3. So Δ_P is closed in $P \times P$, and P is weak Hausdorff by the criterion of Proposition A.6.

Now we treat the general case. The pushout P can be constructed as a quotient space of the disjoint union $Y \amalg Z$, which means that the lower horizontal

map in the commutative square

$$\begin{tikzcd}
X \amalg Z \arrow[r, "f+\mathrm{Id}"] \arrow[d, "i\amalg\mathrm{Id}"'] & Z \arrow[d, "j"] \\
Y \amalg Z \arrow[r, "g+j"'] & P
\end{tikzcd}$$

is a proclusion. Since the original square is a pushout, so is this new square. The spaces $Y \amalg Z$ and Z are compactly generated by hypothesis and Proposition A.5 (iv), and the left vertical map is a closed embedding. So we can apply the special case above to this new square, and conclude that P is compactly generated. □

The following example of Lewis [96, App. A, p.168] shows that a pushout of compactly generated spaces calculated in the ambient category **Spc** need not be compactly generated, and that a cobase change in **T** of a continuous injection need not be injective. We consider the diagram

$$\{-1, 1\} \longleftarrow [-1, 0) \cup (0, 1] \xrightarrow{\text{inclusion}} [-1, 1]$$

where all three spaces have the subspace topology of $\mathbb{R}$, and the left map takes $[-1, 0)$ to -1 and it takes $(0, 1]$ to 1. All three spaces are compactly generated, and the pushout P in the categories **Spc** or **K** has three points, only one of which is closed. Any continuous map from P to a weak Hausdorff space must be constant, so the space $w(P)$ (which is a pushout in **T**) has only one point.

We recall that a poset $(P, \leq)$ is *filtered* if for all elements $i, j \in P$ there is a $k \in P$ such that $i \leq k$ and $j \leq k$. The associated category has object set P and a unique morphism $(i, j) : i \longrightarrow j$ for every pair of elements such that $i \leq j$. The filtered poset we mostly care about is $(\mathbb{N}, \leq)$, the set of natural numbers under it usual ordering; a functor $F : \mathbb{N} \longrightarrow C$ from the associated poset category is determined by the sequence of morphisms

$$F(0) \xrightarrow{F(0,1)} F(1) \xrightarrow{F(1,2)} F(2) \xrightarrow{F(2,3)} \ldots \xrightarrow{F(n-1,n)} F(n) \longrightarrow \ldots$$

So colimits of such functors are sequential colimits. Other filtered posets that come up are the set of finite subsets of an infinite set, and the set of finite-dimensional vector subspaces of an infinite-dimensional vector spaces, both ordered by inclusion.

Proposition A.14. *Let P be a filtered poset and $F : P \longrightarrow$ **Spc** a functor from the associated poset category to the category of topological spaces. Let F_∞ be a colimit of F in the category* **Spc** *with respect to continuous maps $\kappa_i : F(i) \longrightarrow F_\infty$.*

(i) *If the map $F(i, j) : F(i) \longrightarrow F(j)$ is a closed embedding for every comparable pair of elements of P, then all the maps $\kappa_i : F(i) \longrightarrow F_\infty$ are closed embeddings.*

(ii) *If the space $F(i)$ is compactly generated for every element $i \in P$ and the map $F(i, j) : F(i) \longrightarrow F(j)$ in injective for every comparable pair of elements, then the colimit F_∞ is compactly generated. Hence F_∞ is also a colimit of the functor F in the category* **T**.

Proof (i) Colimits in **Spc** are created on underlying sets, so the map $\kappa_i : F(i) \longrightarrow F_\infty$ is injective because all the maps $F(i, j)$ are injective. Now we let $A \subset F(i)$ be a closed subset. If $j \in P$ is another element, we can choose $k \in P$ such that $i \leq k$ and $j \leq k$. Then

$$\kappa_k^{-1}(\kappa_k(F(i,k)(A))) \; = \; F(i,k)(A)$$

because the map κ_k is injective. Hence

$$\kappa_j^{-1}(\kappa_i(A)) \; = \; F(j,k)^{-1}(\kappa_k^{-1}(\kappa_k(F(i,k)(A)))) \; = \; F(j,k)^{-1}(F(i,k)(A)) \, .$$

Since $F(i,k)$ is a closed map, this shows that $\kappa_j^{-1}(\kappa_i(A))$ is closed in $F(j)$ for all $j \in P$. The map $\coprod_{j \in P} F(j) \longrightarrow F_\infty$ given by κ_j on $F(j)$ is a proclusion, so we conclude that $\kappa_i(A)$ is closed in F_∞. So κ_i is a closed embedding.

(ii) The map $\kappa : \coprod_{j \in P} F(j) \longrightarrow F_\infty$ given by κ_j on $F(j)$ is a proclusion, and the source is compactly generated by Proposition A.5 (iv). We show that the equivalence relation

$$E \; = \; (\kappa \times \kappa)^{-1}(\Delta_{F_\infty}) \; \subset \; \left(\amalg_{j \in P} F(j)\right) \times \left(\amalg_{j \in P} F(j)\right)$$

that gives rise to F_∞ is closed; the criterion of Proposition A.7 then shows that F_∞ is compactly generated. The product of $\coprod_{j \in P} F(j)$ with itself is the disjoint union of the subspaces $F(i) \times F(j)$ for all $i, j \in P$, so it suffices to show that the intersection of E with each $F(i) \times F(j)$ is closed. We choose $k \in P$ such that $i \leq k$ and $j \leq k$. Then

$$E \cap (F(i) \times F(j)) \; = \; (F(i,k) \times F(j,k))^{-1}(\Delta_{F(k)})$$

because the map $\kappa_k : F(k) \longrightarrow F_\infty$ is injective. The diagonal is closed in $F(k) \times F(k)$ since $F(k)$ is compactly generated (by Proposition A.7). So $E \cap (F(i) \times F(j))$ is closed in $F(i) \times F(j)$. This verifies the criterion of Proposition A.7 and concludes the proof that the colimit F_∞ is compactly generated. □

Many functors $F : P \longrightarrow \mathbf{T}$ out of filtered posets that arise in practice have the property that a continuous map $f : K \longrightarrow F_\infty$ from a compact space factors through $F(i)$ for some $i \in P$. Such filtered colimits then tend to preserve weak equivalences. We now formulate a precise version of this property for certain

lattices. The most common special case is the poset $(\mathbb{N}, \leq)$ of natural numbers under the usual linear ordering.

We recall that a poset $(P, \leq)$ is a *lattice* if every pair of elements $p, q \in P$ has a join $p \vee q$ (least upper bound) and a meet $p \wedge q$ (greatest lower bound). Every lattice is in particular a filtered poset.

Proposition A.15. *Let $(P, \leq)$ be a lattice with the following property: for every element $q \in P$ and every countable chain*

$$p_1 \ \leq \ p_2 \ \leq \ \ldots \ \leq \ p_n \ \leq \ \ldots$$

in P with $p_n \leq q$ for all $n \geq 1$, the sequence is eventually constant.

(i) *Let $F : P \longrightarrow \mathbf{T}$ be a functor to the category of compactly generated spaces with the following properties:*

  (a) *the map $F(i, j) : F(i) \longrightarrow F(j)$ is a closed embedding for all comparable pairs $i \leq j$ of elements of P, and*

  (b) *for all elements $p, q \in P$ the following square is a pullback:*

$$\begin{tikzcd} F(p \wedge q) \arrow[r] \arrow[d] & F(q) \arrow[d] \\ F(p) \arrow[r] & F(p \vee q) \end{tikzcd}$$

Let F_∞ be a colimit of F with respect to the continuous maps $\kappa_i : F(i) \longrightarrow F_\infty$. Then for every continuous map $\alpha : K \longrightarrow F_\infty$ from a compact space K there exists an element $i \in P$ and a continuous map $\alpha' : K \longrightarrow F(i)$ such that $\alpha = \kappa_i \circ \alpha'$.

(ii) *Let $F' : P \longrightarrow \mathbf{T}$ be another functor satisfying conditions* (a) *and* (b) *and $\psi : F \longrightarrow F'$ a natural transformation. Suppose that the map $\psi(i) : F(i) \longrightarrow F'(i)$ is m-connected for all $i \in P$, for some $m \geq 0$. Then the map $\psi_\infty : F_\infty \longrightarrow F'_\infty$ induced on colimits is m-connected.*

(iii) *Let $F : P \longrightarrow \mathbf{T}$ satisfy conditions* (a) *and* (b) *and suppose that the map $F(i, j) : F(i) \longrightarrow F(j)$ is m-connected for all pairs of comparable elements. Then the map $\kappa_i : F(i) \longrightarrow F_\infty$ is m-connected for every $i \in P$.*

Proof (i) The canonical maps $\kappa_i : F(i) \longrightarrow F_\infty$ are closed embeddings by Proposition A.14 (i). Hence we can, and will, pretend that $F(i)$ is a closed subspace of F_∞ and the maps $F(i, j) : F(i) \longrightarrow F(j)$ are inclusions.

The image of the continuous map $f : K \longrightarrow F_\infty$ is compact in the subspace topology by Proposition A.4 (v). So we may replace K by its image and pretend that f is the inclusion of a compact subspace of F_∞. We then have to show that $K \subset F(i)$ for some $i \in P$.

We argue by contradiction and assume that K is not contained in $F(i)$ for any $i \in P$. We can then choose comparable elements

$$p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$$

of P and elements $x_i \in K$ with $x_i \in F(p_i)\backslash F(p_{i-1})$. Indeed, we start with any choice $x_1 \in F(p_1)$ and continue inductively: since K is not contained in $F(p_n)$, there is an element $x_{n+1} \in K\backslash F(p_n)$. We must have $x_{n+1} \in F(q)$ for some $q \in P$, and then $p_{n+1} = p_n \vee q$ can serve for the inductive step.

We set $C = \{x_1, x_2, x_3, \ldots\}$, a countably infinite subset of K. Now we claim that for every $q \in P$ the intersection $C \cap F(q)$ is finite. If this were not the case, then there would be infinitely many indices $i_1 < i_2 < i_3 < \ldots$ such that $x_{i_j} \in F(q)$ for all $j \geq 1$. The commutative square

$$\begin{tikzcd}
F(p_{i_j} \wedge q) \arrow[r] \arrow[d] & F(q) \arrow[d] \\
F(p_{i_j}) \arrow[r] & F(p_{i_j} \vee q)
\end{tikzcd}$$

is a pullback by hypothesis, and so $x_{i_j} \in F(p_{i_j} \wedge q)$ for all $j \geq 1$. Also by hypothesis there is an $N \geq 1$ such that

$$p_N \wedge q = p_{N+1} \wedge q = p_{N+2} \wedge q = \cdots .$$

So $F(p_N \wedge q)$, and hence also $F(p_N)$, contains infinitely many of the elements of the set C. This contradicts the construction of C, and we can conclude that the intersection $C \cap F(q)$ is finite.

Now we let T be any subset of C. Since $T \cap F(q)$ is finite, it is a closed subset of $F(q)$ by Proposition A.4 (iii). The map

$$\coprod \kappa_q \ : \ \coprod_{q \in P} F(q) \longrightarrow F_\infty$$

is a proclusion, i.e., the space F_∞ carries the quotient topology. Since $T \cap F(q)$ is closed for all $q \in P$, the set T is closed in F_∞. Since T was any subset of C, the subset C is discrete in the subspace topology from F_∞. So altogether C is an infinite discrete subset of the compact space K, which is a contradiction. This proves that K is contained in $F(i)$ for some $i \in P$.

(ii) We consider a commutative square of continuous maps for $k \leq m$:

$$\begin{tikzcd}
\partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F_\infty \arrow[d, "\psi_\infty"] \\
D^k \arrow[r, "\beta"'] & F'_\infty
\end{tikzcd}$$

Part (i) provides $i, j \in P$ and continuous maps $\alpha' : \partial D^k \longrightarrow F(i)$ and β' :

$D^k \longrightarrow F(j)$ such that $\kappa_i \circ \alpha' = \alpha$ and $\kappa'_j \circ \beta' = \beta$. By replacing i and j by their join we may suppose that $i = j$. Then

$$\kappa'_i \circ \psi(i) \circ \alpha' \;=\; \psi_\infty \circ \kappa_i \circ \alpha' \;=\; \psi_\infty \circ \alpha \;=\; \beta|_{\partial D^k} \;=\; \kappa'_i \circ \beta'|_{\partial D^k} \ .$$

Since κ'_i is injective this shows that $\psi(i) \circ \alpha' = \beta'|_{\partial D^k}$. Since $\psi(i)$ is m-connected, there is a continuous map $\lambda : D^k \longrightarrow F(i)$ that restricts to α' on ∂D^k and such that $\psi(i) \circ \lambda$ is homotopic, relative ∂D^k, to β'. Then $\kappa_i \circ \lambda : D^k \longrightarrow F_\infty$ solves the original lifting problem. So the map F_∞ is m-connected.

(iii) We consider a commutative square of continuous maps for $k \leq m$:

$$\begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F(i) \arrow[d, "\kappa_i"] \\ D^k \arrow[r, "\beta"'] & F_\infty \end{tikzcd}$$

Part (i) provides a $j \in P$ and a continuous map $\beta' : D^k \longrightarrow F(j)$ such that $\kappa_j \circ \beta' = \beta$. By replacing j by the join $i \vee j$ and β' by the map $F(i, i \vee j) \circ \beta' : D^k \longrightarrow F(i \vee j)$, we can assume without loss of generality that $i \leq j$. Then

$$\kappa_j \circ F(i, j) \circ \alpha \;=\; \kappa_i \circ \alpha \;=\; \beta|_{\partial D^k} \;=\; \kappa_j \circ \beta'|_{\partial D^k} \ .$$

Since κ_j is injective this shows that $F(i, j) \circ \alpha = \beta'|_{\partial D^k}$. Since $F(i, j)$ is m-connected, there is a continuous map $\lambda : D^k \longrightarrow F(i)$ that restricts to α on ∂D^k and such that $F(i, j) \circ \lambda$ is homotopic, relative ∂D^k, to β'. Then λ also solves the original lifting problem. So the map κ_i is a m-connected. □

Example A.16. We let $(P, \leq)$ be a *linear order*, i.e., a poset in which every two elements are comparable. Such a poset is in particular a lattice. We observe that for every linear order, condition (b) of Proposition A.15 (i) is trivially satisfied for every functor. Indeed, any two elements $p, q \in P$ are comparable, and we suppose that $p \leq q$, the other case being analogous. Then $p \wedge q = p$ and $p \vee q = q$, and the two vertical maps in the commutative square of condition (b) are identity maps. Hence the square is a pullback.

The most familiar special case of Proposition A.15 is the poset $(\mathbb{N}, \leq)$ of natural numbers under the usual ordering. This poset is a linear order, so as we explained in Example A.16, condition (b) of Proposition A.15 (i) is automatically satisfied for every functor. For $(\mathbb{N}, \leq)$, part (i) of Proposition A.15 can be found in many topology textbooks, for example [80, Prop. 2.4.2]. For easier reference we spell out the content of part (ii) and (iii) of Proposition A.15 for the poset $(\mathbb{N}, \leq)$ and for $m = \infty$, i.e., for weak equivalences.

Proposition A.17.

(i) *Let* $e_n : X_n \longrightarrow X_{n+1}$ *and* $f_n : Y_n \longrightarrow Y_{n+1}$ *be closed embeddings between compactly generated spaces, for* $n \geq 0$. *Let* $\psi_n : X_n \longrightarrow Y_n$ *be weak equivalences that satisfy* $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ *for all* $n \geq 0$. *Then the induced map* $\psi_\infty : X_\infty \longrightarrow Y_\infty$ *between the colimits of the sequences is a weak equivalence.*

(ii) *Let* $f_n : Y_n \longrightarrow Y_{n+1}$ *be a closed embedding of compactly generated spaces that is also a weak equivalence, for* $n \geq 0$. *Then the canonical map* $Y_0 \longrightarrow Y_\infty$ *to the colimit of the sequence is a weak equivalence.*

Besides the linear order $(\mathbb{N}, \leq)$, further examples of relevant posets that satisfy the hypotheses of Proposition A.15 are the set of finite subsets of a given set (which features in the following proposition), and the set of finite-dimensional vector subspaces of a given vector space.

Proposition A.18. *Let* $\{X_i\}_{i\in I}$ *be a family of based compactly generated spaces. Then the wedge (one-point union)* $\bigvee_{i\in I} X_i$ *is compactly generated, and thus a coproduct in the category* $\mathbf{T}_*$ *of based compactly generated spaces. Moreover, for every compact space* K *and every continuous map* $f : K \longrightarrow \bigvee_{i\in I} X_i$ *there is a finite subset* J *of* I *such that* f *factors through the subwedge* $\bigvee_{j\in J} X_j$.

Proof The disjoint union $\coprod_{i\in I} X_i$ is compactly generated by Proposition A.5 (iv). Moreover, the equivalence relation that identifies all the basepoints is closed in $(\coprod_{i\in I} X_i) \times (\coprod_{i\in I} X_i)$. So the quotient space $\bigvee_{i\in I} X_i$ is compactly generated by Proposition A.7.

We let P be the poset of finite subsets of I, ordered by inclusion; this is a lattice satisfying the hypotheses of Proposition A.15. We consider the functor $F : P \longrightarrow \mathbf{T}$ sending a finite subset $J \subset I$ to the finite wedge $\bigvee_{j\in J} X_j$. For $J' \subset J \subset I$, the map $\bigvee_{j\in J'} X_j \longrightarrow \bigvee_{j\in J} X_j$ is a closed embedding (by direct inspection, or by Proposition A.12), and property (b) of Proposition A.15 (i) is satisfied. Proposition A.15 (i) thus provides the desired factorization through $\bigvee_{j\in J} X_j$ for some finite subset J of I. □

Now we let X and Y be compactly generated topological spaces equipped with basepoints $x_0 \in X$ and $y_0 \in Y$, respectively. We define the *smash product* as

$$X \wedge Y = X \times Y / (X \times \{y_0\} \cup \{x_0\} \times Y) ,$$

the quotient of the product (with the k-topology) by the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$. We write $x \wedge y$ for the class of $(x, y) \in X \times Y$ in the quotient space. Since points in compactly generated spaces are closed (Proposition A.4 (iii)), the map

$$X \vee Y \longrightarrow X \times Y$$

from the coproduct to the product in $\mathbf{T}_*$ is a closed embedding with image the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$; so $X \wedge Y$ is also a cokernel of this map.

The next proposition is another important reason for working in the category $\mathbf{T}$ of compactly generated spaces. Indeed, in the larger category **Spc** of all topological spaces, the smash product is not generally associative: if smash products are defined as quotients of the usual product topology, then the canonical set-theoretic bijection from $(X \wedge Y) \wedge Z$ to $X \wedge (Y \wedge Z)$ need not be continuous. An explicit example is mentioned without proof in [132, 5.8], namely that $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}$ and $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N})$ are not homeomorphic in **Spc**, where $\mathbb{Q}$ has the subspace topology of $\mathbb{R}$, and $\mathbb{N}$ has the discrete topology. A proof can be found in [116, Thm. 1.7.1].

Proposition A.19. *Let X, Y, and Z be based compactly generated spaces.*

(i) *The space $X \wedge Y$ is compactly generated in the quotient topology of $X \times Y$.*
(ii) *The maps*

$$X \wedge Y \longrightarrow Y \wedge X\,, \quad x \wedge y \longmapsto y \wedge x$$

and

$$(X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z)\,, \quad (x \wedge y) \wedge z \longmapsto x \wedge (y \wedge z)$$

are homeomorphisms.

Proof (i) In weak Hausdorff spaces all points are closed (Proposition A.4 (iii)), hence $X \times \{y_0\} \cup \{x_0\} \times Y$ is a closed subspace of $X \times Y$. So the quotient topology is compactly generated by Corollary A.9.

The first claim of part (ii) is straightforward from the fact that the twist homeomorphism $X \times Y \longrightarrow Y \times X$ sending (x, y) to (y, x) takes the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ to the subspace $Y \times \{x_0\} \cup \{y_0\} \times X$ and hence descends to a homeomorphism on quotient spaces.

Since the projection $p_{X,Y} : X \times Y \longrightarrow X \wedge Y$ is a proclusion, so is $p_{X,Y} \times Z : X \times Y \times Z \longrightarrow (X \wedge Y) \times Z$, by Proposition A.3. Hence the map

$$p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z) \;:\; X \times Y \times Z \longrightarrow (X \wedge Y) \wedge Z\,, \quad (x, y, z) \longmapsto (x \wedge y) \wedge z$$

is a proclusion as the composite of two proclusions. The composite

$$X \times Y \times Z \xrightarrow{p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z)} (X \wedge Y) \wedge Z \xrightarrow{(x \wedge y) \wedge z \mapsto x \wedge (y \wedge z)} X \wedge (Y \wedge Z)$$

is the map $p_{X, Y \wedge Z} \circ (X \times p_{Y,Z})$, and hence continuous. Since $p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z)$ is a proclusion, the second map $(X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z)$ is continuous. By the same reasoning the inverse map $X \wedge (Y \wedge Z) \longrightarrow (X \wedge Y) \wedge Z$ is continuous. Hence the two maps are mutually inverse homeomorphisms. □

Another key advantage of the categories **K** and **T** over the category of all topological spaces is that they are cartesian closed, i.e., categorical product with a fixed object is a left adjoint. The discovery of these facts was predated by similar result in closely related categories; for example, Brown [30, §3] discusses internal function spaces in the category of Hausdorff k-spaces, and this was popularized by Steenrod in [161, Sec. 5]. The first reference to cartesian closedness for **K** and **T** that I am aware of is the appendix of Lewis' thesis [96, App. A, Thm. 5.5]; we will closely follow Lewis' exposition. Internal function spaces in **K** and **T** are given by the set of all continuous maps endowed with the Kelleyfication of a slight modification of the compact-open topology. For weak Hausdorff spaces, the 'modified' compact-open topology actually coincides with the compact-open topology.

Construction A.20. For topological spaces X and Y, we let $C(X, Y)$ denote the set of continuous maps from X to Y. For a continuous map $h : K \longrightarrow X$ from a compact space K and an open subset U of Y we define a subset of $C(X, Y)$ by

$$N(h, U) \ = \ \{\varphi \in C(X, Y) \mid \varphi(h(K)) \subset U\} \ .$$

We endow the set $C(X, Y)$ of continuous maps with the topology generated by the subbasis consisting of all these sets $N(h, U)$. This topology is covariantly functorial in Y and contravariantly functorial in X. Indeed, for every continuous map $f : Y' \longrightarrow Y$ we have

$$C(X, f)^{-1}(N(h, U)) \ = \ N(h, f^{-1}(U)) \ ,$$

which is open in $C(X, Y')$, and so $C(X, f) : C(X, Y') \longrightarrow C(X, Y)$ is continuous. Similarly, for a continuous maps $g : X \longrightarrow X'$ and $h : K \longrightarrow X$, with K a compact space, we have

$$C(g, Y)^{-1}(N(h, U)) \ = \ N(gh, U) \ ,$$

which is open in $C(X', Y)$, and so $C(g, Y) : C(X', Y) \longrightarrow C(X, Y)$ is continuous.

Now we let X and Y be k-spaces. Then $C(X, Y)$ need not be a k-space. We fix this by Kelleyfication, i.e., we define the internal mapping space in **K** as

$$\mathrm{map}(X, Y) \ = \ kC(X, Y) \ .$$

Remark A.21 (Relation to the compact-open topology)**.** We recall that a subbasis for the compact-open topology on $C(X, Y)$ is given by all sets

$$W(K, U) \ = \ \{\varphi \in C(X, Y) \mid \varphi(K) \subset U\} \ ,$$

where K is compact subset of X and U is an open subset of Y. Because $W(K, U) = N(\mathrm{incl} : K \longrightarrow X, U)$, the topology generated by the sets $N(h, U)$ is finer than

the compact-open topology. In general, the compact-open topology may be strictly coarser.

For weak Hausdorff spaces X, the topology on $C(X,Y)$ coincides with the compact-open topology. Indeed, if X is weak Hausdorff, then the image of every continuous map $h : K \longrightarrow X$ is compact in the subspace topology (Proposition A.4 (v)). So in this case $N(h,U) = W(h(K),U)$, and the two subbases coincide.

We recall that a category is *cartesian closed* if it has finite products and product with a fix object is a left adjoint. The proof of the following theorem can be found in [96, App. A, Thm. 5.5].

Theorem A.22. *For all k-spaces X, Y and Z, the natural map*

$$\Psi \;:\; \mathrm{map}(X\times Y, Z) \;\longrightarrow\; \mathrm{map}(X, \mathrm{map}(Y,Z))\ , \quad \Psi(f)(x)(y) \;=\; f(x,y)$$

is a homeomorphism. In particular, the category of k-spaces is cartesian closed.

Proof We reproduce Lewis' arguments. We start by checking that the map

$$\eta_X \;:\; X \;\longrightarrow\; C(Y, X\times Y)\ , \quad \eta_X(x)(y) \;=\; (x,y)$$

is continuous. Since X is a k-space, it suffices to check that the composite $\eta_X \circ f$ is continuous for every continuous map $f : L \longrightarrow X$ from a compact space. Since $\eta_X \circ f = C(Y, f\times Y)\circ \eta_L$ and $C(Y, f\times Y)$ is continuous, we can assume without loss of generality that X is compact. If X is compact, then $X\times Y = X\times_0 Y$ by Proposition A.2 (vi), i.e., $X\times Y$ is already a k-space in the usual product topology.

We let $h : K \longrightarrow Y$ be a continuous map from a compact space and U an open subset of $X\times Y = X\times_0 Y$. Then

$$\eta_X^{-1}(N(h,U)) \;=\; \{x\in X \mid \{x\}\times K \subset (X\times h)^{-1}(U)\}\ .$$

For every $x \in \eta_X^{-1}(N(h,U))$, the 'tube lemma' (see for example [139, Ch. 8, Lemma 8.9']) provides an open neighborhood O of x in X such that $O\times K \subset (X\times h)^{-1}(U)$, i.e., such that $O \subset \eta_X^{-1}(N(h,U))$. So the set $\eta_X^{-1}(N(h,U))$ contains a neighborhood of each of its points, and is thus open. Altogether this shows that inverse images of the subbasis of the topology on $C(Y, X\times Y)$ are open; hence η_X is continuous. Since X is a k-space, the map η_X is also continuous with respect to the Kelleyfied topology on the target, i.e., when considered as a map to $\mathrm{map}(Y, X\times Y)$.

We denote the evaluation map by

$$\epsilon_Z \;:\; \mathrm{map}(Y,Z)\times Y \;=\; k(C(Y,Z)\times_0 Y) \;\longrightarrow\; Z\ , \quad \epsilon(f,y) \;=\; f(y)\ .$$

We show next that for every continuous map $h : K \longrightarrow Y$ from a compact

space the composite $\epsilon_Z \circ (C(Y,Z) \times_0 h) : C(Y,Z) \times_0 K \longrightarrow Z$ is continuous. We let U be an open subset of Z and consider a point $(\varphi, k) \in C(Y,Z) \times_0 K$ in $(\epsilon_Z \circ (C(Y,Z) \times_0 h))^{-1}(U)$; this simply means that $k \in (\varphi \circ h)^{-1}(U)$. Since K is compact and $(\varphi \circ h)^{-1}(U)$ is open in K, there is an open neighborhood V of k in K whose closure $\bar{V}$ is contained in $(\varphi \circ h)^{-1}(U)$. This closure $\bar{V}$ is compact in the subspace topology, so the set $N(h|_{\bar{V}} : \bar{V} \longrightarrow Y, U)$ is open in $C(Y,Z)$, and it contains the map φ. Hence $N(h|_{\bar{V}} : \bar{V} \longrightarrow Y, U) \times V$ is open in $C(Y,Z) \times_0 K$, and contains the point (φ, k). Since this open set is also contained in $(\epsilon_Z \circ (C(Y,Z) \times_0 h))^{-1}(U)$, this latter set is a neighborhood of each of its points, and hence open in $C(Y,Z) \times_0 K$. This completes the proof that the map $\epsilon_Z \circ (C(Y,Z) \times_0 h)$ is continuous.

Now we show that the evaluation map ϵ_Z itself is continuous. At this point the Kelleyfication of the source is crucial, as ϵ_Z need not be continuous with respect to the topology $C(Y,Z) \times_0 Y$. So it suffices to show that the composite $\epsilon_Z \circ (g, h) : K \longrightarrow Z$ is continuous for every continuous map $(g, h) : K \longrightarrow C(Y,Z) \times_0 Y$ from a compact space. The map $\epsilon_Z \circ (g, h)$ coincides with the composite

$$K \xrightarrow{(g,\mathrm{Id}_K)} C(Y,Z) \times_0 K \xrightarrow{C(Y,Z)\times_0 h} C(Y,Z) \times_0 Y \xrightarrow{\epsilon_Z} Z .$$

The composite of the last two maps is continuous by the previous paragraph, so the whole composite $\epsilon_Z \circ (g, h)$ is continuous.

At this point we know that the two maps

$$\eta_X : X \longrightarrow \mathrm{map}(Y, X \times Y) \quad \text{and} \quad \epsilon_Z : \mathrm{map}(Y,Z) \times Y \longrightarrow Z$$

are continuous. The rest of the argument is formal. Since $\epsilon_{X\times Y} \circ (\eta_X \times Y)$ is the identity of $X \times Y$ and $\mathrm{map}(Y, \epsilon_Z) \circ \eta_{\mathrm{map}(Y,Z)}$ is the identity of $\mathrm{map}(Y,Z)$, the natural transformations η and ϵ are the unit and counit of an adjunction $(- \times Y, \mathrm{map}(Y, -))$. In particular, the map

$$\Psi : C(X \times Y, Z) \longrightarrow C(X, \mathrm{map}(Y,Z)) , \quad \Psi(f)(x)(y) = f(x, y)$$

is bijective. We let T be another k-space. Replacing X by $T \times X$ shows that

$$\Psi : C(T \times X \times Y, Z) \longrightarrow C(T \times X, \mathrm{map}(Y,Z))$$

is bijective. The adjunction just established translates this into the fact that

$$C(T, \Psi) : C(T, \mathrm{map}(X \times Y, Z)) \longrightarrow C(T, \mathrm{map}(X, \mathrm{map}(Y,Z)))$$

is bijective. Since T is an arbitrary k-space, Ψ is a homeomorphism. □

Now we turn to compactly generated spaces.

Theorem A.23. (i) *If Y is a weak Hausdorff space and X is any topological*

space, then the spaces $C(X,Y)$ *and* $\mathrm{map}(X,Y)$ *are weak Hausdorff. In particular, if X and Y are compactly generated, then so is* $\mathrm{map}(X,Y)$.

(ii) *For all compactly generated spaces* X, Y *and* Z*, the natural map*

$$\Psi \;:\; \mathrm{map}(X\times Y, Z) \longrightarrow \mathrm{map}(X, \mathrm{map}(Y,Z))\,, \quad \Psi(f)(x)(y) \;=\; f(x,y)$$

is a homeomorphism. In particular, the category of compactly generated spaces is cartesian closed.

Proof (i) For every point $x \in X$ the evaluation map

$$\mathrm{ev}_x \;:\; C(X,Y) \longrightarrow Y\,, \quad f \longmapsto f(x)$$

is continuous. Indeed, for an open set U of Y we have $\mathrm{ev}_x^{-1}(U) = N(\mathrm{incl} : \{x\} \longrightarrow X, U)$ which is open because a one-point space is compact.

The evaluation map

$$C(X,Y) \longrightarrow \prod\nolimits^0_{x\in X} Y\,, \quad f \longmapsto \{f(x)\}_{x\in X}$$

is injective, and it is continuous by the above. Any product, with the usual product topology, of copies of Y is again weak Hausdorff by Proposition A.4 (vii). So $C(X,Y)$ is weak Hausdorff by Proposition A.4 (i). The space $\mathrm{map}(X,Y) = kC(X,Y)$ is then still weak Hausdorff by Proposition A.4 (viii), and hence compactly generated.

(ii) Part (i) shows that the category of compactly generated spaces is closed under the internal function objects in the category **K** of k-spaces. Since products in **T** are also formed in the ambient category **K**, part (ii) becomes a special case of Theorem A.22. □

Construction A.24 (Based mapping spaces)**.** Since we work a lot with *based* spaces, we record that the based version of the mapping space is right adjoint to the smash product in $\mathbf{T}_*$. In combination with the commutativity and associativity property of the smash product (Proposition A.19), and the natural homeomorphisms $X \wedge S^0 \cong X \cong S^0 \wedge X$, this shows that the smash product provides a closed symmetric monoidal structure on the category $\mathbf{T}_*$ of based compactly generated spaces.

The based adjunction is a straightforward consequence of the unbased version in Theorem A.23. We let $(X, x_0), (Y, y_0)$ and (Z, z_0) be based compactly generated spaces. We denote by $\mathrm{map}_*(Y,Z)$ the subspace of $\mathrm{map}(Y,Z)$ consisting of all based continuous maps f, i.e., such that $f(y_0) = z_0$; the basepoint is the constant map with value z_0. Since points are closed in compactly generated spaces and $\mathrm{map}_*(Y,Z)$ is the preimage of $\{z_0\}$ under the continuous evaluation map

$$\mathrm{map}(Y,Z) \longrightarrow Z\,, \quad f \longmapsto f(y_0)\,,$$

we see that $\mathrm{map}_*(Y,Z)$ is a closed subset of $\mathrm{map}(Y,Z)$. So $\mathrm{map}_*(Y,Z)$ is compactly generated in the subspace topology (Proposition A.5 (i)). The composite

$$X \xrightarrow{\eta_X} \mathrm{map}(Y, X \times Y) \xrightarrow{\mathrm{map}(Y,\mathrm{proj})} \mathrm{map}(Y, X \wedge Y)$$

takes values in the subspace $\mathrm{map}_*(Y, X \wedge Y)$, so it restricts to a continuous map

$$\eta'_X \ : \ X \longrightarrow \mathrm{map}_*(Y, X \wedge Y) ,$$

which is moreover based. The composite

$$\mathrm{map}_*(Y,Z) \times Y \xrightarrow{\mathrm{incl}\times Y} \mathrm{map}(Y,Z) \times Y \xrightarrow{\epsilon_Z} Z$$

takes the subspace $\{\mathrm{const}\} \times Y \cup \mathrm{map}_*(Y,Z) \times \{y_0\}$ to the basepoint of Z, so it factors over the quotient space though a continuous based map

$$\epsilon'_Z \ : \ \mathrm{map}_*(Y,Z) \wedge Y \longrightarrow Z .$$

By direct verification, the composites

$$\epsilon'_{X\wedge Y} \circ (\eta'_X \wedge Y) \quad \text{and} \quad \mathrm{map}(Y, \epsilon'_Z) \circ \eta'_{\mathrm{map}_*(Y,Z)}$$

are the identity of $X \wedge Y$ and $\mathrm{map}_*(Y,Z)$. So η' and ϵ' are the unit and counit of an adjunction for the pair of endofunctors $(- \wedge Y, \mathrm{map}_*(Y,-))$ on the category $\mathbf{T}_*$.

Now we specialize to the space of continuous homomorphisms between Lie groups, with compact source. If (Y,d) is a metric space and K a compact topological space, then the supremum metric

$$d(f,g) \ = \ \sup_{x\in K} \{d(f(x), g(x))\}$$

is a metric on the set $C(K,Y)$ of continuous maps. The topology induced by the supremum metric agrees with the compact-open topology, see for example [71, Prop. A.13], and hence also with the topology of Construction A.20. Since the topology on $C(K,Y)$ is metrizable, the space $C(X,Y)$ is in particular compactly generated, by Proposition A.5 (iii). Hence $\mathrm{map}(X,Y) = C(X,Y)$, with topology induced by the supremum metric.

The topology of every second countable smooth manifold is metrizable, so the previous discussion applies in particular when K is a compact Lie group and G is a Lie group. So in that case the compact-open topology on the space $C(K,G)$ of continuous maps is metrizable, and hence compactly generated. Hence $C(K,G) = \mathrm{map}(K,G)$, i.e., the compact-open topology is the topology of the internal mapping space in the category $\mathbf{T}$ of compactly generated spaces.

We let $\hom(K,G)$ denote the set of continuous homomorphisms with the subspace topology of $\mathrm{map}(K,G)$. Since $\mathrm{map}(K,G)$ is metrizable, so is its subspace $\hom(K,G)$, which is then compactly generated by Proposition A.5 (iii).

The next proposition in particular shows that the space $\hom(K, G)$ is the topological disjoint union of the orbits, under conjugation, of the identity component group G°. The key input is a theorem of Montgomery and Zippin [122, Thm. 1 and Corollary] from 1942 which roughly says that in a Lie group 'nearby compact subgroups are conjugate'. The following consequence of Montgomery and Zippin's result appears in [39, III, Lemma 38.1].

Proposition A.25. *Let K and G be Lie groups, and suppose that K is compact. Then every orbit of the conjugation action by G° on* $\hom(K, G)$ *is an open subset of the space* $\hom(K, G)$*. In particular, the connected components of the space* $\hom(K, G)$ *coincide with its path components, and with the G°-orbits under the conjugation action.*

Proof We reproduce the argument from [39, III, Lemma 38.1], in somewhat expanded form. We start by showing that the G°-orbits are open subsets of $\hom(K, G)$. Let $\alpha : K \longrightarrow G$ be a continuous homomorphism. Its graph $\Gamma_\alpha = \{(k, \alpha(k)) \mid k \in K\}$ is then a compact subgroup of the Lie group $K \times G$. By [122, Thm. 1 and Corollary], there is an open subset O of $K \times G$ containing Γ_α with the following property: for every closed subgroup Δ of $K \times G$ with $\Delta \subseteq O$ there is an element $(k, g) \in K^\circ \times G^\circ$ such that $(k, g)^{-1} \cdot \Delta \cdot (k, g) \subseteq \Gamma_\alpha$. We set

$$U \;=\; \{\beta \in \hom(K, G) \mid \Gamma_\beta \subseteq O\} \;;$$

then $\alpha \in U$, and U is contained in the G°-orbit of α because

$$(k, g)^{-1} \cdot \Gamma_\beta \cdot (k, g) \;=\; \Gamma_{c_g \circ c_{\beta(k)^{-1}} \circ \beta} \;.$$

We claim that U is an open subset of $\hom(K, G)$. To see this, we let $\beta : K \longrightarrow G$ be a continuous homomorphism such that $\Gamma_\beta \subset O$, and $k \in K$. Since O is open and $(k, \beta(k)) \in O$, there are open subsets $U_k \subset K$ and $V_k \subset G$ with

$$(k, \beta(k)) \;\subseteq\; U_k \times V_k \;\subseteq\; O \;.$$

Since β is continuous, the set $\beta^{-1}(V_k)$ is open in K, hence so is $U_k \cap \beta^{-1}(V_k)$; moreover, $k \in U_k \cap \beta^{-1}(V_k)$. Since K is compact, there is a compact neighborhood L_k of k that is contained in $U_k \cap \beta^{-1}(V_k)$. Since L_k is a neighborhood of k and K is compact, there are finitely many points $k_1, \ldots, k_n$ such that $K = L_{k_1} \cup \cdots \cup L_{k_n}$. Since $\beta(L_{k_i}) \subset V_{k_i}$ for all $1 \le i \le n$, the homomorphism β is contained in the set

$$W \;=\; \bigcap_{i=1,\ldots,n} N(\mathrm{incl} : L_{k_i} \longrightarrow K, V_{k_i})$$

which is an open subset of $\mathrm{map}(K, G)$. So we are done if we can show that every $\beta \in W \cap \hom(K, G)$ satisfies $\Gamma_\beta \subset O$. Given $k \in K$, there is an $i \in \{1, \ldots, n\}$

with $k \in L_{k_i}$. Thus $\beta(k) \in V_{k_i}$ by hypothesis on β, and so

$$(k, \beta(k)) \in L_{k_i} \times V_{k_i} \subseteq O .$$

Since k was arbitrary, this proves that $\Gamma_\beta \subseteq O$. Since the set U is open in $\hom(K, G)$ and contained in the G°-orbit of α, the G°-orbit of α is open in $\mathrm{map}(K, G)$.

Now we show that every G°-orbit is path connected. For every continuous homomorphism $\alpha : K \longrightarrow G$, the map

$$G \times K \longrightarrow G , \quad (g, k) \longmapsto g^{-1} \cdot \alpha(k) \cdot g$$

is continuous since it can be written as a composite of a diagonal map, an evaluation map and data from the group structure. The adjoint

$$G \longrightarrow \hom(K, G) , \quad g \longmapsto c_g \circ \alpha$$

is thus continuous as well. Since the component group G° is path connected and conjugation is continuous, every G°-orbit is path connected, so in particular connected. The G°-orbits are open by the previous paragraph. Since the complement of an orbit is a union of other orbits, the G°-orbits are also closed. So the G°-orbits are open, closed and path connected; hence they coincide with the path components and the connected components. □

For a closed subgroup H of a compact Lie group G, we let $C_G H$ and $N_G H$ denote the centralizer and the normalizer of H in G, and we write $W_G H = (N_G H)/H$ for the Weyl group.

Proposition A.26. *For every closed subgroup H of a compact Lie group G, the composite*

$$(C_G H)^\circ \xrightarrow{\text{incl}} (N_G H)^\circ \xrightarrow{\text{proj}} (W_G H)^\circ$$

is surjective.

Proof We consider a continuous path $\omega : [0, 1] \longrightarrow W_G H$ starting at $\omega(0) = eH$. The projection $N_G H \longrightarrow (N_G H)/H = W_G H$ is a locally trivial fiber bundle, so the path can be lifted to a continuous path $\bar{\omega} : [0, 1] \longrightarrow N_G H$ with $\bar{\omega}(0) = e$. Proposition A.25 shows that the image of the map

$$H^\circ \longrightarrow \hom(H, H) , \quad h \longmapsto c_h \tag{A.27}$$

is the path component of the identity. Since H° is compact and $\hom(H, H)$ is a Hausdorff space, this map factors over a homeomorphism from $H^\circ/(H^\circ \cap Z(H))$ onto the path component of the identity, where $Z(H)$ is the center of

H. In particular, the map (A.27) is a locally trivial fiber bundle over the path component of the identity. The path

$$[0,1] \longrightarrow \hom(H,H)\ , \quad t \longmapsto c_{\bar{\omega}(t)}$$

can thus be lifted to a continuous path $\nu : [0,1] \longrightarrow H^\circ$ satisfying

$$\nu(0) = e \qquad \text{and} \qquad c_{\bar{\omega}(t)} \;=\; c_{\nu(t)}$$

for all $t \in [0,1]$. The second relation means that $\bar{\omega}(t)\cdot\nu(t)^{-1}$ centralizes H, so

$$\bar{\omega}\cdot\nu^{-1} \;:\; [0,1] \longrightarrow C_G H$$

is a path in the centralizer of H that starts at the identity element. Moreover,

$$\bar{\omega}(1)\cdot\nu^{-1}(1)\cdot H \;=\; \bar{\omega}(1)\cdot H \;=\; \omega(1)\ . \qquad \square$$

The *h-cofibrations* are the morphisms with the homotopy extension property. We will use this concept in various categories, for example in the category of G-spaces, orthogonal spaces and orthogonal spectra. So we recall some basic properties of h-cofibrations in the context of categories enriched over the category of spaces. The arguments are all standard and well known, and we include them for completeness and convenience.

For the discussion of h-cofibrations we work in a cocomplete category C that is tensored and cotensored over the category $\mathbf{T}$ of compactly generated spaces. We write '$\times$' for the pairing and X^K for the cotensor of an object X with a compactly generated space K. A *homotopy* is then a morphism $H : A\times[0,1] \longrightarrow X$ defined on the pairing of a C-object with the unit interval. For a homotopy and any $t \in [0,1]$ we denote by $H_t : A \longrightarrow X$ the composite morphism

$$A \;\cong\; A\times\{t\} \xrightarrow{A\times\text{incl}} A\times[0,1] \xrightarrow{\ H\ } X\ .$$

Definition A.28. Let C be a category tensored over the category $\mathbf{T}$ of spaces. A C-morphism $f : A \longrightarrow B$ is an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism $\varphi : B \longrightarrow X$ and a homotopy $H : A\times[0,1] \longrightarrow X$ such that $H_0 = \varphi f$, there is a homotopy $\bar{H} : B\times[0,1] \longrightarrow X$ such that $\bar{H}\circ(f\times[0,1]) = H$ and $\bar{H}_0 = \varphi$.

There is a universal test case for the homotopy extension problem, namely when X is the pushout:

$$\begin{tikzcd} A \arrow[r, "(-{,}0)"] \arrow[d, "f"'] & A\times[0,1] \arrow[d, "H"] \\ B \arrow[r, "\varphi"'] & B\cup_f (A\times[0,1]) \end{tikzcd}$$

So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism

$$B \cup_f (A \times [0,1]) \longrightarrow B \times [0,1] \tag{A.29}$$

has a retraction. Also, the adjunction between $- \times [0,1]$ and $(-)^{[0,1]}$ lets us rewrite any homotopy extension data (φ, H) in adjoint form as a commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "f"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r, "\varphi"'] & X \end{tikzcd}$$

A solution to the homotopy extension problem is adjoint to a lifting, i.e., a morphism $\lambda : B \longrightarrow X^{[0,1]}$ such that $\lambda f = \hat{H}$ and $\mathrm{ev}_0 \circ \lambda = \varphi$. So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if it has the left lifting property with respect to the morphisms $\mathrm{ev}_0 : X^{[0,1]} \longrightarrow X$ for all objects in C.

The three equivalent characterizations of h-cofibrations quickly imply various closure properties.

Corollary A.30. *Let C be a cocomplete category tensored and cotensored over the category* **T** *of compactly generated spaces.*

(i) *The class of h-cofibrations in C is closed under retracts, cobase change, coproducts and sequential compositions.*
(ii) *Let C' be another category tensored over the category* **T**, *and $F : C \longrightarrow C'$ a continuous functor that commutes with colimits and tensors with* $[0,1]$. *Then F takes h-cofibrations in C to h-cofibrations in C'.*
(iii) *If C is a topological model category in which every object is fibrant, then every cofibration is an h-cofibration.*

Proof (i) Every class of morphisms that can be characterized by the left lifting property with respect to some other class has the closure properties listed.

(ii) Let $f : A \longrightarrow B$ be a cofibration in C and $r : B\times[0,1] \longrightarrow B\cup_f(A\times[0,1])$ a retraction to the canonical morphism. The composite

$$FB \times [0,1] \;\cong F(B \times [0,1]) \xrightarrow{Fr} F(B \cup_f (A \times [0,1])) \;\cong\; FB \cup_{Ff} (FA \times [0,1])$$

is then a retraction to the canonical morphism for $Ff : FA \longrightarrow FB$. So Ff is an h-cofibration.

(iii) Since the model structure is topological, for every cofibration $f : A \longrightarrow B$ the canonical morphism (A.29) is an acyclic cofibration. Since every object is fibrant, this morphism has a retraction, and so f is an h-cofibration. □

We turn to a key technical property of h-cofibrations in the category **T** of compactly generated spaces, namely that these are always closed embeddings. The same conclusion also holds for h-cofibrations in the category $\mathbf{T}_*$ of based compactly generated spaces.

The forgetful functor $\mathbf{T}_* \longrightarrow \mathbf{T}$ does *not* preserve the tensors with unbased spaces, and it does *not* preserve h-cofibrations. Indeed, the 'based tensor' of a based space (A, a_0) with an unbased space K is $A \wedge K_+$, whereas the unbased tensor of the underlying space is simply $A \times K$. For every based space (A, a_0) the inclusion $\{a_0\} \longrightarrow A$ is an h-cofibration in the based sense, but not generally in the unbased sense. By definition, (A, a_0) is well-pointed if this inclusion is an h-cofibration of unbased spaces.

The following proposition is Lemma 8.2 in [96, App. A].

Proposition A.31. *Every h-cofibration between compactly generated spaces is a closed embedding. Every based h-cofibration between compactly generated based spaces is a closed embedding.*

Proof Let $f : A \longrightarrow B$ be an h-cofibration between compactly generated spaces. Since the map $(-, 0) : A \longrightarrow A \times [0, 1]$ is a closed embedding, Proposition A.13 says that the pushout $B \cup_f (A \times [0, 1])$ in the category **T** can in fact be calculated in the ambient category **Spc**. So the map

$$(-, 1) \ : \ A \ \longrightarrow \ B \cup_f (A \times [0, 1]) \ , \quad a \ \longmapsto \ (a, 1)$$

is a closed embedding by direct inspection of the topology on the pushout.

The universal example of the homotopy extension property provides a continuous retraction to the canonical map

$$i \ = \ (-, 0) \cup (f \times [0, 1]) \ : \ B \cup_f (A \times [0, 1]) \ \longrightarrow \ B \times [0, 1] \ .$$

So i is a closed embedding by Proposition A.12. In the commutative square

$$\begin{tikzcd}
A \arrow[r, "(-{,}1)"] \arrow[d, "f"'] & B \cup_f (A \times [0,1]) \arrow[d, "i"] \\
B \arrow[r, "(-{,}1)"'] & B \times [0,1]
\end{tikzcd}$$

the two horizontal and the right vertical maps are thus closed embeddings. So the map f is a closed embedding as well.

The based case, albeit logically independent, proceeds along the same lines. We let $f : (A, a_0) \longrightarrow (B.b_0)$ be a based h-cofibration between compactly generated based spaces. Since points in weak Hausdorff spaces are closed (Propo-

sition A.4 (iii)), the set $\{a_0\} \times [0,1]$ is closed in $A \times [0,1]$, and so the map

$$(-,t) \; : \; A \longrightarrow A \wedge [0,1]_+ \, , \quad a \longmapsto a \wedge t$$

is a closed embedding for every $t \in [0,1]$. Any pushout of based spaces can be calculated in the ambient category of unbased spaces; so Proposition A.13 says that the pushout $B \cup_f (A \wedge [0,1]_+)$ in the category $\mathbf{T}_*$ can in fact be calculated in the ambient category **Spc**. So the map

$$(-,1) \; : \; A \longrightarrow B \cup_f (A \wedge [0,1]_+) \, , \quad a \longmapsto a \wedge 1$$

is a closed embedding by direct inspection.

The universal example of the homotopy extension property provides a continuous retraction to the canonical based map

$$i \; = \; (-,0) \cup (f \wedge [0,1]_+) \; : \; B \cup_f (A \wedge [0,1]_+) \longrightarrow B \wedge [0,1]_+ \, .$$

So i is a closed embedding by Proposition A.12. In the commutative square

$$\begin{tikzcd} A \arrow[r, "{(-,1)}"] \arrow[d, "f"'] & B \cup_f (A \wedge [0,1]_+) \arrow[d, "i"] \\ B \arrow[r, "{(-,1)}"'] & B \wedge [0,1]_+ \end{tikzcd}$$

the two horizontal and the right vertical maps are thus closed embeddings. So the map f is a closed embedding as well. □

Now we turn to geometric realization of simplicial spaces, which is a frequent tool to construct interesting homotopy types.

Construction A.32. We recall the geometric realization of a simplicial space. Geometric realization was originally introduced by Milnor [121] for simplicial sets (which were called 'semi-simplicial complexes' at that time); the version for simplicial spaces is a straightforward generalization. We let $\mathbf{\Delta}$ denote the simplicial indexing category, with objects the finite totally ordered sets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for $n \geq 0$. Morphisms in $\mathbf{\Delta}$ are all weakly monotone maps. We let

$$\Delta^n \; = \; \{(t_1, \ldots, t_n) \in [0,1]^n \mid t_1 \leq t_2 \leq \cdots \leq t_n\}$$

be the topological n-simplex. As n varies, these topological simplices assemble into a covariant functor

$$\Delta^\bullet \; : \; \mathbf{\Delta} \longrightarrow \mathbf{Spc} \, , \quad [n] \longmapsto \Delta^n \, ;$$

the coface maps are given by

$$(d_i)_*(t_1,\dots,t_n) \ = \begin{cases} (0,t_1,\dots,t_n) & \text{for } i=0,\\ (t_1,\dots,t_i,t_i,\dots,t_n) & \text{for } 0<i<n,\\ (t_1,\dots,t_n,1) & \text{for } i=n. \end{cases}$$

The codegeneracy map $(s_i)_* : \Delta^n \longrightarrow \Delta^{n-1}$ drops the entry t_{i+1}.

A *simplicial space* is functor $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$, i.e., a contravariant functor from $\mathbf{\Delta}$ to the category of topological spaces. We use the customary notation $X_n = X([n])$ for the value of a simplicial space at $[n]$. The *geometric realization* of X is

$$|X| \ = \ \left(\coprod\nolimits_{n\geq 0} X_n \times_0 \Delta^n\right) / \ \sim \ ,$$

the quotient space of the disjoint union by the equivalence relation generated by

$$(x,\alpha_*(t)) \ \sim \ (\alpha^*(x),t) \tag{A.33}$$

for all morphisms $\alpha : [n] \longrightarrow [m]$ in $\mathbf{\Delta}$ and all $(x,t) \in X_m \times_0 \Delta^n$. A more categorical way to say this is that $|X|$ is a coend of the functor

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \ \longrightarrow \ \mathbf{Spc} \ , \quad ([m],[n]) \ \longmapsto \ X_m \times_0 \Delta^n \ . \tag{A.34}$$

We recall that the equivalence relation generated by (A.33) is well understood in terms of ‘minimal representatives’:

- every equivalence class has a unique representative $(x,t) \in X_l \times \Delta^l$ of minimal dimension l;
- an element $(x,t) \in X_l \times \Delta^l$ is the minimal representative in its equivalence class if and only if x is non-degenerate (i.e., not in the image of any degeneracy map $s_i^* : X_{l-1} \longrightarrow X_l$) and t belongs to the interior of Δ^l; and
- if $(x,t) \in X_l \times \Delta^l$ is a minimal representative and $(y,s) \in X_m \times \Delta^m$ is equivalent to (x,t), then there is a unique surjective morphism $\sigma : [k] \longrightarrow [l]$, a unique injective morphism $\delta : [k] \longrightarrow [m]$ and a unique $u \in \Delta^k$ such that

$$\delta^*(y) \ = \ \sigma^*(x) \ , \quad s \ = \ \delta_*(u) \quad \text{and} \quad t = \sigma_*(u) \ .$$

These facts go back all the way to Milnor [121, Lemma 3].

The next proposition shows that compactly generated spaces are ‘closed under geometric realization’; more precisely, we consider a simplicial space $X_\bullet$ such that X_n is compactly generated for every $n \geq 0$. The geometric realization, formed in the ambient category $\mathbf{Spc}$, is an enriched coend, hence automatically a k-space. Since colimits of compactly generated spaces are not, in general, formed in the ambient category, it is not completely obvious, though, that the ambient realization is again a weak Hausdorff space. I learned about this fact as a parenthetical remark in the introduction of the paper [156].

Proposition A.35. *Let $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}$ be a simplicial k-space.*

(i) *The geometric realization $|X|$ is a k-space, and hence also a coend internal to the category* $\mathbf{K}$*, of the functor* (A.34).

(ii) *If X_n is compactly generated for every $n \geq 0$, then the geometric realization $|X|$ is compactly generated, and hence also a coend internal to the category* $\mathbf{T}$*, of the functor* (A.34).

(iii) *Let X_n be compactly generated for all $n \geq 0$. Let Y be a simplicial subspace of X such that Y_n is closed in X_n for all $n \geq 0$. Then the inclusion induces a closed embedding $|Y| \longrightarrow |X|$.*

Proof (i) Since Δ^n is compact and X_n is a k-space, $X_n \times_0 \Delta^n = X \times \Delta^n$ is a k-space in the product topology by Proposition A.2 (vi), and so the disjoint union $\coprod_{n\geq 0} X_n \times \Delta^n$ is a k-space. As a quotient space, the geometric realization $|X|$ is then a k-space by Proposition A.2 (i).

(ii) As we argued in part (i), the space $\coprod_{n\geq 0} X_n \times_0 \Delta^n = \coprod_{n\geq 0} X_n \times \Delta^n$ is a k-space. We can thus use the criterion given by Proposition A.7 to show that the quotient space $|X|$ is compactly generated. We let $E \subset (\coprod_{n\geq 0} X_n \times \Delta^n)^2$ be the equivalence relation generated by (A.33), which we had simply denoted '$\sim$' above. We will show that E is closed in the k-topology of $(\coprod_{n\geq 0} X_n \times \Delta^n)^2$. Since products distribute over disjoint unions, we may show that

$$E_{m,n} = E \cap (X_m \times \Delta^m \times X_n \times \Delta^n)$$

is closed in $X_m \times \Delta^m \times X_n \times \Delta^n$ for all $m, n \geq 0$. If $(y, s, \bar{y}, \bar{s}) \in E_{m,n}$, then (y, s) and $(\bar{y}, \bar{s})$ have the same minimal representative $(x, t) \in X_l \times \Delta^l$. So there are surjective morphisms $\sigma : [k] \longrightarrow [l]$ and $\bar{\sigma} : [\bar{k}] \longrightarrow [l]$, injective morphisms $\delta : [k] \longrightarrow [m]$ and $\bar{\delta} : [\bar{k}] \longrightarrow [n]$, and $u \in \Delta^k$, $\bar{u} \in \Delta^{\bar{k}}$ such that

$$\delta^*(y) = \sigma^*(x)\,, \quad s = \delta_*(u) \quad \text{and} \quad t = \sigma_*(u)$$

and

$$\bar{\delta}^*(\bar{y}) = \bar{\sigma}^*(x)\,, \quad \bar{s} = \bar{\delta}_*(\bar{u}) \quad \text{and} \quad t = \bar{\sigma}_*(\bar{u})\,.$$

Hence $E_{m,n}$ is the union, indexed over $l, \sigma, \bar{\sigma}, \delta, \bar{\delta}$ as above, of the finite number of sets

$$(\delta^* \times \Delta^m \times \bar{\delta}^* \times \Delta^n)^{-1} \Big((\sigma^* \times \delta_* \times \bar{\sigma}^* \times \bar{\delta}_*) \Big((X_l \times \sigma_* \times X_l \times \bar{\sigma}_*)^{-1} (\Delta_{X_l \times \Delta^l}) \Big) \Big)\,.$$

The diagonal $\Delta_{X_l \times \Delta^l}$ is closed in $(X_l \times \Delta^l)^2$ because X_l, and hence $X_l \times \Delta^l$, is compactly generated. So its inverse image under the continuous map $X_l \times \sigma_* \times X_l \times \bar{\sigma}_*$ is closed in $X_l \times \Delta^k \times X_l \times \Delta^{\bar{k}}$. Every surjective morphism in $\mathbf{\Delta}$ has a section, so $\sigma^* : X_l \longrightarrow X_k$ and $\bar{\sigma}^* : X_l \longrightarrow X_{\bar{k}}$ have continuous retractions. The maps $\delta_* : \Delta^k \longrightarrow \Delta^m$ and $\bar{\delta}_* : \Delta^{\bar{k}} \longrightarrow \Delta^n$ also have continuous retractions,

hence so does $\sigma^* \times \delta_* \times \bar{\sigma}^* \times \bar{\delta}_*$. This map is then a closed embedding by Proposition A.12. So the set

$$(\sigma^* \times \delta_* \times \bar{\sigma}^* \times \bar{\delta}_*)\left((X_l \times \sigma_* \times X_l \times \bar{\sigma}_*)^{-1}(\Delta_{X_l \times \Delta^l})\right)$$

is closed in $X_k \times \Delta^m \times X_{\bar{k}} \times \Delta^n$. Since $E_{m,n}$ is the inverse image of this latter closed set under a continuous map, this show the claim that $E_{m,n}$ is a closed subset of $X_m \times \Delta^m \times X_n \times \Delta^n$.

(iii) We write $\iota : Y \longrightarrow X$ for the inclusion. Our first claim is that the induced map $|\iota| : |Y| \longrightarrow |X|$ is injective. We consider a point $(y, s) \in Y_m \times \Delta^m$ and we let $(x, t) \in X_l \times \Delta^l$ be the minimal representative of the equivalence class of (y, s) in the ambient space $\coprod_{n \geq 0} X_n \times \Delta^n$. Then there is a surjective morphism $\sigma : [k] \longrightarrow [l]$, an injective morphism $\delta : [k] \longrightarrow [m]$ and $u \in \Delta^k$ such that

$$\delta^*(y) \ = \ \sigma^*(x) \ , \quad s \ = \ \delta_*(u) \quad \text{and} \quad t = \sigma_*(u) \ .$$

We let $\bar{\delta} : [l] \longrightarrow [k]$ be a morphism in $\mathbf{\Delta}$ such that $\sigma\bar{\delta} = \mathrm{Id}_{[l]}$. Then $\bar{\delta}^*(\delta^*(y)) = \bar{\delta}^*(\sigma^*(x)) = x$. Since Y is a simplicial subspace of X, this shows that x belongs to Y_l. If $(\bar{y}, \bar{s}) \in Y_n \times \Delta^n$ is another pair that represents the same point in $|X|$ as (y, s), then by the above, the minimal representative of the equivalence class belongs to $\coprod_{n \geq 0} Y_n \times \Delta^n$, and so (y, s) and $(\bar{y}, \bar{s})$ already represent the same point in $|Y|$. So the map $|\iota|$ is injective.

It remains to show that the continuous injection $|\iota|$ is a closed map. We consider the commutative square

$$\begin{tikzcd}
\coprod_{n\geq 0} Y_n \times \Delta^n \arrow[r, "\coprod \iota_n \times \Delta^n"] \arrow[d, "p"'] & \coprod_{n\geq 0} X_n \times \Delta^n \arrow[d, "q"] \\
{|Y|} \arrow[r, "|\iota|"'] & {|X|}
\end{tikzcd}$$

where the vertical maps are the quotient maps. We let $(y, s) \in X_m \times \Delta^m$ be a point whose equivalence class lies in the image of $|\iota| : |Y| \longrightarrow |X|$. As we showed in the previous paragraph, the representative $(x, t) \in X_l \times \Delta^l$ of minimal dimension in the equivalence class of (y, s) must then lie in the simplicial subspace Y, i.e., we must have $x \in Y_l$. Moreover, there is a surjective morphism $\sigma : [k] \longrightarrow [l]$, an injective morphism $\delta : [k] \longrightarrow [m]$ and $u \in \Delta^k$ such that

$$\delta^*(y) \ = \ \sigma^*(x) \ , \quad s \ = \ \delta_*(u) \qquad \text{and} \qquad t = \sigma_*(u) \ .$$

So for every subset $A \subset |Y|$, we have

$$\begin{aligned}
&(X_m \times \Delta^m) \cap q^{-1}(|\iota|(A)) \\
&\quad = \ \bigcup\nolimits_{\sigma,\delta} (\delta^* \times \Delta^m)^{-1} \left((\sigma^* \times \delta_*)((X_l \times \sigma_*)^{-1}((Y_l \times \Delta^l) \cap p^{-1}(A)))\right) \ .
\end{aligned}$$

The union is over the finite set of pairs consisting of a surjective morphism $\sigma : [k] \longrightarrow [l]$ and an injective morphism $\delta : [k] \longrightarrow [m]$.

Now we assume that A is closed inside $|Y|$. Because p is continuous and Y_l is closed in X_l, the set $(Y_l \times \Delta^l) \cap p^{-1}(A)$ is closed inside $X_l \times \Delta^l$. So $(X_l \times \sigma_*)^{-1}((Y_l \times \Delta^l) \cap p^{-1}(A))$ is a closed subset of $X_l \times \Delta^k$. Since the map $\sigma^* \times \delta_* : X_l \times \Delta^k \longrightarrow X_k \times \Delta^m$ has a continuous retraction, it is a closed embedding by Proposition A.12. So the set $(\sigma^* \times \delta_*)((X_l \times \sigma_*)^{-1}((Y_l \times \Delta^l) \cap p^{-1}(A)))$ is closed in $X_k \times \Delta^m$. As the inverse image under a continuous map, the set

$$(\delta^* \times \Delta^m)^{-1}((\sigma^* \times \delta_*)((X_l \times \sigma_*)^{-1}((Y_l \times \Delta^l) \cap p^{-1}(A))))$$

is then closed in $X_m \times \Delta^m$.

So each set in the finite union above is closed inside $X_m \times \Delta^m$. We conclude that $(X_m \times \Delta^m) \cap q^{-1}(|\iota|(A))$ is closed in $X_m \times \Delta^m$ for every $m \geq 0$, hence the set $q^{-1}(|\iota|(A))$ is closed. Since q is a quotient map, this shows that $|\iota|(A)$ is closed in $|X|$. □

The next proposition is about the interaction of geometric realization and products for simplicial k-spaces and simplicial compactly generated spaces. The essential input here is that the categories $\mathbf{K}$ and $\mathbf{T}$ are cartesian closed, hence product with a fixed object is a left adjoint and commutes with colimits and coends. A direct consequence is that geometric realization commutes with colimits and products with a fixed space, see part (i) and (ii) of Proposition A.37 below.

We also recall that different ways to realize a multi-simplicial space are homeomorphic. To this end we consider a *bisimplicial k-space*, i.e., a functor $Z : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}$. The diagonal simplicial k-space $\operatorname{diag} Z$ is the composite with the diagonal functor $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$. On the other hand, for every fixed 'horizontal' dimension m we can define the simplicial space $\partial_m Z$ as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{(\mathrm{Id}_{[m]},-)} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{Z} \mathbf{K} ,$$

so that $(\partial_m Z)_n = Z_{m,n}$. For varying m, these define a simplicial object in the category of simplicial k-spaces; so the assignment

$$[m] \longmapsto |\partial_m Z|$$

becomes a simplicial k-space. We can geometrically realize this simplicial space and arrive at the *iterated realization* that we denote

$$|Z|^{\mathrm{it}} = |[m] \mapsto |\partial_m Z|| . \tag{A.36}$$

The diagonal realization and the iterated realization are related by a natural

continuous map, see (A.38) below. We show in Proposition A.37 (iii) that this preferred natural map is a homeomorphism.

I am not sure who deserves credit for parts (ii) and (iii) of the following proposition. For preservation of products, Milnor [121, Thm. 2] already treats the special case of simplicial sets (which we can view as discrete simplicial spaces), where he imposes size conditions to ensure that the ordinary product of the two realizations is a k-space. The earliest reference in the present generality I know of is [149, §1, p. 106], where Segal states the result without proof. May [110, Thm. 11.5] gives a proof in the category of Hausdorff k-spaces (which he calls 'compactly generated Hausdorff spaces'), but the proof does not use the Hausdorff property.

Proposition A.37. (i) *For every simplicial k-space $Y : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}$ and every k-space K, the canonical map*

$$|K \times Y| \longrightarrow K \times |Y| , \quad [k, y, t] \longmapsto (k, [y, t])$$

is a homeomorphism.

(ii) *The realization functors*

$$|-| : \mathbf{\Delta}^{\mathrm{op}}\mathbf{K} \longrightarrow \mathbf{K} \qquad \textit{and} \qquad |-| : \mathbf{\Delta}^{\mathrm{op}}\mathbf{T} \longrightarrow \mathbf{T}$$

for simplicial k-spaces and simplicial compactly generated spaces preserves colimits and finite products.

(iii) *For every bisimplicial k-space $Z : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}$, the map*

$$\delta : |\operatorname{diag} Z| \longrightarrow |Z|^{\mathrm{it}} \quad \textit{given by} \quad \delta[z, t] = [[z, t], t] \tag{A.38}$$

is a homeomorphism.

Proof (i) The category of k-spaces is cartesian closed (Theorem A.22), so product with K is a left adjoint and commutes with coends. Hence the canonical map

$$|K \times Y| = \int^{[n]\in\mathbf{\Delta}} (K \times Y_n \times \Delta^n) \longrightarrow K \times \left(\int^{[n]\in\mathbf{\Delta}} Y_n \times \Delta^n \right) = K \times |Y|$$

is a homeomorphism.

(ii) We start by showing that the realization functors preserve colimits. The argument is entirely formal, and literally the same for the categories **K** and **T**, so we only treat the former case. We let I be a small category and $F : I \longrightarrow \mathbf{\Delta}^{\mathrm{op}}\mathbf{K}$ a functor. The category of k-spaces is cartesian closed (Theorem A.22), so product with Δ^n is a left adjoint and commutes with coends. Moreover,

colimits commute with coends, so the canonical maps

$$\operatorname{colim}_I (|-|\circ F) \;=\; \operatorname{colim}_I\left(\int^{[n]\in\mathbf{\Delta}} F_n\times\Delta^n\right) \xrightarrow{\cong} \int^{[n]\in\mathbf{\Delta}} \operatorname{colim}_I(F_n\times\Delta^n)$$
$$\xrightarrow{\cong} \int^{[n]\in\mathbf{\Delta}} (\operatorname{colim}_I F_n)\times\Delta^n \;=\; \int^{[n]\in\mathbf{\Delta}} (\operatorname{colim}_I F)_n\times\Delta^n \;=\; |\operatorname{colim}_I F|$$

are homeomorphisms.

We postpone the proof that the realization functors preserve finite products until later.

(iii) In any presheaf category with values in $\mathbf{K}$, every functor is canonically a colimit of representable functors times a fixed space. For the category $\mathbf{\Delta}\times\mathbf{\Delta}$, this means that every bisimplicial k-space can be expressed as a colimit of bisimplicial spaces of the form $(\mathbf{\Delta}\times\mathbf{\Delta})(-,([m],[n]))\times K$ for $m,n\geq 0$ and k-spaces K. Passage to the diagonal clearly commutes with colimits and products, so source and target of the map δ commute with colimits in $\mathbf{K}$ and products with a fixed k-space, by parts (i) and (ii). This reduces the claim to the special case $Z=\Delta[m,n]=(\mathbf{\Delta}\times\mathbf{\Delta})(-,([m],[n]))$, the bisimplicial discrete space represented by the object $([m],[n])$. The diagonal of this bisimplicial space is $\Delta[m]\times\Delta[n]$, the product of the represented simplicial sets $\Delta[m]$ and $\Delta[n]$.

On the other hand, for fixed $k\geq 0$, the simplicial discrete space $\partial_k\Delta[m,n]$ is isomorphic to $\mathbf{\Delta}([k],[m])\times\Delta[n]$, and so its realization is homeomorphic to $\mathbf{\Delta}([k],[m])\times|\Delta[n]|$. The simplicial space $|\partial_\bullet\Delta[m,n]|$ is thus isomorphic to $\Delta[m]\times|\Delta[n]|$. Since product with $|\Delta[n]|$ commutes with realization by part (i), we conclude that the iterated realization $|\Delta[m,n]|^{\mathrm{it}}$ is homeomorphic to the product $|\Delta[m]|\times|\Delta[n]|$. Moreover, under these identifications, the map δ for $Z=\Delta[m,n]$ specializes to the canonical map $|\Delta[m]\times\Delta[n]|\longrightarrow|\Delta[m]|\times|\Delta[n]|$. It is a classical fact, already observed by Milnor [121, Thm. 2], that this canonical map is a homeomorphism; other references are [58, Ch. III 3.4] and [80, Lemma 3.1.8].

We still need to show that the realization functors preserve finite products. The category $\mathbf{T}$ of compactly generated space is closed under products in the ambient category $\mathbf{K}$ of k-spaces (because the inclusion $\mathbf{K}\longrightarrow\mathbf{T}$ is a left adjoint), and closed under realization by Proposition A.35 (ii). So it suffices to treat realization of simplicial k-spaces.

We consider the bisimplicial k-space $X\bar{\times}Y$ defined as the composite

$$\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X\times Y} \mathbf{K}\times\mathbf{K} \xrightarrow{\times} \mathbf{K}\,;$$

then $X\times Y=\operatorname{diag}(X\bar{\times}Y)$ is its diagonal. The canonical map

$$|\partial_m(X\bar{\times}Y)| \;=\; |X_m\times Y| \;\longrightarrow\; X_m\times\left(\int^{[n]\in\mathbf{\Delta}} Y_n\times\Delta^n\right) \;=\; X_m\times|Y|$$

is a homeomorphism by part (i). Taking realizations over varying m and using part (i) again gives two homeomorphisms

$$|X\bar{\times}Y|^{\mathrm{it}} \;=\; |[m] \mapsto |X_m \times Y|| \;\xrightarrow{\cong}\; |X \times |Y|| \;\xrightarrow{\cong}\; |X| \times |Y| \,.$$

Under these identifications, the canonical map $(|p_X|, |p_Y|) : |X \times Y| \longrightarrow |X| \times |Y|$ becomes the map $\delta : |\operatorname{diag}(X\bar{\times}Y)| \longrightarrow |X\bar{\times}Y|^{\mathrm{it}}$ of part (iii). Since δ is a homeomorphism, this shows the claim. □

For general simplicial topological spaces, i.e., functors $X, Y : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow$ **Spc** geometric realization need not commute with products in **Spc**. Examples already arise when X and Y are simplicial sets (considered as simplicial discrete spaces), in which case $|X \times Y|$ is compactly generated. However, if X and Y are sufficiently large (i.e., neither countable nor locally finite), then the product topology need not make $|X| \times_0 |Y|$ a k-space. A specific example is given by taking both X and Y as wedges of the simplicial 1-sphere, where X has countably infinitely many copies, and Y has uncountably many copies, compare [43, III.5, p. 563].

Now we discuss the latching spaces of a simplicial space X, of which there are competing definitions in the literature. The definition of the nth latching object that we adopt is as a colimit of a certain functor whose values are the subspaces X_i for $i < n$. A possible point of confusion is that different references work in slightly different categories (such as general topological spaces, k-spaces, Hausdorff k-spaces or weak Hausdorff k-spaces), and the presence of a Hausdorff or weak Hausdorff condition affects the meaning of 'colimit'. Some authors define the nth latching space of X as a subspace of X_n, namely the union of the subspaces $s_i^*(X_{n-1})$ for $i = 0, \ldots, n-1$, where $s_i^* : X_{n-1} \longrightarrow X_n$ is the ith degeneracy map. I suspect that the different definitions are not generally homeomorphic in the context of simplicial topological spaces, i.e., functors $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow$ **Spc**.

As we will discuss now, all these definitions in fact coincide for simplicial compactly generated spaces, i.e., functors $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{T}$. We already explained in Proposition A.35 (ii) that then the geometric realization (formed in the ambient category **Spc** of all topological spaces) is automatically compactly generated, and hence also a coend internal to the category **T**. We will now show that also the latching objects, defined in the ambient category **Spc**, are automatically compactly generated, and hence also a colimit in the category **T**. Moreover, the latching map $l_n : L_n X \longrightarrow X_n$ is a closed embedding, with the expected image.

Construction A.39. For $n \geq 0$ we let $\mathbf{\Delta}(n)$ denote category with objects the weakly monotone surjections $\sigma : [n] \longrightarrow [k]$; a morphism from $\sigma : [n] \longrightarrow [k]$

to $\sigma' : [n] \longrightarrow [k']$ is a morphism $\alpha : [k] \longrightarrow [k']$ in $\mathbf{\Delta}$ with $\alpha \circ \sigma = \sigma'$. Such a morphism α, if it exists, is necessarily unique and also surjective. We let $\mathbf{\Delta}(n)_\circ$ denote the full subcategory with all objects *except* the identity of $[n]$.

A simplicial topological space $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ can be restricted along the forgetful functor

$$\mathbf{\Delta}(n)_\circ^{\mathrm{op}} \xrightarrow{u} \mathbf{\Delta}^{\mathrm{op}} , \quad (\sigma : [n] \longrightarrow [k]) \longmapsto [k] , \quad \alpha \longmapsto \alpha .$$

The nth *latching space* of X is the colimit of the composite functor $X \circ u : \mathbf{\Delta}(n)_\circ^{\mathrm{op}} \longrightarrow \mathbf{Spc}$. As σ ranges over the objects of $\mathbf{\Delta}(n)_\circ^{\mathrm{op}}$, the maps

$$\sigma^* \ : \ X_k \ \longrightarrow \ X_n$$

assemble into a continuous map $l_n : L_nX \longrightarrow X$, the nth *latching map*.

Remark A.40. The category $\mathbf{\Delta}(n)^{\mathrm{op}}$ is isomorphic to the poset category $\mathcal{P}(n)$ of subsets of the set $\{1, \ldots, n\}$. Indeed, an isomorphism $\kappa : \mathbf{\Delta}(n)^{\mathrm{op}} \longrightarrow \mathcal{P}(n)$ is given on objects by

$$\kappa(\sigma : [n] \longrightarrow [k]) \ = \ \{i \in \{1, \ldots, n\} \ : \ \sigma(i) > \sigma(i-1)\} .$$

In the other direction, a subset $U \subset \{1, \ldots, n\}$ is taken to the monotone surjection $\kappa^{-1}(U) : [n] \longrightarrow [|U|]$ defined by

$$\kappa^{-1}(U)(i) \ = \ |U \cap \{1, \ldots, i\}| .$$

Since $\kappa(\mathrm{Id}_{[n]}) = \{1, \ldots, n\}$, the subcategory $\mathbf{\Delta}(n)_\circ^{\mathrm{op}}$ is taken to the poset $\mathcal{P}(n)^\circ$ of proper subsets of $\{1, \ldots, n\}$.

The maximal elements of the poset $\mathcal{P}(n)^\circ$ correspond to the morphisms

$$s_i \ : \ [n] \ \longrightarrow \ [n-1] , \quad s_i(j) \ = \ \begin{cases} j & \text{for } 0 \leq j \leq i, \text{ and} \\ j-1 & \text{for } i+1 \leq j \leq n. \end{cases}$$

So the latching space L_nX can also be presented as a coequalizer of two continuous maps

$$\coprod\nolimits_{0 \leq i < j \leq n-1} X_{n-2} \ \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \ \coprod\nolimits_{0 \leq i \leq n-1} X_{n-1} . \tag{A.41}$$

The map α sends the (i, j)th summand to the i-summand by the degeneracy map $s_{j-1}^* : X_{n-2} \longrightarrow X_{n-1}$. The map β sends the (i, j)th summand to the j-summand by the degeneracy map $s_i^* : X_{n-2} \longrightarrow X_{n-1}$. For example, the category $\mathbf{\Delta}(0)_\circ$ is empty, so $L_0(X)$ is empty. The category $\mathbf{\Delta}(1)_\circ$ has a unique object $s_0 : [1] \longrightarrow [0]$, so $L_1(X) = X_0$ and the latching map is given by $s_0^* : X_0 \longrightarrow X_1$. The category $\mathbf{\Delta}(2)_\circ$ has three objects and two non-identity morphisms, and $L_2(X)$ is a pushout of the diagram

$$X_1 \xleftarrow{s_0^*} X_0 \xrightarrow{s_0^*} X_1 .$$

Proposition A.42. *Let $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow$ **Spc** be a simplicial topological space and $n \geq 0$.*

(i) *The continuous latching map $l_n : L_n X \longrightarrow X_n$ is injective and its image is the union of the sets $s_i^*(X_{n-1})$ for $i = 0, \dots, n-1$.*
(ii) *Suppose that the degeneracy map $s_i^* : X_{n-1} \longrightarrow X_n$ is a closed embedding for all $0 \leq i \leq n-1$. Then the latching map $l_n : L_n X \longrightarrow X_n$ is a closed embedding.*
(iii) *Suppose that the space X_n is compactly generated for every $n \geq 0$. Then the nth latching space $L_n X$ formed in the ambient category* **Spc** *of topological spaces is compactly generated, and hence a latching object of X internal to the category* **T**. *Moreover, the latching map $l_n : L_n X \longrightarrow X_n$ is a closed embedding.*

Proof (i) This argument is purely combinatorial and does not use the topology in any way. For easier book-keeping we label the different summands in the source and target of the coequalizer diagram (A.41) as $X_{n-2}^{[i,j]}$ and $X_{n-1}^{[i]}$, respectively. We let $q : \coprod_{0 \leq i \leq n-1} X_{n-1}^{[i]} \longrightarrow L_n X$ denote the quotient map. We consider $x \in X_{n-1}^{[i]}$ and $y \in X_{n-1}^{[j]}$ such that $l_n(q(x)) = l_n(q(y))$ in X_n. This means that $s_i^*(x) = s_j^*(y)$. We assume without loss of generality that $i \leq j$. We write the elements as degeneracies of non-degenerate elements, i.e., $x = \sigma^*(z)$ and $y = \bar{\sigma}^*(\bar{z})$ for surjective morphisms $\sigma : [n-1] \longrightarrow [k], \bar{\sigma} : [n-1] \longrightarrow [\bar{k}]$ and non-degenerate simplices z and $\bar{z}$. Then

$$(\sigma s_i)^*(z) \;=\; s_i^*(\sigma^*(z)) \;=\; s_i^*(x) \;=\; s_j^*(y) \;=\; s_j^*(\bar{\sigma}^*(\bar{z})) \;=\; (\bar{\sigma} s_j)^*(\bar{z}) \, .$$

By the 'Eilenberg-Zilber lemma' ([49, (8.3)], see also [58, Sec. II.3]), the representation of a simplex as a degeneracy of a non-degenerate element is unique, so $k = \bar{k}$, $z = \bar{z}$ and $\sigma s_i = \bar{\sigma} s_j$. Because $i \leq j$, the second relation means that

$$\bar{\sigma}(a) \;=\; \begin{cases} \sigma(a) & \text{for } 0 \leq a \leq i, \\ \sigma(a-1) & \text{for } i+1 \leq a \leq j, \text{ and} \\ \sigma(a) & \text{for } j+1 \leq a \leq n-1. \end{cases}$$

We define $\tau : [n-2] \longrightarrow [k]$ by

$$\tau(a) \;=\; \begin{cases} \bar{\sigma}(a) & \text{for } 0 \leq a \leq i, \\ \bar{\sigma}(a+1) & \text{for } i+1 \leq a \leq n-2. \end{cases}$$

Then

$$\sigma \;=\; \tau s_{j-1} \qquad \text{and} \qquad \bar{\sigma} \;=\; \tau s_i \, .$$

Setting $w = \tau^*(z)$ then yields

$$x \;=\; \sigma^*(z) \;=\; s_{j-1}^*(w) \qquad \text{and} \qquad y \;=\; \bar{\sigma}^*(z) \;=\; s_i^*(w) \, .$$

This means that x and y are equivalent, and hence $q(x) = q(y)$. So the latching map $l_n : L_n X \longrightarrow X_n$ is injective.

(ii) The latching map is continuous and injective by part (i), so it remains to show that l_n is also a closed map. The composite

$$\coprod_{0 \le i \le n-1} X_{n-1}^{[i]} \xrightarrow{q} L_n X \xrightarrow{l_n} X_n$$

of the quotient map and the latching map is the disjoint union of the degeneracy maps $s_i^* : X_{n-1} \longrightarrow X_n$. So for every subset A of $L_n X$, we have

$$l_n(A) = \bigcup_{0 \le i \le n-1} s_i^* \left(q^{-1}(A) \cap X_{n-1}^{[i]} \right) .$$

Now we suppose that A is closed in $L_n X$. Since each s_i^* is a closed embedding, $l_n(A)$ is a finite union of closed subsets of X_n, and hence itself closed.

(iii) Since the morphism $s_i : [n] \longrightarrow [n-1]$ has a retraction, the degeneracy map $s_i^* : X_{n-1} \longrightarrow X_n$ has a continuous retraction. Since X_{n-1} and X_n are compactly generated, $s_i^* : X_{n-1} \longrightarrow X_n$ is a closed embedding by Proposition A.12. So part (ii) applies and shows that $l_n : L_n X \longrightarrow X_n$ is a closed embedding. Since X_n is compactly generated, so is $L_n X$, by Proposition A.5 (i). □

The proof of part (iii) of the previous proposition makes critical use of the fact that all degeneracy maps in a simplicial compactly generated space are closed embeddings. This is not the case more generally for simplicial k-spaces or simplicial topological spaces; so while a latching map is always a continuous injection, it need not be a homeomorphism onto its image. So if latching spaces are ever considered in this generality, one has to beware the possible difference between $L_n X$ defined as the colimit or as the subspace $s_0^*(X_{n-1}) \cup \cdots \cup s_{n-1}^*(X_{n-1})$ of X_n.

Now we recall Reedy cofibrancy, which will be our main condition to ensure good homotopical behavior of geometric realization.

Definition A.43. A simplicial topological space $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ is *Reedy cofibrant* is the latching morphism $l_n : L_n X \longrightarrow X_n$ is a cofibration in the Quillen model structure on the category of topological spaces.

The next proposition is the key reason why we care about Reedy cofibrancy.

Proposition A.44. *Let $f : X \longrightarrow Y$ be a morphism between Reedy cofibrant simplicial topological spaces. If $f_n : X_n \longrightarrow Y_n$ is a weak equivalence for every $n \ge 0$, then the geometric realization $|f| : |X| \longrightarrow |Y|$ is a weak equivalence.*

Proof We use the Reedy model structure on the category of simplicial topological spaces. We endow the category **Spc** of topological space with the Quillen

model structure, see [134, II.3, Thm. 1], which makes it a simplicial model category. Geometric realization thus takes level-wise weak equivalences between Reedy cofibrant objects to weak equivalences by [63, VII Prop. 3.6]. □

Remark A.45. By the precious proposition, the condition that a simplicial space is Reedy cofibrant ensures that its geometric realization is 'homotopically invariant'. There are two other widely used conditions for ensuring that a geometric realization is well-behaved, the notions of a *good* and *proper* simplicial spaces. A simplicial space X is *proper* in the sense of May [110, §11] if the latching map $\nu_n : L_n X \longrightarrow X_n$ is a closed h-cofibration for every $n \geq 0$. A simplicial space X is *good* in the sense of Segal [153, Def. A.4] if the degeneracy map $s_i^* : X_{n-1} \longrightarrow X_n$ is a closed h-cofibration for all $0 \leq i \leq n-1$. One should beware that in the generality of arbitrary topological spaces, the homotopy extension property does not imply that a map is a closed embedding, hence 'closed h-cofibration' is a stronger condition. For compactly generated spaces, however, a continuous map with the homotopy extension property is automatically a closed embedding, by Proposition A.31.

For proper simplicial spaces, homotopical invariance of geometric realization is proved in [111, Thm. A.4]. The case of good simplicial spaces then follows from the fact that 'goodness' implies 'properness'; this is implicit in the proof of [153, Lemma A.5], and stated explicitly in the proof of Lewis' [97, Cor. 2.4 (b)]. The argument makes essential use of Lillig's 'union theorem' [101], so it is *not* of a formal, model category theoretical nature.

Since cofibrations of compactly generated spaces are in particular closed h-cofibrations, every Reedy cofibrant simplicial compactly generated space is in particular proper and good. Whenever we want to appeal to homotopy invariance of geometric realization, our simplicial spaces are Reedy cofibrant; so we do not use the more general conditions 'good' and 'proper', and we don't elaborate on these any further.

We recall that the *singular complex* $\mathrm{Sing}(K)$ of a topological space K is the simplicial set defined as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{(\Delta^\bullet)^{\mathrm{op}}} \mathbf{Spc}^{\mathrm{op}} \xrightarrow{\mathbf{Spc}(-,K)} (\text{sets}) .$$

In particular, the set of n-simplices $\mathrm{Sing}(K)_n$ is the set of continuous maps from Δ^n to K. Now we consider a simplicial topological space X. Applying the singular complex functor level-wise yields a simplicial simplicial set (i.e., a bisimplicial set) $\mathrm{Sing} \circ X$, sending $[n]$ to the simplicial set $\mathrm{Sing}(X_n)$. We apply geometric realization to the simplicial sets $\mathrm{Sing}(X_n)$ for all $n \geq 0$, and end up with a new simplicial space $|-| \circ \mathrm{Sing} \circ X$.

The singular complex functor is right adjoint the geometric realization (restricted from simplicial spaces to simplicial sets), and the adjunction counit is

the continuous map

$$\epsilon_K \; : \; |\operatorname{Sing}(K)| \longrightarrow K \; , \quad [f, t] \longmapsto f(t) \; .$$

As was already noticed Milnor when he introduced the geometric realization of simplicial sets [121, Thm. 4], the map ϵ_X is a weak equivalence. Since ϵ_K is natural, these maps assemble into a morphism of simplicial spaces

$$\epsilon_X \; : \; |-| \circ \operatorname{Sing} \circ X \longrightarrow X \; .$$

We observe that $\| - | \circ \operatorname{Sing} \circ X| = |\operatorname{Sing} \circ X|^{\text{it}}$ is the iterated realization (A.36) of the bisimplicial set $\operatorname{Sing} \circ X$.

Proposition A.46. *Let $X : \mathbf{\Delta}^{\text{op}} \longrightarrow$ **Spc** be a Reedy cofibrant simplicial topological space.*

(i) *The map $|\epsilon_X| : |\operatorname{Sing} \circ X|^{\text{it}} \longrightarrow |X|$ is a weak equivalence.*
(ii) *If the spaces $X_0, \ldots, X_n$ consist only of a single point each, then the realization $|X|$ is n-connected.*

Proof (i) We claim that this simplicial space $|-| \circ \operatorname{Sing} \circ X$ sending $[m]$ to the realization of the simplicial set $\operatorname{Sing}(X_m)$ is automatically Reedy cofibrant. Indeed, for every simplicial set A the latching maps $L_n A \longrightarrow A_n$ are automatically injective (a special case of Proposition A.42 (i)). Hence for fixed $n \geq 0$ and varying $[m] \in \mathbf{\Delta}^{\text{op}}$, the maps

$$L_n^{\text{Sing}}(\operatorname{Sing}(X_m)) \longrightarrow \operatorname{Sing}(X_m)_n$$

(with latching object taken in the direction of the singular complex) are a monomorphism of simplicial sets. Geometric realization takes monomorphisms of simplicial sets to relative CW-inclusions, and hence to cofibrations of spaces, and it commutes with colimits. So the simplicial space $|-| \circ \operatorname{Sing} \circ X$ is Reedy cofibrant. Since we assumed that X is also Reedy cofibrant, the map $|\epsilon_X|$ is a weak equivalence by Proposition A.44.

(ii) By part (i) we may show that the space $|\operatorname{Sing} \circ X|^{\text{it}}$ is n-connected. The iterated realization is homeomorphic to the diagonal realization, by Proposition A.37 (iii); so we may show that the space $|\operatorname{diag}(\operatorname{Sing} \circ X)|$ is n-connected. But the simplicial set $\operatorname{diag}(\operatorname{Sing} \circ X)$ has only a single k-simplex for $0 \leq k \leq n$, by hypothesis. So its geometric realization admits a CW-structure with one 0-cell and no cells in dimension 1 through n. Hence the space $|\operatorname{diag}(\operatorname{Sing} \circ X)|$ is n-connected. $\square$

In [22, Thm. B.4], Bousfield and Friedlander establish a fairly general criterion to ensure that a dimensionwise homotopy cartesian square remains homotopy cartesian after realization. The criterion involves the 'π_*-Kan condition' introduced in [22, B.3]. We now translate the condition into the context of

simplicial spaces; it roughly says that the simplicial sets obtained by taking tth homotopy groups dimensionwise satisfy the Kan extension condition. The precise definition is more elaborate because one needs to properly take care of varying basepoints for homotopy groups.

Definition A.47 (Bousfield-Friedlander)**.** Let $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ be a simplicial topological space, $m, t \geq 1$ and $a \in X_m$. Then X satisfies the *π_t-Kan condition at a* if the following condition holds: for all tuples of elements $x_i \in \pi_t(X_{m-1}, d_i^*(a))$ for $i \in \{0, 1, \ldots, k-1, k+1, \ldots, m\}$ satisfying

$$d_i^*(x_j) = d_{j-1}^*(x_i) \quad \text{in} \quad \pi_t(X_{m-2}, (d_j d_i)^*(a))$$

for all $0 \leq i < j \leq m$ with $i \neq k$, $j \neq k$, there is an element $y \in \pi_t(X_m, a)$ such that

$$d_i^*(y) = x_i$$

for all $0 \leq i \leq m$ with $i \neq k$.

The simplicial space X satisfies the *π_*-Kan condition* if it satisfies the π_t-Kan condition for all $m, t \geq 1$ and all $a \in X_m$.

Example A.48. As Bousfield and Friedlander already remark, the π_t-Kan condition is automatically satisfied at $a \in X_m$ for all $t \geq 1$ whenever $a = \sigma_m^*(b)$ is the degeneracy of some point $b \in X_0$, where $\sigma_m : [m] \longrightarrow [0]$ is the unique morphism. Indeed, in that special case, the collection of homotopy groups $\pi_t(X_n, \sigma_n^*(b))$ forms a simplicial group as n varies over the objects of $\mathbf{\Delta}$; the simplicial sets underlying simplicial groups always satisfy the Kan extension condition, an observation due to Moore, see for example [63, I Lemma 3.4].

Bousfield and Friedlander work with simplicial sets, whereas we need their result for topological spaces. The next theorem does the straightforward translation using the singular complex functor. The reader may observe that [22, Thm. B.4] has no Reedy cofibrancy condition; this should not be surprising because latching morphisms of simplicial objects in simplicial sets are automatically injective, and hence cofibrations. Put differently, bisimplicial spaces are automatically Reedy cofibrant when considered as simplicial objects in simplicial sets.

A *homotopy fiber sequence* is a pair of composable continuous maps

$$A \xrightarrow{i} X \xrightarrow{f} Y$$

whose composite $fi : A \longrightarrow Y$ is constant with value y_0, and such that the induced map

$$\begin{aligned} A &\longrightarrow F(f) = \{(\lambda, x) \in Y^{[0,1]} \times X \mid \lambda(0) = y_0,\ \lambda(1) = f(x)\} \\ a &\longmapsto \quad (\mathrm{const}_{y_0}, i(a)) \end{aligned}$$

from A to the homotopy fiber of f is a weak equivalence.

Theorem A.49. *Let*

$$A \xrightarrow{i} X \xrightarrow{f} Y$$

be morphisms of Reedy cofibrant simplicial topological spaces such that the composite $f \circ i$ is constant. Suppose that the following conditions hold:

(i) *For every $n \geq 0$ the sequence (i_n, f_n) is a homotopy fiber sequence.*
(ii) *The simplicial spaces X and Y satisfy the π_*-Kan condition.*
(iii) *The morphism of simplicial sets $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ is a Kan fibration.*

Then the sequence $(|i|, |f|)$ is a homotopy fiber sequence.

Proof We transfer the question into the context of simplicial sets by use of the singular complex functor, and then quote the theorem of Bousfield and Friedlander. In the commutative diagram

$$\begin{tikzcd}
{|\mathrm{Sing}\circ A|^{\mathrm{it}}} \arrow[r, "{|\mathrm{Sing}\circ i|^{\mathrm{it}}}"] \arrow[d, "\epsilon_A"'] & {|\mathrm{Sing}\circ X|^{\mathrm{it}}} \arrow[r, "{|\mathrm{Sing}\circ f|^{\mathrm{it}}}"] \arrow[d, "\epsilon_X"] & {|\mathrm{Sing}\circ Y|^{\mathrm{it}}} \arrow[d, "\epsilon_Y"] \\
{|A|} \arrow[r, "{|i|}"'] & {|X|} \arrow[r, "{|f|}"'] & {|Y|}
\end{tikzcd}$$

all vertical maps are weak equivalences by Proposition A.46, because we assumed that the simplicial spaces A, X and Y are Reedy cofibrant. The property of being a homotopy fiber sequence is invariant under pointwise weak equivalences, so we may show that the upper row is a homotopy fiber sequence. This sequence arises from a commutative square of bisimplicial sets

$$\begin{tikzcd}
\mathrm{Sing}\circ A \arrow[r, "\mathrm{Sing}\circ i"] \arrow[d] & \mathrm{Sing}\circ X \arrow[d, "\mathrm{Sing}\circ f"] \\
* \arrow[r] & \mathrm{Sing}\circ Y
\end{tikzcd}$$

by taking iterated realizations. In every fixed simplicial dimension, the square of simplicial sets is a homotopy fiber square in the sense of [22, Def. A.2], by hypothesis (i). Moreover, the bisimplicial sets $\mathrm{Sing}\circ X$ and $\mathrm{Sing}\circ Y$ satisfy the π_*-Kan condition by assumption (ii), and the morphism of simplicial sets $\pi_0^v(\mathrm{Sing}\circ f) : \pi_0^v(\mathrm{Sing}\circ X) \longrightarrow \pi_0^v(\mathrm{Sing}\circ Y)$ is a Kan fibration by assump-

tion (iii). So [22, Thm. B.4] shows that the square of diagonal simplicial sets

$$\begin{tikzcd} \mathrm{diag}(\mathrm{Sing}\circ A) \arrow[r, "\mathrm{diag}(\mathrm{Sing}\circ i)"] \arrow[d] & \mathrm{diag}(\mathrm{Sing}\circ X) \arrow[d, "\mathrm{diag}(\mathrm{Sing}\circ f)"] \\ * \arrow[r] & \mathrm{diag}(\mathrm{Sing}\circ Y) \end{tikzcd}$$

is homotopy cartesian. After geometric realization we can identify the diagonal realization with the iterated realization as in Proposition A.37 (iii), so this proves the claim. □

At some later stage we will need to know that the product of two Reedy cofibrant simplicial spaces is again Reedy cofibrant. The proof of this fact is rather formal and works in a much broader context, as the following proposition shows.

Proposition A.50. *Let C be a model category that is equipped with a monoidal product $\boxtimes$ satisfying the pushout product property for cofibrations. Then for all Reedy cofibrant simplicial objects $X, Y : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow C$, the objectwise monoidal product $X \boxtimes Y : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow C$ is Reedy cofibrant.*

Proof I learned this proof from Cary Malkiewich. We let $\mathcal{P}(m)$ denote the power set of the set $\{1, \dots, m\}$, i.e., the set of subsets. We also write $\mathcal{P}(m)$ for the associated poset category, under inclusion. An *m-cube* in C is simply a functor $F : \mathcal{P}(m) \longrightarrow C$. We call such an m-cube *cofibrant* if for every subset B of $\{1, \dots, m\}$ the canonical morphism

$$l_B \; : \; \mathrm{colim}_{A \subsetneq B}\; F(A) \;\longrightarrow\; F(B)$$

is a cofibration.

We claim that for every cofibrant m-cube F and all sets $\mathcal{Y} \subset \mathcal{Z} \subset \mathcal{P}(m)$ that are both closed under passage to subsets, the canonical morphism

$$\mathrm{colim}_{A \in \mathcal{Y}}\; F(A) \;\longrightarrow\; \mathrm{colim}_{A \in \mathcal{Z}}\; F(A)$$

is a cofibration. We start with the special case where $\mathcal{Z} = \mathcal{Y} \cup \{B\}$ for some subset B of $\{1, \dots, m\}$ that does not belong to $\mathcal{Y}$. Since $\mathcal{Z}$ is closed under taking subsets, every proper subset of B belongs to $\mathcal{Z}$, and hence to $\mathcal{Y}$. Then the square

$$\begin{tikzcd} \mathrm{colim}_{A \subsetneq B}\; F(A) \arrow[r, "l_B"] \arrow[d] & F(B) \arrow[d] \\ \mathrm{colim}_{A \in \mathcal{Y}}\; F(A) \arrow[r] & \mathrm{colim}_{A \in \mathcal{Z}}\; F(A) \end{tikzcd}$$

is a pushout in C. The upper horizontal latching morphism is a cofibration by hypothesis; so the lower horizontal morphism is a cofibration.

In the general case we choose a chain of intermediate subsets

$$\mathcal{Y} = \mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \ldots \subset \mathcal{Y}_k = \mathcal{Z}$$

such that each $\mathcal{Y}_i$ is closed under taking subsets and $\mathcal{Y}_i$ has exactly one element more than $\mathcal{Y}_{i-1}$. The claim then holds for each pair $(\mathcal{Y}_i, \mathcal{Y}_{i-1})$. Since the composite of two cofibrations is a cofibration, this proves the general case.

Now we turn to the proof of the proposition. In Remark A.40 we specified an isomorphism of categories $\kappa : \mathbf{\Delta}(m)^{\mathrm{op}} \cong \mathcal{P}(m)$. Since X is Reedy cofibrant, the m-cube

$$\mathcal{P}(m) \xrightarrow[\cong]{\kappa^{-1}} \mathbf{\Delta}(m)^{\mathrm{op}} \xrightarrow{u} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} C$$

is cofibrant for every $m \geq 0$, and the same is true for Y. We observe that the $(m+n)$-cube

$$X \otimes Y \; : \; \mathcal{P}(m+n) \longrightarrow C \, , \quad U + V \longmapsto X_{|U|} \boxtimes Y_{|V|}$$

is then again cofibrant, where '+' denotes disjoint union of sets. Indeed, for $B \in \mathcal{P}(m)$ and $B' \in \mathcal{P}(n)$, the latching object

$$\mathrm{colim}_{A+A' \subsetneq B+B'} \, X_{|A|} \boxtimes Y_{|A'|}$$

is isomorphic to the pushout

$$(L_B X) \boxtimes Y_{|B'|} \cup_{(L_B X) \boxtimes (L_{B'} Y)} X_{|B|} \boxtimes (L_{B'} Y)$$

because $\boxtimes$ preserves colimits in both variables. Under this isomorphism, the latching morphism

$$l_{B+B'} \; : \; \mathrm{colim}_{A+A' \subsetneq B+B'} \, X_{|A|} \boxtimes Y_{|A'|} \longrightarrow X_{|B|} \boxtimes Y_{|B'|}$$

becomes the pushout product $l^X_B \Box l^Y_{B'}$ of the latching morphisms for X and Y. This latter morphism is a cofibration by the pushout product property.

Now we take $m = n$ and let $\mathcal{X}$ be the subposet of $\mathcal{P}(n+n)$ of proper diagonal elements, i.e., the sets $U + U$ for a proper subset U of $\{1, \ldots, n\}$. The latching object $L_n(X \boxtimes Y)$ is then a colimit of the functor $X \otimes Y$ over the poset $\mathcal{X}$. We let $\mathcal{Y}$ be the subposet of $\mathcal{P}(n+n)$ consisting of those sets $U + V$ such that $U \cup V \neq \{1, \ldots, n\}$. The latching map for $X \boxtimes Y$ factors as the composite

$$\begin{aligned} L_n(X \boxtimes Y) \; &= \; \mathrm{colim}_{U+U \in \mathcal{X}} \, X_{|U|} \boxtimes Y_{|U|} \\ &\longrightarrow \mathrm{colim}_{U+V \in \mathcal{Y}} \, X_{|U|} \boxtimes Y_{|V|} \longrightarrow X_n \boxtimes Y_n \, . \end{aligned} \tag{A.51}$$

We observe that the inclusion $\mathcal{X} \longrightarrow \mathcal{Y}$ is final, i.e., for every $U \in \mathcal{Y}$ the comma category $U \downarrow \mathcal{X}$ is non-empty and connected. So the first morphism in

(A.51) is an isomorphism. We set $\mathcal{Z} = \mathcal{P}(n+n)$; since $\mathcal{Z}$ has a terminal object, the object $X_n \boxtimes Y_n$ is a colimit of the functor $X \otimes Y$ over $\mathcal{Z}$. Both $\mathcal{Y}$ and $\mathcal{Z}$ are closed under passage to subsets, so the first paragraph shows that the second morphism in (A.51) is a cofibration. This proves the claim. $\square$

Appendix B
Equivariant spaces

In this appendix we collect basic results about the equivariant homotopy theory of G-spaces. Initially G can be any topological group, but we will eventually specialize to compact Lie groups. We start out by checking that taking fixed-points commutes with certain kinds of colimits, namely pushouts and sequential colimits along closed embeddings, smash products, geometric realization and latching objects, see Proposition B.1. Proposition B.7 provides a self-contained proof of the standard model structure on the category of G-spaces, relative to a set of closed subgroups. In the realm of compact Lie groups, various change of group functors preserve cofibrations, namely fixed-points (see Proposition B.12), restriction along a continuous homomorphism, induction, and orbits (see Proposition B.14). Proposition B.17 is a useful decomposition result for $(G\backslash X)^K$, the K-fixed-points of the G-orbits of a G-free $(K\times G)$-space.

Then we turn to equivariant $\mathbf{\Gamma}$-spaces. We start with the observation that compactly generated spaces are 'closed under prolongation of $\mathbf{\Gamma}$-spaces'. More precisely, if $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$ is a $\mathbf{\Gamma}$-k-space such that $F(n_+)$ is compactly generated for every $n \geq 0$, then it is not completely obvious whether the prolongation $F(K)$ to a general compactly generated space K, defined as a certain quotient space of $\coprod_{n\geq 0} F(n_+)\times K^n$, is weak Hausdorff. We show in Proposition B.26 that this is automatically the case. Because no 'weak Hausdorffication' is necessary, there are no unexpected identifications, and the underlying set of $F(K)$ is what one first thinks of.

For us, the main purpose of $\mathbf{\Gamma}$-spaces is to provide spectra by evaluation on spheres. We want to do this equivariantly, in the presence of an action of a compact Lie group. Proposition B.42 gives a way to calculate the fixed-points of a prolonged $\mathbf{\Gamma}$-G-space under the action of a *connected* compact Lie group. Since G-fixed-points can be obtained by first taking fixed-points of the identity component G°, and then fixed-points of the finite group $\pi_0 G = G/G^\circ$, this formula effectively reduces questions about fixed-points to the case of finite groups.

In order to analyze prolonged $\mathbf{\Gamma}$-G-spaces homotopically, we need a cofibrancy condition that we introduce in Definition B.33. This kind of cofibrancy is stable under passage to closed subgroups and taking fixed-points (see Proposition B.35) and guarantees that evaluation on spheres produces a G-spectrum that is equivariantly connective (see Proposition B.43). Moreover, for G-cofibrant $\mathbf{\Gamma}$-G-spaces, prolongation is homotopical, i.e., strict equivalences prolong to G-weak equivalences on finite G-CW-complexes (see Proposition B.48).

We conclude this appendix with a reformulation and generalization of the Segal-Shimakawa delooping formalism for equivariant $\mathbf{\Gamma}$-spaces, see Theorems B.61 and B.65. When restricted to finite groups, these theorems show that evaluation of a G-cofibrant $\mathbf{\Gamma}$-G-space F on spheres provides a positive G-Ω-spectrum if F is 'special' (compare Definition B.49), and a full fledged G-Ω-spectrum if F is 'very special' (compare Definition B.57). As we explain in Remark B.66, there is no hope to obtain a G-Ω-spectrum for compact Lie groups of positive dimension. However, we do have partial delooping results for non-finite compact Lie groups: Theorem B.65 effectively says that evaluating a G-cofibrant special $\mathbf{\Gamma}$-G-space on spheres yields a 'G°-trivial positive G-Ω-spectrum', where G° is the identity component of G. In other words, evaluating on spheres provides equivariant deloopings with respect to all those G-representations on which G° acts trivially. If G is not finite, this is of course a very restricted class of representations; but for trivial representations we can at least conclude that evaluation on spheres gives a 'naive' G-Ω-spectrum.

We let G be a topological group, which we take to mean a group object in the category $\mathbf{T}$ of compactly generated spaces. So a topological group is a compactly generated space equipped with an associative and unital multiplication

$$\mu \,:\, G\times G \longrightarrow G$$

that is continuous with respect to the compactly generated product topology, and such that the shearing map

$$G\times G \longrightarrow G\times G\,, \quad (g,h) \longmapsto (g,gh)$$

is a homeomorphism (again for the compactly generated product topology). This implies in particular that inverses exist in G, and that the inverse map $g \mapsto g^{-1}$ is continuous. A G-space is then a compactly generated space X equipped with an associative and unital action

$$\alpha \,:\, G\times X \longrightarrow X$$

that is continuous with respect to the compactly generated product topology. We write $G\mathbf{T}$ for the category of G-spaces and continuous G-maps.

The forgetful functor from G-spaces to compactly generated spaces has both

a left and a right adjoint, and hence limits and colimits of G-spaces are created in the underlying category $\mathbf{T}$. This has nothing to do with topology, and is entirely formal, using only that the underlying category $\mathbf{T}$ is cartesian closed, complete and cocomplete.

Since colimits of G-spaces are created in the underlying category $\mathbf{T}$ of compactly generated spaces, the earlier caveat applies as well. A colimit in G-spaces is calculated by first forming a colimit in the category **Spc** of all topological spaces (or equivalently, in the full subcategory $\mathbf{K}$ of k-spaces); the result is a k-space, but it need not be weak Hausdorff. In that case an application of the functor $w : \mathbf{K} \longrightarrow \mathbf{T}$ left adjoint to the inclusion produces a colimit in $\mathbf{T}$. This colimit comes with a preferred G-action making it a colimit in the category of G-spaces. One has to beware that whenever the colimit in **Spc** is not weak Hausdorff, the functor w changes the underlying set; in particular, the forgetful functor to sets does *not* preserve such colimits.

Now we consider a closed subgroup H of a topological group G. Then H is compactly generated in the subspace topology by Proposition A.5 (i), and hence a topological group (internal to the category $\mathbf{T}$) in its own right. For a G-space X we denote by

$$X^H \ = \ \{x \in X \mid hx = x \text{ for all } h \in H\}$$

the subspace of H-fixed-points. For an individual element $h \in H$ the h-fixed subspace $\{x \in X \mid hx = x\}$ is the preimage of the diagonal under the continuous map $(\mathrm{Id}, h \cdot -) : X \longrightarrow X \times X$, so it is a closed subspace of X by Proposition A.6. As an intersection of closed subsets, X^H is then closed in X, and hence compactly generated in the subspace topology, by Proposition A.5 (i).

The following proposition records that fixed-points preserve certain kinds of colimits. In part (iv) we consider a simplicial G-space $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow G\mathbf{T}$, and we write X^G for the simplicial space consisting of the fixed-points of X, i.e., the composite functor

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\ X\ } G\mathbf{T} \xrightarrow{(-)^G} \mathbf{T} \,.$$

Proposition B.1. *Let G be a topological group.*

(i) *For every pushout square of G-spaces on the left*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "h"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A^G \arrow[r, "f^G"] \arrow[d, "i^G"'] & C^G \arrow[d, "j^G"] \\ B^G \arrow[r, "h^G"'] & D^G \end{tikzcd}$$

in which the map i is a closed embedding, the square of fixed-point spaces on the right is a pushout.

(ii) *Taking G-fixed-points commutes with filtered colimits along continuous G-maps that are closed embeddings.*

(iii) *For all based G-spaces X and Y the canonical map $X^G \wedge Y^G \longrightarrow (X \wedge Y)^G$ is a homeomorphism.*

(iv) *For every simplicial G-space $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow G\mathbf{T}$ and every $n \geq 0$, the canonical maps*

$$|X^G| \longrightarrow |X|^G \qquad \textit{and} \qquad L_n(X^G) \longrightarrow (L_n X)^G$$

are homeomorphisms.

Proof (i) Pushouts in G-spaces are formed on underlying compactly generated spaces. Since i is a closed embedding, so is its restriction to fixed-points i^G. So by Proposition A.13 both pushouts are 'as expected', i.e., formed in the ambient category of all topological spaces. In particular, D is the set-theoretic disjoint union of the images of $B - i(A)$ and C, which are both G-invariant. So D^G is the set-theoretic disjoint union of the images of $(B - i(A))^G = B^G - (i^G)(A^G)$ and C^G. The canonical map $B^G \cup_{A^G} C^G \longrightarrow D^G$ is thus a continuous bijection.

Now we show that the canonical map is closed. We consider the commutative square:

$$\begin{tikzcd}
B^G \amalg C^G \arrow[r, "\mathrm{incl}"] \arrow[d, "p"'] & B \amalg C \arrow[d, "h+j"] \\
B^G \cup_{A^G} C^G \arrow[r, "h^G \cup j^G"'] & D
\end{tikzcd}$$

If O is a closed subset of $B^G \cup_{A^G} C^G$, then $p^{-1}(O)$ is closed in $B^G \amalg C^G$, and hence in $B \amalg C$ (since fixed-points are closed in the ambient space). The relation

$$p^{-1}(O) \;=\; (h + j)^{-1}((h^G \cup j^G)(O))$$

holds in $B \amalg C$; since the right vertical map $h + j$ is a proclusion, $(h^G \cup j^G)(O)$ is closed in D, and hence also in D^G.

(ii) We let P be a filtered poset and $F : P \longrightarrow G\mathbf{T}$ a functor to the category of G-spaces. We must show that the canonical continuous map

$$\kappa \;:\; \mathrm{colim}_P\, F^G \;\longrightarrow\; (\mathrm{colim}_P\, F)^G$$

is a homeomorphism, where $F^G = (-)^G \circ F$. Proposition A.14 ensures that both colimits are formed in the ambient category of sets, so the map is bijective. To

see that the canonical map is also closed, we consider the commutative square

$$\begin{tikzcd} \coprod_{j\in P} F(j)^G \arrow[r, "\mathrm{incl}"] \arrow[d, "p"'] & \coprod_{j\in P} F(j) \arrow[d, "q"] \\ \mathrm{colim}_P\ F^G \arrow[r, "\kappa"'] & \mathrm{colim}_P F \end{tikzcd}$$

in which the vertical maps are proclusions. The canonical maps $\kappa_j : F(j) \longrightarrow \mathrm{colim}_P F$ are injective, so every point of $F(j)$ that becomes G-fixed in the colimit is already G-fixed in $F(j)$, i.e., the square is a pullback. If A is a closed subset of $\mathrm{colim}_P F^G$, then $p^{-1}(A)$ is closed in the coproduct of the $F(j)^G$'s. Since $F(j)^G$ is closed in $F(j)$, the set $q^{-1}(\kappa(A)) = p^{-1}(A)$ is closed in the coproduct of the $F(j)$'s. Since q is a proclusion, the set $\kappa(A)$ is closed in $\mathrm{colim}_P F$, and hence also in $(\mathrm{colim}_P F)^G$. This shows that κ is a closed map.

(iii) Points of compactly generated spaces are closed (Proposition A.4 (iii)), so the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ is closed in $X \times Y$. The claim then follows by applying part (i) to the pushout of the diagram

$$* \longleftarrow X \times \{y_0\} \cup \{x_0\} \times Y \longrightarrow X \times Y .$$

(iv) Since X_n^G is closed inside X_n, the inclusion $X^G \subset X$ induces a closed embedding $|X^G| \longrightarrow |X|$ by Proposition A.35 (iii). The image of this map is clearly G-fixed. So it remains to show that every G-fixed-point of $|X|$ is in the image of $|X^G|$. We let $(x, t) \in X_l \times \Delta^l$ be the minimal representative of a given G-fixed-point in $|X|^G$. Then for every $g \in G$ the point (gx, t) is equivalent to (x, t). Since the minimal representative is unique, this forces $gx = x$. Hence $x \in X_l^G$, which proves the claim.

The latching map $l_n : L_n X \longrightarrow X_n$ is a closed embedding by Proposition A.42 (iii). Hence the restriction to fixed-points $(l_n)^G : (L_n X)^G \longrightarrow (X_n)^G$ is also a closed embedding. The composite

$$L_n(X^G) \longrightarrow (L_n X)^G \xrightarrow{(l_n)^G} (X_n)^G$$

is the latching map for the simplicial compactly generated space X^G, hence also a closed embedding, again by Proposition A.42 (iii). So the map in question is a closed embedding.

It remains to show that every G-fixed-point of $L_n X$ is the image of a point in $L_n(X^G)$. By Proposition A.42, the latching map identifies $L_n X$ with the union of the subspaces $s_i^*(X_{n-1})$ for $i = 0, \ldots, n-1$. So we may show that the composite

$$L_n(X^G) \longrightarrow (L_n X)^G \xrightarrow[\cong]{(l_n)^G} \Big(\bigcup_{i=0,\ldots,n-1} s_i^*(X_{n-1})\Big)^G$$

is surjective. If $y \in X_{n-1}$ is such that $x = s_i^*(y)$ is G-fixed, then y is G-fixed

because every degeneracy map is injective and G-equivariant. So $y \in (X_{n-1})^G$ represents a point in $L_n(X^G)$ that maps to x. □

Proposition B.2. *Let H be a closed subgroup of a topological group G.*

(i) *The orbit space G/H is compactly generated in the quotient topology, and hence a G-space.*

(ii) *Let*

$$X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow \ldots$$

be a sequence of closed embeddings of G-spaces and X_∞ a colimit of the sequence in the category of G-spaces. Then for every compact space K, every continuous G-map from $G/H \times K$ to X_∞ factors through X_n for some n, and that factorization is a continuous G-map.

Proof (i) The equivalence relation $E \subset G \times G$ that gives rise to G/H is the inverse image of H under the continuous map

$$G \times G \longrightarrow G\,, \quad (g, \bar{g}) \longmapsto g^{-1} \cdot \bar{g}\,.$$

Because H is closed, so is the equivalence relation, and hence the quotient topology is compactly generated by the criterion given by Proposition A.7.

(ii) We let $f : G/H \times K \longrightarrow X_\infty$ be a continuous G-map. Colimits of G-spaces are created in the underlying category $\mathbf{T}$ of compactly generated spaces; since the maps in question are closed embeddings, the sequential colimit is created in the ambient category **Spc** of all topological spaces, by Proposition A.14. The composite

$$K \xrightarrow{(eH,-)} G/H \times K \xrightarrow{f} X_\infty \tag{B.3}$$

factors through a continuous map $\bar{f} : K \longrightarrow X_n$ for some $n \geq 0$, by Proposition A.15 (i). The canonical map $X_n \longrightarrow X_\infty$ is injective by Proposition A.14. Since the image of the composite (B.3) is contained in the H-fixed-points of X_∞, the image of the factorization $\bar{f} : K \longrightarrow X_n$ is contained in the H-fixed-points of X_n. So the composite

$$G \times K \xrightarrow{G \times \bar{f}} G \times X_n \xrightarrow{\text{act}} X_n \tag{B.4}$$

factors through a well-defined set theoretic G-equivariant map

$$f' \;:\; G/H \times K \longrightarrow X_n\,.$$

The map $\mathrm{proj} \times K : G \times K \longrightarrow G/H \times K$ is a proclusion by Proposition A.3, so the continuity of (B.4) implies that f' is also continuous, and hence the desired factorization of the original morphism f. □

An argument that we need several times in this book proves that a certain model structure is topological. To avoid repeating the same kind of argument, we axiomatize it. The 'classical' model structure on the category of all topological spaces was established by Quillen in [134, II.3 Thm. 1]. We use the straightforward adaptation of this model structure to the category of compactly generated spaces, which is described for example in [80, Thm. 2.4.25]. In this model structure on the category $\mathbf{T}$, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of generalized CW-complexes, i.e., relative cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions.

We consider a model category $\mathcal{M}$ that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of compactly generated spaces. We denote the tensor by $\times$. Given a morphism $f : X \longrightarrow Y$ in $\mathcal{M}$ and a continuous map of spaces $g : A \longrightarrow B$, we denote by $f\Box g$ the *pushout product* morphism defined as

$$f\Box g = (f \times B) \cup (Y \times g) \ : \ X \times B \cup_{X\times A} Y \times A \ \longrightarrow \ Y \times B \, .$$

We recall that the model structure is called *topological* if the following two conditions hold:

- if f is a cofibration in $\mathcal{M}$ and g is a cofibration of spaces, then the pushout product morphism $f\Box g$ is a cofibration; and
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.

The pushout product condition can also be stated in two different, but equivalent, adjoint forms, compare [80, Lemma 4.2.2]. In the next proposition, we denote by

$$i_k \ : \ \partial D^k \ \longrightarrow \ D^k \qquad \text{and} \qquad j_k \ : \ D^k \times \{0\} \ \longrightarrow \ D^k \times [0,1]$$

the inclusions. Then $\{i_k\}_{k\geq 0}$ is the standard set of generating cofibrations for the Quillen model structure on the category of spaces, and $\{j_k\}_{k\geq 0}$ is the standard set of generating acyclic cofibrations, compare [80, Thm. 2.4.25].

Proposition B.5. *Let $\mathcal{M}$ be a model category that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of spaces. Suppose that there is a set of cofibrant objects $\mathcal{G}$ and a set of acyclic cofibrations $\mathcal{Z}$ of $\mathcal{M}$ with the following properties:*

(a) *The acyclic fibrations are characterized by the right lifting property with respect to the morphisms $K \times i_k$ for all $K \in \mathcal{G}$ and $k \geq 0$.*

(b) *The fibrations are characterized by the right lifting property with respect to the union of the morphisms $K \times j_k$ for all $K \in \mathcal{G}$ and $k \geq 0$ and the pushout products $c\Box i_k$ for all $c \in \mathcal{Z}$ and $k \geq 0$.*

Then the model structure is topological.

Proof Since the tensor bifunctor $\times$ has an adjoint in each variable, it preserves colimits in each variable. So it suffices to check the pushout product properties when the maps f and g are from the sets of generating (acyclic) cofibrations, compare [80, Cor. 4.2.5]. The set of inclusions of spheres into discs is closed under pushout product, in the sense that $i_k\Box i_l$ is homeomorphic to i_{k+l}. So pushout product with i_l preserves the set of generating cofibrations $\{K \times i_k\}_{K\in\mathcal{G},k\geq 0}$ (up to isomorphism). This takes care of the part of the pushout product property that involves the cofibrations only.

Similarly, the pushout product of j_k with i_l is isomorphic to j_{k+l}. So pushout product with i_l preserves the set of generating acyclic cofibrations $\{K\times j_k\}_{K\in\mathcal{G},k\geq 0}$, and it preserves the set of generating acyclic cofibrations $\{c \times i_k\}_{c\in\mathcal{Z},k\geq 0}$. This shows that the pushout product of an acyclic cofibration in $\mathcal{M}$ with a cofibration of spaces is again an acyclic cofibration.

Finally, pushout product with j_l takes the generating cofibrations $K \times i_k$ to morphisms of the form $K \times j_m$ with $K \in \mathcal{G}$ and $m \geq 0$, which are acyclic cofibrations. This shows that the pushout product of a cofibration in $\mathcal{M}$ with an acyclic cofibration of spaces is again an acyclic cofibration. □

Now we let C be a set of closed subgroups of a topological group G. We call a morphism $f : X \longrightarrow Y$ of G-spaces a C-*equivalence* (or C-*fibration*) if the restriction $f^H : X^H \longrightarrow Y^H$ to H-fixed-points is a weak equivalence (or Serre fibration) of spaces for all subgroups H of G in C. A C-*cofibration* is a morphism with the left lifting property with respect to all morphisms that are simultaneously C-equivalences and C-fibrations.

Proposition B.6 (Gluing lemma)**.** *Let G be a topological group and C a set of closed subgroups of G. Consider a commutative diagram of G-spaces*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ \bar{C} & \bar{A} \arrow[l, "\bar{g}"] \arrow[r, "\bar{f}"'] & \bar{B} \end{tikzcd}$$

such that f and $\bar{f}$ are h-cofibrations of G-spaces. Suppose that the maps α,β and γ are C-equivalences. Then the induced map of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ \bar{C} \cup_{\bar{A}} \bar{B}$$

is a C-equivalence.

Proof We let H be a closed subgroup from the set C, and we contemplate the commutative diagram of fixed-points:

$$\begin{tikzcd} C^H \arrow[d, "\gamma^H"'] & A^H \arrow[l, "g^H"'] \arrow[r, "f^H"] \arrow[d, "\alpha^H"] & B^H \arrow[d, "\beta^H"] \\ \bar{C}^H & \bar{A}^H \arrow[l, "\bar{g}^H"] \arrow[r, "\bar{f}^H"'] & \bar{B}^H \end{tikzcd}$$

Since f and $\bar{f}$ are h-cofibrations of G-spaces, f^H and $\bar{f}^H$ are h-cofibrations of spaces. The three vertical maps are weak equivalences by hypothesis. The gluing lemma for weak equivalences and pushouts along h-cofibrations then shows that the induced map on horizontal pushouts

$$\gamma^H \cup \beta^H \;:\; C^H \cup_{A^H} B^H \longrightarrow \bar{C}^H \cup_{\bar{A}^H} \bar{B}^H$$

is a weak equivalence, see for example [17, Appendix, Prop. 4.8 (b)]. Since f and $\bar{f}$ are h-cofibrations of G-spaces, they are in particular h-cofibrations of underlying spaces, and hence closed embeddings (Proposition A.31 (ii)). So taking H-fixed-points commutes with the horizontal pushout (by Proposition B.1 (i)), and we conclude that also the map

$$(\gamma \cup \beta)^H \;:\; (C \cup_A B)^H \longrightarrow (\bar{C} \cup_{\bar{A}} \bar{B})^H$$

is a weak equivalence. This proves the claim. □

The following C-projective model structure is well known and fairly standard, and mentioned in various places in the literature, for example in [52, Prop. 2.11] and [108, III Thm. 1.8]. However, I do not know a reference that is both self-contained and complete, so I provide the proof.

Proposition B.7. *Let G be a topological group and C a set of closed subgroups of G. Then the C-equivalences, C-cofibrations and C-fibrations form a model structure, the* C-projective model structure *on the category of G-spaces. This model structure is proper, cofibrantly generated and topological.*

Proof We number the model category axioms as in [48, 3.3]. The category of G-spaces is complete and cocomplete and all limits and colimits are created in the underlying category of compactly generated spaces. Model category axioms MC2 (2-out-of-3) and MC3 (closure under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of C-cofibrations. The proof of the remaining axioms uses Quillen's small object argument, originally given in [134, II p. 3.4], and later axiomatized in various places, for example in [48, 7.12] or [80, Thm. 2.1.14]. In the category of (non-equivariant) spaces, the set

$\{i_k : \partial D^k \longrightarrow D^k\}_{k\geq 0}$ of inclusions of spheres into discs detects Serre fibrations that are simultaneously weak equivalences. By adjointness, the set

$$I_C \;=\; \{G/H \times i_k \;:\; G/H \times \partial D^k \;\longrightarrow\; G/H \times D^k\}_{k\geq 0, H\in C} \tag{B.8}$$

then detects acyclic fibrations in the C-projective model structure on G-spaces. Similarly, the set of inclusions $\{j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]\}_{k\geq 0}$ detects Serre fibrations; so by adjointness, the set

$$J_C \;=\; \{G/H \times j_k\}_{k\geq 0, H\in C} \tag{B.9}$$

detects fibrations in the C-projective model structure on G-spaces.

All morphisms in I_C and J_C are closed embeddings, and this property is preserved by coproducts, cobase change and sequential colimits in the category of G-spaces. Proposition B.2 guarantees that sources and targets of all morphisms in I_C and J_C are finite (sometimes called 'finitely presented') with respect to sequences of closed embeddings of G-spaces. In particular, the sources of all these morphisms are finite with respect to sequences of I_C-cell complexes and J_C-cell complexes.

Now we can prove the factorization axiom MC5. Every morphism in I_C and J_C is a C-cofibration by adjointness. Hence every I_C-cofibration or J_C-cofibration is a C-cofibration of G-spaces. The small object argument applied to the set I_C gives a factorization of any morphism of G-spaces as a C-cofibration followed by a morphism with the right lifting property with respect to I_C. Since I_C detects the C-acyclic C-fibrations, this provides the factorizations as cofibrations followed by acyclic fibrations.

For the other half of the factorization axiom MC5 we apply the small object argument to the set J_C; we obtain a factorization of any morphism of G-spaces as a J_C-cell complex followed by a morphism with the right lifting property with respect to J_C. Since J_C detects the C-fibrations, it remains to show that every J_C-cell complex is a C-equivalence. To this end we observe that the morphisms in J_C are inclusions of deformation retracts internal to the category of G-spaces. This property is inherited by coproducts and cobase changes, so every morphism obtained by cobase changes of coproducts of morphisms in J_C is a homotopy equivalence of G-spaces, hence also a C-equivalence. We also need to pass to sequential colimits, which is fine because J_C-cell complexes are closed embeddings, and taking H-fixed-points commutes with sequential colimits over closed embeddings (Proposition B.1 (ii)).

It remains to prove the other half of MC4, i.e., that every C-acyclic C-cofibration $f : A \longrightarrow B$ has the left lifting property with respect to C-fibrations. The small object argument provides a factorization

$$A \xrightarrow{\;j\;} W \xrightarrow{\;q\;} B$$

as a J_C-cell complex followed by a C-fibration. In addition, q is a C-equivalence since f and j are. Since f is a C-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus f is a retract of the morphism j that has the left lifting property for C-fibrations. So f itself has the left lifting property for C-fibrations.

The model structure is topological by Proposition B.5. Right properness of the model structure is a straightforward consequence of right properness of the model structure on spaces, since the H-fixed-point functor preserves pullbacks and takes C-fibrations to Serre fibrations. Since the projective C-model structure is topological and all objects are fibrant, every cofibration is an h-cofibration by Corollary A.30 (iii). So left properness follows from the gluing lemma for C-equivalences (Proposition B.6). □

Definition B.10. Let G be a topological group. A *universal space* for a set C of closed subgroups of G is a C-cofibrant G-space E such that for every subgroup $H \in C$ the fixed-point space E^H is weakly contractible.

Any cofibrant replacement of the one-point G-space in the C-projective model structure is a universal space for the set C, so universal spaces exist for any set of closed subgroups. Moreover, any two universal spaces for the same subgroups are G-equivariantly homotopy equivalent:

Proposition B.11. *Let G be a topological group, C a set of closed subgroups of G, and E a universal G-space for the set C.*

(i) *Every C-cofibrant G-space admits a continuous G-map to E, and any two such maps are homotopic as G-maps.*
(ii) *If E' is another universal G-space for C, then every continuous G-map from E' to E is a G-equivariant homotopy equivalence.*

Proof (i) This all follows from the existence of the C-projective model structure described in Proposition B.7. We let A be a C-cofibrant G-space. The unique map $E \longrightarrow *$ to a one-point G-space is a C-equivalence and a C-fibration. So the unique morphism from A to the one-point G-space lifts to a continuous G-map $A \longrightarrow E$. The C-projective model structure is topological, so the inclusion $A \times \{0, 1\} \longrightarrow A \times [0, 1]$ is a C-cofibration, and thus has the left lifting property with respect to $E \longrightarrow *$. Given two G-maps $f, f' : A \longrightarrow E$,

any solution to the lifting problem

$$\begin{tikzcd} A \times \{0,1\} \arrow[r, "f+f'"] \arrow[d] & E \arrow[d] \\ A \times [0,1] \arrow[r] & * \end{tikzcd}$$

is a G-homotopy from f to f'. Since universal G-spaces are $\mathcal{C}$-cofibrant, part (ii) is a consequence of (i). □

A morphism of G-spaces is a *G-cofibration* if it has the left lifting property with respect to all morphisms that are simultaneously weak equivalences and Serre fibrations on the fixed-points for all closed subgroups of G. Equivalently, G-cofibrations are the $\mathcal{A}ll$-cofibrations in the sense of Proposition B.7 for the maximal set of all closed subgroups of G. Relative G-CW-complexes are special kinds of $I_{\mathcal{A}ll}$-cell complexes (with $I_{\mathcal{A}ll}$ defined in (B.8)), namely those where all equivariant cells of the same dimension are attachable at once, and in order of increasing dimensions. Thus we have the following implications between the various kinds of 'nice equivariant embeddings':

$$\text{relative } G\text{-CW-complex} \implies G\text{-cofibration} \implies \text{h-cofibration of } G\text{-spaces}$$

The second implication is Corollary A.30, applied to the $\mathcal{A}ll$-projective model structure. Both implications are strict.

Proposition B.12. *Let N be a closed normal subgroup of a topological group G. Then for every G-cofibration $i : A \longrightarrow B$ the map*

$$i^N \ : \ A^N \ \longrightarrow \ B^N$$

is a G/N-cofibration, and the map

$$i \cup \text{incl} \ : \ A \cup_{A^N} B^N \ \longrightarrow \ B$$

is a G-cofibration.

Proof The class of G-cofibrations for which the claim holds is clearly closed under coproducts and retracts. We consider a pushout square of G-spaces on the left

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "j"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A^N \arrow[r, "i^N"] \arrow[d] & B^N \arrow[d] \\ C^N \arrow[r, "j^N"'] & D^N \end{tikzcd}$$

such that i is a G-cofibration for which the claim holds. Then i is a closed embedding, and the square on the right is also a pushout of G-spaces (by Proposition B.1 (i)). In particular, j^N is a G-cofibration whenever i^N is. This means that $C \cup_{C^N} D^N$ is also a pushout of C and B^N over A^N, and hence the square

$$\begin{tikzcd} A \cup_{A^N} B^N \arrow[r, "i \cup \mathrm{incl}"] \arrow[d] & B \arrow[d] \\ C \cup_{C^N} D^N \arrow[r, "j \cup \mathrm{incl}"'] & D \end{tikzcd}$$

is another pushout of G-spaces. The upper horizontal map is a G-cofibration by hypothesis, hence so is the lower horizontal map. So the class of G-cofibrations satisfying the claim is closed under cobase change. Because N-fixed-points preserve sequential colimits along closed embeddings (Proposition B.1 (ii)), the class of G-cofibrations satisfying the claim is also closed under sequential composites.

Given these closure properties, it suffices to verify the claim for the generating G-cofibrations of the form

$$G/H \times i_k \ : \ G/H \times \partial D^k \ \longrightarrow \ G/H \times D^k \ .$$

Because $(G/H)^N$ is either empty (whenever N is not contained in H) or all of G/H (whenever $N \leq H$), the map in question is either the map $G/H \times i_k$ or the identity of $G/H \times D^k$; in either case it is a G-cofibration. This proves the claim for the generating cofibrations, and thus concludes the proof. □

Now we specialize from general topological groups to the class of compact topological groups. The first useful special feature is that in this restricted context, orbit spaces are always compactly generated without any need to force the weak Hausdorff condition. The arguments involved in the following proposition are well-known, see for example [131, 1.1.3] or [26, I Cor. 1.3] for closely related statements. Since the references I know of work in slightly different categories, I spell out the proof.

Proposition B.13. *Let G be a compact topological group and X a G-space.*

(i) *The orbit space $G\backslash X$ is compactly generated in the quotient topology.*
(ii) *The projection $\Pi_X : X \longrightarrow G\backslash X$ is both open and closed.*
(iii) *For every G-invariant closed subset Y of X, the tautological map $G\backslash Y \longrightarrow G\backslash X$ is a closed embedding.*

Proof (i) Proposition A.7 reduces the claim to showing that the orbit equivalence relation

$$E \ = \ \{(gx, x) \mid g \in G, \ x \in X\}$$

is closed in $X \times X$. Since G is compact, projection away from G is a closed map; so the composite

$$G \times X \times X \xrightarrow{(g,x,y)\mapsto(g,gx,y)} G \times X \times X \xrightarrow{\text{proj}} X \times X$$

is a closed map since the first map is a homeomorphism. Since X is weak Hausdorff, the diagonal Δ_X is closed in $X \times X$. So $G \times \Delta_X$ is closed in $G \times X \times X$. The orbit relation E is the image of $G \times \Delta_X$ under the above composite, so E is closed in $X \times X$.

(ii) If O is an open subset of X, then gO is open for every $g \in G$, since left translation by g is a homeomorphism. So

$$\Pi_X^{-1}(\Pi_X(O)) \;=\; \bigcup_{g\in G} gO$$

is open as a union of open subsets. Hence $\Pi_X(O)$ is open in the quotient topology. The shearing map

$$\chi \;:\; G \times X \;\longrightarrow\; G \times X\,, \quad (g, x) \;\longmapsto\; (g, gx)$$

is a homeomorphism, and the composite

$$G \times X \xrightarrow{\;\chi\;} G \times X \xrightarrow{\text{proj}} X$$

is the action map $\alpha : G \times X \longrightarrow X$. Since G is compact, projection away from G is a closed map; so the action map is also a closed map. If A is closed in X, then $G \times A$ is closed in $G \times X$. So the set

$$\Pi_X^{-1}(\Pi_X(A)) \;=\; \alpha(G \times A)$$

is closed in X. Hence $\Pi_X(A)$ is closed in the quotient topology.

(iii) The tautological map $G\backslash Y \longrightarrow G\backslash X$ is continuous and injective. We show that the map is also closed. We denote by $\Pi_Y : Y \longrightarrow G\backslash Y$ the quotient map. We let $A \subset G\backslash Y$ be any closed subset. Then $\Pi_Y^{-1}(A)$ is closed in Y, hence also in X. Since Π_X is a closed map by part (ii), $\Pi_X(A)$ is closed in $G\backslash X$. $\square$

Now we specialize our discussion to compact Lie groups. If K is a closed subgroup of a compact Lie group G of smaller dimension, then the underlying K-space of a G-CW-complex need not admit a K-CW-structure – an example is given by Illman in [85, Sec. 2]. Nevertheless, the underlying K-space of a G-CW-complex is always K-homotopy equivalent to a K-CW-complex, which can be chosen to be compact if the original G-space is, see [85, Thm. A].

The next proposition shows that with respect to restriction of group actions, the class of cofibrant equivariant spaces is better behaved than equivariant CW-complexes. An additional advantage of cofibrant spaces over CW-complexes is that 'cofibrant' is a property, whereas CW-structures are additional data.

Proposition B.14. *Let G be a compact Lie group.*

(i) *For every compact Lie group K and every continuous homomorphism $\alpha : K \longrightarrow G$ the restriction functor $\alpha^* : G\mathbf{T} \longrightarrow K\mathbf{T}$ takes G-cofibrations to K-cofibrations.*
(ii) *For every closed subgroup H of G the induction functor $G \times_H - : H\mathbf{T} \longrightarrow G\mathbf{T}$ takes H-cofibrations to G-cofibrations.*
(iii) *For every closed normal subgroup N of G the orbit functor $N\backslash - : G\mathbf{T} \longrightarrow (G/N)\mathbf{T}$ takes G-cofibrations to G/N-cofibrations.*

Proof (i) The restriction functor α^* preserves colimits, so we may show that it takes the generating G-cofibrations $G/H \times i_k : G/H \times \partial D^k \longrightarrow G/H \times D^k$ to K-cofibrations, for any closed subgroup H of G. The K-action on the smooth compact manifold G/H is by left translation through α; continuous homomorphisms between Lie groups are automatically smooth, compare [28, Prop. I.3.12], so the K-action on G/H is smooth. Illman's theorem [84, Cor. 7.2] provides a finite K-CW-structure on G/H, so in particular G/H is cofibrant as a K-space. Since the projective model structure on K-spaces (for the set of all closed subgroups) is topological, the map $G/H \times i_k$ is a K-cofibration.

(ii) Since $G \times_H -$ preserves colimits, it suffices to show that it takes the generating H-cofibrations $H/J \times i_k : H/J \times \partial D^k \longrightarrow H/J \times D^k$ to G-cofibrations, for any closed subgroup J of H. This in turn is clear since $G \times_H (H/J)$ is G-homeomorphic to G/J.

(iii) Since $N\backslash -$ preserves colimits, it suffices to show that it takes the generating G-cofibrations $G/H \times i_k : G/H \times \partial D^k \longrightarrow G/H \times D^k$ to G/N-cofibrations, for any closed subgroup H of G. This in turn is clear since $N\backslash(G/H)$ is (G/N)-homeomorphic to $(G/N)/(H/H \cap N)$. □

Now we show that taking cartesian product preserves equivariant cofibrations. There are two related questions, namely 'external products' of a G-space and a K-space, and 'internal products' of two G-spaces with diagonal action.

Proposition B.15. *Let G and K be topological groups.*

(i) *The pushout product of a G-cofibration with a K-cofibration is a $(G \times K)$-cofibration.*
(ii) *If G is a compact Lie group, then the pushout product of two G-cofibrations is a G-cofibration with respect to the diagonal G-action.*

Proof (i) The product functor

$$\times \; : \; G\mathbf{T} \times K\mathbf{T} \;\longrightarrow\; (G \times K)\mathbf{T}$$

preserves colimits in each variable, so it suffices to check the pushout product of a generating G-cofibration $G/H \times i_k$ with a generating K-cofibration $K/L \times$

i_m, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. The pushout product of these is isomorphic to

$$(G \times K)/(H \times L) \times i_{k+m}$$

and hence a cofibration of $(G \times K)$-spaces.

(ii) By (i) the pushout product of two G-cofibrations is a $(G \times G)$-cofibration. Since G is compact Lie, restriction along the diagonal embedding $G \longrightarrow G \times G$ preserves cofibrations by Proposition B.14 (i). □

Now we prove a decomposition result for certain kinds of fixed-points. We let G and K be topological groups and X a $(K \times G)$-space. We want to describe the K-fixed-points $(G\backslash X)^K$ of the G-orbit space $G\backslash X$. For a continuous homomorphism $\alpha : K \longrightarrow G$ we set

$$X^\alpha \;=\; \{x \in X \mid (k, \alpha(k))x = x \text{ for all } k \in K\} \; .$$

Equivalently, X^α is the fixed-point space of the graph of α, which is a closed subgroup of $K \times G$. The subspace X^α of X is invariant under the action of the centralizer of the image of α, i.e., the group

$$C(\alpha) \;=\; \{g \in G \mid g\alpha(k) = \alpha(k)g \text{ for all } k \in K\} \; .$$

The inclusion $X^\alpha \longrightarrow X$ then passes to a continuous map

$$\alpha^\flat \;:\; C(\alpha)\,\backslash X^\alpha \;\longrightarrow\; G\backslash X$$

on orbit spaces. For $x \in X^\alpha$ and $k \in K$ the relation

$$k(Gx) \;=\; G(kx) \;=\; G(\alpha(k)^{-1}x) \;=\; Gx$$

shows that $\alpha^\flat$ takes values in the K-fixed-points $(G\backslash X)^K$. Moreover, for $g \in G$ we have

$$g \cdot X^\alpha \;=\; X^{c_g \circ \alpha} \tag{B.16}$$

as subspaces of X. So the maps $\alpha^\flat$ and $(c_g \circ \alpha)^\flat$ arising from conjugate homomorphisms have the same image in the orbit space $G\backslash X$. It is relatively straightforward to see that the coproduct of all the maps $\alpha^\flat$ is bijective if the G-action is free; some point-set topology is involved in showing that this continuous bijection is in fact a homeomorphism if in addition G and K are compact Lie groups.

Proposition B.17. *Let G and K be compact Lie groups. Let X be a $(K \times G)$-space such that G-action is free. Then the map*

$$\coprod \alpha^\flat \;:\; \coprod\nolimits_{\langle\alpha\rangle} C(\alpha)\backslash X^\alpha \;\longrightarrow\; (G\backslash X)^K$$

is a homeomorphism. Here the disjoint union runs over conjugacy classes of

continuous homomorphisms $\alpha : K \longrightarrow G$ and $C(\alpha)$ is the centralizer, in G, of the image of α.

Proof We let $\Pi : X \longrightarrow G\backslash X$ denote the quotient map. We set

$$\bar{X} \ = \ \Pi^{-1}((G\backslash X)^K) \ .$$

Since X is compactly generated, so is $G\backslash X$ by Proposition B.13 (i). Since $(G\backslash X)^K$ a closed subset of the orbit space, $\bar{X}$ is a $(K \times G)$-invariant closed subspace of X. In particular, $\bar{X}$ is compactly generated in the subspace topology.

For a given continuous homomorphism $\alpha : K \longrightarrow G$, we set

$$X^{(\alpha)} \ = \ G \cdot X^{\alpha} \ ,$$

the smallest G-subspace of $\bar{X}$ containing X^{α}. The relation (B.16) shows that $X^{(\alpha)}$ is the union of the subsets $X^{\alpha'}$ as α' runs over all conjugates of α. We factor the map in question as a composite

$$\coprod\nolimits_{\langle\alpha\rangle} C(\alpha)\backslash X^{\alpha} \ \longrightarrow \ \coprod\nolimits_{\langle\alpha\rangle} G\backslash X^{(\alpha)} \ \longrightarrow \ G\backslash\bar{X} \ \longrightarrow \ (G\backslash X)^K \ ,$$

induced by the inclusions $X^{\alpha} \longrightarrow X^{(\alpha)} \longrightarrow \bar{X} \longrightarrow X$; we show that each of the three maps is a homeomorphism. The third map $G\backslash\bar{X} \longrightarrow (G\backslash X)^K$ is a homeomorphism by Proposition B.13 (iii).

For every $x \in \bar{X}$ and every $k \in K$ we have $k \cdot (Gx) = G(kx) = Gx$, so there exists a $g \in G$ such that $kx = g^{-1}x$. Since the G-action is free, the element g is uniquely determined by this property. So we can define a map $\beta_x : K \longrightarrow G$ by the property $kx = \beta_x(k)^{-1}x$; equivalently, the characterizing condition for β_x is that $(k, \beta_x(k)) \cdot x = x$. It is straightforward to see that β_x is a group homomorphism.

By definition, the stabilizer group of $x \in \bar{X}$ inside $K \times G$ is precisely the graph of the homomorphism $\beta_x : K \longrightarrow G$. Since X is compactly generated, this stabilizer group is a closed subset of $K \times G$, which means that the homomorphism β_x is continuous. Our next claim is that the assignment

$$\beta \ : \ \bar{X} \ \longrightarrow \ \mathrm{hom}(K, G) \ , \quad x \ \longmapsto \ \beta_x$$

is continuous. Since $\mathrm{hom}(K, G)$ has the subspace topology of $\mathrm{map}(K, G)$, it suffices to show that the adjoint map

$$\bar{\beta} \ : \ \bar{X} \times K \ \longrightarrow \ G \ , \quad (x, k) \ \longmapsto \ \beta_x(k)$$

is continuous. This, in turn, is equivalent to the claim that the graph of $\bar{\beta}$ is closed as a subset of $\bar{X} \times K \times G$. The graph of $\bar{\beta}$ is the inverse image of the diagonal under the continuous map

$$\bar{X} \times K \times G \ \longrightarrow \ \bar{X} \times \bar{X} \ , \quad (x, k, g) \ \longmapsto \ (kx, gx) \ .$$

So the graph of $\bar{\beta}$ is closed, hence $\bar{\beta}$ and β are continuous.

By Proposition A.25 the space $\hom(K, G)$ is the topological disjoint union of the G°-orbits under the conjugation action. In particular, every G°-conjugacy class is open and closed. A G-conjugacy class is a finite union of G°-conjugacy classes, so every G-conjugacy class in $\hom(K, G)$ is also open and closed. Since $\beta : \bar{X} \longrightarrow \hom(K, G)$ is continuous, $\bar{X}$ is the topological disjoint union of the subsets $X^{(\alpha)}$ as α ranges over all conjugacy classes in $\hom(K, G)$. Taking orbits commutes with disjoint unions, so the canonical map from $\coprod_{(\alpha)} G\backslash X^{(\alpha)}$ to $G\backslash\bar{X}$ is a homeomorphism.

The final step is to show that for every continuous homomorphism $\alpha : K \longrightarrow G$ the canonical continuous map

$$C(\alpha)\backslash X^\alpha \longrightarrow G\backslash X^{(\alpha)} \tag{B.18}$$

is a homeomorphism. The map is surjective because $X^{(\alpha)} = G \cdot X^\alpha$. The map is also injective; if $x, y \in X^\alpha$ satisfy $Gx = Gy$, then $x = gy$ for some $g \in G$, and so $x \in X^\alpha \cap X^{c_g \circ \alpha}$. This implies that $\alpha = c_g \circ \alpha$, and so $g \in C(\alpha)$. Hence $C(\alpha)x = C(\alpha)gy = C(\alpha)y$. Finally, the map (B.18) is closed: if $O \subset C(\alpha)\backslash X^\alpha$ is a closed subset, then $\Pi_{X^\alpha}^{-1}(O)$ is closed in X^α. Since X^α is closed in X, hence also in $X^{(\alpha)}$, the set $\Pi_{X^\alpha}^{-1}(O)$ is also closed in $X^{(\alpha)}$. Since the projection $X^{(\alpha)} \longrightarrow G\backslash X^{(\alpha)}$ is a closed map, the image of O in $G\backslash X^{(\alpha)}$ is closed. This completes the proof that the map (B.18) is a homeomorphism. □

Proposition B.19. *Let G be a compact Lie group and A a free cofibrant G-space. Then the functor $A \times_G -$ takes G-maps that are non-equivariant weak equivalences to weak equivalences.*

Proof We let $f : X \longrightarrow Y$ be a continuous G-map that is a non-equivariant weak equivalence. We start with the special case where A is a G-CW-complex with skeleton filtration

$$\emptyset = A^{-1} \subset A^0 \subset A^1 \subset \ldots \subset A^n \subset \ldots .$$

We show by induction over n that the map $A^n \times_G f : A^n \times_G X \longrightarrow A^n \times_G Y$ is a weak equivalence. The induction start with $n = -1$, where there is nothing to show. Now we consider $n \geq 0$ and assume the claim for smaller values of n. Since G acts freely on A, only free G-cells occur in the equivariant CW-structure. So there is an index set I and a pushout square of G-spaces:

$$\begin{tikzcd} G \times I \times \partial D^n \arrow[r, "\mathrm{incl}"] \arrow[d, "\alpha"'] & G \times I \times D^n \arrow[d] \\ A^{n-1} \arrow[r] & A^n \end{tikzcd}$$

The map $A^n \times_G f$ can thus be obtained by passing to pushouts in the horizontal direction of the commutative diagram

$$\begin{tikzcd} A^{n-1} \times_G X \arrow[d, "A^{n-1}\times_G f"'] & I \times \partial D^n \times X \arrow[l, "\alpha\times_G X"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "I\times\partial D^n\times f"] & I \times D^n \times X \arrow[d, "I\times D^n\times f"] \\ A^{n-1} \times_G Y & I \times \partial D^n \times Y \arrow[l, "\alpha\times_G Y"] \arrow[r, "\mathrm{incl}"'] & I \times D^n \times Y \end{tikzcd}$$

The left vertical map is a weak equivalence by the inductive hypothesis, and the middle and right vertical maps are weak equivalences because f is. The upper and lower right horizontal maps are h-cofibrations, so the gluing lemma for weak equivalences and pushouts along h-cofibrations (see for example [17, Appendix, Prop. 4.8 (b)]) shows that then the map on pushouts $A^n \times_G f$ is also a weak equivalence. Instead of the gluing lemma, we can alternatively quote [46, Lemma A.1].

The space $A\times_G X$ is the sequential colimit, along h-cofibrations, of the spaces $A^n \times_G X$, and similarly for $A \times_G Y$. In the category $\mathbf{T}$, all h-cofibrations are closed embeddings (Proposition A.31), and sequential colimits over sequences of closed embeddings in $\mathbf{T}$ preserve weak equivalences (Proposition A.17 (i)). So the map $A \times_G f$ is a weak equivalence. This completes the proof in the special case when A admits a G-CW-structure. A general cofibrant G-space is G-homotopy equivalent to a G-CW-complex, and the functor $- \times_G X$ takes G-homotopy equivalences to non-equivariant homotopy equivalences. This reduces the general case to the case of G-CW-complexes. □

Now we turn to $\mathbf{\Gamma}$-spaces and study the categorical and homotopical properties of prolongation of a $\mathbf{\Gamma}$-space to a continuous functor defined on all based spaces. In the body of this book, we always work in the category $\mathbf{T}$ of compactly generated spaces. However, this full subcategory is not closed under quotient spaces nor coends inside the ambient category of all topological spaces; since the construction of the prolongation involves a coend (quotient space), some care needs to be taken. So while we are mostly interested $\mathbf{\Gamma}$-spaces with values in the full subcategory $\mathbf{T}$ of compactly generated spaces, we define and discuss prolongation for $\mathbf{\Gamma}$-k-spaces.

Construction B.20. We let $\mathbf{\Gamma}$ denote the category whose objects are the based sets $n_+ = \{0, 1, \dots, n\}$, with basepoint 0, and with morphisms all based maps. A *$\mathbf{\Gamma}$-k-space* is a functor from $\mathbf{\Gamma}$ to the category of k-spaces that is reduced, i.e., the value at 0_+ is a one-point space.

A $\mathbf{\Gamma}$-k-space $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$ can be 'evaluated' at a based k-space K as follows. We set

$$F(K) \;=\; \Big(\coprod\nolimits_{n\geq 0} F(n_+) \times K^n\Big) / \sim \;\; .$$

In more detail, $F(K)$ is the quotient space of the disjoint union of the spaces $F(n_+) \times K^n$ by the equivalence relation generated by

$$(F(\alpha)(x);\, k_1, \dots, k_n) \;\sim\; (x;\, k_{\alpha(1)}, \dots, k_{\alpha(m)}) \tag{B.21}$$

for all $x \in F(m_+)$, all $(k_1, \dots, k_n)$ in K^n, and all morphisms $\alpha : m_+ \longrightarrow n_+$ in $\mathbf{\Gamma}$. Here $k_{\alpha(i)}$ is to be interpreted as the basepoint of K whenever $\alpha(i) = 0$. We emphasize that we use the Kelleyfied product topology on $F(n^+) \times K^n$, in order to remain inside the category of k-spaces. We mostly care about the special case when K is compact, and then this coincides with the ordinary product topology $F(m^+) \times_0 K^{\times_0 n}$ by Proposition A.2 (vi). The class of k-spaces is closed under disjoint unions and passage quotient spaces (by Proposition A.2 (i)); so $F(K)$ is indeed a k-space. We refer to the extension as the *prolongation* of the $\mathbf{\Gamma}$-space F. A more categorical way to describe $F(K)$ is as a coend of the functor

$$\mathbf{\Gamma} \times \mathbf{\Gamma}^{\mathrm{op}} \;\longrightarrow\; \mathbf{K}\,, \quad (m^+, n^+) \;\longmapsto\; F(m^+) \times K^n\,. \tag{B.22}$$

Remark B.23. We want to justify the abuse of notation of not distinguishing between the original $\mathbf{\Gamma}$-space and its prolongation: the value of the prolongation on m_+ is canonically homeomorphic to the original value. Slightly more is true, namely that the coend description exhibits the prolongation as the left Kan extension of F along the inclusion $\mathbf{\Gamma} \longrightarrow \mathbf{K}_*$; we won't use this, however.

Only for this argument we denote the prolongation of F by $\hat{F}$. The continuous maps

$$F(n_+) \times (m_+)^n \;\longrightarrow\; F(m_+)\,, \quad (x;\, i_1, \dots, i_n) \;\longmapsto\; F(i)(x)$$

are compatible with the equivalence relation defining $\hat{F}(m_+)$; here $i : n_+ \longrightarrow m_+$ is the based map with $i(j) = i_j$. So these maps assemble into a continuous map $\hat{F}(m_+) \longrightarrow F(m_+)$ that is inverse to the continuous map

$$F(m_+) \;\longrightarrow\; \hat{F}(m_+)\,, \quad x \;\longmapsto\; [x;\, 1, 2, \dots, m]\,.$$

We have to analyze the equivalence relation generated by (B.21) more closely. This part of the argument has nothing to do with topology, is purely combinatorial, and of a very similar flavor as the analysis of the equivalence relation defining the geometric realization of a simplicial set. We call an element $x \in F(m_+)$ *degenerate* if it is in the image of the map $F(\alpha) : F((m-1)_+) \longrightarrow F(m_+)$ for some morphism $\alpha : (m-1)_+ \longrightarrow m_+$ in $\mathbf{\Gamma}$. We call x *non-degenerate* if it is not degenerate. We let $C_m(K) \subset K^m$ be the set of those tuples $(k_1, \dots, k_m)$ whose coordinates k_i are pairwise distinct and all different from the basepoint of K. Part (i) of the following proposition is a special case of [12, Prop. 6.9]; part (ii) also ought to be well known, but I am not aware of a reference. The next proposition in particular implies that the reduction map (B.25) defined in the proof of part (ii) is a bijection from $F(K)$ to the set-theoretic disjoint

union, for $m \geq 0$, of the sets $F(m_+)^{\mathrm{nd}} \times_{\Sigma_m} C_m(K)$, where $F(m_+)^{\mathrm{nd}}$ is the set of non-generate elements of $F(m_+)$.

Proposition B.24. *Let $F : \mathbf{\Gamma} \longrightarrow$ (sets) be a functor such that $F(0_+)$ has one element.*

(i) *For every element $y \in F(m_+)$ there exists an injective morphism $\delta : l_+ \longrightarrow m_+$ and a non-degenerate element $x \in F(l_+)^{\mathrm{nd}}$ such that $y = F(\delta)(x)$. If moreover $y = F(\bar{\delta})(\bar{x})$ for another injective morphism $\bar{\delta} : \bar{l}_+ \longrightarrow m_+$ and a non-degenerate element $\bar{x} \in F(\bar{l}_+)^{\mathrm{nd}}$, then $l = \bar{l}$ and there exists a bijective morphism $\lambda : l_+ \longrightarrow l_+$ such that $\delta = \bar{\delta}\lambda$ and $\bar{x} = F(\lambda)(x)$.*

(ii) *Let K be a based set. Let $(x, t) \in F(l_+) \times K^l$ be an element of minimal dimension l within its equivalence class. Let $(y, s) \in F(m_+) \times K^m$ be equivalent to (x, t). Then there exists a surjective morphism $\sigma : m_+ \longrightarrow k_+$, an injective morphism $\delta : l_+ \longrightarrow k_+$ and $u \in C_k(K)$ such that*

$$F(\sigma)(y) = F(\delta)(x) , \quad s = \sigma^*(u) \quad \textit{and} \quad t = \delta^*(u) .$$

(iii) *If (x, t) and (y, s) are equivalent elements, both of minimal dimension l in their equivalence class, then there is a unique isomorphism $\alpha : l_+ \longrightarrow l_+$ such that $(F(\alpha)(y), s) = (x, \alpha^*(t))$.*

Proof Part (i) is an analog of the 'Eilenberg-Zilber lemma' [49, (8.3)] for simplicial sets, and a special case of [12, Prop. 6.9]. For the convenience of the reader, we give a direct proof. The existence part is proved by induction on m, starting with $m = 0$, where there is nothing to show. If m is positive and y is non-degenerate, then $x = y$ and $\delta = \mathrm{Id}$ do the job. Otherwise $y = F(\alpha)(z)$ for some morphism $\alpha : (m-1)_+ \longrightarrow m_+$ and $z \in F((m-1)_+)$. We factor $\alpha = \delta' \circ \beta$ such that $\beta : (m-1)_+ \longrightarrow k_+$ is surjective and $\delta' : k_+ \longrightarrow m_+$ is injective. We must have $k < m$, so the inductive hypothesis provides an injective morphism $\delta : l_+ \longrightarrow k_+$ and a non-degenerate element $x \in F(l_+)^{\mathrm{nd}}$ such that $F(\beta)(z) = F(\delta)(x)$. But then $\delta'\delta$ is also injective and

$$y = F(\alpha)(z) = F(\delta')(F(\beta)(z)) = F(\delta')(F(\delta)(x)) = F(\delta'\delta)(x) .$$

This proves the first statement.

For the second statement we consider injective morphisms $\delta : l_+ \longrightarrow m_+$ and $\bar{\delta} : \bar{l}_+ \longrightarrow m_+$, and non-degenerate elements $x \in F(l_+)^{\mathrm{nd}}$ and $\bar{x} \in F(\bar{l}_+)^{\mathrm{nd}}$ such that $F(\delta)(x) = F(\bar{\delta})(\bar{x})$. We let $\bar{\sigma} : m_+ \longrightarrow \bar{l}_+$ be the unique morphism that sends all elements not in the image of $\bar{\delta}$ to the basepoint 0 and satisfies $\bar{\sigma}\bar{\delta} = \mathrm{Id}$. Then

$$F(\bar{\sigma}\delta)(x) = F(\bar{\sigma})(F(\delta)(x)) = F(\bar{\sigma})(F(\bar{\delta})(\bar{x})) = F(\bar{\sigma}\bar{\delta})(\bar{x}) = \bar{x} .$$

If $\bar{\sigma}\delta : l_+ \longrightarrow \bar{l}_+$ were not surjective, then it could be factored through $(\bar{l} - 1)_+$,

and $\bar{x}$ would be degenerate, contradicting the assumptions. So $\bar{\sigma}\delta$ is surjective, and hence $l \geq \bar{l}$. Reversing the roles of (x, δ) and $(\bar{x}, \bar{\delta})$ gives $l \leq \bar{l}$, and hence $l = \bar{l}$. Since $l = \bar{l}$ and $\bar{\sigma}\delta$ is surjective, it must in fact be bijective.

Now we claim that the morphisms $\delta, \bar{\delta} : l_+ \longrightarrow m_+$ have the same image. If this were not the case, then $\bar{\sigma}\delta : l_+ \longrightarrow l_+$ would not be surjective, which again would contradict the assumption that $\bar{x}$ is non-degenerate. Since δ and $\bar{\delta}$ have the same image, there is a bijection $\lambda : l_+ \longrightarrow l_+$ such that $\delta = \bar{\delta}\lambda$. Hence

$$F(\lambda)(x) \ = \ F(\bar{\sigma}\bar{\delta}\lambda)(x) \ = \ F(\bar{\sigma})(F(\delta)(x)) \ = \ F(\bar{\sigma})(F(\bar{\delta})(\bar{x})) \ = \ \bar{x}\ .$$

So λ is the bijection with the desired properties.

(ii) We call a quadruple (σ, δ, u, x) consisting of a surjective morphism $\sigma : m_+ \longrightarrow k_+$, an injective morphism $\delta : l_+ \longrightarrow k_+$, a tuple $u \in C_k(K)$ and a non-degenerate element $x \in F(l_+)$ a *reduction datum* for $(y, s) \in F(m_+) \times K^m$ if

$$F(\sigma)(y) = F(\delta)(x) \qquad \text{and} \qquad s = \sigma^*(u)\ .$$

Since δ is injective, the map $\delta^* : K^k \longrightarrow K^l$ omits some coordinates and reorders the remaining ones. In particular, δ^* does not introduce duplicates or basepoints, so it sends the subset $C_k(K)$ to the subset $C_l(K)$, and hence $\delta^*(u) \in C_l(K)$.

Claim 1: Every pair (y, s) has a reduction datum. We let $u = (u_1, \ldots, u_k)$ be the non-basepoint elements in the set $\{s_1, \ldots, s_m\}$, without repetition and in some chosen order. We define a surjective morphism $\sigma : m_+ \longrightarrow k_+$ by

$$\sigma(i) \ = \ \begin{cases} j & \text{if } s_i = u_j\text{, and} \\ 0 & \text{if } s_i \text{ is the basepoint of } K. \end{cases}$$

Then $u \in C_k(K)$ and $s = \sigma^*(u)$. Part (i) now provides an injective morphism $\delta : l_+ \longrightarrow k_+$ and $x \in F(l_+)^{\mathrm{nd}}$ such that $F(\sigma)(y) = F(\delta)(x)$. So (σ, δ, u, x) is a reduction datum for (y, s).

Claim 2: We show that the reduction datum is 'essentially unique' in the following sense. If

$$(\sigma : m_+ \longrightarrow k_+, \delta : l_+ \longrightarrow k_+, u, x) \quad \text{and} \quad (\bar{\sigma} : m_+ \longrightarrow \bar{k}_+, \bar{\delta} : \bar{l}_+ \longrightarrow \bar{k}_+, \bar{u}, \bar{x})$$

are reduction data for the same element (y, s), then $k = \bar{k}$, $l = \bar{l}$ and there are bijective morphisms $\lambda : l_+ \longrightarrow l_+$ and $\beta : k_+ \longrightarrow k_+$ such that

$$\bar{\sigma} = \beta\sigma\ , \quad \bar{\delta}\lambda = \beta\delta\ , \quad u = \beta^*(\bar{u}) \qquad \text{and} \qquad \bar{x} = F(\lambda)(x)\ .$$

Indeed, because $s = \sigma^*(u) = \bar{\sigma}^*(\bar{u})$ and u and $\bar{u}$ don't contain duplicates or basepoints, we must have $k = \bar{k}$ and u and $\bar{u}$ can only differ by the ordering.

So there is a bijective morphism $\beta : k_+ \longrightarrow k_+$ such that $u = \beta^*(\bar{u})$. For all $i \in \{1, \dots, m\}$ we then have

$$\bar{u}_{\bar{\sigma}(i)} \;=\; s_i \;=\; u_{\sigma(i)} \;=\; \bar{u}_{\beta(\sigma(i))} \;.$$

Since the coordinates of $\bar{u}$ are all distinct, this implies that $\bar{\sigma} = \beta\sigma$. Using that both quadruples are reduction data, we know that

$$F(\bar{\delta})(\bar{x}) \;=\; F(\bar{\sigma})(y) \;=\; F(\beta)(F(\sigma)(y)) \;=\; F(\beta)(F(\delta)(x)) \;=\; F(\beta\delta)(x) \;.$$

Since $\bar{\delta}$ and $\beta\delta$ are both injective and x and $\bar{x}$ are both non-degenerate, then essential uniqueness statement in part (i) shows that $l = \bar{l}$ and provides a bijective morphism $\lambda : l_+ \longrightarrow l_+$ such that $\beta\delta = \bar{\delta}\lambda$ and $F(\lambda)(x) = \bar{x}$.

We can now define a *reduction map*

$$\rho \;:\; \coprod_{n\geq 0} F(n_+) \times K^n \;\longrightarrow\; \coprod_{l\geq 0} F(l_+)^{\mathrm{nd}} \times_{\Sigma_l} C_l(K) \tag{B.25}$$

by choosing a reduction datum (σ, δ, u, x) for a given element (y, s) and setting

$$\rho(y, s) \;=\; [x, \delta^*(u)] \;,$$

where $[-,-]$ denotes the Σ_l-orbit. If (σ, δ, u, x) is another reduction datum for (y, s), then

$$[x, \delta^*(u)] \;=\; [x, \delta^*(\beta^*(\bar{u}))] \;=\; [x, \lambda^*(\bar{\delta}^*(\bar{u}))] \;=\; [F(\lambda)(x), \bar{\delta}^*(\bar{u})] \;=\; [\bar{x}, \bar{\delta}^*(\bar{u})] \;,$$

by Claim 2. So $\rho(y, s)$ is well-defined.

Claim 3: If $(y, s) \in F(m_+) \times K^m$ and $(\bar{y}, \bar{s}) \in F(\bar{m}_+) \times K^{\bar{m}}$ are equivalent, then $\rho(y, s) = \rho(\bar{y}, \bar{s})$. It suffices to show the claim whenever (y, s) and $(\bar{y}, \bar{s})$ are related by a generating relation (B.21), i.e., we can assume that $y = F(\alpha)(\bar{y})$ and $\bar{s} = \alpha^*(s)$ for some morphism $\alpha : \bar{m}_+ \longrightarrow m_+$. We let (σ, δ, u, x) be a reduction datum for (y, s). We choose a factorization

$$\sigma \circ \alpha \;=\; \bar{\delta} \circ \bar{\sigma}$$

as a surjective morphism $\bar{\sigma} : \bar{m}_+ \longrightarrow \bar{k}_+$ followed by an injective morphism $\bar{\delta} : \bar{k}_+ \longrightarrow k_+$. Using part (i) we write

$$F(\bar{\sigma})(\bar{y}) \;=\; F(\delta')(\bar{x})$$

for an injective morphism $\delta' : l'_+ \longrightarrow \bar{k}_+$ and a non-degenerate element $\bar{x} \in F(l'_+)^{\mathrm{nd}}$. Then

$$\begin{aligned} F(\delta)(x) \;&=\; F(\sigma)(y) \;=\; F(\sigma)(F(\alpha)(\bar{y})) \\ &=\; F(\bar{\delta})(F(\bar{\sigma})(\bar{y})) \;=\; F(\bar{\delta})(F(\delta')(\bar{x})) \;=\; F(\bar{\delta}\delta')(\bar{x}) \;. \end{aligned}$$

Since x and $\bar{x}$ are non-degenerate and δ and $\bar{\delta}\delta'$ are injective, the essential

uniqueness of part (i) shows that $l = l'$ and provides a bijection $\lambda : l_+ \longrightarrow l_+$ such that $\delta = \bar{\delta}\delta'\lambda$ and $\bar{x} = F(\lambda)(x)$. We conclude that

$$F(\bar{\sigma})(\bar{y}) \;=\; F(\delta')(\bar{x}) \;=\; F(\delta')(F(\lambda)(x)) \;=\; F(\delta'\lambda)(x)$$

and

$$\bar{s} \;=\; \alpha^*(s) \;=\; \alpha^*(\sigma^*(u)) \;=\; \bar{\sigma}^*(\bar{\delta}^*(u))\;.$$

Since $u \in C_k(K)$ and $\bar{\delta}$ is injective, $\bar{\delta}^*(u)$ belongs to $C_{\bar{k}}(K)$. This shows that $(\bar{\sigma}, \delta'\lambda, \bar{\delta}^*(u), x)$ is a reduction datum for $(\bar{y}, \bar{s})$. So

$$\rho(\bar{y}, \bar{s}) \;=\; [x, (\delta'\lambda)^*(\bar{\delta}^*(u))] \;=\; [x, \delta^*(u)] \;=\; \rho(y, s)\;.$$

This finishes the proof of Claim 3.

Now we can prove part (ii) of the proposition. If $(x, t) \in F(l_+)\times K^l$ is of minimal dimension in its equivalence class, then x is non-degenerate and $t \in C_l(K)$, for otherwise (x, t) would be equivalent to an element of smaller dimension. So $(\mathrm{Id}, \mathrm{Id}, t, x)$ is a reduction datum for (x, t), and hence $\rho(x, t) = [x, t]$. Now we let (y, s) be equivalent to (x, t), and we let (σ, δ, u, x') be a reduction datum for (y, s). Then

$$[x', \delta^*(u)] \;=\; \rho(y, s) \;=\; \rho(x, t) \;=\; [x, t]\;,$$

where the second equality is Claim 3. So there is a bijective morphism $\lambda : l_+ \longrightarrow l_+$ such that

$$(x', \lambda^*(\delta^*(u))) \;=\; (F(\lambda)(x), t)\;.$$

Hence

$$F(\sigma)(y) \;=\; F(\delta)(x') \;=\; F(\delta\lambda)(x)\;, \qquad s \;=\; \sigma^*(u) \quad \text{and} \quad t = (\delta\lambda)^*(u)\;.$$

(iii) By the minimality assumptions, the data provided by part (ii) must satisfy $m = k = l$, and the morphisms σ and δ must both be isomorphisms. So $(F(\alpha)(y), s) = (x, \alpha^*(t))$ with $\alpha = \delta^{-1}\sigma : l_+ \longrightarrow l_+$. The coordinates of t are distinct, by minimality of dimension, so there is only one permutation with $s = \alpha^*(t)$. □

Now that we understand the equivalence relation generated by (B.21), we can analyze the topology of the space $F(K)$ when F is a $\mathbf{\Gamma}$-space and K is a based topological space. In particular, we show now that the prolongation of a $\mathbf{\Gamma}$-space with values in the category $\mathbf{T}$ of compactly generated spaces to a compactly generated space is automatically compactly generated, with the weak Hausdorff property being the issue. The statement and proof of the following proposition are analogous to the ones in Proposition A.35.

Proposition B.26. *Let $F : \mathbf{\Gamma} \longrightarrow \mathbf{T}$ be a $\mathbf{\Gamma}$-space with values in the category of compactly generated spaces, and K a compactly generated based space.*

(i) *The space* $F(K)$ *is compactly generated, and hence a coend internal to the category* $\mathbf{T}$*, of the functor* (B.22)*.*

(ii) *Let* E *be a* $\mathbf{\Gamma}$*-subspace of* F *such that* $E(n_+)$ *is closed in* $F(n_+)$ *for every* $n \geq 0$*. Let* L *be a closed based subset of* K*. Then the inclusions* $E \longrightarrow F$ *and* $L \longrightarrow K$ *induce a closed embedding* $\iota : E(L) \longrightarrow F(K)$*.*

Proof (i) The category $\mathbf{T}$ is closed under products inside the category $\mathbf{K}$, so the functor (B.22) takes values in $\mathbf{T}$; the issue is that a priori, the quotient topology need not be weak Hausdorff. We use Proposition A.7 to show that the quotient space $F(K)$ is compactly generated. We let $E \subset (\coprod_{n\geq 0} F(n_+)\times K^n)^2$ be the equivalence relation generated by (B.21), which we had simply denoted '$\sim$' above. We will show that E is closed in the k-topology of $(\coprod_{n\geq 0} F(n_+) \times K^n)^2$. Since products distribute over disjoint unions, we may show that

$$E_{m,n} \ = \ E \cap (F(m_+) \times K^m \times F(n_+) \times K^n)$$

is closed in $F(m_+)\times K^m \times F(n_+)\times K^n$ for all $m, n \geq 0$. We consider $(y, s, \bar{y}, \bar{s}) \in E_{m,n}$, i.e., the pairs (y, s) and $(\bar{y}, \bar{s})$ are equivalent. Proposition B.24 (ii) provides surjective morphisms $\sigma : m_+ \longrightarrow k_+$ and $\bar{\sigma} : n_+ \longrightarrow \bar{k}_+$, injective morphisms $\delta : l_+ \longrightarrow k_+$ and $\bar{\delta} : l_+ \longrightarrow \bar{k}_+$ and $u \in K^k$, $\bar{u} \in K^{\bar{k}}$ and $(x, t) \in F(l_+) \times K^l$ such that

$$F(\sigma)(y) \ = \ F(\delta)(x) \ , \quad s \ = \ \sigma^*(u) \qquad \text{and} \qquad t = \delta^*(u)$$

and

$$F(\bar{\sigma})(\bar{y}) \ = \ F(\bar{\delta})(x) \ , \quad \bar{s} \ = \ \bar{\sigma}^*(\bar{u}) \qquad \text{and} \qquad t = \bar{\delta}^*(\bar{u}) \ .$$

Equivalently, $E_{m,n}$ is the union, indexed over $l, \sigma, \bar{\sigma}, \delta, \bar{\delta}$ as above, of the finite number of sets

$$(F(\sigma)\times K^m \times F(\bar{\sigma}) \times K^n)^{-1} \\ \Big((F(\delta) \times \sigma^* \times F(\bar{\delta}) \times \bar{\sigma}^*) \Big((F(l_+) \times \delta^* \times F(l_+) \times \bar{\delta}^*)^{-1}(\Delta_{F(l_+)\times K^l})\Big)\Big) \ .$$

The diagonal $\Delta_{F(l_+)\times K^l}$ is closed in $(F(l_+)\times K^l)^2$ because $F(l_+)\times K^l$ is compactly generated. So its inverse image under the continuous map $F(l_+)\times\delta^*\times F(l_+)\times\bar{\delta}^*$ is closed in $F(l_+) \times K^k \times F(l_+) \times K^{\bar{k}}$. The map $F(\delta) \times \sigma^* \times F(\bar{\delta}) \times \bar{\sigma}^*$ has a continuous retraction, and is thus a closed embedding by Proposition A.12. So the set

$$(F(\delta) \times \sigma^* \times F(\bar{\delta}) \times \bar{\sigma}^*) \Big((F(l_+) \times \delta^* \times F(l_+) \times \bar{\delta}^*)^{-1}(\Delta_{F(l_+)\times K^l})\Big)$$

is closed in $F(k_+) \times K^m \times F(\bar{k}_+) \times K^n$. Since $E_{m,n}$ is the inverse image of this latter closed set under a continuous map, this show the claim that $E_{m,n}$ is a closed subset of $F(m_+) \times K^m \times F(n_+) \times K^n$.

(ii) Our first claim is that the map $\iota : E(L) \longrightarrow F(K)$ is injective. We let

$(x; l_1, \dots, l_m) \in E(m_+) \times L^m$ be a minimal representative of an element of $E(L)$. So the element x is non-degenerate, and the l_i's are pairwise distinct and different from the basepoint of L. We claim that x is also non-degenerate when viewed as an element of the ambient $\mathbf{\Gamma}$-space F. To see this, we argue by contradiction and assume that there was an injective based map $\delta : k_+ \longrightarrow n_+$ with $k < n$ and an element $y \in F(k_+)$ such that $F(\delta)(y) = x$. We let $\sigma : n_+ \longrightarrow k_+$ be a retraction to δ. Then $y = F(\sigma)(F(\delta)(y)) = F(\sigma)(x)$. Since E is a $\mathbf{\Gamma}$-subspace of F, we conclude that $y \in E(k_+)$. Then $x = E(\delta)(y)$, contradicting the non-degeneracy of x. Altogether this shows that a representative of minimal dimension for a class in $E(L)$ remains a minimal representative for its class in $F(K)$. Because minimal representatives are unique up to a permutation (Proposition B.24 (iii)), this implies that the map $\iota : E(L) \longrightarrow F(K)$ is injective.

It remains to show that the continuous injection ι is a closed map. We consider the commutative square

$$\begin{tikzcd}
\coprod_{n\geq 0} E(n_+) \times L^n \arrow[r, "\mathrm{incl}"] \arrow[d, "p"'] & \coprod_{n\geq 0} F(n_+) \times K^n \arrow[d, "q"] \\
E(L) \arrow[r, "\iota"'] & F(K)
\end{tikzcd}$$

where the vertical maps are the quotient maps. We consider a point $(y, s) \in F(m_+) \times K^m$ whose class in $F(K)$ is in the image of ι. Then (y, s) is equivalent to an element $(x, t) \in E(l_+) \times L^l$, which we can take as a minimal representative in its (E, L)-equivalence class. As we argued in the injectivity statement, (x, t) is then also a minimal representative in its (F, K)-equivalence class.

Proposition B.24 (ii) provides a surjective morphism $\sigma : m_+ \longrightarrow k_+$, an injective morphism $\delta : l_+ \longrightarrow k_+$ and $u \in K^k$ such that

$$F(\sigma)(y) = F(\delta)(x) , \quad s = \sigma^*(u) \qquad \text{and} \qquad t = \delta^*(u) .$$

So for every subset $A \subset E(L)$, we have

$$\begin{aligned}
&(F(m_+) \times K^m) \cap q^{-1}(\iota(A)) = \\
&\bigcup\nolimits_{\sigma,\delta} (F(\sigma) \times K^m)^{-1} \Big((F(\delta) \times \sigma^*)((F(l_+) \times \delta^*)^{-1}((E(l_+) \times L^l) \cap p^{-1}(A)))\Big) .
\end{aligned}$$

The union is over the finite set of pairs $(\sigma : m_+ \longrightarrow k_+, \delta : l_+ \longrightarrow k_+)$ as above.

Now we assume that A is closed inside $E(L)$. Because p is continuous, $E(l_+)$ is closed in $F(l_+)$ and L is closed in K, the set $(E(l_+) \times L^l) \cap p^{-1}(A)$ is then closed inside $F(l_+) \times K^l$. So $(F(l_+) \times \delta^*)^{-1}((E(l_+) \times L^l) \cap p^{-1}(A))$ is a closed subset of $F(l_+) \times K^k$. Since $F(\delta) \times \sigma^*$ has a continuous retraction, it is a closed embedding by Proposition A.12. So the set $(F(\delta) \times \sigma^*)((F(l_+) \times \delta^*)^{-1}((E(l_+) \times L^l) \cap p^{-1}(A)))$ is closed in $F(k_+) \times K^m$. As the inverse image of a closed set under a continuous map, each set in the finite union above is closed inside

$F(m_+) \times K^m$. We conclude that $(F(m_+) \times K^m) \cap q^{-1}(\iota(A))$ is closed in $F(m_+) \times K^m$ for every $m \geq 0$, hence the set $q^{-1}(\iota(A))$ is closed. Since q is a quotient map, this shows that $\iota(A)$ is closed in $F(K)$. $\square$

We let $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$ be a $\mathbf{\Gamma}$-k-space and K and L two based k-spaces. The prolongation comes with a continuous, based *assembly map*

$$\alpha \; : \; K \wedge F(L) \longrightarrow F(K \wedge L) \, , \; \alpha(k \wedge [x; \, l_1, \dots, l_n]) \; = \; [x; \, k \wedge l_1, \dots, k \wedge l_n] \, .$$

The assembly map is natural in all three variables and associative and unital. We define a 'shifted' $\mathbf{\Gamma}$-space $F_K = F \circ (K \wedge -)$ as the composite

$$\mathbf{\Gamma} \xrightarrow{K \wedge -} \mathbf{K}_* \xrightarrow{F} \mathbf{K}_* \, .$$

Then for every based k-space L, we consider the maps

$$\phi_n \; : \; F_K(n_+) \times L^n \; = \; F(K \wedge n_+) \times L^n \; \longrightarrow \; F(K \wedge L)$$

defined by

$$\phi_n([x; k_1 \wedge i_1, \dots, k_m \wedge i_m]; \, l_1, \dots, l_n) \; = \; [x; \, k_1 \wedge l_{i_1}, \dots, k_m \wedge l_{i_m}] \, ,$$

where $x \in F(m_+)$, $k_1, \dots, k_m \in K$ and $i_1, \dots, i_m \in \{1, \dots, n\}$. Another way to say this is that ϕ_n is the composite

$$F(K \wedge n_+) \times L^n \xrightarrow{\text{proj}} F(K \wedge n_+) \wedge L^n \xrightarrow{\text{ass}} F(K \wedge n_+ \wedge L^n) \xrightarrow{F(K \wedge \epsilon)} F(K \wedge L) \, ,$$

where

$$\epsilon \; : \; n_+ \wedge L^n \longrightarrow L \, , \quad i \wedge (l_1, \dots, l_n) \longmapsto l_i \, .$$

is the evaluation map. These maps ϕ_n are compatible with the equivalence relation defining $F_K(L)$, so they combine into a continuous map

$$\phi \; : \; F_K(L) \longrightarrow F(K \wedge L) \, .$$

The map ϕ is natural in F, K and L.

Proposition B.27. *Let $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$ be a $\mathbf{\Gamma}$-k-space and K and L based k-spaces. Then the natural map $\phi : F_K(L) \longrightarrow F(K \wedge L)$ is a homeomorphism.*

Proof We let $q : K \times L \longrightarrow K \wedge L$ denote the projection. In the category $\mathbf{K}$, product with any space preserves proclusions (Proposition A.3), so the map

$$F(m_+) \times q^m \; : \; F(m_+) \times (K \times L)^m \longrightarrow F(m_+) \times (K \wedge L)^m$$

is a proclusion for every $m \geq 0$. The continuous map

$$\psi_m \; : \; F(m_+) \times (K \times L)^m \longrightarrow F_K(L)$$

defined by

$$\psi_m(x;(k_1,l_1),\ldots,(k_m,l_m)) \;=\; [[x;\, k_1\wedge 1,\ldots,k_m\wedge m];\, l_1,\ldots,l_m]$$

is constant on the fibers of $F(m_+)\times q^m$, so it factors over a continuous map

$$\bar\psi_m \;:\; F(m_+)\times(K\wedge L)^m \;\longrightarrow\; F_K(L)\;.$$

These maps are compatible with the equivalence relation defining $F(K\wedge L)$, so they combine into a continuous map

$$\psi \;:\; F(K\wedge L) \;\longrightarrow\; F_K(L)\;.$$

Since ϕ and ψ are inverse to each other, this completes the proof. □

Construction B.28. Now we prove an interchange relation between geometric realization and prolongation of a $\mathbf{\Gamma}$-space. We let $A : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}_*$ be a simplicial based k-space and $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$ a $\mathbf{\Gamma}$-k-space. The composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\ A\ } \mathbf{K}_* \xrightarrow{\ F\ } \mathbf{K}_*$$

is a simplicial space, where the second functor is the prolongation of F, denoted by the same symbol. The composite has a geometric realization

$$|F\circ A| \;=\; |[n]\longmapsto F(A_n)|\;.$$

We exhibit a natural homeomorphism from $|F\circ A|$ to $F(|A|)$, the value of F at the realization of A. We let $\kappa_n : A_n\wedge\Delta^n_+ \longrightarrow |A|$ be the canonical based map sending a point (a,t) to its equivalence class in A. Then the maps

$$F(A_n)\times\Delta^n \xrightarrow{\text{assembly}} F(A_n\wedge\Delta^n_+) \xrightarrow{F(\kappa_n)} F(|A|)$$

are compatible with the equivalence relation defining $|F\circ A|$, as $[n]$ varies over the objects of $\mathbf{\Delta}$. So the maps define a continuous map

$$\kappa \;:\; |F\circ A| \;\longrightarrow\; F(|A|)\;.$$

In terms of elements, κ is thus given by

$$\kappa[[x;\, a_1,\ldots,a_m],t] \;=\; [x;\, [a_1,t],\ldots,[a_m,t]]\;,$$

for $x\in F(m_+)$, $a_1,\ldots,a_m\in A_n$ and $t\in\Delta^n$.

The following proposition is [192, Lemma 1.9].

Proposition B.29. *For every simplicial based k-space $A : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{K}_*$ and every $\mathbf{\Gamma}$-space $F : \mathbf{\Gamma} \longrightarrow \mathbf{K}$, the map $\kappa : |F\circ A| \longrightarrow F(|A|)$ is a homeomorphism.*

Proof This is a special case of the 'Fubini theorem' for coends. In more detail we consider the functor

$$\mathbf{\Gamma} \times \mathbf{\Gamma}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \longrightarrow \mathbf{K}\,, \quad (k_+, l_+, [m], [n]) \longmapsto F(k_+) \times A^l_m \times \Delta^n\,.$$

If we fix objects $[m]$ and $[n]$ of $\mathbf{\Delta}$, and exploit the fact that coends commute with product with Δ^n, we obtain a homeomorphism

$$\int^{k_+\in\mathbf{\Gamma}} F(k_+) \times A^k_m \times \Delta^n \cong \left(\int^{k_+\in\mathbf{\Gamma}} F(k_+) \times A^k_m\right) \times \Delta^n \cong F(A_m) \times \Delta^n\,.$$

These homeomorphisms are natural as $([m], [n])$ vary in $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}$. So taking coends over $\mathbf{\Delta}$ gives a homeomorphism

$$\int^{[m]\in\mathbf{\Delta}} \int^{k_+\in\mathbf{\Gamma}} F(k_+) \times A^k_m \times \Delta^m \cong \int^{[m]\in\mathbf{\Delta}} F(A_m) \times \Delta^m = |F \circ A|\,.$$

On the other hand, if we fix k and l and exploit the fact that geometric realization commutes with product with the space $F(k_+)$ and with products (see parts (i) and (ii) of Proposition A.37), we obtain a homeomorphism

$$\begin{aligned}\int^{[m]\in\mathbf{\Delta}} F(k_+) \times A^l_m \times \Delta^m &\cong F(k_+) \times \int^{[m]\in\mathbf{\Delta}} A^l_m \times \Delta^m \\ &= F(k_+) \times |A^l| \cong F(k_+) \times |A|^l\,.\end{aligned}$$

These homeomorphisms are natural as (k_+, l_+) vary in $\mathbf{\Gamma}\times\mathbf{\Gamma}^{\mathrm{op}}$. So taking coends over $\mathbf{\Gamma}$ gives a homeomorphism

$$\int^{k_+\in\mathbf{\Gamma}} \int^{[m]\in\mathbf{\Delta}} F(k_+) \times A^k_m \times \Delta^m \cong \int^{k_+\in\mathbf{\Gamma}} F(k_+) \times |A|^k = F(|A|)\,.$$

The 'Fubini theorem' for iterated coends (the dual to [105, IX.8, Corollary]) says that these two ways of iteratively taking coends are canonically isomorphic; the isomorphism is in fact the map κ of the proposition. □

Now we add group actions to the discussion of $\mathbf{\Gamma}$-spaces. By a *$\mathbf{\Gamma}$-G-space*, for a topological group G, we simply mean a reduced functor from $\mathbf{\Gamma}$ to the category $G\mathbf{T}$ of G-spaces. The previous proposition reduces the study of prolonged $\mathbf{\Gamma}$-G-spaces on the realizations of G-simplicial sets to the study of simplicial G-spaces of the form $F \circ A$. A sufficient condition for the realization of a simplicial G-space to be homotopically meaningful is Reedy cofibrancy. This explains why we look for a practical condition to ensure that simplicial G-spaces of the form $F \circ A$ are Reedy cofibrant; the concept of 'G-cofibrancy' introduced in Definition B.33 below does the job.

Construction B.30. We let $\mathcal{P}(n)$ denote the power set of $\{1, \ldots, n\}$, i.e., the set of subsets. We also write $\mathcal{P}(n)$ for the associated poset category, i.e., with

object set $\mathcal{P}(n)$ and exactly one morphism $U \longrightarrow T$ whenever $U \subseteq T$. Given a $\mathbf{\Gamma}$-space F, we obtain a functor from the poset category $\mathcal{P}(n)$ to based spaces by sending a subset U to $F(U_+)$, with the maps $F(U_+) \longrightarrow F(T_+)$ induced by the inclusions. We obtain a *latching map*

$$l_n \ : \ \mathrm{colim}_{U \subsetneq \{1,\dots,n\}} \ F(U_+) \ \longrightarrow \ F(n_+) \ . \tag{B.31}$$

The symmetric group Σ_n acts on the target of the latching map by functoriality of F. The symmetric group also acts on the source of l_n, by letting $\sigma \in \Sigma_n$ send $F(U_+)$ to $F(\sigma(U)_+)$ via the map

$$F((\sigma \cdot -)_+) \ : \ F(U_+) \ \longrightarrow \ F(\sigma(U)_+) \ .$$

The latching map is equivariant for these two Σ_n-actions.

We will often consider a $\mathbf{\Gamma}$-G-space for a topological group G. Then G also acts on source and target of the latching map l_n, which is thus $(G \times \Sigma_n)$-equivariant.

Now we observe that the latching map for a $\mathbf{\Gamma}$-space F is also a latching map for a certain simplicial space, whence the terminology and notation. We recall that the 'simplicial circle' $\mathbf{S}^1 : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{\Gamma}$ is given on objects by $\mathbf{S}^1_n = n_+$, with face maps $d_i^* : n_+ \longrightarrow (n-1)_+$ given by

$$d_i^*(j) \ = \ \begin{cases} j-1 & \text{for } i < j, \text{ and} \\ j & \text{for } i \geq j \text{ and } j \neq n, \\ 0 & \text{for } i = j = n, \end{cases}$$

and degeneracy maps $s_i^* : n_+ \longrightarrow (n+1)_+$ given by

$$s_i^*(j) \ = \ \begin{cases} j+1 & \text{for } i < j, \text{ and} \\ j & \text{for } i \geq j. \end{cases}$$

The simplicial set $\mathbf{S}^1$ is isomorphic to $\Delta[1]/\partial\Delta[1]$, and its realization is homeomorphic to a circle, whence the name.

Proposition B.32. *Let F be a $\mathbf{\Gamma}$-space and $n \geq 0$.*

(i) *There is a homeomorphism*

$$L_n(F \circ \mathbf{S}^1) \ \xrightarrow{\cong} \ \mathrm{colim}_{U \subsetneq \{1,\dots,n\}} \ F(U_+)$$

whose composite with the latching map (B.31) *is the latching map $l_n : L_n(F \circ \mathbf{S}^1) \longrightarrow F(\mathbf{S}^1_n) = F(n_+)$ of the simplicial space $F \circ \mathbf{S}^1$.*

(ii) *The latching map l_n* (B.31) *is a closed embedding.*

(iii) *Suppose that F is a $\mathbf{\Gamma}$-G-space for a topological group G. Then the canonical map*

$$\mathrm{colim}_{U\subsetneq\{1,\dots,n\}}\,(F(U_+))^G \;\longrightarrow\; (\mathrm{colim}_{U\subsetneq\{1,\dots,n\}}\;F(U_+))^G$$

is a homeomorphism.

Proof (i) We recall that $\mathbf{\Delta}(n)$ is the category with objects the weakly monotone surjections $\sigma : [n] \longrightarrow [k]$; a morphism from σ to σ' is a morphism $\alpha : [k] \longrightarrow [k']$ in $\mathbf{\Delta}$ with $\alpha \circ \sigma = \sigma'$. Moreover $\mathbf{\Delta}(n)_\circ$ is the full subcategory with all objects except the identity of $[n]$.

As we recalled in Remark A.40, the category $\mathbf{\Delta}(n)^{\mathrm{op}}$ is isomorphic to the poset category $\mathcal{P}(n)$. Indeed, an isomorphism is given on objects by

$$\kappa(\sigma : [n] \longrightarrow [k]) \;=\; \{i \in \{1,\dots,n\} \;:\; \sigma(i) > \sigma(i-1)\}\,.$$

In the other direction, a subset $U \subset \{1,\dots,n\}$ is taken to the monotone surjection $\sigma_U : [n] \longrightarrow [|U|]$ defined by

$$\sigma_U(i) \;=\; |U \cap \{1,\dots,i\}|\,.$$

The structure map $\sigma^* : \mathbf{S}^1_k \longrightarrow \mathbf{S}^1_n = n_+$ is injective with image

$$\sigma^*(\mathbf{S}^1_k) \;=\; \kappa(\sigma) \cup \{0\}\,.$$

This means that the maps $\sigma^* : \mathbf{S}^1_k = k_+ \longrightarrow \sigma^*(k_+) = \kappa(\sigma)_+$ define a natural isomorphism between the two composites in the square of functors:

$$\begin{tikzcd}
\mathbf{\Delta}(n)^{\mathrm{op}} \arrow[r, "\kappa", "\cong"'] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\sigma^*}{\nearrow}"] & \mathcal{P}(n) \arrow[d, "U\mapsto U_+"] \\
\mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\mathbf{S}^1"'] & \mathbf{\Gamma}
\end{tikzcd}$$

So as σ varies over the objects of $\mathbf{\Delta}(n)_\circ$, the homeomorphisms

$$(F \circ \mathbf{S}^1 \circ u)(\sigma) \;=\; F(k_+) \;\xrightarrow[\cong]{F(\sigma^*)}\; F(\kappa(\sigma)_+) \;\longrightarrow\; \mathrm{colim}_{U\subsetneq\{1,\dots,n\}}\;F(U_+)$$

assemble into the desired isomorphism from $L_n(F \circ \mathbf{S}^1)$.

(ii) Part (i) shows that the latching map (B.31) is an instance of the latching map of a simplicial compactly generated space. Claim (ii) is then a special case of Proposition A.42 (iii).

(iii) We contemplate the commutative square:

$$\begin{tikzcd}
L_n\left((F \circ \mathbf{S}^1)^G\right) \arrow[r] \arrow[d, "\cong"'] & (L_n(F \circ \mathbf{S}^1))^G \arrow[d, "\cong"] \\
\mathrm{colim}_{U\subsetneq\{1,\dots,n\}}\,(F(U_+))^G \arrow[r] & (\mathrm{colim}_{U\subsetneq\{1,\dots,n\}}\;F(U_+))^G
\end{tikzcd}$$

The left vertical map is the homeomorphism of part (i) for the non-equivariant $\mathbf{\Gamma}$-space $(F \circ \mathbf{S}^1)^G$. The right vertical map is the effect on G-fixed-points of the homeomorphism of part (i). The upper horizontal map is a homeomorphism by Proposition B.1 (iv), for the simplicial G-space $F \circ \mathbf{S}^1$. So the lower horizontal map is a homeomorphism. □

Definition B.33. Let G be a compact Lie group. A $\mathbf{\Gamma}$-G-space F is *G-cofibrant* if for every $n \geq 1$ the latching map

$$l_n \ : \ \mathrm{colim}_{U \subsetneq \{1,\dots,n\}} F(U_+) \ \longrightarrow \ F(n_+)$$

is a $(G \times \Sigma_n)$-cofibration.

We discuss a relevant example of a cofibrant $\mathbf{\Gamma}$-G-space in Example 6.3.16, namely the $\mathbf{\Gamma}$-G-space of pairwise orthogonal, finite-dimensional subspaces of a complete complex G-universe.

Example B.34 (Equivariant $\mathbf{\Gamma}$-simplicial sets)**.** We let G be a finite group. In the literature about equivariant $\mathbf{\Gamma}$-spaces with values in simplicial sets, no cofibrancy condition is needed to ensure good homotopical behavior. One way to explain this is to observe that the geometric realization of a $\mathbf{\Gamma}$-G-simplicial set $E : \mathbf{\Gamma} \longrightarrow G\mathbf{sset}$ is automatically G-cofibrant. Indeed, applying Proposition B.32 (ii) to the $\mathbf{\Gamma}$-set in any given simplicial dimension shows that the latching morphism

$$l_n \ : \ \mathrm{colim}_{U \subsetneq \{1,\dots,n\}} E(U_+) \ \longrightarrow \ E(n_+) \ ,$$

taken internal to simplicial sets, is dimensionwise injective. Geometric realization commutes with colimits and takes injective morphisms of $(G \times \Sigma_n)$-simplicial sets to relative $(G \times \Sigma_n)$-CW-inclusions, which are in particular $(G \times \Sigma_n)$-cofibrations. So the map $|l_n|$ is a $(G \times \Sigma_n)$-cofibration without any hypotheses on E.

Our notion of 'G-cofibrant' should not be confused with cofibrancy in the strict model structure that Bousfield and Friedlander introduce for non-equivariant $\mathbf{\Gamma}$-simplicial sets in [22, Thm. 3.5], and generalized to $\mathbf{\Gamma}$-G-simplicial sets by Ostermayr [129, Thm. 4.12]. Being 'strictly cofibrant' includes the condition that the symmetric group Σ_n acts freely on the complement of the image of the latching map l_n; for our purposes, no such freeness is necessary.

As we shall now see, the notion of cofibrancy for equivariant $\mathbf{\Gamma}$-spaces is stable under passage to closed subgroups and fixed-points by normal subgroups. If F is a $\mathbf{\Gamma}$-G-space and H a closed normal subgroup of G, we obtain a $\mathbf{\Gamma}$-G/H-space F^H by taking H-fixed-points objectwise. In other words, F^H is the composite functor

$$\mathbf{\Gamma} \ \xrightarrow{F} \ G\mathbf{T} \ \xrightarrow{(-)^H} \ (G/H)\mathbf{T} \ .$$

Proposition B.35. *Let H be a closed subgroup of a compact Lie group G and F a G-cofibrant $\mathbf{\Gamma}$-G-space.*

(i) *The underlying $\mathbf{\Gamma}$-H-space of F is H-cofibrant.*
(ii) *If H is normal, then the $\mathbf{\Gamma}$-G/H-space F^H is (G/H)-cofibrant.*

Proof Part (i) is clear because restriction from $G \times \Sigma_n$ to $H \times \Sigma_n$ preserves colimits, and it preserves equivariant cofibrations by Proposition B.14 (i).

(ii) The nth latching map for F^H factors as the composite

$$\mathrm{colim}_{U \subsetneq \{1,\dots,n\}} F^H(U_+) \longrightarrow (\mathrm{colim}_{U \subsetneq \{1,\dots,n\}} F(U_+))^H \xrightarrow{(l_n)^H} F(n_+)^H$$

of the canonical map and the restriction of the latching map for F to H-fixed-points. The first map is an isomorphism of $(G/H \times \Sigma_n)$-spaces by Proposition B.32 (iii). Since l_n is a $(G \times \Sigma_n)$-cofibration, the second map $(l_n)^H$ is a $(G/H \times \Sigma_n)$-cofibration by Proposition B.12. □

Proposition B.36. *Let G be a compact Lie group and F a G-cofibrant $\mathbf{\Gamma}$-G-space. Let K be a finite group and T a finite K-set. Let $\mathcal{Y} \subset \mathcal{P}(T)$ be a K-invariant set of subsets of T that is closed under passage to subsets. Then the canonical map*

$$\mathrm{colim}_{A \in \mathcal{Y}} F(A_+) \longrightarrow F(T_+)$$

is a $(G \times K)$-cofibration.

Proof We prove the following more general statement. We let $\mathcal{Y} \subset \mathcal{Z} \subset \mathcal{P}(T)$ be two K-invariant sets that are both closed under passage to subsets. We show that then the canonical morphism

$$\mathrm{colim}_{A \in \mathcal{Y}} F(A_+) \longrightarrow \mathrm{colim}_{A \in \mathcal{Z}} F(A_+)$$

is a $(G \times K)$-cofibration. The claim is the special case $\mathcal{Z} = \mathcal{P}(T)$, which has a terminal object T.

We start with the special case where $\mathcal{Z} = \mathcal{Y} \cup \{k \cdot B \,:\, k \in K\}$ for some subset B of T that does not belong to $\mathcal{Y}$. Since $\mathcal{Z}$ is closed under taking subsets, every proper subset of B then belongs to $\mathcal{Z}$, and hence to $\mathcal{Y}$. We let $L \leq K$ be the stabilizer group of B, i.e., the subgroup of those $k \in K$ that map B to itself. Then the square

$$\begin{tikzcd}
K \times_L \mathrm{colim}_{U \subsetneq B} F(U_+) \arrow[r, "K \times_L l_B"] \arrow[d] & K \times_L F(B_+) \arrow[d] \\
\mathrm{colim}_{A \in \mathcal{Y}} F(A_+) \arrow[r] & \mathrm{colim}_{A \in \mathcal{Z}} F(A_+)
\end{tikzcd}$$

is a pushout of $(G \times K)$-spaces. The latching map

$$l_B \ : \ \mathrm{colim}_{U \subsetneq B} \ F(U_+) \ \longrightarrow \ F(B_+)$$

is a $(G \times \Sigma_B)$-cofibration by the hypothesis that F is G-cofibrant. The L-action on B specifies a homomorphism $L \longrightarrow \Sigma_B$, so the latching map l_B is a $(G \times L)$-cofibration by Proposition B.14 (i). Hence the upper horizontal map in the square is a $(G \times K)$-cofibration by Proposition B.14 (ii). Since equivariant cofibrations are stable under cobase change, the lower horizontal morphism is a $(G \times K)$-cofibration.

In the general case we choose a chain of intermediate K-invariant subsets

$$\mathcal{Y} \ = \ \mathcal{Y}_0 \ \subset \ \mathcal{Y}_1 \ \subset \ \ldots \ \subset \mathcal{Y}_n \ = \ \mathcal{Z}$$

such that each $\mathcal{Y}_i$ is closed under taking subsets and $\mathcal{Y}_i$ has exactly one K-orbit more than $\mathcal{Y}_{i-1}$. The claim then holds for each pair $(\mathcal{Y}_i, \mathcal{Y}_{i-1})$. Since $(G \times K)$-cofibrations are stable under composition, this proves the general case. □

Proposition B.37. *Let G be a finite group, F a G-cofibrant $\mathbf{\Gamma}$-G-space, and A a simplicial finite based G-set.*

(i) *For all $m, n \geq 0$ the 'double latching map'*

$$L_m \left(F \circ (A \wedge n_+)\right) \cup_{\mathrm{colim}_{V \subsetneq \{1,\ldots,n\}} L_m(F \circ (A \wedge V_+))} \left(\mathrm{colim}_{V \subsetneq \{1,\ldots,n\}} F(A_m \wedge V_+)\right) \longrightarrow \ F(A_m \wedge n_+) \qquad \text{(B.38)}$$

is a $(G \times \Sigma_n)$-cofibration.

(ii) *The shifted $\mathbf{\Gamma}$-G-space $F_{|A|}$ is G-cofibrant.*

(iii) *The simplicial G-space $F \circ A$ is Reedy G-cofibrant.*

(iv) *For every subgroup H of G the simplicial space $(F \circ A)^H$ is Reedy cofibrant.*

Proof (i) The double latching space is the colimit of the functor

$$(\mathbf{\Delta}(m)^{\mathrm{op}} \times \mathcal{P}(n))^{\circ} \ \xrightarrow{(\sigma:[m]\longrightarrow[k],V)\mapsto A_k \wedge V_+} \ \mathbf{\Gamma} \ \xrightarrow{F} \ G\mathbf{T}_* \ ;$$

here $(\mathbf{\Delta}(m)^{\mathrm{op}} \times \mathcal{P}(n))^{\circ}$ is the 'punctured' poset category with objects those pairs $(\sigma : [m] \longrightarrow [k], V)$ that are not both maximal, i.e., such that $k < m$ or V is a proper subset of $\{1, \ldots, n\}$. As we explained in Remark A.40, the category $\mathbf{\Delta}(m)^{\mathrm{op}}$ is isomorphic to the poset category $\mathcal{P}(m)$, with $U \subset \{1, \ldots, m\}$ corresponding to $\sigma_U : [m] \longrightarrow [|U|]$ defined as

$$\sigma_U(i) \ = \ |U \cap \{1, \ldots, i\}| \ .$$

Under this isomorphism of categories, the first functor above becomes isomorphic to the functor

$$(\mathcal{P}(m) \times \mathcal{P}(n))^{\circ} \ \xrightarrow{(U,V)\mapsto \sigma_U^*(A_k) \wedge V_+} \ \mathbf{\Gamma} \ . \qquad \text{(B.39)}$$

We rewrite the functor (B.39). We set $S = A_m \backslash \{*\}$, identify S_+ with A_m, and set

$$I_U = \mathrm{Im}(\sigma_U^* : A_{|U|} \longrightarrow A_m) \backslash \{*\} ,$$

which is a G-invariant subset of S. We let $\mathcal{Y}$ denote the subposet of $\mathcal{P}(S \times \{1, \dots, n\})$ consisting of all subsets that are contained in $I_U \times \{1, \dots, n\}$ for some $(U, V) \in (\mathcal{P}(m) \times \mathcal{P}(n))^\circ$. Then the functor (B.39) factors as the composite

$$(\mathcal{P}(m) \times \mathcal{P}(n))^\circ \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\mathrm{incl}} \mathcal{P}(S \times \{1, \dots, n\}) \xrightarrow{(-)_+} \mathbf{\Gamma} ,$$

where $\varphi(U, V) = I_U \times V$. Altogether this exhibits the double latching space as the colimit of the composite

$$(\mathcal{P}(m) \times \mathcal{P}(n))^\circ \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{(-)_+} \mathbf{\Gamma} \xrightarrow{F} G\mathbf{T}_* .$$

We claim that the poset map φ is final, i.e., for every subset $B \in \mathcal{Y}$ the comma category $B \downarrow \varphi$ is non-empty and connected. The comma category $B \downarrow \varphi$ is non-empty by the very definition of $\mathcal{Y}$. Now we let (U, V) and (U', V') be elements of the poset $(\mathcal{P}(m) \times \mathcal{P}(n))^\circ$ such that

$$B \subseteq I_U \times V \qquad \text{and} \qquad B \subseteq I_{U'} \times V' .$$

We let $\sigma = \sigma_U : [m] \longrightarrow [k]$ and $\sigma' = \sigma_{U'} : [m] \longrightarrow [k']$ be the corresponding monotone surjections. There is then a unique pushout in the category $\mathbf{\Delta}$:

$$\begin{tikzcd} {[m]} \arrow[r, "\sigma'"] \arrow[d, "\sigma"'] & {[k']} \arrow[d, "\alpha'"] \\ {[k]} \arrow[r, "\alpha"'] & {[l]} \end{tikzcd} \tag{B.40}$$

The morphisms α and α' are again surjective. We set $\tau = \alpha \circ \sigma = \alpha' \circ \sigma'$ and claim that

$$\tau^*(A_l) = \sigma^*(A_k) \cap (\sigma')^*(A_{k'}) . \tag{B.41}$$

This is essentially the content of [58, 3.2], slightly reformulated; we reproduce the argument for the convenience of the reader. Since $\tau^*(A_l) = \sigma^*(\alpha^*(A_l))$, the set $\tau^*(A_l)$ is contained in $\sigma^*(A_k)$, and similarly for $(\sigma')^*(A_{k'})$. Conversely, we let $a \in A_m$ be a simplex such that $a = \sigma^*(x) = (\sigma')^*(y)$ for some $x \in A_k$ and $y \in A_{k'}$. We write $x = \beta^*(z)$ and $y = (\beta')^*(\bar{z})$ for surjective homomorphisms $\beta : [k] \longrightarrow [p], \beta' : [k'] \longrightarrow [p']$ and non-degenerate simplices z and $\bar{z}$. Then

$$(\beta\sigma)^*(z) = a = (\beta'\sigma')^*(\bar{z}) .$$

By the 'Eilenberg-Zilber lemma' ([49, (8.3)], see also [58, Sec. II.3]), the representation of a simplex as a degeneracy of a non-degenerate element is unique,

so $p = p'$, $z = \bar{z}$ and $\beta\sigma = \beta'\sigma'$. Since the square (B.40) is a pushout, there is a unique morphism $\lambda : [l] \longrightarrow [p]$ such that $\lambda\alpha = \beta$ and $\lambda\alpha' = \beta'$. So

$$a \;=\; (\beta\sigma)^*(z) \;=\; (\lambda\alpha\sigma)^*(z) \;=\; (\lambda\tau)^*(z) \;=\; \tau^*(\lambda^*(z)) \;\in\; \tau^*(A_l)\,.$$

This completes the proof of the relation (B.41).

We define

$$T \;=\; \{i \in \{1,\dots,m\} \;:\; \tau(i) > \tau(i-1)\}\,,$$

so that $\tau^*(A_l\backslash\{*\}) = I_T$. We now observe that

$$B \;\subseteq\; (I_U \times V) \cap (I_{U'} \times V') \;=\; \tau^*(A_l\backslash\{*\}) \times (V \cap V') \;=\; I_T \times (V \cap V')\,.$$

We have thus found an element $(T, V \cap V')$ in the poset $(\mathcal{P}(m) \times \mathcal{P}(n))^\circ$ that is less than or equal to both (U, V) and (U, V'), and such that $B \subseteq \varphi(T, V \cap V')$. So the comma category $B \downarrow \varphi$ is connected. Since the functor φ is final, we can conclude that the canonical $(G \times \Sigma_n)$-equivariant map

$$\mathrm{colim}_{(U,V)\in(\mathcal{P}(m)\times\mathcal{P}(n))^\circ}\; F((I_U \times V)_+) \;\longrightarrow\; \mathrm{colim}_{B\in\mathcal{Y}}\; F(B_+)$$

is a homeomorphism. On the other hand, the set $\mathcal{Y}$ is closed under passage to subsets and $(G \times \Sigma_n)$-invariant inside $\mathcal{P}(S \times \{1,\dots,n\})$. So we can apply Proposition B.36 with $K = G \times \Sigma_n$ and $T = S \times \{1,\dots,n\}$. We conclude that the canonical map

$$\mathrm{colim}_{B\in\mathcal{Y}}\; F(B_+) \;\longrightarrow\; F((S \times \{1,\dots,n\})_+) \;\cong\; F(A_m \wedge n_+)$$

is a $(G \times G \times \Sigma_n)$-cofibration. When we restrict to the diagonal G-action, the same map is a $(G \times \Sigma_n)$-cofibration by Proposition B.14 (i). Combining these two facts shows that the double latching morphism (B.38) is a $(G \times \Sigma_n)$-cofibration.

(ii) Part (i) says that the morphism of simplicial $(G \times \Sigma_n)$-spaces

$$\mathrm{colim}_{V\subsetneq\{1,\dots,n\}}\, F \circ (A \wedge V_+) \;\longrightarrow\; F \circ (A \wedge n_+)$$

is a Reedy $(G \times \Sigma_n)$-cofibration. Geometric realization is a left Quillen functor, so it takes Reedy $(G\times\Sigma_n)$-cofibrations of simplicial $(G\times\Sigma_n)$-spaces to $(G\times\Sigma_n)$-cofibrations. So the map

$$|\,\mathrm{colim}_{V\subsetneq\{1,\dots,n\}}\, F \circ (A \wedge V_+)| \;\longrightarrow\; |F \circ (A \wedge n_+)|$$

is a $(G \times \Sigma_n)$-cofibration. Realization commutes with colimits, so the source is a colimit, over proper subsets of $\{1,\dots,n\}$, of the spaces $|F \circ (A \wedge V_+)|$. The Fubini isomorphism of Proposition B.29 identifies the G-space $|F \circ (A \wedge V_+)|$ with $F(|A \wedge V_+|) \cong F(|A| \wedge V_+) = F_{|A|}(V_+)$. This proves the claim.

Part (iii) is the special case of part (i) for $n = 1$.

(iv) The restriction functor from G-spaces to H-spaces preserves colimits, and hence latching objects, and takes G-cofibrations to H-cofibrations by Proposition B.14 (i); so the latching map $l_n : L_n(F \circ A) \longrightarrow F(A_n)$ is an H-cofibration by part (iii). The H-fixed-point map

$$(l_n)^H \ : \ (L_n(F \circ A))^H \ \longrightarrow \ (F(A_n))^H$$

is then a non-equivariant cofibration by Proposition B.12. Taking H-fixed-points commutes with taking latching objects, by Proposition B.1 (iv). So the nth latching map for the simplicial space $(F \circ A)^H$ is a cofibration. □

The following proposition provides a way to reduce certain questions about $\mathbf{\Gamma}$-G-spaces for compact Lie groups to the special case of finite groups.

Proposition B.42. *Let G be a connected compact Lie group and F a $\mathbf{\Gamma}$-G-space. Then for every based G-space K the map*

$$(F^G)(K^G) \ \longrightarrow \ (F(K))^G$$

induced by the fixed-point inclusions $F^G \longrightarrow F$ and $K^G \longrightarrow K$ is a homeomorphism.

Proof Since $F(n_+)^G$ is closed inside $F(n_+)$ and K^G is a closed subset of K, Proposition B.26 (ii) shows that the inclusions induce a closed embedding $(F^G)(K^G) \longrightarrow F(K)$. The image of this map is contained in $F(K)^G$, so it only remains to show that every G-fixed-point of $F(K)$ is the image of a point in $(F^G)(K^G)$. We consider a point of $F(K)$ represented by a tuple $(x;\ k_1, \ldots, k_n)$ in $F(n_+) \times K^n$. We assume that the number n has been chosen minimally, so that x is non-degenerate and the entries k_i are pairwise distinct and different from the basepoint of K. If the point $[x;\ k_1, \ldots, k_n]$ of $F(K)$ is G-fixed, then for every group element g the tuple $(gx;\ gk_1, \ldots, gk_n)$ is equivalent to the original tuple. By Proposition B.24 (iii) there is a unique permutation $\sigma(g) \in \Sigma_n$ such that

$$(gx;\ gk_1, \ldots, gk_n) \ = \ (F(\sigma(g)^{-1})(x);\ k_{\sigma(g)(1)}, \ldots, k_{\sigma(g)(n)}) \ .$$

The map $\sigma : G \longrightarrow \Sigma_n$ is a homomorphism, and it is continuous since G acts continuously on K. Since G is connected, the homomorphism must be trivial, i.e., $\sigma(g) = 1$ for all $g \in G$. Thus the points x and $k_1, \ldots, k_n$ are all G-fixed. □

For us, the main purpose of equivariant $\mathbf{\Gamma}$-spaces is to construct equivariant spectra by evaluation on spheres. In more detail, we let G be a compact Lie group and F a $\mathbf{\Gamma}$-G-space. We define an orthogonal G-spectrum $F(\mathbb{S})$ by

$$F(\mathbb{S})(V) \ = \ F(S^V) \ .$$

The structure map $\sigma_{V,W} \ : \ S^V \wedge F(\mathbb{S})(W) \ \longrightarrow \ F(\mathbb{S})(V \oplus W)$ is the assembly

map for $K = S^V$ and $L = S^W$, followed by the effect of F on the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$. The $O(V)$-action on $F(\mathbb{S})(V)$ is via the action on S^V and the continuous functoriality of F. If V is a G-representation, then $F(S^V)$ has the diagonal G-action, from its actions on F and on V.

Proposition B.43. *Let G be a compact Lie group and F a G-cofibrant $\mathbf{\Gamma}$-G-space.*

(i) *For every G-representation V, the fixed-point space $F(S^V)^G$ is $(\dim(V^G)-1)$-connected.*
(ii) *The orthogonal G-spectrum $F(\mathbb{S})$ is equivariantly connective.*

Proof (i) We start with the special case where G is a finite group. We set $d = \dim(V^G)$. A choice of linear isometry $V^G \cong \mathbb{R}^d$ induces a G-equivariant homeomorphism between $F(S^V)$ and $F(S^d \wedge S^{V^\perp})$, where $V^\perp$ is the orthogonal complement of V^G inside V. So $F(S^V)^G$ is homeomorphic to $F(S^d \wedge S^{V^\perp})^G$.

The sphere S^d is homeomorphic to the geometric realization of the based simplicial set $\Delta[d]/\partial\Delta[d]$, the represented simplicial set $\Delta[d] = \mathbf{\Delta}(-,[d])$ modulo its simplicial boundary. The representation sphere $S^{V^\perp}$ is homeomorphic to the geometric realization of a finite based G-simplicial set. Indeed, $S^{V^\perp}$ admits the structure of a smooth G-manifold; Illman's triangulation theorem [83] then provides a G-equivariant triangulation. Passing to the barycentric subdivision provides a finite G-simplicial set B that realizes to $S^{V^\perp}$. Altogether, S^V is G-homeomorphic to the geometric realization of the G-simplicial set $A = (\Delta[d]/\partial\Delta[d]) \wedge B$.

Proposition B.29 provides a G-equivariant homeomorphism

$$F(S^V) \cong F(|A|) \cong |F \circ A| .$$

Taking G-fixed-points commutes with realization by Proposition B.1 (iv), so $F(S^V)^G$ is homeomorphic to the realization of the simplicial space $(F \circ A)^G$ that takes $[m] \in \mathbf{\Delta}^{\mathrm{op}}$ to the space $F(A_m)^G$. In dimensions below d, the simplicial set $\Delta[d]/\partial\Delta[d]$, and hence also the simplicial set $A = (\Delta[d]/\partial\Delta[d]) \wedge B$, consists only of the base point. So for $m < d$, the space $F(A_m)^G$ is a single point. Moreover, the simplicial space $(F \circ A)^G$ is Reedy cofibrant by Proposition B.37 (iv). So its realization is $(d-1)$-connected by Proposition A.46 (ii). Altogether this establishes the claim that the space $F(S^V)^G$ is $(d-1)$-connected.

Now we treat the case of a general compact Lie group. We let G° denote the connected component of the identity and $\bar{G} = G/G^\circ$ the finite group of components of G. The $\mathbf{\Gamma}$-$\bar{G}$-space F^{G° is $\bar{G}$-cofibrant by Proposition B.35 (ii). Proposition B.42 provides a homeomorphism

$$F(S^V)^G = (F(S^V)^{G^\circ})^{\bar{G}} \cong ((F^{G^\circ})(S^{V^{G^\circ}}))^{\bar{G}} .$$

Because $(V^{G^\circ})^{\bar{G}} = V^G$, the right-hand side is $(\dim(V^G) - 1)$-connected by the special case above, for the finite group $\bar{G}$, the $\mathbf{\Gamma}$-$\bar{G}$-space F^{G° and the $\bar{G}$-representation V^{G°.

(ii) We let H be any closed subgroup of G and show that the group $\pi^H_{-k}(F(\mathbb{S}))$ is trivial for all $k \geq 1$. We let V be any H-representation, K a closed subgroup of H, and we set $d_K = \dim(V^K)$. The underlying $\mathbf{\Gamma}$-K-space of F is K-cofibrant by Proposition B.35 (i). So $F(S^{\mathbb{R}^k \oplus V})^K$ is $(k + d_K - 1)$-connected by part (i).

On the other hand, the cellular dimension of S^V at K, in the sense of [179, II.2, p. 106], is at most d_K. Because k is positive, the cellular dimension of S^V at K does not exceed the connectivity of $F(S^{\mathbb{R}^k \oplus V})^K$. So every based continuous H-map $S^V \longrightarrow F(S^{\mathbb{R}^k \oplus V})$ is equivariantly null-homotopic by [179, II Prop. 2.7], and the set $[S^V, F(S^{\mathbb{R}^k \oplus V})]^H$ has only one element. Passage to the colimit over $V \in s(\mathcal{U}_H)$ proves the claim. □

We can also show that prolongation of G-cofibrant $\mathbf{\Gamma}$-G-spaces is homotopical in the $\mathbf{\Gamma}$-space variable, as long as we evaluate on finite based G-CW-complexes.

Definition B.44. Let G be a compact Lie group. A morphism $\psi : E \longrightarrow F$ of $\mathbf{\Gamma}$-G-spaces is a *strict equivalence* if for every $n \geq 1$ the map of $(G \times \Sigma_n)$-spaces $\psi(n_+) : E(n_+) \longrightarrow F(n_+)$ is an $\mathcal{F}(G; \Sigma_n)$-weak equivalence, where $\mathcal{F}(G; \Sigma_n)$ is the family of graph subgroups of $G \times \Sigma_n$.

Remark B.45. A strict equivalence $\psi : E \longrightarrow F$ of $\mathbf{\Gamma}$-G-spaces also satisfies the following condition: for every closed subgroup H of G and every finite H-set S, the map $\psi(S_+) : E(S_+) \longrightarrow F(S_+)$ is an H-weak equivalence with respect to the diagonal H-actions. Indeed, if S has n elements, we may suppose that $S = \{1, \ldots, n\}$ with H-action specified by a continuous homomorphism $\rho : H \longrightarrow \Sigma_n$. We let K be a closed subgroup of H and Γ the graph of $\rho|_K : K \longrightarrow \Sigma_n$. The map $\psi(S_+)^\Gamma : E(S_+)^\Gamma :\longrightarrow F(S_+)^\Gamma$ is a weak equivalence by the hypothesis on ψ. Moreover,

$$F(S_+)^K = F(S_+)^\Gamma ,$$

where the fixed-points on the left-hand side are with respect to the diagonal K-action. This proves the claim.

The following result will allow us to extend equivariant homotopical properties from the class of finite G-simplicial sets to the class of finite G-CW-complexes. The non-equivariant case is treated for example in [71, Thm. 2C.5]. I am sure that also the equivariant result is well-known; however, I don't know a reference, so I provide a proof.

Proposition B.46. *Let G be a finite group.*

(i) *Let X be a finite G-CW-complex, A a finite G-simplicial set and $f : |A| \longrightarrow X$ a continuous G-map. Then there is a finite G-simplicial set B, a monomorphism of G-simplicial sets $i : A \longrightarrow B$ and a G-homotopy equivalence $h : |B| \longrightarrow X$ such that*

$$h \circ |i| = f : |A| \longrightarrow X .$$

(ii) *Every finite based G-CW-complex is based G-homotopy equivalent to the realization of a finite based G-simplicial set.*

Proof (i) We let $cY = (Y \times [0,1])/(Y \times \{1\})$ denote the unreduced cone of a space Y. We start with a very special case, namely when there is a pushout square of G-spaces

$$\begin{tikzcd} G/H \times |\partial\Delta[k]| \arrow[r, "\mathrm{incl}"] \arrow[d, "\alpha"'] & G/H \times c|\partial\Delta[k]| \arrow[d] \\ |A| \arrow[r, "f"'] & X \end{tikzcd}$$

for some subgroup H of G and some $k \geq 0$; in other words, we suppose that X is obtained from $|A|$ by attaching one equivariant cell.

The continuous map

$$\alpha(eH, -) : |\partial\Delta[k]| \longrightarrow |A|$$

lands in the H-fixed-points. Since fixed-points commute with realization (Proposition B.1 (iv)), we may view it as a continuous map to $|A^H|$. Now we use a 'simplicial approximation', by which we mean the following data:

- a finite simplicial set D,
- a morphism of simplicial sets $\Phi : D \longrightarrow A^H$, and
- a homotopy equivalence $\varphi : |D| \longrightarrow |\partial\Delta[k]|$ such that $\alpha(eH, -) \circ \varphi$ is homotopic to the realization of Φ.

For example, we can use $D = \mathrm{Sd}^m(\partial\Delta[k])$ for a suitably large $m \geq 0$, where Sd is Kan's subdivision functor [87, Sec. 7]; the remaining data is then provided by Lemma 7.5 and Theorem 8.5 of [87]. Alternatively, we can take $D = SD^m(\partial\Delta[k])$ for a suitably large $m \geq 0$, where SD is the 'double simplicial subdivision' of [41, Def. (12.5)]; the other data is provided by [41, Thm. (12.7)].

We let $\tilde{\Phi} : G/H \times D \longrightarrow A$ be the G-equivariant extension of Φ, i.e., $\tilde{\Phi}_n(gH, x) = g \cdot \Phi(x)$ for $g \in G$ and $x \in D_n$. The geometric realization of $\tilde{\Phi}$ is then G-equivariantly homotopic to the composite

$$G/H \times |D| \xrightarrow{G/H \times \varphi} G/H \times |\partial\Delta[k]| \xrightarrow{\alpha} |A| .$$

We choose a G-equivariant homotopy

$$K \ : \ G/H \times |D| \times [0,1] \ \longrightarrow \ |A|$$

from $|\tilde{\Phi}|$ to the map $\alpha \circ (G/H \times \varphi)$. We absorb the homotopy into the mapping cylinder of the map $|\tilde{\Phi}| : G/H \times |D| \longrightarrow |A|$, and obtain a commutative diagram of G-spaces:

$$\begin{tikzcd}
|A| \cup_{|\tilde{\Phi}|} (G/H \times |D| \times [0,1]) \arrow[d, "\mathrm{Id}_{|A|} \cup K"'] & G/H \times |D| \arrow[l, "(-,1)"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "G/H \times \varphi"] & G/H \times c|D| \arrow[d, "G/H \times \bar{\varphi}"] \\
|A| & G/H \times |\partial\Delta[k]| \arrow[l, "\alpha"] \arrow[r, "\mathrm{incl}"'] & G/H \times c|\partial\Delta[k]|
\end{tikzcd}$$

All vertical maps in the diagram are G-homotopy equivalences, and the right horizontal maps are G-cofibrations. The gluing lemma (Proposition B.6) then shows that the induced map on pushouts

$$\begin{aligned} &|A| \cup_{|\tilde{\Phi}|} (G/H \times |D| \times [0,1]) \cup_{G/H \times |D|} (G/H \times c|D|) \\ &\qquad \longrightarrow \ |A| \cup_{\alpha} (G/H \times c|\partial\Delta[k]|) \ \cong \ X \end{aligned} \tag{B.47}$$

is a G-weak equivalence; moreover, its restriction to $|A|$ is the original map $f : |A| \longrightarrow X$. The source of (B.47) is homeomorphic to the unreduced mapping cone of the map $|\tilde{\Phi}|$. Mapping cones can also be formed in the category of G-simplicial sets, so the source of (B.47) is equivariantly homeomorphic to the realization of the unreduced mapping cone of $\tilde{\Phi} : G/H \times D \longrightarrow A$. This simplicial mapping cone is the desired G-simplicial set B.

The rest of the argument is now straightforward. Induction over the number of relative equivariant cells proves the case where $(X, |A|)$ is a finite relative G-CW-pair and where $f : |A| \longrightarrow X$ is the inclusion. In the general case, the filtration of A by simplicial skeleta induces a filtration on the geometric realization that gives $|A|$ the structure of a finite G-CW-complex. If the map f is cellular for this structure, then the mapping cylinder $Z = |A| \times [0,1] \cup_f X$ inherits a finite G-CW-structure in which $|A|$ is an equivariant subcomplex. So the previous case provides a finite G-simplicial set B, a monomorphism of G-simplicial sets $i : A \longrightarrow B$ and a G-homotopy equivalence $h : |B| \longrightarrow Z$ such that the composite

$$h \circ |i| \ : \ |A| \ \longrightarrow \ Z$$

is the 'front inclusion' $(-,0) : |A| \longrightarrow Z$. The projection $Z \longrightarrow X$ is a G-equivariant homotopy equivalence and $f = p \circ (-,0) : |A| \longrightarrow X$. So the triple $(B, i, p \circ h)$ has the desired properties. If the map f is arbitrary, we use the equivariant cellular approximation theorem (see for example [25, II Prop. 5.6], [109,

Thm. 4.4] or [179, Ch. II, Thm. 2.1]) and the equivariant homotopy extension property of the G-map $|i| : |A| \longrightarrow |B|$ to reduce to the cellular case.

(ii) We let X be a finite based G-CW-complex. We apply part (i) to the terminal simplicial G-set $A = *$ and the inclusion of the basepoint $|*| \longrightarrow X$; part (i) provides a finite based G-simplicial set B and a based G-homotopy equivalence $h : |B| \longrightarrow X$. □

Now we have all tools ready to show that prolongation preserves strict equivalences between cofibrant $\mathbf{\Gamma}$-G-spaces, at least on finite G-CW-complexes.

Proposition B.48. *Let G be a finite group and $\psi : E \longrightarrow F$ a strict equivalence between cofibrant $\mathbf{\Gamma}$-G-spaces. Then for every finite based G-CW-complex X, the map $\psi(X)^G : E(X)^G \longrightarrow F(X)^G$ is a weak equivalence.*

Proof We start with the special case where $X = |B|$ is the geometric realization of a finite based G-simplicial set B. In this situation, the morphism of simplicial spaces $(\psi \circ B)^G : (E \circ B)^G \longrightarrow (F \circ B)^G$ is level-wise a weak equivalence by Remark B.45. Moreover, source and target are Reedy cofibrant by Proposition B.37 (iv). Geometric realization takes level-wise weak equivalences between Reedy cofibrant simplicial spaces to weak equivalences (Proposition A.44). This proves the special case.

In the general case we choose a finite based G-simplicial set B and a based G-homotopy equivalence $h : |B| \longrightarrow X$, as provided by Proposition B.46 (ii). Prolonged $\mathbf{\Gamma}$-spaces are continuous functors, so they preserve equivariant based homotopies. In the commutative square

$$\begin{tikzcd}
E(|B|)^G \arrow[r, "\psi(|B|)^G"] \arrow[d, "E(h)^G"'] & F(|B|)^G \arrow[d, "F(h)^G"] \\
E(X)^G \arrow[r, "\psi(X)^G"'] & F(X)^G
\end{tikzcd}$$

both vertical maps are thus homotopy equivalences. The upper map is a weak equivalence by the previous paragraph, hence so is the lower map. □

Now we move on to the analysis of special and very special $\mathbf{\Gamma}$-G-spaces. We will eventually assume that the $\mathbf{\Gamma}$-G-space is G-cofibrant, in order to have homotopical control over its prolongation. The final aim is to show that for finite G and very special (or special) F, the evaluation on spheres $F(\mathbb{S})$ is a G-Ω-spectrum (or 'positive' G-Ω-spectrum), see Theorem B.61 and Theorem B.65 below.

If F is any $\mathbf{\Gamma}$-space and S a finite set, then we define the map

$$P_S \ : \ F(S_+) \ \longrightarrow \ \mathrm{map}(S, F(1_+))$$

by $P_S(x)(s) = F(p_s)(x)$, where $p_s : S_+ \longrightarrow 1_+$ sends s to 1 and all other elements of S_+ to the basepoint. If a group G acts on F and S, then the map P_S is G-equivariant for the diagonal G-action on the source and the conjugation action on the target.

Definition B.49. Let G be a compact Lie group. A $\mathbf{\Gamma}$-G-space F is *special* if for every closed subgroup H of G and every finite H-set S the map

$$(P_S)^H \ : \ F(S_+)^H \ \longrightarrow \ \mathrm{map}^H(S, F(1_+))$$

is a weak equivalence.

We showed in Proposition B.35 that the notion of cofibrancy for equivariant $\mathbf{\Gamma}$-spaces is stable under passage to closed subgroups and fixed-points by normal subgroups. Now we show the analogous statement for specialness.

Proposition B.50. *Let H be a closed subgroup of a compact Lie group G and F a special $\mathbf{\Gamma}$-G-space.*

(i) *The underlying $\mathbf{\Gamma}$-H-space of F is special.*
(ii) *If H is normal, then the $\mathbf{\Gamma}$-G/H-space F^H is special.*

Proof Part (i) is clear by definition. For part (ii) we consider a closed subgroup of G/H, which must be of the form Δ/H for a closed subgroup Δ of G with $H \leq \Delta$. We let S be a finite Δ/H-set, which we can also view as a finite Δ-set by restriction along the projection $\Delta \longrightarrow \Delta/H$. The claim then follows from the hypothesis that F is special and the relations

$$(F^H(S))^{\Delta/H} \ = \ F(S)^{\Delta} \quad \text{and} \quad \mathrm{map}^{\Delta/H}(S, F^H(1_+)) \ = \ \mathrm{map}^{\Delta}(S, F(1_+)) \ . \qquad \square$$

The functor obtained by prolonging a $\mathbf{\Gamma}$-G-space comes with 'Wirthmüller type' maps, defined as follows. We let H be a closed subgroup of finite index in G and Z a based H-space. We recall that $l_Z : G \ltimes_H Z \longrightarrow Z$ denotes the H-equivariant projection to the wedge summand indexed by the preferred coset eH, i.e.,

$$l_Z[g, z] \ = \ \begin{cases} g \cdot z & \text{if } g \in H, \text{ and} \\ * & \text{else.} \end{cases}$$

The H-equivariant map $F(l_Z) : F(G \ltimes_H Z) \longrightarrow F(Z)$ is then adjoint to a G-equivariant Wirthmüller map

$$\omega_Z \ : \ F(G \ltimes_H Z) \ \longrightarrow \ \mathrm{map}^H(G, F(Z)) \ . \tag{B.51}$$

We recall some equivalent characterizations of special $\mathbf{\Gamma}$-G-spaces. The equivalence of the first two conditions was first noted in [158, Cor.].

Proposition B.52. *Let G be a compact Lie group and F a $\boldsymbol{\Gamma}$-G-space. Then the following conditions are equivalent.*

(i) *The $\boldsymbol{\Gamma}$-G-space F is special.*

(ii) *For every $n \geq 1$ the map*

$$P_n \ : \ F(n_+) \ \longrightarrow \ \mathrm{map}(\{1,\dots,n\}, F(1_+)) \ = \ F(1_+)^n$$

is an $\mathcal{F}(G;\Sigma_n)$-equivalence, where $\mathcal{F}(G;\Sigma_n)$ is the family of graph subgroups.

(iii) *For all pairs of finite G-sets T and U, the map*

$$(F(p_T), F(p_U)) \ : \ F((T \amalg U)_+) \ \longrightarrow \ F(T_+) \times F(U_+)$$

is a G-weak equivalence, and for every finite index subgroup H of G and every finite H-set S, the Wirthmüller map $\omega_{S_+} : F((G \times_H S)_+) \longrightarrow \mathrm{map}^H(G, F(S_+))$ is a G-weak equivalence.

Proof (i)$\Longleftrightarrow$ (ii) We let H be a closed subgroup of G. Every finite H-set is isomorphic to $\alpha^*\{1,\dots,n\}$ for some continuous homomorphism $\alpha : H \longrightarrow \Sigma_n$. If $\Gamma \leq G \times \Sigma_n$ denotes the graph of α, then $F(n_+)^\Gamma = F(\alpha^*\{1,\dots,n\})^H$ and $(F(1_+)^n)^\Gamma = \mathrm{map}^H(\alpha^*\{1,\dots,n\}, F(1_+))$. So the map $(P_n)^\Gamma$ is a weak equivalence if and only if the map $(P_{\alpha^*\{1,\dots,n\}})^H$ is a weak equivalence.

(i)$\Longrightarrow$ (iii) In the commutative square

$$\begin{tikzcd}
F((T \amalg U)_+) \arrow[r, "{(F(p_T),F(p_U))}"] \arrow[d, "P_{T\amalg U}"', "\simeq"] & F(T_+) \times F(U_+) \arrow[d, "\simeq"', "p_T \times p_U"] \\
\mathrm{map}(T \amalg U, F(1_+)) \arrow[r, "\cong"'] & \mathrm{map}(T, F(1_+)) \times \mathrm{map}(U, F(1_+))
\end{tikzcd} \tag{B.53}$$

both vertical maps are G-weak equivalences because F is special, and the lower horizontal map is a homeomorphism. So the upper horizontal map is a G-weak equivalence.

Now we suppose that H has finite index in G and we let S be a finite H-set. Then the Wirthmüller map participates in the commutative diagram:

$$\begin{tikzcd}
F((G \times_H S)_+) \arrow[r, "\omega_{S_+}"] \arrow[d, "P_{G\times_H S}"'] & \mathrm{map}^H(G, F(S_+)) \arrow[d, "\mathrm{map}^H(G, P_S)"] \\
\mathrm{map}(G \times_H S, F(1_+)) \arrow[r, "\cong"'] & \mathrm{map}^H(G, \mathrm{map}(S, F(1_+)))
\end{tikzcd}$$

The left and right vertical maps are G-weak equivalences by specialness, applied to the G-set $G \times_H S$ and the H-set S, respectively. The lower horizontal homeomorphism sends $f : G \times_H S \longrightarrow F(1_+)$ to the adjoint H-map

$\hat{f} : G \longrightarrow \mathrm{map}(S, F(1_+))$ with $\hat{f}(g)(s) = f[g, s]$. Altogether this shows that the Wirthmüller map ω_{S_+} is a G-weak equivalence.

(iii)$\Longrightarrow$ (i) We start by showing that for every finite G-set S the map $P_S : F(S_+) \longrightarrow \mathrm{map}(S, F(1_+))$ is a G-weak equivalence. If $S = T \amalg U$ is the disjoint union of two finite G-sets T and U, we contemplate the commutative square (B.53). The upper horizontal map is a G-weak equivalence by hypothesis. The lower horizontal map is an equivariant homeomorphism. So if the claim holds for the G-sets T and U, then it holds for their disjoint union. This reduces the claim to the case of transitive G-sets. For every finite index subgroup H of G, there is a commutative square of G-maps

$$\begin{tikzcd}
F((G/H)_+) \arrow[r, "P_{G/H}"] \arrow[d, "\cong"'] & \mathrm{map}(G/H, F(1_+)) \arrow[d, "\cong"] \\
F(G \ltimes_H 1_+) \arrow[r, "\omega_{1_+}"'] & \mathrm{map}^H(G, F(1_+))
\end{tikzcd}$$

where the right vertical map is adjoint to the H-map

$$\mathrm{map}(G/H, F(1_+)) \longrightarrow F(1_+) , \quad f \longmapsto f(eH) .$$

The two vertical maps are homeomorphisms. The lower horizontal map is the Wirthmüller map for the H-set $\{1\}$; this map is a G-weak equivalence by hypothesis. So the map P_S is a G-weak equivalence for transitive G-sets; this completes the proof of the first claim.

Now we let H be a finite index subgroup of G and S a finite H-set. The H-maps $[1,-] : S_+ \longrightarrow (G\times_H S)_+$ and $l_S : (G\times_H S)_+ \longrightarrow S_+$ express S_+ as an H-equivariant retract of the based G-set $(G \times_H S)_+$. The middle vertical map in the commutative diagram

$$\begin{tikzcd}
F(S_+)^H \arrow[r, "F({[1,-]})^H"] \arrow[d, "(P_S)^H"'] & F((G\times_H S)_+)^H \arrow[r, "F(l_S)^H"] \arrow[d, "(P_{G\times_H S})^H"'] & F(S_+)^H \arrow[d, "(P_S)^H"] \\
\mathrm{map}^H(S, F(1_+)) \arrow[r, "(l_S)^*"'] & \mathrm{map}^H(G\times_H S, F(1_+)) \arrow[r, "{[1,-]^*}"'] & \mathrm{map}^H(S, F(1_+))
\end{tikzcd}$$

is a weak equivalence by the previous paragraph, and both horizontal composites are the identity maps. Since weak equivalences are closed under retracts, the map $(P_S)^H$ is a weak equivalence. This proves that the $\mathbf{\Gamma}$-G-space F is special. □

Proposition B.54. *Let G be a compact Lie group and F a G-cofibrant special $\mathbf{\Gamma}$-G-space.*

(i) *For all finite based G-CW-complexes X and Y the map*

$$(F(p_X), F(p_Y)) \;:\; F(X \vee Y) \longrightarrow F(X) \times F(Y)$$

is a G-weak equivalence.

(ii) *For every finite index subgroup H of G and every finite based H-CW-complex Z, the Wirthmüller map*

$$\omega_Z \;:\; F(G \ltimes_H Z) \longrightarrow \mathrm{map}^H(G, F(Z))$$

is a G-weak equivalence.

(iii) *For every finite based G-CW-complex X, the shifted $\mathbf{\Gamma}$-G-space F_X is special.*

Proof (i) We wish to show that for every closed subgroup H of G the map

$$(F(p_X)^H, F(p_Y)^H) \;:\; F(X \vee Y)^H \longrightarrow F(X)^H \times F(Y)^H$$

is a weak equivalence. The underlying $\mathbf{\Gamma}$-H-space of F is H-cofibrant by Proposition B.35 (i) and special by Proposition B.50 (i). Moreover, the underlying H-spaces of X and Y are H-homotopy equivalent to finite H-CW-complexes, by [85, Cor. B]. So we may assume without loss of generality that $H = G$.

We start with the special case where the group G is finite and $X = |A|$ and $Y = |B|$ are realizations of two finite based G-simplicial sets A and B. For every $n \geq 0$, the map

$$(F(p_{A_n})^G, F(p_{B_n})^G) \;:\; F(A_n \vee B_n)^G \longrightarrow F(A_n)^G \times F(B_n)^G$$

is a weak equivalence by Proposition B.52 (iii). The map of simplicial spaces

$$((F \circ p_A)^G, (F \circ p_B)^G) \;:\; (F \circ (A \vee B))^G \longrightarrow (F \circ A)^G \times (F \circ B)^G$$

is thus level-wise a weak equivalence. The simplicial spaces $(F \circ (A \vee B))^G$, $(F \circ A)^G$ and $(F \circ B)^G$ are Reedy cofibrant by Proposition B.37 (iv). The product simplicial space $(F \circ A)^G \times (F \circ B)^G$ is then Reedy cofibrant by Proposition A.50. So the map induced on geometric realizations

$$|(F \circ (A \vee B))^G| \longrightarrow |(F \circ A)^G \times (F \circ B)^G|$$

is a weak equivalence by Proposition A.44. Realization commutes with fixed-points (Proposition B.1 (iv)) and products (Proposition A.37 (ii)), so the map

$$(|F \circ p_A|, |F \circ p_B|)^G \;:\; |F \circ (A \vee B)|^G \longrightarrow (|F \circ A| \times |F \circ B|)^G$$

is a weak equivalence. The Fubini isomorphism $|F \circ A|^G \cong F(|A|)^G$ of Proposition B.29 then translates this into the claim for the geometric realizations of A and B.

Now we continue to assume that G is finite, but X and Y are arbitrary finite

based G-CW-complexes. We choose based G-homotopy equivalences $|A| \simeq X$ and $|B| \simeq Y$ for suitable finite based G-simplicial sets A and B as provided by Proposition B.46 (ii). Prolonged $\mathbf{\Gamma}$-G-spaces preserve equivariant homotopy equivalences, so the general case follows from the special case.

It remains to treat the general case of a compact Lie group. We let G° be the identity component of G and we write $\bar{G} = G/G^\circ$ for the finite group of path components. The $\mathbf{\Gamma}$-$\bar{G}$-space F^{G° is $\bar{G}$-cofibrant by Proposition B.35 (ii) and special by Proposition B.50 (ii). Since X and Y are finite G-CW-complexes, X^{G° and Y^{G° are finite $\bar{G}$-CW-complexes. So the map

$$(F^{G^\circ}(p_{X^{G^\circ}}), F^{G^\circ}(p_{Y^{G^\circ}})) \ : \ F^{G^\circ}(X^{G^\circ} \vee Y^{G^\circ}) \ \longrightarrow \ F^{G^\circ}(X^{G^\circ}) \times F^{G\circ}(Y^{G^\circ})$$

induces a weak equivalence on $\bar{G}$-fixed-points by the previous paragraph. Moreover,

$$(F^{G^\circ}(X^{G^\circ}))^{\bar{G}} \ \cong \ (F(X)^{G^\circ})^{\bar{G}} \ = \ F(X)^G$$

by Proposition B.42, and similarly for the G-fixed-points of $F(Y)$ and $F(X \vee Y)$. This completes the proof.

(ii) We start by showing that the G-fixed-point map

$$(\omega_Z)^G \ : \ F(G \ltimes_H Z)^G \ \longrightarrow \ (\mathrm{map}^H(G, F(Z)))^G$$

is a weak equivalence. Evaluation at $1 \in G$ identifies the target of $(\omega_Z)^G$ with $F(Z)^H$, so we may show that the composite map

$$F(G \ltimes_H Z)^G \ \xrightarrow{\text{incl}} \ F(G \ltimes_H Z)^H \ \xrightarrow{F(l_Z)^H} \ F(Z)^H \tag{B.55}$$

is a weak equivalence.

We start with the special where G is finite and $Z = |B|$ is the geometric realization of a finite based H-simplicial set B; then $G \ltimes_H Z$ is G-homeomorphic to the geometric realization of the finite based G-simplicial set $G \ltimes_H B$. By Proposition B.29 and A.37 (ii), the space $F(|G \ltimes_H B|)^G$ is homeomorphic to $|(F \circ (G \ltimes_H B))^G|$. Similarly, $F(|B|)^H$ is homeomorphic to $|(F \circ B)^H|$. Under these homeomorphisms, the map (B.55) becomes the geometric realization of the morphism of simplicial spaces

$$(F \circ l_B)^H \circ \text{incl} \ : \ (F \circ (G \ltimes_H B))^G \ \longrightarrow \ (F \circ B)^H \tag{B.56}$$

whose value at the object $[n]$ of $\mathbf{\Delta}^{\text{op}}$ is the map $F(l_{B_n})^H \circ \text{incl}$. The morphism of simplicial spaces (B.56) is a weak equivalence in every simplicial dimension by Proposition B.52 (iii); moreover, source and target are Reedy cofibrant by Proposition B.37 (iv). As a level-wise weak equivalence between Reedy cofibrant simplicial spaces, the morphism (B.56) induces a weak equivalence on geometric realizations, by Proposition A.44. This completes the proof that the map (B.55) is a weak equivalence in the special case $Z = |B|$.

Now we treat the case where G is finite, but Z is any finite based H-CW-complex. Proposition B.46 (ii) provides a finite based H-simplicial set B and a based H-homotopy equivalence $h : |B| \longrightarrow Z$. The map $G \ltimes_H h : G \ltimes_H |B| \longrightarrow G \ltimes_H Z$ is then a based G-homotopy equivalence. In the commutative square

$$\begin{tikzcd} F(G \ltimes_H |B|)^G \arrow[r, "F(l_{|B|})^H \circ \mathrm{incl}"] \arrow[d, "F(G \ltimes_H h)^G"'] & F(|B|)^H \arrow[d, "F(h)^H"] \\ F(G \ltimes_H Z)^G \arrow[r, "F(l_Z)^H \circ \mathrm{incl}"'] & F(Z)^H \end{tikzcd}$$

both vertical maps are then homotopy equivalences. The upper map is a weak equivalence by the previous paragraph, hence so is the lower map. This completes the proof that the map (B.55) is a weak equivalence in the special case when G is finite.

It remains to treat the case of a general compact Lie group. We let G° be the identity component of G and we write $\bar{G} = G/G^\circ$ for the finite group of path components. The $\mathbf{\Gamma}$-$\bar{G}$-space F^{G° is $\bar{G}$-cofibrant by Proposition B.35 (ii) and special by Proposition B.50 (ii). Since H has finite index in G, the identity components of H and G are the same, i.e., $H^\circ = G^\circ$. Moreover, $\bar{H} = H/H^\circ$ is a subgroup of $\bar{G}$.

Since Z is a finite H-CW-complex, X^{H° is a finite $\bar{H}$-CW-complex. So the map

$$(F^{G^\circ}(l_{Z^{H^\circ}}))^{\bar{H}} \circ \mathrm{incl} \ : \ (F^{G^\circ}(\bar{G} \ltimes_{\bar{H}} Z^{H^\circ}))^{\bar{G}} \ \longrightarrow \ (F^{G^\circ}(Z^{H^\circ}))^{\bar{H}}$$

is a weak equivalence by the previous paragraph. Proposition B.42 lets us rewrite source and target of this map as

$$\begin{aligned} (F^{G^\circ}(\bar{G} \ltimes_{\bar{H}} Z^{H^\circ}))^{\bar{G}} \ &\cong \ (F^{G^\circ}((G \ltimes_H Z)^{G^\circ}))^{\bar{G}} \\ &\cong \ (F(G \ltimes_H Z)^{G^\circ})^{\bar{G}} \ = \ F(G \ltimes_H Z)^G \end{aligned}$$

and

$$(F^{G^\circ}(Z^{H^\circ}))^{\bar{H}} \ = \ (F^{H^\circ}(Z^{H^\circ}))^{\bar{H}} \ \cong \ (F(Z)^{H^\circ})^{\bar{H}} \ \cong \ F(Z)^H \ .$$

Under these homeomorphisms, the map $(F^{G^\circ}(l_{Z^{H^\circ}}))^{\bar{H}} \circ \mathrm{incl}$ becomes the map (B.55). This completes the proof that the map (B.55) is a weak equivalence for all compact Lie groups G and all finite based H-CW-complexes Z. Hence the map $(\omega_Z)^G$ is a weak equivalence.

Now we let K be any closed subgroup of G. Since H has finite index in G, the set G/H is finite, and there is a finite set $\{\gamma_i\}_{i=1,\dots,n}$ of K-H-double coset representatives. These determine a K-equivariant wedge decomposition

$$D \ : \ \bigvee_{i=1,\dots,n} K \ltimes_{K \cap {}^{\gamma_i}H} c^*_{\gamma_i}(Z) \ \cong \ G \ltimes_H Z \ , \quad [k,z]_i \ \longmapsto \ [k\gamma_i, z] \ .$$

We obtain a commutative square:

$$\begin{tikzcd}
F(G \ltimes_H Z)^K \arrow[r, "(\omega_Z)^K"] \arrow[d, "\simeq"'] & (\mathrm{map}^H(G, F(Z)))^K \arrow[d, "\cong"] \\
\prod_{i=1}^n F(K \ltimes_{K \cap^{\gamma_i} H} c^*_{\gamma_i}(Z))^K \arrow[r, "\prod \omega_{c^*_{\gamma_i}(Z)}"'] & \prod_{i=1}^n F(Z)^{K^{\gamma_i} \cap H}
\end{tikzcd}$$

The right vertical map is evaluation at the chosen coset representatives, and it is a homeomorphism. The left map is $F(D^{-1})^K$ followed by $F(-)^K$ applied to the projections to the wedge summands. The latter is a weak equivalence by the additivity property of part (i), applied to the underlying $\mathbf{\Gamma}$-K-space of F, and the spaces $K \ltimes_{K \cap^{\gamma_i} H} c^*_{\gamma_i}(Z)$; there is a slight caveat, namely that the underlying $(K^{\gamma_i} \cap H)$-space of Z need not admit the structure of an equivariant CW-complex. However, it is always $(K^{\gamma_i} \cap H)$-homotopy equivalent to a finite $(K^{\gamma_i} \cap H)$-CW-complex, by [85, Cor. B]; this is enough to run the argument because prolonged $\mathbf{\Gamma}$-K-spaces preserve equivariant homotopy equivalences. The lower horizontal map in the above square is a weak equivalence by the previous paragraph, applied to the underlying $\mathbf{\Gamma}$-K-space of F, and the subgroups $K \cap {}^{\gamma_i}H$. So we can conclude that the map $(\omega_Z)^K$ is a weak equivalence. Since K was an arbitrary closed subgroup of G, this completes the proof that the Wirthmüller map ω_Z is a G-weak equivalence.

(iii) We verify the criterion of Proposition B.52 (iii) for the shifted $\mathbf{\Gamma}$-G-space F_X. Given two finite G-sets T and U, part (i) shows that the map

$$\begin{aligned}
(F_X(p_T), F_X(p_U)) \ : \ F_X((T \amalg U)_+) \ &= \ F(X \wedge (T \amalg U)_+) \\
\longrightarrow \ F(X \wedge T_+) \times F(X \wedge U_+) \ &= \ F_X(T_+) \times F_X(U_+)
\end{aligned}$$

is a G-weak equivalence. For every finite index subgroup H of G and every finite H-set S, the effect of the shearing isomorphism

$$X \wedge (G \times_H S)_+ \ \cong \ G \ltimes_H (X \wedge S_+)$$

identifies the Wirthmüller map

$$\begin{aligned}
\omega_{S_+} \ : \ F_X((G \times_H S)_+) \ &= \ F(X \wedge (G \times_H S)_+) \\
\longrightarrow \ \mathrm{map}^H(G, F(X \wedge S_+)) \ &= \ \mathrm{map}^H(G, F_X(S_+))
\end{aligned}$$

for the shifted $\mathbf{\Gamma}$-G-space and the H-space S_+ with the Wirthmüller map

$$\omega_{X \wedge S_+} \ : \ F(G \ltimes_H (X \wedge S_+)) \ \longrightarrow \ \mathrm{map}^H(G, F(X \wedge S_+))$$

of the original $\mathbf{\Gamma}$-G-space and the H-space $X \wedge S_+$. The latter map is a G-weak equivalence by part (ii), hence so is the former. Proposition B.52 now applies and shows that the shifted $\mathbf{\Gamma}$-G-space F_X is special. □

We still have to recall the notion of a 'very special' $\mathbf{\Gamma}$-G-space. We let G be a compact Lie group and F a special $\mathbf{\Gamma}$-G-space. We let $p_1, p_2 : 2_+ \longrightarrow 1_+$ denote the two projections. The map

$$(F(p_1), F(p_2)) \ : \ F(2_+) \ \longrightarrow \ F(1_+) \times F(1_+)$$

is a G-weak equivalence by specialness. We let $\nabla : 2_+ \longrightarrow 1_+$ denote the fold map. We obtain a diagram of set maps

$$\pi_0(F(1_+)^G)\times\pi_0(F(1_+)^G) \xleftarrow[\cong]{(\pi_0(F(p_1)^G),\pi_0(F(p_2)^G))} \pi_0(F(2_+)^G) \xrightarrow{\pi_0(F(\nabla)^G)} \pi_0(F(1_+)^G)$$

the left of which is bijective. So the map

$$\pi_0(F(\nabla)^G) \circ (\pi_0(F(p_1)^G), \pi_0(F(p_2)^G))^{-1} \ : \\ \pi_0(F(1_+)^G) \times \pi_0(F(1_+)^G) \ \longrightarrow \ \pi_0(F(1_+)^G)$$

is a binary operation on the set $\pi_0(F(1_+)^G)$.

If $\tau : 2_+ \longrightarrow 2_+$ is the involution that interchanges 1 and 2, then composition with τ fixes ∇ and interchanges p_1 and p_2; this implies that the operation + is commutative. Contemplating the different ways to fold and project from the based set 3_+ leads to the proof that the operation is also associative, and hence an abelian monoid structure on the set $\pi_0(F(1_+)^G)$.

For every closed subgroup H of G, the underlying $\mathbf{\Gamma}$-H-space is again special. So the same argument provides an abelian monoid structure on $\pi_0(F(1_+)^H)$.

Definition B.57. Let G be a compact Lie group. A $\mathbf{\Gamma}$-G-space F is *very special* if it is special and for every closed subgroup H of G the abelian monoid $\pi_0(F(1_+)^H)$ is an abelian group.

Remark B.58. We let G be a compact Lie group and F a special $\mathbf{\Gamma}$-G-space. Then the construction of the abelian monoid structure on $\pi_0(F(1_+)^H)$ generalizes as follows. We let H be a closed subgroup of G and S a finite G-set. We let $p_1, p_2 : (S \amalg S)_+ \longrightarrow S_+$ denote the two projections. The map

$$(\pi_0(F(p_1)^H), \pi_0(F(p_2)^H)) \ : \ \pi_0(F((S \amalg S)_+)^H) \ \longrightarrow \ \pi_0(F(S_+)^H)\times\pi_0(F(S_+)^H)$$

is bijective by Proposition B.52 (iii); inverting this map and composing with the effect of the fold map $\nabla : (S \amalg S)_+ \longrightarrow S_+$ on $\pi_0(F(-)^H)$ yields a binary operation on the set $\pi_0(F(S_+)^H)$. The same arguments as in the special case $S = \{1\}$ show that this operation defines an abelian monoid structure on $\pi_0(F(S_+)^H)$, which is moreover natural for G-maps in S. If K is a closed subgroup of H, then the inclusion $F(S_+)^H \subset F(S_+)^K$ induces a monoid homomorphism $\pi_0(F(S_+)^H) \longrightarrow \pi_0(F(S_+)^K)$.

We claim that for very special F, all the abelian monoids $\pi_0(F(S_+)^H)$ are in fact groups. To see this, we choose a set of representatives $s_1, \ldots, s_n$ for the

H-orbits of S, and we let K_i be the stabilizer group of s_i inside H. We consider the composite

$$F(S_+)^H \xrightarrow{(P_S)^H} \mathrm{map}^H(S, F(1_+)) \cong \prod_{i=1}^n F(1_+)^{K_i} ,$$

where the first map is a weak equivalence by specialness, and the homeomorphism is evaluation at the elements $s_1, \ldots, s_n$. The induced bijection

$$\pi_0(F(S_+)^H) \cong \prod_{i=1}^n \pi_0(F(1_+)^{K_i})$$

is a homomorphism of abelian monoids by the previous paragraph. The target is an abelian group since F is very special. So the abelian monoid $\pi_0(F(S_+)^H)$ is also an abelian group.

Example B.59. We recall how very special $\mathbf{\Gamma}$-G-spaces give rise to simplicial spaces that satisfy the π_*-Kan condition (see [22, B.3] or Definition A.47). We let G be a finite group and $F : \mathbf{\Gamma} \longrightarrow G\mathbf{T}_*$ a very special $\mathbf{\Gamma}$-G-space. We let A be a simplicial finite based G-set. Then for every subgroup H of G we obtain a simplicial space $(F \circ A)^H$ whose space of n-simplices is $F(A_n)^H$ (where are usual, $F(A_n)$ has the diagonal H-action by restriction of the G-actions on F and A_n). We claim that this simplicial space $(F \circ A)^H$ satisfies the π_*-Kan condition. To see this, we borrow the argument from Bousfield and Friedlander [22, Proof of Lemma 4.3]: since F is special, the space $F(A_n)^H$ comes with a homotopy-commutative H-space structure, so each of its path components is a simple space. Moreover, the map of simplicial sets $\beta : \pi_t((F \circ A)^H)_{\mathrm{free}} \longrightarrow \pi_0((F \circ A)^H)$ discussed in B.3 of [22] is underlying a surjective morphism of simplicial groups, and is thus a Kan fibration by [41, Lemma 3.2]. So [22, (B.3.1)] proves that $(F \circ A)^H$ satisfies the π_*-Kan condition.

Proposition B.60. *Let H be a closed subgroup of a compact Lie group G and F a very special $\mathbf{\Gamma}$-G-space.*

(i) *The underlying $\mathbf{\Gamma}$-H-space of F is very special.*
(ii) *If H is normal, then the $\mathbf{\Gamma}$-G/H-space F^H is very special.*

Proof Part (i) is clear by definition. For part (ii) we recall from Proposition B.50 (ii) that the $\mathbf{\Gamma}$-G/H-space F^H is special. Any closed subgroup of G/H is of the form Δ/H for a closed subgroup Δ of G with $H \leq \Delta$. Then $(F^H(1_+))^{\Delta/H} = F(1_+)^\Delta$. Since the monoid $\pi_0(F(1_+)^\Delta)$ is a group by hypothesis, we conclude that the monoid $\pi_0(F^H(1_+)^{\Delta/H})$ is a group. □

The following Theorem B.61 is essentially a reformulation of the equivariant delooping results of Segal [155] and of Shimakawa [157]; however, I am not aware that the result has been established in this form. The difference is that we formulate the result for the prolongation (i.e., categorical Kan extension),

whereas Segal and Shimakawa work with a bar construction (also known as a homotopy coend or homotopy Kan extension) instead. We also give a partial extension of the machinery to compact Lie groups.

A minor technical difference is that in Shimakawa's paper [157], an equivariant $\mathbf{\Gamma}$-space is assumed to take values in non-degenerately based G-spaces with the homotopy type of a based G-CW-complex; similarly, Segal [155] is implicitly assuming that the values of equivariant $\mathbf{\Gamma}$-spaces are G-ANRs. Our proof includes the observation that the key arguments work in the larger category of compactly generated G-spaces and G-weak equivalences, a long as we restrict to G-cofibrant $\mathbf{\Gamma}$-G-spaces.

Our proof verifies that the prolonged cofibrant $\mathbf{\Gamma}$-G-space satisfies the delooping criterion of Blumberg [14, Thm. 1.2]. An advantage of this approach is that Blumberg does not quote Segal or Shimakawa; rather, he works in the category of compactly generated G-spaces throughout, and adapts the relevant arguments to this context.

The paper [115] by May, Merling and Osorno also contains a modern perspective on the work of Segal and Shimakawa, and there is some overlap with our present discussion. In fact, our exposition of equivariant $\mathbf{\Gamma}$-spaces partially arose from discussions with Mona Merling and Peter May, to whom I am indepted for valuable feedback. In [115] the method of producing an equivariant spectrum by evaluating a $\mathbf{\Gamma}$-space on spheres is referred to as the *conceptual Segal machine*. In various respects, [115] goes further than we do here; for example, it contains detailed comparisons of prolongation (the 'conceptual Segal machine'), bar construction (the 'homotopical Segal machine') and the operadic approach to equivariant delooping (via a 'generalized Segal machine'). Moreover, for special $\mathbf{\Gamma}$-G-spaces, the value at S^1 is identified as an equivariant group completion; this is particularly relevant since many $\mathbf{\Gamma}$-G-spaces that arise in practice are special, but *not* very special.

Theorem B.61. *Let G be a compact Lie group and F a G-cofibrant very special $\mathbf{\Gamma}$-G-space.*

(i) *The prolonged functor F takes G-homotopy pushout squares of finite based G-CW-complexes to G-homotopy pullback squares.*
(ii) *For every finite based G-CW-complex X and every G-representation V on which the identity component G° acts trivially, the adjoint assembly map*

$$\tilde{\alpha} \ : \ F(X) \ \longrightarrow \ \mathrm{map}_*(S^V, F(X \wedge S^V))$$

is a G-weak equivalence.
(iii) *If G is finite, then the orthogonal G-spectrum $F(\mathbb{S})$ is a G-Ω-spectrum.*

Proof (i) We start with the special case where G is finite, and we consider a

monomorphism of finite based G-simplicial sets $i : A \longrightarrow B$. We show that then the sequence

$$F(|A|) \xrightarrow{F(|i|)} F(|B|) \xrightarrow{F(|q|)} F(|B/A|) \tag{B.62}$$

is a G-homotopy fiber sequence, where $q : B \longrightarrow B/A$ is the projection. We adapt an argument that Bousfield and Friedlander use for non-equivariant $\mathbf{\Gamma}$-spaces of simplicial sets in [22, Lemma 4.3]. The space $F(|A|)$ is homeomorphic to the geometric realization of the simplicial space $F \circ A$, by Proposition B.29. Moreover, for $H \leq G$, taking H-fixed-points commutes with geometric realization by Proposition B.1 (iv), so $F(|A|)^H$ is homeomorphic to the geometric realization of the simplicial space $(F \circ A)^H$, sending $[n]$ to $F(A_n)^H$. The same applies to B and B/A, so we may show that the sequence

$$|(F \circ A)^H| \xrightarrow{|(F\circ i)^H|} |(F \circ B)^H| \xrightarrow{|(F\circ q)^H|} |(F \circ (B/A))^H| \tag{B.63}$$

is a homotopy fiber sequence. We let $r : B_n \longrightarrow A_n$ be the retraction to i_n that sends the complement of $i_n(A_n)$ to the basepoint. For fixed $n \geq 0$ the lower row in the commutative diagram

$$\begin{tikzcd}
F(A_n)^H \arrow[r, "F(i_n)^H"] \arrow[d, equal] & F(B_n)^H \arrow[r, "F(q_n)^H"] \arrow[d, "(F(r)^H, F(q_n)^H)", "\simeq"'] & F(B_n/A_n)^H \arrow[d, equal] \\
F(A_n)^H \arrow[r] & F(A_n)^H \times F(B_n/A_n)^H \arrow[r, "\mathrm{proj}"'] & F(B_n/A_n)^H
\end{tikzcd}$$

is a homotopy fiber sequence. The middle vertical map is a weak equivalence by Proposition B.52 (iii). So the upper row in the diagram is a homotopy fiber sequence.

Because F is G-cofibrant, the three simplicial spaces $(F \circ A)^H$, $(F \circ B)^H$ and $(F \circ (B/A))^H$ are Reedy cofibrant by Proposition B.37 (iv). So we are considering a sequence of simplicial spaces that is a homotopy fiber sequence in every simplicial dimension. Moreover, the simplicial spaces $(F \circ B)^H$ and $(F \circ (B/A))^H$ satisfy the π_*-Kan condition by Example B.59. The morphism of simplicial sets $\pi_0((F \circ q)^H) : \pi_0((F \circ B)^H) \longrightarrow \pi_0((F \circ (B/A))^H)$ underlies a homomorphism of simplicial abelian groups by Remark B.58; this morphism is split surjective in every simplicial dimension. Every surjective homomorphism of simplicial groups is a Kan fibration, see for example [41, Lemma 3.2] or [63, V Lemma 2.6]. So the sequence $(F\circ A)^H \longrightarrow (F\circ B)^H \longrightarrow (F\circ(B/A))^H$ satisfies the hypotheses of Bousfield and Friedlander's Theorem A.49; we conclude that the sequence (B.63) is a homotopy fiber sequence. This completes the proof that the sequence (B.62) is a G-homotopy fiber sequence.

Now we consider a G-cofibration $j : X \longrightarrow Y$ between finite based G-CW-complexes, where G is still a finite group. We claim that then the sequence

$$F(X) \xrightarrow{F(j)} F(Y) \xrightarrow{F(p)} F(Y/X) \tag{B.64}$$

is a G-homotopy fiber sequence. Proposition B.46 provides a finite G-simplicial set A and a G-homotopy equivalence $k : |A| \longrightarrow X$. Another application of Proposition B.46 provides a finite G-simplicial set B, a monomorphism of G-simplicial sets $i : A \longrightarrow B$ and a G-homotopy equivalence $h : |B| \longrightarrow Y$ such that $h \circ |i| = j \circ k : |A| \longrightarrow Y$. The induced map on cokernels

$$h/k \; : \; |B/A| \;\cong\; |B|/|A| \;\longrightarrow\; Y/X$$

is then a G-weak equivalence by the gluing lemma (Proposition B.6). Source and target of h/k are cofibrant as G-spaces, so this G-weak equivalence is even a G-homotopy equivalence. The upper row in the commutative diagram of G-spaces

$$\begin{tikzcd}
F(|A|) \arrow[r, "F(|i|)"] \arrow[d, "F(k)"', "\simeq"] & F(|B|) \arrow[r, "F(|q|)"] \arrow[d, "F(h)"', "\simeq"] & F(|B/A|) \arrow[d, "\simeq"', "F(h/k)"] \\
F(X) \arrow[r, "F(j)"'] & F(Y) \arrow[r, "F(p)"'] & F(Y/X)
\end{tikzcd}$$

is a G-homotopy fiber sequence by the previous paragraph. The prolongation of every $\mathbf{\Gamma}$-G-space preserves based G-homotopy equivalences; so all vertical maps are G-homotopy equivalences. Hence the lower row in the diagram is a G-homotopy fiber sequence.

Now we let G be an arbitrary compact Lie group, and $j : X \longrightarrow Y$ a G-cofibration between finite based G-CW-complexes. We claim that then the sequence (B.64) is a G-homotopy fiber sequence. We let H be a closed subgroup of G with identity component H° and group of path components $\bar{H} = H/H^\circ$. The $\mathbf{\Gamma}$-$\bar{H}$-space F^{H° is $\bar{H}$-cofibrant by Proposition B.35 and very special by Proposition B.60. The map $j^{H^\circ} : X^{H^\circ} \longrightarrow Y^{H^\circ}$ is an $\bar{H}$-cofibration by Propositions B.14 (i) and B.12. Since j is in particular a closed embedding, the canonical map $X^{H^\circ}/Y^{H^\circ} \longrightarrow (X/Y)^{H^\circ}$ is a homeomorphism by Proposition B.1 (i). So by the previous paragraph, the sequence

$$F^{H^\circ}(X^{H^\circ}) \xrightarrow{F^{H^\circ}(j^{H^\circ})} F^{H^\circ}(Y^{H^\circ}) \xrightarrow{F^{H^\circ}(p^{H^\circ})} F^{H^\circ}((Y/X)^{H^\circ})$$

is an $\bar{H}$-homotopy fiber sequence. We may thus take $\bar{H}$-fixed-points and obtain a non-equivariant homotopy fiber sequence. Proposition B.42 provides a homeomorphism $(F^{H^\circ}(X^{H^\circ}))^{\bar{H}} \cong (F(X)^{H^\circ})^{\bar{H}} = F(X)^H$, and similarly for the

other two terms. So we conclude that the sequence

$$F(X)^H \xrightarrow{F(j)^H} F(Y)^H \xrightarrow{F(p)^H} F(Y/X)^H$$

is a non-equivariant homotopy fiber sequence. This completes the proof that the sequence (B.64) is a G-homotopy fiber sequence.

Now we treat a general homotopy cocartesian square of finite based G-CW-complexes:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "k"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "i"'] & P \end{tikzcd}$$

By replacing Y and P by the reduced mapping cylinders of j and i, respectively, we may assume that the horizontal maps are G-cofibrations. Because the square is homotopy cocartesian, the induced map $h/k : Y/X \longrightarrow P/Z$ is a G-weak equivalence, and hence a G-homotopy equivalence. We obtain a commutative diagram of based G-spaces:

$$\begin{tikzcd} F(X) \arrow[r, "F(j)"] \arrow[d, "F(k)"'] & F(Y) \arrow[r] \arrow[d, "F(h)"'] & F(Y/X) \arrow[d, "\simeq"', "F(h/k)"] \\ F(Z) \arrow[r, "F(i)"'] & F(P) \arrow[r] & F(P/Z) \end{tikzcd}$$

Because F preserves equivariant homotopy equivalences, the right vertical map $F(h/k)$ is a G-homotopy equivalence. Both rows are G-homotopy fiber sequences by the previous paragraph; so the left square is G-homotopy cartesian. This completes the proof that the prolonged functor takes G-homotopy cocartesian squares to G-homotopy cartesian squares.

(ii) As usual we start with the special case where G is finite. Restricting the prolonged functor F to finite G-CW-complexes gives a $\mathscr{W}_G$-space in the sense of [14]. This $\mathscr{W}_G$-space satisfies conditions (1) and (2') of [14, Thm. 1.2], by part (i) of this theorem and Proposition B.54 (ii), respectively. So [14, Thm. 1.2] applies to the prolonged functor, and gives the desired conclusion.

Now we let G be a general compact Lie group. We want to show that for every closed subgroup H of G the map

$$\tilde{\alpha}^H \ : \ F(X)^H \ \longrightarrow \ \mathrm{map}_*^H(S^V, F(X \wedge S^V))$$

is a weak equivalence. The underlying $\mathbf{\Gamma}$-H-space of F is H-cofibrant by Proposition B.35 (i) and very special by Proposition B.60 (i). Moreover, the underlying H-space of X is H-homotopy equivalent to a finite H-CW-complex, by

[85, Cor. B]. Finally, the identity component of H is contained in G°, so it acts trivially on V. So we may assume without loss of generality that $H = G$.

As usual we write $\bar{G} = G/G^\circ$ for the finite group of path components. The $\mathbf{\Gamma}$-$\bar{G}$-space F^{G° is $\bar{G}$-cofibrant by Proposition B.35 (ii) and very special by Proposition B.60 (ii). Since X is a finite G-CW-complex, X^{G° is a finite $\bar{G}$-CW-complex. Since G° acts trivially on V, we can view V as a representation of the factor group $\bar{G}$. The previous paragraph shows that the map

$$\tilde{\alpha} \ : \ F^{G^\circ}(X^{G^\circ}) \ \longrightarrow \ \mathrm{map}_*(S^V, F^{G^\circ}(X^{G^\circ} \wedge S^V))$$

induces a weak equivalence on $\bar{G}$-fixed-points. Proposition B.42 provides a homeomorphism

$$(F^{G^\circ}(X^{G^\circ}))^{\bar{G}} \ \cong \ (F(X)^{G^\circ})^{\bar{G}} \ = \ F(X)^G \ ;$$

similarly,

$$\begin{aligned} \mathrm{map}_*^{\bar{G}}(S^V, F^{G^\circ}(X^{G^\circ} \wedge S^V)) \ &\cong \ \mathrm{map}_*^{\bar{G}}(S^V, F(X \wedge S^V)^{G^\circ}) \\ &= \ \mathrm{map}_*^{G}(S^V, F(X \wedge S^V)) \ . \end{aligned}$$

Under these identifications, the map $\tilde{\alpha}^{\bar{G}}$ corresponds to the G-fixed-points of the adjoint assembly map for the $\mathbf{\Gamma}$-G-space F. This completes the proof.

Part (iii) is simply the special case of (ii) with $X = S^W$ for some G-representation W. Finite groups have a trivial identity component, which must necessarily act trivially on V. □

Now we let F be a cofibrant $\mathbf{\Gamma}$-G-space that is special (but not necessarily very special). Then Theorem B.61 need not hold for F, as the adjoint assembly map $F(1_+) \longrightarrow \Omega F(S^1)$ need not be a weak equivalence. However, there is a standard argument to deduce from Theorem B.61 that the adjoint assembly map is still a G-weak equivalence on suspensions; this argument uses the shift of the given $\mathbf{\Gamma}$-G-space by S^1. Since we work with prolongations throughout, there is a new ingredient, not already present in the classical references, namely that the S^1-shift of a G-cofibrant $\mathbf{\Gamma}$-G-space is again G-cofibrant.

Theorem B.65. *Let G be a compact Lie group and F a G-cofibrant, special $\mathbf{\Gamma}$-G-space. Then for every pair of G-representations V and W such that $W^G \neq 0$ and G° acts trivially on V, the adjoint assembly map*

$$\tilde{\alpha} \ : \ F(S^W) \ \longrightarrow \ \mathrm{map}_*(S^V, F(S^{V \oplus W}))$$

is a G-weak equivalence.

Proof One more time we start with the special case when G is finite. We consider the shifted $\mathbf{\Gamma}$-G-space $F_{S^1} = F \circ (S^1 \wedge -)$, which is G-cofibrant by

Proposition B.37 (ii) and special by Proposition B.54 (iii). Moreover, for every subgroup H of G the space

$$(F_{S^1}(1_+))^H \;\cong\; F(S^1)^H$$

is path connected by Proposition B.43 (i). So the abelian monoid $\pi_0(F_{S^1}(1_+)^H)$ has only one element, and is thus an abelian group. Hence F_{S^1} is very special.

So Theorem B.61 applies to the G-cofibrant and very special $\mathbf{\Gamma}$-G-space F_{S^1}. Since W has a non-zero G-fixed-point, there is a G-representation U and a G-equivariant linear isometry $\varphi : \mathbb{R} \oplus U \cong W$. We let $\phi : F_{S^1}(L) \longrightarrow F(S^1 \wedge L)$ denote the natural homeomorphism established in Proposition B.27. In the commutative diagram

$$\begin{tikzcd}
F_{S^1}(S^U) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\phi"', "\cong"] & \mathrm{map}_*(S^V, F_{S^1}(S^{V\oplus U})) \arrow[d, "\cong"', "\mathrm{map}_*(S^V,\phi)"] \\
F(S^1 \wedge S^U) \arrow[r, "\tilde{\alpha}"] \arrow[d, "F(S^\varphi)"', "\cong"] & \mathrm{map}_*(S^V, F(S^1 \wedge S^{V\oplus U})) \arrow[d, "\cong"', "\mathrm{map}_*(S^V,F(S^{V\oplus\varphi}))"] \\
F(S^W) \arrow[r, "\tilde{\alpha}"'] & \mathrm{map}_*(S^V, F(S^{V\oplus W}))
\end{tikzcd}$$

all vertical maps are then homeomorphisms, and the upper horizontal map is a G-weak equivalence by Theorem B.61. So the lower horizontal map is a G-weak equivalence. This completes the proof in the case of finite groups.

The case of general compact Lie groups can be reduced to the case of finite groups by taking fixed-points with respect to the identity component G°, and using the residual action of the finite group G/G°; the argument is almost literally the same as in the last paragraph of the proof of Theorem B.61 (ii), and we omit it. □

Remark B.66. The previous Theorems B.61 and B.65 indicate why equivariant $\mathbf{\Gamma}$-spaces are usually considered for finite groups only – this is where the equivariant delooping machine reveals its full power. We want to briefly explain why the spectrum $F(\mathbb{S})$ cannot be a G-Ω-spectrum when G has positive dimension, except in degenerate cases. The reason is that when H is a closed subgroup of strictly smaller dimension than G, the 'dimension shifting Wirthmüller map' cannot be a G-weak equivalence, except in degenerate cases. Given a based H-space Z, the H-equivariant collapse map

$$l_Z \;:\; G \ltimes_H Z \;\longrightarrow\; Z \wedge S^L$$

was introduced in (3.2.2); here $L = T_{eH}(G/H)$ is the tangent H-representation. The dimension shifting Wirthmüller map

$$\omega_Z \;:\; F(G \ltimes_H Z) \;\longrightarrow\; \mathrm{map}^H(G, F(Z \wedge S^L))$$

is the adjoint of the H-map $F(l_Z) : F(G \ltimes_H Z) \longrightarrow F(Z \wedge S^L)$.

Blumberg's recognition theorem [14, Thm. 1.2] says that his conditions (1) and (2') are not only sufficient, but also necessary for $F(\mathbb{S})$ to be a G-Ω-spectrum. If F is G-cofibrant and very special, then the prolonged functor always satisfies conditions (1), by Theorem B.61 (i). But condition (2') requires that the dimension shifting Wirthmüller map ω_Z is a G-weak equivalence; Proposition B.54 (ii) shows that this is true for compact Lie groups, provided H has finite index in G (in which case $L = 0$). However, the map ω_Z is typically *not* a G-weak equivalence when H has strictly smaller dimension than G. To illustrate this we take the extreme case $H = e$ and $Z = 1_+$, in which case $G \ltimes_H Z$ is simply G_+. If the identity component G° of G is non-trivial, then $F(G_+)^{G^\circ}$ is a single point by Proposition B.42. On the other hand, evaluation at a set of representatives of the path components of G is a homeomorphism

$$(\mathrm{map}(G, F(S^L)))^{G^0} \;\cong\; \mathrm{map}(G/G^\circ, F(S^L)) \;\cong\; F(S^L)^m ,$$

where $m = [G : G^\circ]$. So if the Wirthmüller map ω_{1_+} is a G-weak equivalence, then the space $F(S^L)$ is non-equivariantly weakly contractible. Since $F(\mathbb{S})$ is a connective non-equivariant Ω-spectrum, it must be altogether stably trivial.

Appendix C
Enriched functor categories

In this final appendix we review definitions, properties and constructions involving categories of enriched functors. The general setup consists of:

- a complete and cocomplete closed symmetric monoidal category $\mathcal{V}$ (the 'base category'). We denote the monoidal product in $\mathcal{V}$ by $\otimes$;
- a skeletally small $\mathcal{V}$-category $\mathcal{D}$ (the 'index category').

We denote by $\mathcal{D}^*$ the category of covariant $\mathcal{V}$-functors $X : \mathcal{D} \longrightarrow \mathcal{V}$ from the index category to the base category. We are mostly interested in the following special cases of such functor categories:

- orthogonal spaces (where $\mathcal{V} = \mathbf{T}$ and $\mathcal{D} = \mathbf{L}$, see Definition 1.1.1);
- orthogonal spectra (where $\mathcal{V} = \mathbf{T}_*$ and $\mathcal{D} = \mathbf{O}$, see Definition 3.1.3);
- global functors (where $\mathcal{V} = \mathcal{A}b$ and $\mathcal{D} = \mathbf{A}$, see Definition 4.2.2).

We discuss the Day convolution product (Definition C.4) that turns the functor category into a symmetric monoidal category, compare Theorem C.10. Construction C.13 introduces the skeleton filtration, which enters into the level model structure on certain functor categories, see Proposition C.23. For background on symmetric monoidal categories and enriched category theory we refer the reader to [90].

Construction C.1 ($\mathcal{V}$-enrichment of functor categories)**.** The objects of the functor category $\mathcal{D}^*$ are the $\mathcal{V}$-functors $X : \mathcal{D} \longrightarrow \mathcal{V}$, and the morphisms in $\mathcal{D}^*$ are the $\mathcal{V}$-natural transformations between such functors. The functor category $\mathcal{D}^*$ is itself enriched in the base category $\mathcal{V}$, as follows. Given two objects X, Y in $\mathcal{D}^*$, a $\mathcal{V}$-object of $\mathcal{V}$-natural transformations $\underline{\mathcal{D}}^*(X, Y)$ is defined as an enriched end [90, Sec. 2.1], i.e., an equalizer in $\mathcal{V}$ of two morphisms

$$\prod_c \underline{\mathcal{V}}(X(c), Y(c)) \quad \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \quad \prod_{d,e} \underline{\mathcal{V}}(\mathcal{D}(d, e) \otimes X(d), Y(e)) \ .$$

Here the products run over a set of representatives for the isomorphism classes

of $\mathcal{D}$-objects and $\underline{\mathcal{V}}(-,-)$ denotes the internal function objects in the closed monoidal structure. The (d,e)-component of the morphism α is the composite

$$\prod_c \underline{\mathcal{V}}(X(c),Y(c)) \xrightarrow{\text{proj}_d} \underline{\mathcal{V}}(X(d),Y(d)) \longrightarrow \underline{\mathcal{V}}(\mathcal{D}(d,e)\otimes X(d),Y(e))$$

where the second morphism is adjoint to

$$\underline{\mathcal{V}}(X(d),Y(d))\otimes\mathcal{D}(d,e)\otimes X(d) \xrightarrow{\text{eval}} \mathcal{D}(d,e)\otimes Y(d) \xrightarrow{Y} Y(e)\ .$$

The (d,e)-component of the morphism β is the composite

$$\prod_c \underline{\mathcal{V}}(X(c),Y(c)) \xrightarrow{\text{proj}_e} \underline{\mathcal{V}}(X(e),Y(e)) \longrightarrow \underline{\mathcal{V}}(\mathcal{D}(d,e)\otimes X(d),Y(e))$$

where the second morphism is adjoint to

$$\underline{\mathcal{V}}(X(e),Y(e))\otimes\mathcal{D}(d,e)\otimes X(d) \xrightarrow{\text{Id}\otimes X} \underline{\mathcal{V}}(X(e),Y(e))\otimes X(e) \xrightarrow{\text{eval}} Y(e)\ .$$

The *underlying set* of a $\mathcal{V}$-object V is

$$uV \ = \ \mathcal{V}(I,V)\ ,$$

the set of morphisms from the monoidal unit. The functor $\mathcal{V}(I,-) : \mathcal{V} \longrightarrow$ (sets) takes equalizers in $\mathcal{V}$ to equalizers of sets. When applied to the defining equalizer for $\underline{\mathcal{D}}^*(X,Y)$ this shows that the set of morphisms $\mathcal{D}^*(X,Y)$ in the functor category $\mathcal{D}^*$ (i.e., the set of $\mathcal{V}$-natural transformations) can be recovered as the underlying set of the $\mathcal{V}$-object $\underline{\mathcal{D}}^*(X,Y)$, via a natural bijection

$$\mathcal{D}^*(X,Y) \ \cong \ \mathcal{V}(I,\underline{\mathcal{D}}^*(X,Y)) \ = \ u(\underline{\mathcal{D}}^*(X,Y))\ .$$

The $\mathcal{V}$-objects $\underline{\mathcal{D}}^*(X,Y)$ even assemble into an entire $\mathcal{V}$-category $\underline{\mathcal{D}}^*$ of $\mathcal{V}$-functors whose underlying category is $\mathcal{D}^*$. We refer to [90, Sec. 2.1] for details.

The category $\mathcal{D}^*$ is also tensored over the base category $\mathcal{V}$. Given a functor $Y\in\mathcal{D}^*$ and an object A of $\mathcal{V}$, we define a new functor $Y\otimes A\in\mathcal{D}^*$ by objectwise product in $\mathcal{V}$, i.e., $Y\otimes A$ is the composite $\mathcal{V}$-functor

$$\mathcal{D} \xrightarrow{Y} \mathcal{V} \xrightarrow{-\otimes A} \mathcal{V}\ .$$

This construction is an action of $\mathcal{V}$ on $\mathcal{D}^*$, which means preferred natural associativity and unit isomorphisms

$$Y\otimes(A\otimes B) \ \cong \ (Y\otimes A)\otimes B \qquad \text{and} \qquad Y\otimes I \ \cong \ Y\ .$$

Indeed, the value of the above isomorphisms at an object $d\in\mathcal{D}$ is simply the associativity (or unit) isomorphism in the monoidal structure of $\mathcal{V}$, for the triple of objects $(Y(d),A,B)$.

Remark C.2 (Enriched Yoneda lemma)**.** The Yoneda lemma has enriched versions that we use frequently to identify morphisms out of representable $\mathcal{V}$-functors. For every object $d \in \mathcal{D}$ the covariant hom functor

$$d^* \;=\; \mathcal{D}(d,-) \;:\; \mathcal{D} \;\longrightarrow\; \mathcal{V}$$

is a $\mathcal{V}$-functor, hence an object in the functor category $\mathcal{D}^*$. The enriched Yoneda lemma comes in two flavors, a $\mathcal{V}$-valued version and a set-valued version. For a $\mathcal{V}$-functor Y in $\mathcal{D}^*$, the evaluation morphism is the composite

$$\begin{aligned}\underline{\mathcal{D}^*}(d^*,Y) \;\xrightarrow{\mathrm{proj}_d}\; \underline{\mathcal{V}}(d^*(d),Y(d)) \;&=\; \underline{\mathcal{V}}(\mathcal{D}(d,d),Y(d))\\ &\xrightarrow{\underline{\mathcal{V}}(\mathrm{Id}_d,Y(d))}\; \underline{\mathcal{V}}(I,Y(d)) \;\cong\; Y(d)\end{aligned}$$

where $\mathrm{Id}_d : I \longrightarrow \mathcal{D}(d,d)$ is the identity of d. This composite is an isomorphism in $\mathcal{V}$, see for example [90, Sec. 2.4]. Passing to underlying sets produces a bijection

$$\mathcal{D}^*(d^*,Y) \;\xrightarrow{\cong}\; u(Y(d))$$

between the set of $\mathcal{V}$-natural transformations and the underlying set of $Y(d)$. This is the weak form of the enriched Yoneda lemma, compare [90, Sec. 1.9].

Convolution product. We review a general method, due to B. Day [42], for constructing symmetric monoidal structures on certain functor categories. We now make the additional assumption that the index category $\mathcal{D}$ is a *symmetric monoidal* $\mathcal{V}$-category. We denote the monoidal product on $\mathcal{D}$ by $\oplus$.

We get the main cases of interest by specializing as follows:

(i) For $\mathcal{V} = \mathbf{T}$ the category of spaces under cartesian product and $\mathcal{D} = \mathbf{L}$ the topological category of inner product spaces under orthogonal direct sum, $\mathcal{D}^* = spc$ is the category of orthogonal spaces. This yields the box product of orthogonal spaces.

(ii) For $\mathcal{V} = \mathbf{T}_*$ the category of based spaces under smash product and $\mathcal{D} = \mathbf{O}$ the Thom space category of the orthogonal complement bundles, under orthogonal direct sum, $\mathcal{D}^* = \mathcal{S}p$ is the category of orthogonal spectra. This yields the smash product of orthogonal spectra.

(iii) For $\mathcal{V} = \mathcal{A}b$ the category of abelian groups under tensor product and $\mathcal{D} = \mathbf{A}$ the pre-additive Burnside category with the monoidal structure of Theorem 4.2.15, $\mathcal{D}^* = \mathcal{GF}$ is the category of global functors. This yields the box product of global functors.

We denote by $\mathcal{D} \otimes \mathcal{D}$ the $\mathcal{V}$-category whose objects are pairs of $\mathcal{D}$-objects, and with morphism $\mathcal{V}$-objects

$$(\mathcal{D} \otimes \mathcal{D})((d,d'),(e,e')) \;=\; \mathcal{D}(d,e) \otimes \mathcal{D}(d',e')\;.$$

Given two $\mathcal{V}$-functors $X, Y : \mathcal{D} \longrightarrow \mathcal{V}$, the composite

$$X \otimes Y \ : \ \mathcal{D} \otimes \mathcal{D} \xrightarrow{X \otimes Y} \mathcal{V} \otimes \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

is a $\mathcal{V}$-functor.

Definition C.3. A *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of objects (X, Y) of $\mathcal{D}^*$ to another object Z of $\mathcal{D}^*$ is a $\mathcal{V}$-natural transformation $b : X \otimes Y \longrightarrow Z \circ \oplus$ of $\mathcal{V}$-functors $\mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{V}$.

So a bimorphism $b : (X, Y) \longrightarrow Z$ consists of $\mathcal{V}$-morphisms

$$b(d, e) \ : \ X(d) \otimes Y(e) \ \longrightarrow \ Z(d \oplus e)$$

for all objects d, e of $\mathcal{D}$, that form a $\mathcal{V}$-natural transformation. We can then define the box product as a universal recipient of a bimorphism from X and Y.

Definition C.4. A *box product* for objects X and Y of $\mathcal{D}^*$ is a pair $(X \Box Y, i)$ consisting of a $\mathcal{D}^*$-object $X \Box Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \Box Y$, i.e., a bimorphism such that for every $\mathcal{D}^*$-object Z the map

$$\mathcal{D}^*(X \Box Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto f \circ i$$

is bijective.

Proposition C.5. *Every pair of objects of $\mathcal{D}^*$ has a box product.*

Proof The universal property of the box product precisely means that $(X \Box Y, i)$ is an enriched left Kan extension of the $\mathcal{V}$-functor $X \otimes Y : \mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{V}$ along the $\mathcal{V}$-functor $\oplus : \mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D}$. Such a Kan extension exists because $\mathcal{V}$ is cocomplete and $\mathcal{D}$ is skeletally small, see [90, Prop. 4.33]. □

Remark C.6. As we saw in the proof of Proposition C.5, the box product is an enriched Kan extension along the functor $\oplus : \mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D}$. We can make this a little more explicit. Given $X, Y \in \mathcal{D}^*$, a box product $X \Box Y$ is a coequalizer of the two morphisms of $\mathcal{V}$-functors

$$\coprod_{e,e',d,d'} \mathcal{D}(e \oplus e', -) \otimes \mathcal{D}(d, e) \otimes \mathcal{D}(d', e') \otimes X(d) \otimes Y(d')$$

$$\rightrightarrows \ \coprod_{d,d'} \mathcal{D}(d \oplus d', -) \otimes X(d) \otimes Y(d') \ .$$

The left coproduct is indexed over all quadruples (e, e', d, d') in a set of representatives of the isomorphism classes of $\mathcal{D}$-objects. The right coproduct is indexed over pairs (d, d') of such representatives. One of the two morphisms to be coequalized is the coproduct of the monoidal products of

$$\mathcal{D}(e \oplus e', -) \otimes \mathcal{D}(d, e) \otimes \mathcal{D}(d', e') \ \longrightarrow \ \mathcal{D}(d \oplus d', -) \ , \quad (\varphi, \tau, \tau') \ \longmapsto \ \varphi \circ (\tau \oplus \tau')$$

and the identity on $X(d) \otimes Y(d')$. The other morphism is the coproduct of the monoidal products of the identity on $\mathcal{D}(e \oplus e', -)$ and the action maps $\mathcal{D}(d, e) \otimes X(d) \longrightarrow X(e)$ and $\mathcal{D}(d', e') \otimes Y(d') \longrightarrow Y(e')$.

Remark C.7 (Uniqueness of box products)**.** The universal property makes box products of two functors unique up to preferred isomorphism. Indeed, if $(X\Box Y, i)$ and $(X\Box' Y, i')$ are two box products, then the universal properties provide unique morphisms $f : X\Box Y \longrightarrow X\Box' Y$ and $g : X\Box' Y \longrightarrow X\Box Y$ that satisfy

$$f \circ i \;=\; i' \qquad \text{and} \qquad g \circ i' \;=\; i\,.$$

Then $(g \circ f) \circ i = g \circ i' = i = \mathrm{Id}_{X\Box Y} \circ i$, so $g \circ f = \mathrm{Id}_{X\Box Y}$ by the uniqueness part of the universal property. Reversing the roles gives $f \circ g = \mathrm{Id}_{X\Box' Y}$.

In the following we *choose* a box product $(X\Box Y, i)$ for every pair of objects X and Y of $\mathcal{D}^*$. The universal bimorphism i is often omitted from the notation, but one should remember that the pair $(X\Box Y, i)$, and not the object $X\Box Y$ alone, has a universal property. While the box products are choices, the rest of the structure is canonically determined by these.

Construction C.8 (Functoriality)**.** The box product $X\Box Y$ automatically becomes a functor in both variables: if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are morphisms in $\mathcal{D}^*$, then the $\mathcal{V}$-morphisms

$$\Big\{ X(d) \otimes Y(e) \xrightarrow{f(d)\otimes g(e)} X'(d) \otimes Y'(e) \xrightarrow{i'_{d,e}} (X'\Box Y')(d \oplus e) \Big\}_{d,e}$$

form a bimorphism $(X, Y) \longrightarrow X'\Box Y'$. So there is a unique morphism $f\Box g : X\Box Y \longrightarrow X'\Box Y'$ such that $(f\Box g)(d \oplus e) \circ i_{d,e} = i'_{d,e} \circ (f(d) \otimes g(e))$ for all objects d, e of $\mathcal{D}$. The uniqueness part of the universal property implies that this is compatible with identities and composition in both variables.

Now that we have constructed a box product functor $\Box : \mathcal{D}^* \times \mathcal{D}^* \longrightarrow \mathcal{D}^*$, we recall that it 'is' automatically symmetric monoidal. Since 'symmetric monoidal' is extra data, and not a property, we are obliged to construct unit, associativity and symmetry isomorphisms.

Construction C.9 (Coherence isomorphisms)**.** It will be convenient to make the $\mathcal{V}$-functor $0^* = \mathcal{D}(0, -)$ represented by the unit object 0 of $\mathcal{D}$ a strict unit for the box product (as opposed to a unit up to coherent isomorphisms). So we make the following conventions:

- (Right unit) We choose $X\Box 0^* = X$ with universal bimorphism $i : (X, 0^*) \longrightarrow$

X given by the maps

$$X(d) \otimes 0^*(e) \;=\; X(d) \otimes \mathcal{D}(0, e) \xrightarrow{X(d) \otimes (d \oplus -)} X(d) \otimes \mathcal{D}(d \oplus 0, d \oplus e)$$
$$\cong\; X(d) \otimes \mathcal{D}(d, d \oplus e) \xrightarrow{\circ} X(d \oplus e)\;.$$

- (Left unit) We choose $0^* \Box Y = Y$ with universal bimorphism $i : (0^*, Y) \longrightarrow Y$ given by the composite

$$0^*(d) \otimes Y(e) \;=\; \mathcal{D}(0, d) \otimes Y(e) \xrightarrow{(-\oplus e) \otimes Y(e)} \mathcal{D}(0 \oplus e, d \oplus e) \otimes Y(e)$$
$$\cong\; \mathcal{D}(e, d \oplus e) \otimes Y(e) \xrightarrow{\circ} Y(d \oplus e)\;.$$

The unnamed isomorphisms are induced by the unit isomorphisms in $\mathcal{D}$, and '$\circ$' is the functor structure of X and Y. For $X = Y = 0^*$ these two bimorphisms $(0^*, 0^*) \longrightarrow 0^*$ are equal, so there is no ambiguity.

We also obtain the associativity and symmetry isomorphisms from the universal property of the box products. For the associativity isomorphism we notice that the family

$$\left\{ i_{d,e\oplus f} \circ (X(d) \otimes i_{e,f}) \;:\; X(d) \otimes\; Y(e) \otimes Z(f) \;\longrightarrow\; (X\Box(Y\Box Z))(d \oplus e \oplus f) \right\}_{d,e,f}$$

and the family

$$\left\{ i_{d\oplus e,f} \circ (i_{d,e} \otimes Z(f)) \;:\; X(d) \otimes\; Y(e) \otimes Z(f) \;\longrightarrow\; ((X\Box Y)\Box Z)(d \oplus e \oplus f) \right\}_{d,e,f}$$

both have the universal property of a *tri* morphism (whose definition is hopefully clear) out of X, Y and Z. The uniqueness of representing objects gives a unique isomorphism of $\mathcal{V}$-functors

$$\alpha_{X,Y,Z} \;:\; X\Box(Y\Box Z) \;\cong\; (X\Box Y)\Box Z$$

such that $(\alpha_{X,Y,Z})_{d\oplus e\oplus f} \circ i_{d,e\oplus f} \circ (X(d) \otimes i_{e,f}) = i_{d\oplus e,f} \circ (i_{d,e} \otimes Z(f))$.

The symmetry isomorphism $\tau_{X,Y} : X\Box Y \longrightarrow Y\Box X$ corresponds to the bimorphism from (X, Y) to $Y\Box X$ with components

$$X(d) \otimes Y(e) \xrightarrow{\tau_{X(d),Y(e)}} Y(e) \otimes X(d) \xrightarrow{i_{e,d}} (Y\Box X)(e \oplus d) \xrightarrow{(Y\Box X)(\tau_{e,d})} (Y\Box X)(d \oplus e)\;.$$

The following theorem is due to B. Day. In fact, Theorems 3.3 and 4.1 of [42] together imply the 'monoidal' part, and Theorem 3.6 of [42] deals with the symmetries.

Theorem C.10. *The associativity and symmetry isomorphisms make the box product a symmetric monoidal product on the category $\mathcal{D}^*$ with the functor 0^* as a strict unit object.*

The box product also commutes with $\mathcal{V}$-tensors up to a natural isomorphism that we now construct. The family of morphisms

$$X(d) \otimes (Y(e) \otimes A) \xrightarrow{\alpha_{X(d),Y(e),A}} (X(d) \otimes Y(e)) \otimes A \xrightarrow{i_{d,e} \otimes A} (X\square Y)(d \oplus e) \otimes A$$

forms a bimorphism from $(X, Y \otimes A)$ to $(X\square Y) \otimes A$. So the universal property provides a distinguished morphism

$$\alpha_{X,Y,A} \ : \ X\square(Y \otimes A) \longrightarrow (X\square Y) \otimes A \ .$$

Since the monoidal structure on $\mathcal{V}$ is closed, the functor $-\otimes A$ is a left adjoint, so it preserves coends. So the morphism $\alpha_{X,Y,A}$ is an isomorphism. The two types of associativity isomorphisms $X\square(Y\otimes A) \cong (X\square Y)\otimes A$ and $Y\otimes(A\otimes B) \cong (Y \otimes A) \otimes B$ are compatible in the sense of a commuting pentagon:

$$\begin{tikzcd}
& X\square(Y \otimes (A \otimes B)) \arrow[dl, "X\square\alpha_{Y,A,B}"'] \arrow[dr, "\alpha_{X,Y,A\otimes B}"] & \\
X\square((Y \otimes A) \otimes B) \arrow[d, "\alpha_{X,Y\otimes A,B}"'] & & (X\square Y) \otimes (A \otimes B) \arrow[d, "\alpha_{X\square Y,A,B}"] \\
(X\square(Y \otimes A)) \otimes B \arrow[rr, "\alpha_{X,Y,A}\otimes B"'] & & ((X\square Y) \otimes A) \otimes B
\end{tikzcd}$$

A minimally more complicated construction that also involves the symmetry constraint in $\mathcal{V}$ to move the object A past the functor Y provides an isomorphism

$$(X \otimes A)\square Y \ \cong \ (X\square Y) \otimes A \ .$$

Remark C.11 (Convolution product of representable functors)**.** The box product of the functors represented by two objects a, b of $\mathcal{D}$ is related by a preferred isomorphism to the functor represented by $a \oplus b$. Indeed, the $\mathcal{V}$-morphisms

$$a^*(d) \otimes b^*(e) \ = \ \mathcal{D}(a, d) \otimes \mathcal{D}(b, e) \ \xrightarrow{\oplus} \ \mathcal{D}(a \oplus b, d \oplus e) \ = \ (a \oplus b)^*(d \oplus e)$$

form a bimorphism as d and e vary. So there is a unique morphism

$$j_{a,b} \ : \ a^*\square b^* \ \longrightarrow \ (a \oplus b)^*$$

such that $j_{a,b} \circ i = \oplus$, where i is the universal bimorphism from (a^*, b^*) to $a^*\square b^*$. On the other hand, the morphism

$$I \ \cong \ I \otimes I \ \xrightarrow{\mathrm{Id}_a \otimes \mathrm{Id}_b} \ \mathcal{D}(a, a) \otimes \mathcal{D}(b, b) \ = \ a^*(a) \otimes b^*(b) \ \xrightarrow{i_{a,b}} \ (a^*\square b^*)(a \oplus b)$$

is an element of the underlying set of $(a^*\square b^*)(a \oplus b)$. The enriched Yoneda lemma thus provides a unique morphism of $\mathcal{V}$-functors $(a\oplus b)^* \longrightarrow a^*\square b^*$ that evaluates to the element at $a \oplus b$. The universal properties of the convolution

product and of the representable functors show that these two morphisms are inverse to each other.

Remark C.12 (Internal function objects)**.** The box product is a *closed* monoidal product in the sense that for all objects Y and Z of $\mathcal{D}^*$ the functor

$$\mathcal{D}^*(-\Box Y, Z) \ : \ \mathcal{D}^* \ \longrightarrow \ (\text{sets}) \ , \quad X \ \longmapsto \ \mathcal{D}^*(X\Box Y, Z)$$

is representable. In this book we don't make any substantial use of the internal function objects, so we will not elaborate on the construction. The most relevant consequence for us is that the box product preserves colimits in both variables because the functors $X\Box-$ and $-\Box Y$ are left adjoints. This consequence can also be shown without reference to function objects by observing that the original monoidal product in $\mathcal{V}$ preserves colimits in both variables (because the monoidal structure of $\mathcal{V}$ is closed), because $X\Box Y$ can be constructed as an enriched coend, and because enriched coends commute with colimits.

Construction C.13 (Skeleton filtration)**.** For the rest of this appendix we assume that the index category $\mathcal{D}$ is equipped with a dimension function $\dim : \mathrm{ob}(\mathcal{D}) \longrightarrow \mathbb{N}$ to the natural numbers, satisfying the following conditions for all objects d, e of $\mathcal{D}$:

- If $\dim(e) < \dim(d)$, then $\mathcal{D}(d, e)$ is an initial object of the base category $\mathcal{V}$.
- If $\dim(e) = \dim(d)$, then d and e are isomorphic.

The two examples that we care about in this book are the category of orthogonal spaces and the category of orthogonal spectra. In both cases the dimension function is the dimension as an $\mathbb{R}$-vector space.

We denote by $\mathcal{D}_{\leq m}$ the full $\mathcal{V}$-subcategory of $\mathcal{D}$ spanned by all objects of dimension at most m. We denote by $\mathcal{D}^*_{\leq m} = (\mathcal{D}_{\leq m})^*$ the category of enriched functors from $\mathcal{D}_{\leq m}$ to $\mathcal{V}$. The restriction functor

$$\mathcal{D}^* \ \longrightarrow \ \mathcal{D}^*_{\leq m} \ , \quad Y \ \longmapsto \ Y^{\leq m} \ = \ Y|_{\mathcal{D}_{\leq m}}$$

has a left adjoint

$$l_m \ : \ \mathcal{D}^*_{\leq m} \ \longrightarrow \ \mathcal{D}^*$$

given by an enriched Kan extension as follows. For every $k \geq 0$ we choose an object $\mathbf{k}$ of $\mathcal{D}$ of dimension k. The extension $l_m(Z)$ of an enriched functor $Z : \mathcal{D}_{\leq m} \longrightarrow \mathcal{V}$ is a coequalizer of the two morphisms in $\mathcal{D}^*$:

$$\textstyle\coprod_{0\leq j\leq k\leq m} \mathcal{D}(\mathbf{k}, -) \otimes \mathcal{D}(\mathbf{j}, \mathbf{k}) \otimes Z(\mathbf{j}) \rightrightarrows \coprod_{0\leq i\leq m} \mathcal{D}(\mathbf{i}, -) \otimes Z(\mathbf{i}) \qquad \text{(C.14)}$$

One morphism arises from the composition morphisms

$$\mathcal{D}(\mathbf{k}, -) \otimes \mathcal{D}(\mathbf{j}, \mathbf{k}) \ \longrightarrow \ \mathcal{D}(\mathbf{j}, -)$$

and the identity on $Z(\mathbf{j})$; the other morphism arises from the action morphisms

$$\mathcal{D}(\mathbf{j},\mathbf{k}) \otimes Z(\mathbf{j}) \longrightarrow Z(\mathbf{k})$$

and the identity on the represented functor $\mathcal{D}(\mathbf{k},-)$. Colimits in the functor category $\mathcal{D}^*$ are created objectwise, so the value $l_m(Z)(d)$ at an object d can be calculated by plugging d into the variable slot in the coequalizer diagram (C.14). Kan extensions along a fully faithful functor do not change the values on the given subcategory, see for example [90, Prop. 4.23]. For the inclusion $\mathcal{D}_{\leq m} \longrightarrow \mathcal{D}$ this means that the adjunction unit

$$Z \longrightarrow (l_m Z)^{\leq m}$$

is an isomorphism for every functor $Z : \mathcal{D}_{\leq m} \longrightarrow \mathcal{V}$.

Definition C.15. The *m-skeleton*, for $m \geq 0$, of an enriched functor $Y : \mathcal{D} \longrightarrow \mathcal{V}$ is the functor

$$\mathrm{sk}^m Y = l_m(Y^{\leq m}) ,$$

the extension of the restriction of Y to $\mathcal{D}_{\leq m}$. It comes with a natural morphism $i_m : \mathrm{sk}^m Y \longrightarrow Y$, the counit of the adjunction $(l_m, (-)^{\leq m})$. We set $\mathcal{D}(m) = \mathcal{D}(\mathbf{m},\mathbf{m})$, a $\mathcal{V}$-monoid under composition. The *mth latching object* of Y is the $\mathcal{D}(m)$-object

$$L_m Y = (\mathrm{sk}^{m-1} Y)(\mathbf{m}) ;$$

it comes with a natural $\mathcal{D}(m)$-equivariant morphism

$$\nu_m = i_{m-1}(\mathbf{m}) : L_m Y \longrightarrow Y(\mathbf{m}) ,$$

the *mth latching morphism*.

We agree to set $\mathrm{sk}^{-1} Y = \emptyset$, an initial object of $\mathcal{D}^*$; then $L_0 Y = \emptyset$, an initial $\mathcal{V}$-object. The two morphisms $i_{m-1} : \mathrm{sk}^{m-1} Y \longrightarrow Y$ and $i_m : \mathrm{sk}^m Y \longrightarrow Y$ both restrict to isomorphisms on $\mathcal{D}^{\leq m-1}$, so there is a unique morphism $j_m : \mathrm{sk}^{m-1} Y \longrightarrow \mathrm{sk}^m Y$ such that $i_m \circ j_m = i_{m-1}$. The sequence of skeleta stabilizes to Y in a very strong sense. For every object d, the morphisms $j_m(d)$ and $i_m(d)$ are isomorphisms as soon as $m > \dim(d)$. In particular, $Y(d)$ is a colimit, with respect to the morphisms $i_m(d)$, of the sequence of morphisms $j_m(d)$. Since colimits in the functor category $\mathcal{D}^*$ are created objectwise, we deduce that Y is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Example C.16 (Latching objects of represented functors)**.** Let d be an object of $\mathcal{D}$ of dimension $m = \dim(d)$, and A an object of $\mathcal{V}$. Then the functor $\mathcal{D}(d,-) \otimes A$ is 'purely m-dimensional' in the following sense. The evaluation functor

$$\mathrm{ev}_d : \mathcal{D}^* \longrightarrow \mathcal{V}$$

factors through the category $\mathcal{D}^*_{\leq m}$ as the composite

$$\mathcal{D}^* \longrightarrow \mathcal{D}^*_{\leq m} \xrightarrow{\mathrm{ev}_d} \mathcal{V} .$$

So the left adjoint free functor $A \mapsto \mathcal{D}(d,-)\otimes A$ can be chosen as the composite of the two individual left adjoints

$$\mathcal{D}(d,-)\otimes - \;=\; l_m \circ (\mathcal{D}_{\leq m}(d,-)\otimes -) .$$

The object $\mathcal{D}(d,e)\otimes A$ is initial for $\dim(e) < m$, by hypothesis on the dimension function, and hence the latching object $L_n(\mathcal{D}(d,-)\otimes A)$ is initial for $n \leq m$. For $n > m$ the latching morphism $\nu_n : L_n(\mathcal{D}(d,-)\otimes A) \longrightarrow \mathcal{D}(d,\mathbf{n})\otimes A$ is an isomorphism. So the skeleton $\mathrm{sk}^n(\mathcal{D}(d,-)\otimes A)$ is initial for $n < m$ and $\mathrm{sk}^n(\mathcal{D}(d,-)\otimes A) = \mathcal{D}(d,-)\otimes A$ is the entire functor for $n \geq m$.

We denote by

$$G_m \;:\; \mathcal{D}(m)\mathcal{V} \longrightarrow \mathcal{D}^* , \quad A \longmapsto \mathcal{D}(\mathbf{m},-)\otimes_{\mathcal{D}(m)} A$$

the left adjoint to the evaluation functor $Y \mapsto Y(\mathbf{m})$. As a consequence of the previous example, the enriched functor $G_m A$ is purely m-dimensional for every $\mathcal{D}(m)$-object A.

Proposition C.17. *For every enriched functor $Y : \mathcal{D} \longrightarrow \mathcal{V}$ and every $m \geq 0$ the commutative square*

$$\begin{tikzcd} G_m L_m Y \arrow[r, "G_m \nu_m"] \arrow[d] & G_m Y(\mathbf{m}) \arrow[d] \\ \mathrm{sk}^{m-1} Y \arrow[r, "j_m"'] & \mathrm{sk}^m Y \end{tikzcd} \tag{C.18}$$

is a pushout in the category $\mathcal{D}^*$. The two vertical morphisms are adjoint to the identity of $L_m Y$ and $Y(\mathbf{m})$, respectively.

Proof All four functors are 'm-dimensional', i.e., isomorphic to the extensions of their restrictions to $\mathcal{D}_{\leq m}$; for $G_m L_m Y$ and $G_m Y(\mathbf{m})$ this follows from Example C.16. So it suffices to check that the square is a pushout in $\mathcal{V}$ when evaluated at objects d of dimension at most m. When the dimension of d is strictly less than m, then $(G_m L_m Y)(d)$ and $(G_m Y_m)(d)$ are initial objects and the morphism $j_m(d) : (\mathrm{sk}^{m-1} Y)(d) \longrightarrow (\mathrm{sk}^m Y)(d)$ is an isomorphism. So for $\dim(d) < m$ both horizontal morphisms are isomorphisms at d. When $\dim(d) = m$, then d is isomorphic to $\mathbf{m}$, so we may suppose that $d = \mathbf{m}$.

Then the square (C.18) evaluates to

$$\begin{tikzcd} \mathcal{D}(\mathbf{m},\mathbf{m}) \otimes_{\mathcal{D}(m)} L_mY \arrow[r, "\mathcal{D}(\mathbf{m},\mathbf{m})\otimes_{\mathcal{D}(m)}\nu_m"] \arrow[d] & \mathcal{D}(\mathbf{m},\mathbf{m}) \otimes_{\mathcal{D}(m)} Y(\mathbf{m}) \arrow[d] \\ L_mY \arrow[r, "j_m(\mathbf{m})"'] & Y(\mathbf{m}) \end{tikzcd}$$

So for $\dim(d) = m$, both vertical maps are isomorphisms at d. □

Given any morphism $f : A \longrightarrow B$ of enriched functors in $\mathcal{D}^*$, we define a relative skeleton using the commutative square of enriched functors:

$$\begin{tikzcd} \mathrm{sk}^m A \arrow[r, "\mathrm{sk}^m f"] \arrow[d, "i_m"'] & \mathrm{sk}^m B \arrow[d, "i_m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The *relative m-skeleton* of f is the pushout

$$\mathrm{sk}^m[f] \;=\; A \cup_{\mathrm{sk}^m A} \mathrm{sk}^m B\,; \tag{C.19}$$

it comes with a unique morphism $i_m : \mathrm{sk}^m[f] \longrightarrow B$ which restricts to $f : A \longrightarrow B$ and to $i_m : \mathrm{sk}^m B \longrightarrow B$. Since $(\mathrm{sk}^{m-1} A)(\mathbf{m}) = L_mA$ we have

$$(\mathrm{sk}^{m-1}[f])(\mathbf{m}) \;=\; A(\mathbf{m}) \cup_{L_mA} L_mB\,,$$

the mth relative latching object. A morphism $j_m[f] : \mathrm{sk}^{m-1}[f] \longrightarrow \mathrm{sk}^m[f]$ is obtained from the commutative diagram

$$\begin{tikzcd} A \arrow[d, equal] & \mathrm{sk}^{m-1} A \arrow[l, "i_{m-1}"'] \arrow[r, "\mathrm{sk}^{m-1} f"] \arrow[d, "j_m^A"'] & \mathrm{sk}^{m-1} B \arrow[d, "j_m^B"] \\ A & \mathrm{sk}^m A \arrow[l, "i_m"] \arrow[r, "\mathrm{sk}^m f"'] & \mathrm{sk}^m B \end{tikzcd}$$

by taking pushouts. The original morphism $f : A \longrightarrow B$ factors as the composite of the countable sequence

$$A = \mathrm{sk}^{-1}[f] \xrightarrow{j_0[f]} \mathrm{sk}^0[f] \xrightarrow{j_1[f]} \mathrm{sk}^1[f] \longrightarrow \cdots \xrightarrow{j_m[f]} \mathrm{sk}^m[f] \longrightarrow \cdots .$$

If d has dimension n, then the sequence stabilizes to the identity map of $B(d)$ from $(\mathrm{sk}^n[f])(d)$ onward; in particular, the compatible morphisms j_m exhibit B as a colimit of the sequence. The pushout square (C.18) also has a relative

version. We contemplate the following commutative diagram:

$$\begin{tikzcd}
G_m(A(\mathbf{m}) \cup_{L_m A} L_m B) \arrow[r, "G_m(\nu_m f)"] \arrow[d] & G_m B(\mathbf{m}) \arrow[d] \\
\mathrm{sk}^m A \cup_{\mathrm{sk}^{m-1} A} \mathrm{sk}^{m-1} B \arrow[r] \arrow[d] & \mathrm{sk}^m B \arrow[d] \\
\mathrm{sk}^{m-1}[f] \arrow[r, "j_m[f]"'] & \mathrm{sk}^m[f]
\end{tikzcd} \tag{C.20}$$

All three horizontal morphisms are isomorphisms at objects of dimension less than m. At the object $\mathbf{m}$, the two upper vertical morphisms are isomorphisms. All four objects in the upper square are 'm-dimensional', i.e., isomorphic to the extensions of their restrictions to $\mathcal{D}_{\leq m}$; so the upper square is a pushout of enriched functors. The lower square is also a pushout, by definition of the relative skeleta. We conclude that the outer composite square in the diagram (C.20) is a pushout in the category $\mathcal{D}^*$.

The following proposition is an immediate application of the relative skeleton filtration. It is the key ingredient to the lifting properties of the various level model structures in this book. We recall that a pair $(i : A \longrightarrow B, f : X \longrightarrow Y)$ of morphisms in some category has the *lifting property* if for all morphisms $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow Y$ such that $f\alpha = \beta i$ there exists a *lifting*, i.e., a morphism $\lambda : B \longrightarrow X$ such that $\lambda i = \alpha$ and $f\lambda = \beta$. Instead of saying that the pair (i, f) has the lifting property we also say 'i has the left lifting property with respect to f' or 'f has the right lifting property with respect to i'.

Proposition C.21. *Let $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ be morphisms of enriched functors in $\mathcal{D}^*$. If the pair*

$$(\nu_m i = i(\mathbf{m}) \cup \nu_m^B : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m}), \quad f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m}))$$

has the lifting property in the category of $\mathcal{D}(m)$-objects for every $m \geq 0$, then the pair (i, f) has the lifting property in the functor category $\mathcal{D}^*$.

Proof We consider the class f-cof of all morphisms in $\mathcal{D}^*$ that have the left lifting property with respect to f; this class is closed under cobase change and countable composition. Since the pair $(\nu_m i, f_m)$ has the lifting property in the category of $\mathcal{D}(m)$-objects, the morphism $G_m(\nu_m i)$ belongs to the class f-cof by adjointness. The relative skeleton filtration (C.19) shows that i is a countable composite of cobase changes of the morphisms $G_m(\nu_m i)$, so i belongs to the class f-cof. □

Now we discuss a general recipe for constructing level model structures on

the functor category $\mathcal{D}^*$. As input we need, for every $m \geq 0$, a model structure $\mathcal{C}(m)$ on the category of $\mathcal{D}(m)$-objects. We call a morphism $f : X \longrightarrow Y$ in $\mathcal{D}^*$

- a *level equivalence* if $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a weak equivalence in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *level fibration* if the morphism $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a fibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$; and
- a *cofibration* if the latching morphism $\nu_m f : X(\mathbf{m}) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbf{m})$ is a cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$.

Proposition C.23 below shows that if the various model structures $\mathcal{C}(m)$ satisfy the following 'consistency condition', then the level equivalences, level fibrations and cofibrations define a model structure on the functor category $\mathcal{D}^*$.

Definition C.22 (Consistency condition)**.** For all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the model structure $\mathcal{C}(m)$ on $\mathcal{D}(m)$-objects, every cobase change, in the category of $\mathcal{D}(m+n)$-objects, of the morphism

$$\mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} i \ : \ \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} A \ \longrightarrow \ \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} B$$

is a weak equivalence in the model structure $\mathcal{C}(m+n)$.

For example, the consistency conditions holds if the functor

$$\mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} - \ : \ \mathcal{D}(m)\mathcal{V} \ \longrightarrow \ \mathcal{D}(m+n)\mathcal{V}$$

is a left Quillen functor in the given model structures, for all $m, n \geq 0$.

Proposition C.23. *Let $\mathcal{C}(m)$ be a model structure on the category of $\mathcal{D}(m)$-objects, for $m \geq 0$, such that the consistency condition holds.*

(i) *The classes of level equivalences, level fibrations and cofibrations define a model structure on the functor category $\mathcal{D}^*$.*

(ii) *A morphism $i : A \longrightarrow B$ in $\mathcal{D}^*$ is simultaneously a cofibration and a level equivalence if and only if for all $m \geq 0$ the latching morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$.*

(iii) *Suppose that the fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $J(m)$; then the level fibrations are detected by the set of morphisms*

$$\{G_m j \mid m \geq 0, \ j \in J(m)\} .$$

Similarly, if the acyclic fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $I(m)$, then the level acyclic fibrations are detected by the set of morphisms

$$\{G_m i \mid m \geq 0, \ i \in I(m)\} .$$

Proof We start by showing one of the directions of part (ii): we let $i : A \longrightarrow B$ be a morphism such that the latching morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ is an acyclic cofibration in the model structure $C(m)$ for all $m \geq 0$; we show that then i is a level equivalence.

The morphism $i(\mathbf{n}) : A(\mathbf{n}) \longrightarrow B(\mathbf{n})$ is the finite composite

$$A(\mathbf{n}) = (\mathrm{sk}^{-1}[i])(\mathbf{n}) \xrightarrow{(j_0[i])(\mathbf{n})} (\mathrm{sk}^0[i])(\mathbf{n}) \xrightarrow{(j_1[i])(\mathbf{n})} \dots \xrightarrow{(j_n[i])(\mathbf{n})} (\mathrm{sk}^n[i])(\mathbf{n}) = B(\mathbf{n}),$$

so it suffices to show that $j_k[i]$ is a level equivalence for all $k \geq 0$. The pushout square (C.20) in level $m + n$ is a pushout of $\mathcal{D}(m + n)$-objects

$$\begin{tikzcd}
\mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} (A(\mathbf{m}) \cup_{L_m A} L_m B) \arrow[r, "\mathcal{D}(\mathbf{m},\mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} (\nu_m i)"] \arrow[d] & \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \otimes_{\mathcal{D}(m)} B(\mathbf{m}) \arrow[d] \\
(\mathrm{sk}^{m-1}[i])(\mathbf{m}+\mathbf{n}) \arrow[r, "(j_m[i])(\mathbf{m}+\mathbf{n})"'] & (\mathrm{sk}^m[i])(\mathbf{m}+\mathbf{n})
\end{tikzcd}$$

The consistency condition guarantees that the lower horizontal morphism is a $C(m + n)$-weak equivalence.

(i) Several of the axioms are straightforward: the functor category $\mathcal{D}^*$ inherits all small limits and colimits from the base category $\mathcal{V}$; the level equivalences satisfy the 2-out-of-3 property; the classes of level equivalences, cofibrations and fibrations are closed under retracts.

Now we prove the factorization axiom, i.e., we show that every morphism $f : A \longrightarrow X$ in $\mathcal{D}^*$ can be factored as $f = qi$ where q is a level acyclic fibration and i a cofibration; and it can be factored as $f = pj$ where p is a level fibration and j a cofibration and level equivalence. We start with the first factorization and construct an enriched functor B and morphisms $i : A \longrightarrow B$ and $q : B \longrightarrow X$ by induction over the dimension. Since the objects $\mathbf{m}$ for $m \geq 0$ form a skeleton of the category $\mathcal{D}$, it suffices to construct all the relevant data on these objects. In level 0 we choose a factorization

$$A(\mathbf{0}) \xrightarrow{i(\mathbf{0})} B(\mathbf{0}) \xrightarrow{q(\mathbf{0})} X(\mathbf{0})$$

of $f(\mathbf{0})$ in the category of $\mathcal{D}(0)$-objects such that $i(\mathbf{0})$ is a cofibration and $q(\mathbf{0})$ is an acyclic fibration in the model structure $C(0)$. Now we suppose that the enriched functor B and the morphisms i and q have already been constructed for all objects of dimension less than m. Then we have all the data necessary to define the mth latching object $L_m B$; moreover, the 'partial morphism' q :

$B \longrightarrow X$ provides a $\mathcal{D}(m)$-morphism $L_m B \longrightarrow X(\mathbf{m})$ such that the square

$$\begin{array}{ccc} L_m A & \xrightarrow{L_m i} & L_m B \\ {\scriptstyle \nu_m^A}\downarrow & & \downarrow \\ A(\mathbf{m}) & \xrightarrow[f(\mathbf{m})]{} & X(\mathbf{m}) \end{array}$$

commutes. We factor the resulting morphism $A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow X(\mathbf{m})$ in the category of $\mathcal{D}(m)$-objects

$$A(\mathbf{m}) \cup_{L_m A} L_m B \xrightarrow{\nu_m i} B(\mathbf{m}) \xrightarrow{q(\mathbf{m})} X(\mathbf{m}) \tag{C.24}$$

such that $\nu_m i$ is a cofibration and $q(\mathbf{m})$ is an acyclic fibration in the model structure $C(m)$. The intermediate $\mathcal{D}(m)$-object $B(\mathbf{m})$ defines the value of B at the object $\mathbf{m}$, and $q(\mathbf{m})$ defines the mth level of the morphism q. For $0 \leq i < m$, the structure morphism is the composite

$$\mathcal{D}(\mathbf{i}, \mathbf{m}) \otimes B(\mathbf{i}) \longrightarrow L_m B \longrightarrow A(\mathbf{m}) \cup_{L_m A} L_m B \xrightarrow{\nu_m i} B(\mathbf{m})$$

and the composite of $\nu_m i$ with the canonical morphism $A(\mathbf{m}) \longrightarrow A(\mathbf{m}) \cup_{L_m A} L_m B$ is the mth level of the morphism i.

At the end of the day we have indeed factored $f = qi$ in the category $\mathcal{D}^*$ such that q is a level equivalence and level fibration. Moreover, the mth latching morphism comes out to be the morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ in the factorization (C.24), which is a cofibration in the model structure $C(m)$. So the morphism i is indeed a cofibration.

The second factorization $f = pj$ as a cofibration and level equivalence j followed by a level fibration p is similar, but instead of the factorization (C.24) we use a factorization, in the model category $C(m)$, as an acyclic cofibration followed by a fibration. Then the resulting morphism p is a level fibration and the morphism j has the property that all its latching morphisms $\nu_m j$ are acyclic cofibrations. So j is a cofibration (by definition) and a level equivalence (by the part of (ii) established above).

It remains to show the lifting axioms. In each of the model structures $C(m)$ the cofibrations have the left lifting property with respect to the acyclic fibrations; so by Proposition C.21 the cofibrations in $\mathcal{D}^*$ have the left lifting property with respect to level equivalences which are also level fibrations.

We postpone the proof of the other lifting property and prove the remaining direction of (ii) next. We let $i : A \longrightarrow B$ be a cofibration and a level equivalence. The second factorization axiom proved above provides a factorization $i = pj$ where $j : A \longrightarrow D$ is a level equivalence such that each latching morphism $\nu_m j$ is an acyclic cofibration in the model structure $C(m)$, and $p : D \longrightarrow B$ is a level fibration. Since i and j are level equivalences, so is

p. So the cofibration i has the left lifting property with respect to the level equivalence and level fibration p by the previous paragraph. In particular, a lift $\lambda : B \longrightarrow D$ in the square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & D \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dotted, "\lambda"] & B \end{tikzcd}$$

shows that the morphism i is a retract of the morphism j. So the latching morphism $\nu_m i$ is a retract of the latching morphism $\nu_m j$, hence also an acyclic cofibration in the model structure $C(m)$. This proves (ii).

Now we prove the remaining lifting property. We let $i : A \longrightarrow B$ be a cofibration that is also a level equivalence. By (ii), which has just been shown, each latching morphism $\nu_m i$ is an acyclic cofibration in the model structure $C(m)$. So i has the left lifting property with respect to all level fibrations by Proposition C.21.

Property (iii) is a straightforward consequence of the fact that the functor G_m is left adjoint to evaluation at $\mathbf{m}$. □

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