Errata for

Global homotopy theory

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This document lists typos and mistakes that I became aware of after publication of the book *Global homotopy theory* in September 2018. As of today, I know of no serious mathematical issue, but there are a certain number of potentially misleading typos, and minor fixable gaps in the proofs of Propositions 2.4.28, Proposition 3.1.36, Proposition 3.2.16 and Theorem 6.2.24. I would like to thank Benjamin Böhme, Emma Brink, Jack Davies, Urs Flock, Grigory Garkusha, Vincent Grande, Alexander Müller, Phil Pützstück, Jonathan Wassermann and Qi Zhu for discovering some of the mistakes.

All errata below refer to the published version of my book; in the electronic version available on my homepage, I have already corrected the typos, fixed the known mathematical issues, and updated the references.

Chapter 1:

p.23, l.14: the $(K \times G)$-cofibrations should be defined by the *left* lifting property (not the *right* lifting property) with respect to all morphisms of $(K \times G)$-spaces f such that f^Γ is a weak equivalence and Serre fibration for every closed subgroup Γ of $K \times G$.

p.36, l.-10: in the proof of the unstable strong level model structure (Proposition 1.2.10), the last instance of the orthogonal space $L_{H,\mathbb{R}^m}$ should be in mathbold font, i.e., $\mathbf{L}_{H,\mathbb{R}^m}$.

p.64, l.-14: '(...) the $\mathcal{F}$-global model structure lifts to categor<u>ies</u> of modules and algebras (...)' (plural)

p.65, l.1: the $\mathcal{F}(m)$-cofibrations should be defined by the *left* lifting property (not the *right* lifting property) with respect to all morphisms q of $O(m)$-spaces such that the map q^H is a weak equivalence and Serre fibration for all $H \in \mathcal{F}(m)$.

Chapter 2:

p.99, l.-8: 'Moreover, the coequalizer is <u>split</u> in the underlying category (...)' (as opposed to 'reflexive')

p.111, in the central displayline it should read $\mu(V \oplus W)(i_{V,W}(x, y))$, the image of the pair $(x, y) \in R(V) \times R(W)$ under the composite

$$R(V) \times R(W) \xrightarrow{i_{V,W}} (R \boxtimes R)(V \oplus W) \xrightarrow{\mu(V \oplus W)} R(V \oplus W) \ .$$

p.162, l.13: the arguments to $f^\star$ and $(\mathbf{Gr}(\varphi) \circ f)^\star$ are missing. The correct sentence should be: 'So the two G-vector bundles $f^\star(\gamma_V)$ and $(\mathbf{Gr}(\varphi) \circ f)^\star(\gamma_W)$ over A are isomorphic.'

p.166, l.-1: as we shall now explain

p.180, proof of Proposition 2.4.28: contrary to my claim, the diagram

$$\begin{tikzcd} \mathbf{BO}'_{(m)} \arrow[r, "\mathrm{incl}"] \arrow[dr, "i_{\mathbf{BO}'_{(m)}}"'] & \mathbf{BO}'_{(m+1)} \arrow[d, "j"'] \arrow[dr, "i_{\mathbf{BO}'_{(m+1)}}"] & \\ & \mathrm{sh}(\mathbf{BO}'_{(m)}) \arrow[r, "\mathrm{sh}(\mathrm{incl})"'] & \mathrm{sh}(\mathbf{BO}'_{(m+1)}) \end{tikzcd}$$

in the middle of page 180 does *not* commute. This issue can be fixed as follows. We modify the definition of the morphism

$$j \ : \ \mathbf{BO}'_{(m+1)} \longrightarrow \mathrm{sh}(\mathbf{BO}'_{(m)})$$

by using the linear isometric embedding

$$\begin{aligned} V \oplus V \oplus \mathbb{R}^{m+1} \quad &\longrightarrow \quad V \oplus \mathbb{R} \oplus V \oplus \mathbb{R} \oplus \mathbb{R}^m \\ (v, v', (x_1, \dots, x_{m+1})) \ &\longmapsto \ (v, 0, v', x_{m+1}, (x_1, \dots, x_m)) \end{aligned}$$

instead of the one specified in the published book; the difference is a cyclic permutation of the last $m+1$ coordinates. With this modified definition of j, the left triangle in the above diagram commutes. The right triangle still does *not* commute; however, it commutes up to a homotopy of morphisms of orthogonal spaces. Indeed, the two morphisms from $\mathbf{BO}'_{(m+1)}$ to $\mathrm{sh}(\mathbf{BO}'_{m+1})$ are induced by two different linear isometric embeddings from $\mathbb{R}^{m+1}$ to $\mathbb{R} \oplus \mathbb{R}^{m+1}$ that are applied to the last coordinates; the space of such linear isometric embeddings is path connected, and the desired homotopy is induced by any choice of path. The rest of the proof then applies unchanged.

p.182, l.13: $\mathbf{bO}(A_3)$ should be replaced by $\pi_0^{A_3}(\mathbf{bO})$

p.200, l.-4: in the commutative square at the bottom of page 200, the four instances of π_k^G should just be π_k, i.e., without the superscript 'G'

p.216, l.6: 'We emphasize that the behaviour on morphisms (...)'

Chapter 3:

p.236, l.11: in the last line of the proof of Proposition 3.1.16, a superscript 'X' is missing in $(l_g^X)_*(c_g^*[f]) = [f]$.

p.240, l.7; p.284, l.6 and l.8; p.471, l.5; and p.548, l.-8: the term 'antipodal map' for the involution of S^V that gives rise to $\varepsilon_V : \pi_k^G(X \wedge S^V) \longrightarrow \pi_k^G(X \wedge S^V)$ is misleading: the involution $S^{-\mathrm{Id}_V} : S^V \longrightarrow S^V$ fixes the points 0 and ∞ and sends every other vector to its negative. So the restriction to the unit sphere is what is usually called the antipodal map. There is nothing wrong with the mathematics, but the adjective 'antipodal' is poorly chosen here.

p.251, proof of Proposition 3.1.36: As stated, the right triangle in the upper diagram on page 251 does *not* commute up to based G-homotopy. Instead, the right diagonal map $f \wedge S^1 : X \wedge S^1 \longrightarrow Y \wedge S^1$ must be replaced by $f \wedge \tau : X \wedge S^1 \longrightarrow Y \wedge S^1$, where $\tau : S^1 \longrightarrow S^1$ is the sign involution $\tau(x) = -x$. In this corrected form, the commutativity of the right triangle is an instance of Proposition 3.1.35 (i). This mistake also influences the next step in the proof of exactness. Because the degree of the sign involution τ is -1, the right square that compares the sequence for $i : Y \longrightarrow Cf$ with the sequence for $f : X \longrightarrow Y$ only commutes up *up to sign*:

$$\begin{aligned} \partial &= (- \wedge S^{-1}) \circ \pi_k^G(p_i) \\ &= (- \wedge S^{-1}) \circ \pi_k^G(f \wedge \tau) \circ \pi_k^G(* \cup p) \\ &= -(- \wedge S^{-1}) \circ \pi_k^G(f \wedge S^1) \circ \pi_k^G(* \cup p) \\ &= -\pi_{k-1}^G(f) \circ (- \wedge S^{-1}) \circ \pi_k^G(* \cup p)\ . \end{aligned}$$

(There is also a typo, in that $\pi_k^G(f)$ should be $\pi_{k-1}^G(f)$.) Fortunately, changing a homomorphism of abelian groups into its negative does not change kernel nor cokernel. So the extra sign does not influence the question of exactness.

p.273/274, proof of Proposition 3.2.16: in the proof of part (ii) for $k \geq 0$, we use that the upper vertical assembly maps in the first diagram of (i) are isomorphisms. While this is true, we have not proven it. The fix is to observe that the proof of Theorem 3.2.15 for $k = 0$ works in the same way for $k \geq 0$, by simply adding an extra $\mathbb{R}^k$ to the source of the representative f, making it an H-equivariant based map $f \colon S^{U \oplus \mathbb{R}^k} \longrightarrow Y(U) \wedge S^L$.

p.293, l.-9: in the third-to-last line of the proof of Proposition 3.3.8, the part 'for every G-representation V' is redundant.

p.294, proof of Proposition 3.3.10: in both parts of the proof, all instances of *equivalences* of equivariant orthogonal spectra should be replaced by $\underline{\pi}_*$*-isomorphisms.*

p.320, l.10: which amounts to $\underline{\text{a}}$ G-Mackey functor

p.333, l.16: in the target of the symmetric monoidal structure on the category $\mathbf{O}$, the arguments should be $V \oplus V'$ and $W \oplus W'$ (as opposed to $V \oplus W$ and $V' \oplus W'$); so the target should read $\mathbf{O}(V \oplus V', W \oplus W')$.

Chapter 4:

p.367, l.-9: this should read '(...) and K-representation $\bar{U}$ (...)' (as opposed to 'G-representation').

p.367, l.-4/5: the two instances of equivariant homotopy groups should be indexed by the Lie group K (as opposed to G), so they should read $\pi_k^K(\lambda_{G,V,W})$ and $\pi_{-k}^K(\lambda_{G,V,W})$, respectively.

p.371, l.-4: Theorem 4.2.6 tell$\underline{\text{s}}$ us

p.414, l.-15: in the proof of Theorem 4.4.4, it should read '(...) if F preserves sums, then for every object X of $\mathcal{S}$, (...)' (as opposed to: 'for every object X of $\mathcal{T}$')

p.434, Proposition 4.5.4: this is not a mistake, but the hypothesis that $\epsilon_I \colon PI \longrightarrow I$ is an isomorphism is redundant. Indeed, the strong symmetric monoidal structure on P includes an isomorphism $I \longrightarrow PI$. The monoidal structure on the identity functor is implicitly the identity monoidal structure. So the hypothesis that ϵ is a monoidal transformation includes the property that $\epsilon_I \colon PI \longrightarrow I$ is left inverse to the isomorphism from the strong monoidal structure on P.

p.454, l.8: in the displayline, the superscript 'G' is missing from $\mathbb{Q} \otimes \pi_k^G(f)$

Chapter 5:

p.519, l.9: 'where $C_s \subset U(1)$ is <u>the</u> group of sth roots of unity.'

Chapter 6:

p.549, l.21: in Example 6.1.7, the definition of the structure maps of $\mathbf{MOP}$ should have $\varphi^2(x)$ instead of $\mathbf{BOP}(\varphi)(x)$; the correct displayline should read

$$(w, \varphi) \ \wedge \ (x, U) \quad \longmapsto \ ((w, 0) + \varphi^2(x), \mathbf{BOP}(\varphi)(U)).$$

p.560: in line -15, $\mathbf{bO}(V)$ should be $\mathbf{bOP}(V)$; in the displayline -12, the definition of the structure maps of $\mathbf{mOP}$ should have $(\varphi \oplus \mathbb{R}^\infty)(x)$ instead of $\mathbf{bOP}(\varphi)(x)$; the correct displayline should read

$$(w, \varphi) \wedge (x, U) \quad \longmapsto \ ((w, 0) + (\varphi \oplus \mathbb{R}^\infty)(x), \mathbf{bOP}(\varphi)(U)).$$

p.562, l.-13 should read '(...) $\tau_{G,V}$ lies in the homogeneous summand $\mathbf{mOP}^{[-m]}$.' (as opposed to $\mathbf{MOP}^{[-m]}$)

p.579, l.-11: there need to be G-fixed points around $Gr_j(V^\perp)$, so it should read '(...) the wedge of the spaces $\mathbf{mOP}^{[j]}(V^G) \wedge (Gr_j(V^\perp))^G$ for $j \geq 0$.'

p.581, l.-10: the formula for the precursor of the structure map of $\mathbf{mU}$ is wrong: to U we need to add $(V_\mathbb{C} \oplus 0 \oplus 0)$, and not $(0 \oplus W_\mathbb{C} \oplus 0)$

p.585, l.13: remove 'to' in '. . . and to $(H \circ \psi)|'_M$. . . '

p.594, l.-6: in the displayline, it should read $\psi[x, s]$ (as opposed to $q[x, s]$)

p.600, l.-4: the displayline should read $T_x(M^G) = (T_x M)^G$ (and not with di)

p.605, l.15: the C-action on the smooth manifold $[-1, 1] \times_C M$ should be given by $\tau \cdot [x, m] = [-x, m]$ (indeed, because $[x, m] = [-x, \tau m]$, the incorrect formula in the published book describes the trivial C-action).

p.609, l.12: the last instance of M in the display line should be M_+, i.e. it should come with a subscript '+' for a disjoint basepoint.

p.612 ff: The proof of Theorem 6.2.24 contains a gap. The published argument only proves the relation

$$\mathrm{Wirth}_H^G\langle G \times_H M\rangle \ = \ \langle M\rangle \wedge \mathrm{Wirth}_H^G\langle G/H\rangle \ ,$$

which reduces the claim in the general case to the special case $M = *$.

Here is the missing argument for the special case, i.e., the relation $\mathrm{Wirth}_H^G\langle G/H\rangle = \tau_{H,L}$. We choose a G-equivariant wide smooth embedding $i : G/H \longrightarrow V$ into a G-representation. The differential at the coset eH of the embedding i is a linear embedding

$$L \ = \ T_{eH}(G/H) \ \xrightarrow{di} \ V \ ;$$

we define a scalar product on L so that this embedding becomes isometric. As before we let $W = V - (di)_{eH}(L)$ denote the orthogonal complement of the image of L. In this situation there are two different collapse maps:

- the collapse map $c : S^V \longrightarrow G \ltimes_H S^W$ defined in (3.2.10) for the construction of the external transfer isomorphism; and
- the collapse map $c_{G/H} : S^V \longrightarrow \mathbf{MGr}(V) \wedge G/H_+$ used in Construction 6.2.20 to define the normal class $\langle G/H\rangle$.

We also choose a slice as in the construction of the map l_H^G, i.e., a smooth embedding

$$s \ : \ D(L) \ \longrightarrow \ G$$

of the unit disc of L with $s(0) = 1$, such that $s(h \cdot l) = h \cdot s(l) \cdot h^{-1}$ for all $(h, l) \in H \times D(L)$, and such that the differential at 0 of the composite

$$D(L) \ \xrightarrow{\ s\ } \ G \ \xrightarrow{\text{proj}} \ G/H$$

is the identity of L. The various collapse maps participate in the following diagram of H-equivariant based maps:

$$\begin{tikzcd}
S^V \arrow[r, "c", "(3.2.10)"'] \arrow[dr, bend right, "c_{G/H}"'] & G \ltimes_H S^W \arrow[r, "l_{S^W}", "(3.2.2)"'] \arrow[d, "b"] & S^W \wedge S^L \arrow[d, "S^W \wedge t_{H,L}"'] \arrow[dd, bend left=60, "W \diamond t_{H,L}"] \\
& \mathbf{MGr}(V) \wedge G/H_+ \arrow[d, "\mathbf{MGr}(V) \wedge l"] & S^W \wedge \mathbf{MGr}(L) \wedge S^L \arrow[d, "\sigma^{\mathbf{MGr}}_{W,L} \wedge S^L"'] \\
& \mathbf{MGr}(V) \wedge S^L \arrow[r, "\mathbf{MGr}(V) \wedge \epsilon_L"'] & \mathbf{MGr}(V) \wedge S^L
\end{tikzcd}$$

Here the vertical map $b : G \ltimes_H S^W \longrightarrow \mathbf{MGr}(V) \wedge G/H_+$ is defined by $b[g, w] = (gw, gW) \wedge gH$, the map $t_{H,L}$ is the representative from (6.1.2) for the inverse Thom class $\tau_{H,L}$, and $\epsilon_L : S^L \longrightarrow S^L$ is the involution sending l to $-l$. The upper left triangle commutes on the nose, by direct inspection of the explicit formulas for the collapse maps.

We claim that the right part of the diagram commutes up to H-equivariant based homotopy. Both composites starting in $G \ltimes_H S^W$ involve the collapse map l_H^G, so both take the complement of the tubular neighborhood of H in G to the basepoint. So we may specify an H-homotopy of the two composites, precomposed with the tubular neighborhood embedding

$$D(L) \times S^W \ \longrightarrow \ G \ltimes_H S^W \ , \quad (l, w) \ \longmapsto \ [s(l), w] \ ,$$

provided the homotopy is constant on the boundary $S(L) \times S^W$. The following homotopy serves the purpose:

$$\begin{aligned} K \ : \ [0,1] \times D(L) \times S^W \ &\longrightarrow \quad \mathbf{MGr}(V) \wedge S^L \ , \\ K(x, l, w) \quad &= \ (s(xl) \cdot w, s(xl) \cdot W) \wedge \frac{-l}{1 - |l|} \end{aligned}$$

Indeed, $s(0)$ is the multiplicative unit of G, so

$$K(0, l, w) \ = \ (w, W) \wedge \frac{-l}{1 - |l|} \ = \ (W \diamond t_{H,L})((l_{S^W}[s(l), w]) \ .$$

Moreover,

$$K(1, l, w) \ = \ (s(l) \cdot w, s(l) \cdot W) \wedge \frac{-l}{1 - |l|} \ = \ (\mathbf{MGr} \wedge (\epsilon_L \circ l))(b[s(l), w]) \ .$$

By Proposition 3.2.12 (i), the composite $l_{S^W} \circ c : S^V \longrightarrow S^W \wedge S^L$ is H-equivariantly homotopic to the inverse of the H-isometry

$$W \oplus L \ \cong \ V \ , \quad (w, x) \ \longmapsto \ W + (di)_{eH}(x) \ .$$

So the previous homotopy-commutative diagram witnesses the relation

$$(\epsilon_L)_*(l_*(\mathrm{res}^G_H(\langle G/H\rangle))) \;=\; \tau_{H,L}\ .$$

By Proposition 6.1.4 (i), the involution $(\epsilon_L)_*$ of $\mathbf{MGr}^G_0(S^L)$ fixes the inverse Thom class $\tau_{H,L}$, so this is the desired relation $\mathrm{Wirth}^G_H\langle G/H\rangle = \tau_{H,L}$.

p.614, l.9: the middle term in the displayline should be $G\times_H M$ (as opposed to G), and the map into it should be $\bar{s}\times_H M$ (as opposed to $\bar{s}$). So the line should read:

$$D(L)\times M \ \xrightarrow{\bar{s}\times_H M}\ G\times_H M \ \xrightarrow{\psi}\ V\oplus W$$

p.614/615, Example 6.2.25: in the entire argument, $O(m)$ should be embedded into $O(1+m)$ as the subgroup fixing the vector $(1,0)$ of $\mathbb{R}\oplus\nu_m$. So in order to suggest the correct embedding, the notation $O(m+1)$ should be replaced by $O(1+m)$ throughout Example 6.2.25, i.e., $m+1$ should become $1+m$. Correspondingly, the map ψ should be defined as

$$\psi \;:\; O(1+m)/O(m) \;\longrightarrow\; S(\mathbb{R}\oplus\nu_m)\ ,\quad \psi(A\cdot O(m)) \;=\; A\cdot(1,0,\dots,0)\ .$$

p.620, l.-17: in the lowest displayline, the dimensional subscript to the first Grassmannian should be $\dim(V^G)-n+j$ (instead of $\dim(V^G)+j$).

p.655, l.7: it must be $z\colon C_2\longrightarrow U(1)$ (as opposed to $z\colon U(1)\longrightarrow C_2$)

Appendix A:

p.688, l.-15: source and target of the functor w must be exchanged, so that it becomes $w:\mathbf{K}\longrightarrow\mathbf{T}$

p.697, l.13: source and target of the inclusion must be exchanged, so that it becomes $\mathbf{T}\subset\mathbf{K}$

p.697, l.-14f: unlike for **Spc** <u>or</u> **K**, the forgetful functor from **T** to sets need not preserves colimits. More loosely speaking, the underlying set of <u>a</u> colimit in **T** may be smaller than one first thinks.

p.712, l.6: the reference to Conner and Floyd's book [39] should refer to Chapter VIII, so the correct reference is [39, VIII, Lemma 38.1]

p.716, l.-2: $(B.b_0)$ should be (B,b_0) (comma instead of period)

Appendix B:

p.738, l.-1: the third sentence in the proof of Proposition B.1 (ii) contains the phrase 'both colimits are formed in the ambient category of sets, so the map is bijective'. This formulation misleadingly suggest that G-fixed points commute with arbitrary filtered colimits. But for discrete groups G, this is *not* the case unless we assume that G finitely generated. However, this caveat is not relevant for our proof: we consider a filtered system of closed embeddings, which are in particular injective; and for all groups G, filtered colimits over *injective* G-maps commute with G-fixed points.

p.751, Proposition B.17: 'such that <u>the</u> G-action is free.'

References:

The volume number in reference [112] is Vol. 577 (and not Vol. 77).

The former preprints [73], [74] and [145] have been published as follows:

[73] M. Hausmann, *Symmetric spectra model global homotopy theory of finite groups.* Algebr. Geom. Topol. **19** (2019), no. 3, 1413–1452.

[74] M. Hausmann, D. Ostermayr, *Filtrations of global equivariant K-theory.* Math. Z. **295** (2020), 161–210.

[145] S. Schwede, *Orbispaces, orthogonal spaces, and the universal compact Lie group.* Math. Z. **294** (2020), 71–107.