Winter term 2025/26

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Exercises for **Algebraic Topology I**
Sheet 10

This is the last exercise sheet counting towards admission for the final exam.

Exercise 1 (*12 points*)**.** Let $n \geq 2$, let $f\colon S^{2n-1} \to S^n$ be a continuous map, and set $X = D^{2n} \cup_f S^n$. In Topology II, we defined the *Hopf invariant* of f as the unique non-negative integer h_f such that there exist generators $\alpha \in H^n(X;\mathbb{Z}) \cong \mathbb{Z}$ and $\beta \in H^{2n}(X;\mathbb{Z}) \cong \mathbb{Z}$ with $\alpha^2 = h_f\beta$.

Use the Adem relations to show that if there exists a continuous map f with Hopf invariant 1, then n is a power of 2.

Remark. It is a theorem of Adams that maps of Hopf invariant 1 in fact exist precisely for $n = 2, 4, 8$ (given by the classical Hopf maps).

Exercise 2 (*13 points*)**.** Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a pullback square in **SSet** (i.e., the given maps $A \to B$, $A \to C$, and the joint composite $A \to D$ exhibit A as the limit of the diagram $B \to D \leftarrow C$), and assume that B is finite. Show that the induced square

$$\begin{array}{ccc} |A| & \longrightarrow & |B| \\ \downarrow & & \downarrow \\ |C| & \longrightarrow & |D| \end{array}$$

is a pullback in **Top**.

Exercise 3 (*15 points + 5 bonus points*)**.** Recall from Topology I, Sheet 7 the fully faithful *nerve functor* $\mathrm{N}\colon \mathbf{Cat} \to \mathbf{SSet}$ assigning to a small category $\mathcal{C}$ the simplicial set given by $\mathrm{N}(\mathcal{C})_n = \hom_{\mathbf{Cat}}([n], \mathcal{C})$ with simplicial structure maps given via precomposition; the functoriality of N in $\mathcal{C}$ is via postcomposition. Back then, we also saw that we can identify vertices of $\mathrm{N}(\mathcal{C})$ with objects of $\mathcal{C}$ (via evaluation at the unique object $0 \in [0]$) and n-simplices for $n \geq 1$ with length n chains of composable morphisms in $\mathcal{C}$ (via evaluating at the morphisms $i \to i+1$ for $0 \leq i < n$).

$*1$. (*5 bonus points*) Let K be a simplicial set, and let $f_0\colon K_0 \to \mathrm{N}(\mathcal{C})_0$ and $f_1\colon K_1 \to \mathrm{N}(\mathcal{C})_1$ be maps of sets such that $d_i^* f_1 = f_0 d_i^*$ for $i = 0, 1$, and $s^* f_0 = f_1 s^*$. Show that there is at most one extension of f_0, f_1 to a morphism $f\colon K \to \mathrm{N}(\mathcal{C})$ of simplicial sets, and such an extension exists if and only if for every 2-simplex σ of K the relation

$$f_1(d_1^*\sigma) = f_1(d_0^*\sigma) \circ f_1(d_2^*\sigma)$$

holds in $\mathcal{C}$.

2. Use this to construct a left adjoint to $\mathrm{N}\colon \mathbf{Cat} \to \mathbf{SSet}$.
3. Show: if $(F\colon \mathscr{C} \to \mathscr{D}, G\colon \mathscr{D} \to \mathscr{C}, \epsilon\colon FG \to \mathrm{id}, \eta\colon \mathrm{id} \to GF)$ is any adjunction, then the right adjoint G is fully faithful if and only if counit $\epsilon\colon FU \to \mathrm{id}$ is a natural isomorphism.
4. Conclude that **Cat** has all small colimits.

Please hand in your solutions by **January 19, 2pm** via eCampus.